Representations of a Quantum Phase Space with General Degrees of Freedom

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Abstract

For each integer $n \geq 2$ and a parameter $\Lambda = (\theta, \eta)$ with θ and η being $n \times n$ real anti-symmetric matrices, a quantum phase space (QPS) (or a non-commutative phase space) with n degrees of freedom, denoted $\mathrm{QPS}_n(\Lambda)$, is defined, where θ and η are parameters measuring non-commutativity of the QPS. Hilbert space representations of $\mathrm{QPS}_n(\Lambda)$ are considered. A concept of quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$ is introduced. It is shown that there exists a general correspondence between representations of $\mathrm{QPS}_n(\Lambda)$ and those of the canonical commutation relations with n degrees of freedom. Irreducibility of representations of $\mathrm{QPS}_n(\Lambda)$ are investigated. A concept of Weyl representation of $\mathrm{QPS}_n(\Lambda)$ is defined. It is proved that every Weyl representation of $\mathrm{QPS}_n(\Lambda)$ on a separable Hilbert space is unitarily equivalent to a direct sum of a quasi-Schrödinger representation of the $\mathrm{QPS}_n(\Lambda)$ (a uniqueness theorem). Finally representations of $\mathrm{QPS}_n(\Lambda)$ which are not unitarily equivalent to any direct sum of a quasi-Schrödinger representation are described.

Keywords: Quantum phase space; non-commutative phase space; canonical commutation relations.

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1 Introduction

In recent years, there have been increasing interests in studying physical aspects of quantum theory on non-commutative space-times (e.g., [3, 5, 11]), non-commutative spaces (e.g., [7, 8]) and non-commutative phase spaces (e.g., [9, 10, 15, 19]). Each of these non-commutative objects are defined by a non-commutative algebra. It seems, however, that mathematically rigorous analyses of the non-commutative algebras from representation theoretic points of view have not yet fully developed. In this paper we consider Hilbert space representations of a non-commutative phase space with general finite degrees of freedom.

We denote by $\mathbb{N} = \{1, 2, \cdots\}$ the set of natural numbers. Let $n \in \mathbb{N}$ with $n \geq 2$. To define a non-commutative phase space with n degrees of freedom, we take two $n \times n$ real anti-symmetric matrices $\theta = (\theta_{jk})_{j,k=1,\cdots,n}$ and $\eta = (\eta_{jk})_{j,k=1,\cdots,n}$. Then we introduce an algebra generated by $2n$ elements $\hat{Q}_j, \hat{P}_j (j = 1, \cdots, n)$ and a unit element I obeying deformed canonical commutation relations (CCR's) with n degrees of freedom

$$[\hat{Q}_j, \hat{Q}_k] = i\theta_{jk} I, \tag{1.1}$$

$$[\hat{P}_j, \hat{P}_k] = i\eta_{jk} I, \tag{1.2}$$

$$[\hat{Q}_j, \hat{P}_k] = i\delta_{jk} I, \quad j, k = 1, \cdots, n, \tag{1.3}$$

where $[A, B] := AB - BA$, i is the imaginary unit, and δ_{jk} is the Kronecker delta. We call this algebra the *quantum phase space* (QPS) or the *non-commutative phase space with n degrees of freedom* and parameter

$$\Lambda := (\eta, \theta). \tag{1.4}$$

We denote it by $\mathrm{QPS}_n(\Lambda)$.

It is obvious that $\hat{Q}_j$ and $\hat{Q}_k$ (resp. $\hat{P}_j$ and $\hat{P}_k$) with $j \neq k$ do not commute if and only if $\theta_{jk} \neq 0$ (resp. $\eta_{jk} \neq 0$). Hence the parameter Λ "measures" the non-commutativity of $\hat{Q}_j$'s and $\hat{P}_j$'s respectively. Moreover $\mathrm{QPS}_n(\Lambda)$ in the case $\theta = \eta = 0$ reduces to the algebra of the CCR's with n degrees of freedom. Hence $\mathrm{QPS}_n(\Lambda)$ can be regarded as a deformation of the algebra of the CCR's with n degrees of freedom.

As a piece of work closely related to the present one, we mention only [19], where the following case is considered in a heuristic manner: $n = 2$,

$$\theta = \frac{a}{1 + \dfrac{ab}{4}} \epsilon, \quad \eta = \frac{b}{1 + \dfrac{ab}{4}} \epsilon$$

($a > 0$ and $b > 0$ are constants) with

$$\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.5}$$

Our QPS is a generalization of this QPS.

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$ (linear in the second variable) and norm $\| \cdot \|$. For a linear operator A, we denote its domain by $D(A)$. Let $\mathcal{D} \neq \{0\}$ be a subspace of $\mathcal{H}$ (not necessarily dense in $\mathcal{H}$) and $\hat{Q}_j, \hat{P}_j$ be symmetric operators on $\mathcal{H}$.

Definition 1.1 We say that the triple $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ is a representation (on $\mathcal{H}$) of the algebra $\mathrm{QPS}_n(\Lambda)$ if $\mathcal{D} \subset \cap_{j,k=1}^n D(\hat{Q}_j\hat{Q}_k) \cap D(\hat{P}_j\hat{P}_k) \cap D(\hat{Q}_j\hat{P}_k) \cap D(\hat{P}_j\hat{Q}_k)$ and it satisfy (1.1)–(1.3) on $\mathcal{D}$ with I being the identity on $\mathcal{H}$ (we sometimes omit the identity I below).

If all $\hat{Q}_j$ and $\hat{P}_j$ $(j = 1, \cdots, n)$ are self-adjoint, we say that the representation $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ is self-adjoint.

In every representation $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ of $\mathrm{QPS}_n(\Lambda)$, we have commutation relations (1.1)–(1.3) on $\mathcal{D}$. Hence the following Heisenberg uncertainty relations follow: for all $\psi \in \mathcal{D}$ with $\|\psi\| = 1$ and $j, k = 1, \cdots, n$,

$$(\Delta\hat{Q}_j)_\psi(\Delta\hat{Q}_k)_\psi \geq \frac{1}{2}|\theta_{jk}|, \tag{1.6}$$

$$(\Delta\hat{P}_j)_\psi(\Delta\hat{P}_k)_\psi \geq \frac{1}{2}|\eta_{jk}|, \tag{1.7}$$

$$(\Delta\hat{Q}_j)_\psi(\Delta\hat{P}_k)_\psi \geq \frac{1}{2}|\delta_{jk}|, \tag{1.8}$$

where, for a symmetric operator A and a vector $\psi \in D(A)$ with $\|\psi\| = 1$,

$$(\Delta A)_\psi := \|A - \langle\psi, A\psi\rangle\|,$$

the uncertainty of A in the vector state ψ.

The outline of the present paper is as follows. In Section 2, we introduce a concept of normality of the parameter Λ. Using the Schrödinger representation of the CCR's with n degrees of freedom, we show that there exists a general class of self-adjoint representations of $\mathrm{QPS}_n(\Lambda)$ with Λ normal. We call each of them a quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$. As a special case, we introduce a concept of Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$. We also define regularity of Λ and show that, if Λ is regular, then the Schrödinger representation of the CCR's with n degrees of freedom can be recovered from a quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$.

In Section 3, we show that there exists a general correspondence between representations of $\mathrm{QPS}_n(\Lambda)$ and those of the CCR's with n degrees of freedom.

Section 4 is concerned with irreducibility of representations of $\mathrm{QPS}_n(\Lambda)$. We formulate a sufficient condition for a representation of $\mathrm{QPS}_n(\Lambda)$ to be irreducible. As a corollary, we show that every quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$ is irreducible.

In Section 5 we define a concept of Weyl representation of $\mathrm{QPS}_n(\Lambda)$ and prove that each Weyl representation of the CCR's with n degrees freedom produces a Weyl representation of $\mathrm{QPS}_n(\Lambda)$.

In Section 6 we show that every Weyl representation of $\mathrm{QPS}_n(\Lambda)$ on a *separable* Hilbert space with Λ regular is unitarily equivalent to a direct sum of a quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$. This is a QPS version of the celebrated von Neumann uniqueness theorem on Weyl representations of the CCR's with n degrees of freedom [18].

In the last section, we consider representations of $\mathrm{QPS}_n(\Lambda)$ which are not unitarily equivalent to any direct sum of a quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$. Concrete examples of such representations of $\mathrm{QPS}_n(\Lambda)$ are given.

In Appendix, we prove some general facts on self-adjoint operators by which generated strongly continuous one-parameter unitary groups obey Weyl type relations. They may have independent interests.

It would be interesting to develop operator theoretical or spectral analyses for operators constructed from representations of $\mathrm{QPS}_n(\Lambda)$. But, in the present paper, we do not investigate these aspects.

2 A Class of Self-Adjoint Representations of $\mathrm{QPS}_n(\Lambda)$ on $L^2(\mathbb{R}^n)$

In this section, we show that there exist self-adjoint representations of $\mathrm{QPS}_n(\Lambda)$ on $L^2(\mathbb{R}^n)$, the Hilbert space consisting of equivalence classes of square integrable Borel measurable functions on $\mathbb{R}^n = \{x = (x_1, \cdots, x_n) | x_j \in \mathbb{R}, j = 1, \cdots, n\}$ ($\mathbb{R}$ is the set of real numbers). This is done by using the Schrödinger representation of the CCR's with n degrees of freedom.

We denote by $C_0^\infty(\mathbb{R}^n)$ the set of infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

Let $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{q_j, p_j\}_{j=1}^n\right)$ be the Schrödinger representation of the CCR's with n degrees of freedom, namely, q_j is the multiplication operator by the jth variable x_j on $L^2(\mathbb{R}^n)$ and $p_j := -iD_j$ with D_j being the generalized partial differential operator in x_j on $L^2(\mathbb{R}^n)$, so that

$$[q_j, p_k] = i\delta_{jk}, \tag{2.1}$$

$$[q_j, q_k] = 0, \quad [p_j, p_k] = 0, \quad j, k = 1, \cdots, n, \tag{2.2}$$

on the subspace $C_0^\infty(\mathbb{R}^n)$.

For linear operators $L_1, \cdots, L_M$ on a Hilbert space ($M \in \mathbb{N}$), the domain of the sum $\sum_{m=1}^M L_m$ is defined by

$$D\left(\sum_{m=1}^M L_m\right) := \cap_{m=1}^M D(L_m),$$

as usual, unless otherwise stated.

Lemma 2.1 *For all* $a_j, b_j \in \mathbb{R}, j = 1, \cdots, n$, $\sum_{j=1}^n (a_j p_j + b_j q_j)$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^n)$.

Proof. This fact may be well known. But, for completeness, we give a proof. Let $X := \sum_{j=1}^n (a_j p_j + b_j q_j)$. Then $C_0^\infty(\mathbb{R}^n) \subset D(X)$ and X is a symmetric operator. As is well known, the operator $N := \sum_{j=1}^n (p_j^2 + q_j^2) + 1$ is self-adjoint with $N \geq 1$ and $C_0^\infty(\mathbb{R}^n)$ is a core of N. It is easy to see that there exist constants $c, d > 0$ such that

$$\|Xf\| \leq c\|Nf\|,$$
$$|\langle Xf, Nf\rangle - \langle Nf, Xf\rangle| \leq d\|N^{1/2}f\|^2, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Thus, by the Nelson commutator theorem (s.g., [14, Theorem X.37]), X is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. □

For an n-tuple $L = (L_1, \cdots, L_n)$ of linear operators $L_j, j = 1, \cdots, n$, on a Hilbert space and an $n \times n$ matrix $A = (A_{jk})_{j,k=1,\cdots,n}$, we define the n-tuple $AL = ((AL)_1, \cdots, (AL)_n)$ of linear operators by

$$(AL)_j := \sum_{k=1}^n A_{jk} L_k. \tag{2.3}$$

We say that the parameter $\Lambda = (\theta, \eta)$ is *normal* if there exist $n \times n$ real matrices A, B, C and D satisfying

$$A\,{}^t\!D - B\,{}^t\!C = I_n, \tag{2.4}$$
$$A\,{}^t\!B - B\,{}^t\!A = \theta, \tag{2.5}$$
$$C\,{}^t\!D - D\,{}^t\!C = \eta, \tag{2.6}$$

where I_n is the $n \times n$ unit matrix and ${}^t\!A$ denotes the transposed matrix of A.

For a normal parameter Λ with (2.4)–(2.6), we can define a $(2n) \times (2n)$ matrix:

$$G := \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{2.7}$$

Let

$$K(\Lambda) := \begin{pmatrix} \theta & I_n \\ -I_n & \eta \end{pmatrix}, \quad J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{2.8}$$

Then we have

$$GJ_n{}^t\!G = K(\Lambda). \tag{2.9}$$

Conversely, if a $(2n) \times (2n)$ real matrix G of the form (2.7) satisfies (2.9), then A, B, C and D obey relations (2.4)–(2.6).

Thus Λ is normal if and only if there exists a $(2n) \times (2n)$ real matrix G satisfying (2.9). In that case, we call G a generating matrix of Λ.

We remark that, for a normal parameter Λ, its generating matrices are not unique. For example, if G is a generating matrix of Λ, then, for all orthogonal matrix M commuting with $K(\Lambda)$, MG is a generating matrix of Λ too.

Suppose that Λ is normal with (2.4)–(2.6). We set

$$\mathbf{q} = (q_1, \cdots, q_n), \quad \mathbf{p} = (p_1, \cdots, p_n) \tag{2.10}$$

and define

$$\hat{\mathbf{q}} := A\mathbf{q} + B\mathbf{p}, \quad \hat{\mathbf{p}} := C\mathbf{q} + D\mathbf{p}. \tag{2.11}$$

Then, by Lemma 2.1, the operators $\hat{q}_j$ and $\hat{p}_j$ ($j = 1, \cdots, n$) are essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. Hence their closures $\bar{\hat{q}}_j$ and $\bar{\hat{p}}_j$ are self-adjoint[1]. Moreover, we have the following result:

Theorem 2.2 *The set* $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1,\cdots,n}\right)$ *is a self-adjoint representation of* $\mathrm{QPS}_n(\Lambda)$.

Proof. It is easy to see that $\hat{q}_j$ and $\hat{p}_j$ leave $C_0^\infty(\mathbb{R}^n)$ invariant. Then, by direct computations using (2.1) and (2.2), we have

$$[\hat{q}_j, \hat{q}_k] = i \sum_{\ell=1}^{n} (A_{j\ell}B_{k\ell} - B_{j\ell}A_{k\ell}) = i(A^{\mathrm{t}}B - B^{\mathrm{t}}A)_{jk}$$

on $C_0^\infty(\mathbb{R}^n)$. By (2.5), the right hand side is equal to $i\theta_{jk}$ on $C_0^\infty(\mathbb{R}^n)$. Similarly one can prove the other cases. □

We call the representation $\left(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1,\cdots,n}\right)$ *the quasi-Schrödinger representation* of $\mathrm{QPS}_n(\Lambda)$ with generating matrix G of the form (2.7).

Remark 2.3 One can write

$$\begin{pmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \\ \hat{p}_1 \\ \vdots \\ \hat{p}_n \end{pmatrix} = G \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \tag{2.12}$$

on $\cap_{j=1}^n D(q_j) \cap D(p_j)$. Equation (2.9) is rewritten as follows:

$$GJ_n{}^{t}G = J_n + \delta(\Lambda) \tag{2.13}$$

with

$$\delta(\Lambda) := \begin{pmatrix} \theta & 0 \\ 0 & \eta \end{pmatrix}. \tag{2.14}$$

[1]For a closable linear operator T, we denote its closure by $\bar{T}$.

Hence ${}^{\mathrm{t}}G$ is symplectic if and only if $\delta(\Lambda) = 0$ (i.e., $\theta = \eta = 0$). Therefore the matrix $\delta(\Lambda)$ represents a difference from the symplectic relation. Note that the diagonal element θ (resp. η) of $\delta(\Lambda)$ gives the non-commutativity of $\hat{q}_j$'s (resp. $\hat{p}_k$'s) ($j, k = 1, \cdots, n$).

2.1 The Schrödinger representation of QPS

It may be interesting to consider a special case of Λ. Let $a \geq 0, b \geq 0$ be constants and

$$\xi := \frac{1}{\sqrt{1 + \dfrac{ab}{4}}}. \tag{2.15}$$

Let γ be an $n \times n$ real anti-symmetric matrix satisfying

$$\gamma^2 = -I_n. \tag{2.16}$$

Then the parameter

$$\Lambda_{\mathrm{S}} := (\xi^2 a\gamma, \xi^2 b\gamma) \quad (\text{the case } \theta = \xi^2 a\gamma, \eta = \xi^2 b\gamma) \tag{2.17}$$

is normal, since the matirix

$$G_{\mathrm{S}} := \begin{pmatrix} \xi I_n & -\frac{1}{2}\xi a\gamma \\ \frac{1}{2}\xi b\gamma & \xi I_n, \end{pmatrix} \tag{2.18}$$

is a generating matrix of Λ_{S}, as is easily checked. We denote $\bar{\hat{q}}_j$ and $\bar{\hat{p}}_j$ in the present case by $\hat{q}_j^{(\mathrm{S})}$ and $\hat{p}_j^{(\mathrm{S})}$ respectively:

$$\hat{q}_j^{(\mathrm{S})} := \xi\overline{\left(q_j - \frac{1}{2}a(\gamma p)_j\right)}, \quad \hat{p}_j^{(\mathrm{S})} := \xi\overline{\left(p_j + \frac{1}{2}b(\gamma q)_j\right)}, \quad j = 1, \cdots, n. \tag{2.19}$$

As is seen, this representation is simple. We call this self-adjoint representation $\Big(L^2(\mathbb{R}^n)$, $C_0^\infty(\mathbb{R}^n)$, $\{\hat{q}_j^{(\mathrm{S})}, \hat{p}_j^{(\mathrm{S})}\}_{j=1,\cdots,n}\Big)$ of $\mathrm{QPS}_n(\Lambda_{\mathrm{S}})$ the *Schrödinger representation of* $\mathrm{QPS}_n(\Lambda_{\mathrm{S}})$.

Example 2.4 Consider the case $n = 2$ and let ϵ be the 2×2 matrix defined by (1.5). Define operators $\hat{q}_j$ and $\hat{p}_j$ ($j = 1, 2$) on $L^2(\mathbb{R}^2)$ as follows:

$$\hat{q}_j := q_j \ (j = 1, 2), \quad \hat{p}_1 := \overline{p_1 + \frac{B}{2}q_2}, \quad \hat{p}_2 := \overline{p_2 - \frac{B}{2}q_1},$$

where $B \in \mathbb{R} \setminus \{0\}$ is a constant. Then we have

$$[\hat{q}_j, \hat{q}_k] = 0, \quad [\hat{p}_j, \hat{p}_k] = iB\epsilon_{jk}, \quad [\hat{q}_j, \hat{p}_k] = i\delta_{jk}, \quad j, k = 1, 2,$$

on $C_0^\infty(\mathbb{R}^2)$. Hence the set $\{\hat{q}_j, \hat{p}_j\}_{j=1}^2$ in the present example is the Schrödinger representation of $\mathrm{QPS}_2(0, B\epsilon)$ (the case $\Lambda = (0, B\epsilon)$). As is well known, this representation appears in the two dimensional quantum system with a constant magnetic field B.

2.2 Reconstruction of the Schrödinger representation of the CCR's with n degrees of freedom

In this subsection, we consider reconstruction of q_j and p_j in terms of $\hat{q}_j$ and $\hat{p}_j$. By (2.12), this problem may be reduced by the invertibility of the matrix G. From this point of view, we introduce a class of parameters Λ.

We say that Λ is *regular* if it is normal and has an invertible generating matrix. It follows from (2.9) that, if Λ is regular, then every generating matrix of Λ is invertible.

The next lemma characterizes the regularity of Λ:

Lemma 2.5 *Let Λ be normal with a generating matrix G given by (2.7). Then Λ is regular if and only if $I_n + \theta\eta$ and $I_n + \eta\theta$ are invertible. In that case, G is invertible and*

$$ {}^{\mathrm{t}}(G^{-1}) J_n G^{-1} = - \begin{pmatrix} (I_n + \eta\theta)^{-1}\eta & -(I_n + \eta\theta)^{-1} \\ (I_n + \theta\eta)^{-1} & (I_n + \theta\eta)^{-1}\theta \end{pmatrix}. \tag{2.20} $$

Proof. Throughout the proof, we set $K = K(\Lambda)$.

Suppose that Λ is regular. Then (2.9) implies that K is invertible. Let

$$ M_1 := \begin{pmatrix} \eta & -I_n \\ I_n & 0 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 0 & -I_n \\ I_n & \theta \end{pmatrix}. $$

Then M_1 and M_2 are invertible. Hence KM_1 and KM_2 are invertible. On the other hand, by direct computations, we have

$$ KM_1 = \begin{pmatrix} I_n + \theta\eta & -\theta \\ 0 & I_n \end{pmatrix}, \quad KM_2 = \begin{pmatrix} I_n & 0 \\ \eta & I_n + \eta\theta \end{pmatrix}. $$

For a square matrix M, we denote by $\det M$ the determinant of M. Then we have $0 \neq \det(KM_1) = \det(I_n + \theta\eta), 0 \neq \det(KM_2) = \det(I_n + \eta\theta)$, Thus $I_n + \theta\eta$ and $I_n + \eta\theta$ are invertible.

Conversely, suppose that $I_n + \theta\eta$ and $I_n + \eta\theta$ are invertible. By direct computations, we have

$$ K \begin{pmatrix} \eta & -I_n \\ I_n & \theta \end{pmatrix} = \begin{pmatrix} I_n + \theta\eta & 0 \\ 0 & I_n + \eta\theta \end{pmatrix}. $$

Hence

$$ \det K \det \begin{pmatrix} \eta & -I_n \\ I_n & \theta \end{pmatrix} = \det(I_n + \theta\eta) \det(I_n + \eta\theta) \neq 0, $$

Therefore $\det K \neq 0$, implying that K is invertible. Then, by (2.9), $\det G \neq 0$. Hence G is invertible. Hence Λ is regular. Using (2.9) and $J_n^{-1} = -J_n$, we have

$$ ({}^{\mathrm{t}}G)^{-1} J_n G^{-1} = -K^{-1}. $$

It is easy to see that

$$K^{-1} = \begin{pmatrix} (I_n + \eta\theta)^{-1}\eta & -(I_n + \eta\theta)^{-1} \\ (I_n + \theta\eta)^{-1} & (I_n + \theta\eta)^{-1}\theta \end{pmatrix}.$$

Thus (2.20) holds. □

Let Λ be regular with a generating matrix G. Then we can write

$$G^{-1} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}, \tag{2.21}$$

where F_1, F_2, F_3 and F_4 are $n \times n$ real matrices.

Let

$$\hat{\mathbf{q}} := (\hat{q}_1, \cdots, \hat{q}_n), \quad \hat{\mathbf{p}} := (\hat{p}_1, \cdots, \hat{p}_n). \tag{2.22}$$

Theorem 2.6 *The following equations hold:*

$$\mathbf{q} = F_1\hat{\mathbf{q}} + F_2\hat{\mathbf{p}}, \quad \mathbf{p} = F_3\hat{\mathbf{q}} + F_4\hat{\mathbf{p}}. \tag{2.23}$$

on $\cap_{j=1}^n D(q_j) \cap D(p_j)$.

Proof. By (2.12), we have

$$\begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{q}} \\ \hat{\mathbf{p}} \end{pmatrix}$$

on $\cap_{j=1}^n D(q_j) \cap D(p_j)$. Hence (2.23) on $\cap_{j=1}^n D(q_j) \cap D(p_j)$ follows. □

Theorem 2.6 also implies relations of matrix elements of G^{-1}:

Corollary 2.7

$$F_1\theta{}^{\mathrm{t}}F_1 + F_2\eta{}^{\mathrm{t}}F_2 + F_1{}^{\mathrm{t}}F_2 - F_2{}^{\mathrm{t}}F_1 = 0, \tag{2.24}$$

$$F_3\theta{}^{\mathrm{t}}F_3 + F_4\eta{}^{\mathrm{t}}F_4 + F_3{}^{\mathrm{t}}F_4 - F_4{}^{\mathrm{t}}F_3 = 0, \tag{2.25}$$

$$F_1\theta{}^{\mathrm{t}}F_3 + F_2\eta{}^{\mathrm{t}}F_4 + F_1{}^{\mathrm{t}}F_4 - F_2{}^{\mathrm{t}}F_3 = I_n. \tag{2.26}$$

Proof. Using (2.23), one needs only to compute $[q_j, q_k] = 0$ (resp. $[p_j, p_k] = 0, [q_j, p_k] = i$) on $C_0^\infty(\mathbb{R}^n)$. Then one obtains (2.24) (resp. (2.25), (2.26)). □

We now apply Theorem 2.6 to the Schrödinger representation $\{\hat{q}_j^{(\mathrm{S})}, \hat{p}_j^{(\mathrm{S})}\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda_{\mathrm{S}})$:

Corollary 2.8 *Let a, b, ξ and γ be as in Subsection 2.1. Suppose that*

$$\chi := 1 - \frac{1}{4}ab \neq 0. \tag{2.27}$$

Then

$$q_j = \frac{1}{\xi\chi}\left(\hat{q}_j^{(\mathrm{S})} + \frac{1}{2}a(\gamma\hat{p}^{(\mathrm{S})})_j\right), \tag{2.28}$$

$$p_j = \frac{1}{\xi\chi}\left(\hat{p}_j^{(\mathrm{S})} - \frac{1}{2}b(\gamma\hat{q}^{(\mathrm{S})})_j\right), \quad j = 1, \cdots, n, \tag{2.29}$$

on $C_0^\infty(\mathbb{R}^n)$.

Proof. In the present case, we have

$$I_n + \theta\eta = I_n + \eta\theta = (1 - \xi^4 ab)I_n = \xi^4\chi^2 \neq 0.$$

Hence $I_n + \theta\eta$ and $I_n + \eta\theta$ are invertible. By (2.18), we have

$$G_{\mathrm{S}}^{-1} = \frac{1}{\xi\chi}\begin{pmatrix} I_n & \frac{1}{2}a\gamma \\ -\frac{1}{2}b\gamma & I_n \end{pmatrix}.$$

Thus (2.28) and (2.29) follow. □

3 General Correspondence Between a Representation of $\mathrm{QPS}_n(\Lambda)$ and a Representation of the CCR's with n Degrees of Freedom

3.1 Construction of a representation of $\mathrm{QPS}_n(\Lambda)$ from a representation of the CCR's with n degrees of freedom

The contents in Section 2 suggest a general method to construct a representation of $\mathrm{QPS}_n(\Lambda)$ from a representation of the CCR's with n degrees of freedom.

Let $\left(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n\right)$ be a representation of the CCR's with n degrees of freedom, namely, $\mathcal{H}$ is a Hilbert space, $\mathcal{D}$ is a dense subspace of $\mathcal{H}$ and Q_j and P_j ($j = 1, \cdots, n$) are symmetric operators on $\mathcal{H}$ such that $\mathcal{D} \subset \cap_{j,k=1}^n D(Q_jQ_k) \cap D(P_jP_k) \cap D(Q_jP_k) \cap D(P_kQ_j)$ and $\{Q_j, P_j\}_{j=1}^n$ obeys the CCR's with n degrees of freedom on $\mathcal{D}$: for $j, k = 1, \cdots, n$,

$$[Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad [Q_j, P_k] = i\delta_{jk} \tag{3.1}$$

on $\mathcal{D}$. Let

$$\mathbf{Q} = (Q_1, \cdots, Q_n), \quad \mathbf{P} = (P_1, \cdots, P_n).$$

Let Λ be normal and A, B, C, D be $n \times n$ real matrices obeying (2.4)–(2.6). By an analogy with (2.11), we define the n-tuples

$$\hat{\mathbf{Q}} := (\hat{Q}_1, \cdots, \hat{Q}_n), \tag{3.2}$$

and

$$\hat{\mathbf{P}} := (\hat{P}_1, \cdots, \hat{P}_n), \tag{3.3}$$

by

$$\hat{\mathbf{Q}} := A\mathbf{Q} + B\mathbf{P}, \quad \hat{\mathbf{P}} := C\mathbf{Q} + D\mathbf{P}. \tag{3.4}$$

Theorem 3.1 *The set* $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ *defined by (3.4) is a representation of* $\mathrm{QPS}_n(\Lambda)$.

Proof. The symmetry of $\hat{Q}_j$ and $\hat{P}_j$ follows from the density of $\mathcal{D}$ and the symmetry of Q_j and P_j $(j = 1, \cdots, n)$. Commutation relations (1.1)–(1.3) can be proved by direct computations. □

We remark that the representation $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ of $\mathrm{QPS}_n(\Lambda)$ is not necessarily self-adjoint even in the case where all Q_j and P_j $(j = 1, \cdots, n)$ are self-adjoint.

As in the case of quasi-Schrödinger representations of $\mathrm{QPS}_n(\Lambda)$ discussed in Section 2, we have the following fact:

Theorem 3.2 *Let* Λ *be regular with generating matrix* G *given by (2.7) and* F_1, F_2, F_3 *and* F_4 *be as in (2.21). Then*

$$\mathbf{Q} = F_1\hat{\mathbf{Q}} + F_2\hat{\mathbf{P}}, \tag{3.5}$$

$$\mathbf{P} = F_3\hat{\mathbf{Q}} + F_4\hat{\mathbf{P}}. \tag{3.6}$$

on $\mathcal{D}$.

3.2 Construction of a representation of the CCR's with n degrees of dreedom from a representation of $\mathrm{QPS}_n(\Lambda)$

We next consider constructing a representation of the CCR's with n degrees of freedom from a representation of $\mathrm{QPS}_n(\Lambda)$. A method for that is suggested by Theorem 3.2.

Let $\left(\mathcal{H}, \mathcal{D}, \{\hat{Q}_j, \hat{P}_j\}_{j=1}^n\right)$ be a representation of $\mathrm{QPS}_n(\Lambda)$ on a Hilbert space $\mathcal{H}$ with $\mathcal{D}$ dense in $\mathcal{H}$. Throughout this subsection, we assume the following:

(A) The parameter Λ is regular with generating matrix G given by (2.7).

Let F_1, F_2, F_3 and F_4 be as in (2.21). Then we can define $\mathbf{Q}(\Lambda) = (Q_1(\Lambda), \cdots, Q_n(\Lambda))$ and $\mathbf{P}(\Lambda) = (P_1(\Lambda), \cdots, P_n(\Lambda))$ by

$$\mathbf{Q}(\Lambda) := F_1\hat{\mathbf{Q}} + F_2\hat{\mathbf{P}}, \tag{3.7}$$

$$\mathbf{P}(\Lambda) := F_3\hat{\mathbf{Q}} + F_4\hat{\mathbf{P}}. \tag{3.8}$$

Theorem 3.3 *Assume (A). Then* $\left(\mathcal{H}, \mathcal{D}, \{Q_j(\Lambda), P_j(\Lambda)\}_{j=1}^n\right)$ *is a representation of the CCR's with* n *degrees of freedom.*

Proof. The symmetry of $Q_j(\Lambda)$ and $P_j(\Lambda)$ is obvious. By direct computations using (1.1)–(1.3) and (2.24)–(2.26), one can show that $Q_j(\Lambda)$'s and $P_k(\Lambda)$'s satisfy the CCR's with n degrees of freedom. □

The next theorem shows that every representation of $\mathrm{QPS}_n(\Lambda)$ with condition (A) comes from a representation of the CCR's with n degrees of freedom:

Theorem 3.4 *Assume (A). Let* $\mathbf{Q}(\Lambda)$ *and* $\mathbf{P}(\Lambda)$ *be defined by (3.7) and (3.8) respectively. Then*

$$\hat{\mathbf{Q}} = A\mathbf{Q}(\Lambda) + B\mathbf{P}(\Lambda), \quad \hat{\mathbf{P}} = C\mathbf{Q}(\Lambda) + D\mathbf{P}(\Lambda) \tag{3.9}$$

on $\mathcal{D}$.

Proof. Direct computations. □

4 Irreducibility

For a Hilbert space $\mathcal{H}$, we denote by $\mathsf{B}(\mathcal{H})$ the set of all bounded linear operators B on $\mathcal{H}$ with $D(B) = \mathcal{H}$. Let A be a linear operator on $\mathcal{H}$. We say that A *strongly commutes* with $B \in \mathsf{B}(\mathcal{H})$ if $BA \subset AB$ (i.e., for all $\psi \in D(A)$, $B\psi \in D(A)$ and $BA\psi = AB\psi$). For a set A of linear operators on $\mathcal{H}$, we define

$$\mathsf{A}' := \{B \in \mathsf{B}(\mathcal{H}) | BA \subset AB, \forall A \in \mathsf{A}\}. \tag{4.1}$$

We call A' the *strong commutant* of A.

We say that A is *irreducible* if $\mathsf{A}' = \{cI | c \in \mathbb{C}\}$ ($\mathbb{C}$ is the set of complex numbers).

Lemma 4.1 *Let* S *be a self-adjoint operator on a Hilbert space* $\mathcal{H}$ *and* $B \in \mathsf{B}(\mathcal{H})$ *such that* $BS \subset SB$. *Then, for all* $t \in \mathbb{R}$, $Be^{itS} = e^{itS}B$.

Proof. Let $C^\infty(S) := \cap_{n=1}^\infty D(S^n)$. Then, for all $\psi \in C^\infty(S)$ and all $n \in \mathbb{N}$, $B\psi$ is in $D(S^n)$ and $BS^n\psi = S^nB\psi$. Let $E_S(\cdot)$ be the spectral measure of S and

$$\mathcal{D}_0 := \cup_{a \geq 0} \mathrm{Ran}\,(E_S([-a, a])),$$

where, for a linear operator A, $\mathrm{Ran}\,(A)$ denotes the range of A. Then it is easy to see that $\mathcal{D}_0$ is a dense subspace of $\mathcal{H}$ satisfying $\mathcal{D}_0 \subset C^\infty(S)$. For all $\phi, \psi \in \mathcal{D}_0$, $t \in \mathbb{R}$ and $N \in \mathbb{N}$, we have

$$\left\langle B^*\phi, \sum_{n=0}^N \frac{(itS)^n}{n!}\psi \right\rangle = \left\langle \sum_{n=0}^N \frac{(-itS)^n}{n!}\phi, B\psi \right\rangle.$$

Employing spectral representations on S and the Lebesgue dominated convergence theorem to take the limit $N \to \infty$, we obtain $\langle B^*\phi, e^{itS}\psi \rangle = \langle e^{-itS}\phi, B\psi \rangle$, which implies that $Be^{itS}\psi = e^{itS}B\psi$. Since $\mathcal{D}_0$ is dense, the operator equality $Be^{itS} = e^{itS}B$ follows. $\square$

Theorem 4.2 *Assume (A) in Subsection 3.2. Let $\left(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n\right)$ be a representation of the CCR's with n degrees of freedom. Suppose that, for each $j = 1, \cdots, n$, Q_j and P_j are essentially self-adjoint on $\mathcal{D}$ and $\{\bar{Q}_j, \bar{P}_j\}_{j=1}^n$ is irreducible. Then the representation $\left(\mathcal{H}, \mathcal{D}, \{\bar{\hat{Q}}_j, \bar{\hat{P}}_j\}_{j=1}^n\right)$ of* $\mathrm{QPS}_n(\Lambda)$ *given by (3.4) is irreducible.*

Proof. Let $B \in \mathsf{B}(\mathcal{H})$ such that $B\bar{\hat{Q}}_j \subset \bar{\hat{Q}}_j B, \quad B\bar{\hat{P}}_j \subset \bar{\hat{P}}_j B, j = 1, \cdots, n$. Let

$$R_j := \sum_{k=1}^{n} \left((F_1)_{jk}\bar{\hat{Q}}_k + (F_2)_{jk}\bar{\hat{P}}_k \right).$$

Then, $BR_j \subset R_jB$. This implies that $B\bar{R}_j \subset \bar{R}_jB$. On the other hand, by Theorem 3.2, we have $Q_j|\mathcal{D} \subset R_j$. By this fact and the essential self-adjointness of Q_j on $\mathcal{D}$, we have $\bar{Q}_j = \bar{R}_j$. Therefore $B\bar{Q}_j \subset \bar{Q}_jB$. Similarly we can show that $B\bar{P}_j \subset \bar{P}_jB$. It follows from the irreducibility of $\{\bar{Q}_j, \bar{P}_j\}_{j=1}^n$ that $B = cI$ with some $c \in \mathbb{C}$. Thus the desired result follows. $\square$

We can apply Theorem 4.2 to the quasi-Schrödinger representation $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$ discussed in Section 2.

Theorem 4.3 *Assume (A). Then the representation $(L^2(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n), \{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n)$ of* $\mathrm{QPS}_n(\Lambda)$ *is irreducible.*

Proof. We need only to apply Theorem 4.2 to the case where $\mathcal{H} = L^2(\mathbb{R}^n)$, $\mathcal{D} = C_0^\infty(\mathbb{R}^n)$, $Q_j = q_j, P_j = p_j$, $\hat{Q}_j = \hat{q}_j$ and $\hat{P}_j = \hat{p}_j$. It is well known that q_j and p_j are essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ and $\{q_j, p_j\}_{j=1}^n$ is irreducible. Hence, in the present case, the assumption of Theorem 4.2 is satisfied. $\square$

5 Weyl Representations of $\mathrm{QPS}_n(\Lambda)$

5.1 Definition and a basic fact

As is well known, a Weyl representation of the CCR's with n degrees of freedom on a Hilbert space $\mathcal{H}$ is defined to be a set $\{Q_j, P_j\}_{j=1}^n$ of $2n$ self-adjoint operators on $\mathcal{H}$ obeying the Weyl relations:

$$e^{itQ_j}e^{isP_k} = e^{-ist\delta_{jk}}e^{isP_k}e^{itQ_j}, \tag{5.1}$$

$$e^{itQ_j}e^{isQ_k} = e^{isQ_k}e^{itQ_j}, \tag{5.2}$$

$$e^{itP_j}e^{isP_k} = e^{isP_k}e^{itP_j}, \quad j, k = 1, \cdots, n, s, t \in \mathbb{R}. \tag{5.3}$$

Based on an analogy with Weyl representations of CCR's, we introduce a concept of Weyl representation of $\mathrm{QPS}_n(\Lambda)$.

Definition 5.1 Let $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ be a set of self-adjoint operators on a Hilbert space $\mathcal{H}$. We say that $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ is a *Weyl representation of* $\mathrm{QPS}_n(\Lambda)$ if

$$e^{it\hat{Q}_j}e^{is\hat{P}_k} = e^{-ist\delta_{jk}}e^{is\hat{P}_k}e^{it\hat{Q}_j}, \tag{5.4}$$

$$e^{it\hat{Q}_j}e^{is\hat{Q}_k} = e^{-ist\theta_{jk}}e^{is\hat{Q}_k}e^{it\hat{Q}_j}, \tag{5.5}$$

$$e^{it\hat{P}_j}e^{is\hat{P}_k} = e^{-ist\eta_{jk}}e^{is\hat{P}_k}e^{it\hat{P}_j}, \quad j,k = 1,\cdots,n, s,t \in \mathbb{R}. \tag{5.6}$$

We call these relations the *deformed Weyl relations* with parameter Λ.

One can write relations (5.4)–(5.6) in simpler form. Let

$$\hat{A}_j := \begin{cases} \hat{Q}_j \ ; & j = 1,\cdots,n \\ \hat{P}_{j-n} \ ; & j = n+1,\cdots,2n \end{cases} \tag{5.7}$$

and

$$\alpha_{jk} := \begin{cases} \theta_{jk} \ ; & j,k = 1,\cdots,n \\ \eta_{(j-n)(k-n)} \ ; & j,k = n+1,\cdots,2n \\ \delta_{j(k-n)} \ ; & j = 1,\cdots,n; k = n+1,\cdots,2n \\ -\delta_{k(j-n)} \ ; & j = n+1,\cdots,2n; k = 1,\cdots,n. \end{cases} \tag{5.8}$$

Then (5.4)–(5.6) are equivalent to the following relations:

$$e^{it\hat{A}_j}e^{is\hat{A}_k} = e^{-ist\alpha_{jk}}e^{is\hat{A}_k}e^{it\hat{A}_j}, \quad j,k = 1,\cdots,2n. \tag{5.9}$$

For a linear operator A on a Hilbert space, we denote its spectrum by $\sigma(A)$.

Proposition 5.2 *Let $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ be a Weyl representation of $\mathrm{QPS}_n(\Lambda)$. Then it is a self-adjoint representation of $\mathrm{QPS}_n(\Lambda)$. Moreover, for each $j = 1,\cdots,n$, $\hat{Q}_j$ and $\hat{P}_j$ are purely absolutely continuous with*

$$\sigma(\hat{Q}_j) = \mathbb{R}, \quad \sigma(\hat{P}_j) = \mathbb{R}, \quad j = 1,\cdots,n. \tag{5.10}$$

Proof. By (5.9), we can apply the results described in Appendix of the present paper. In the present context, we need only to take $N = 2n$, $a_{jk} = \alpha_{jk}$ and $A_j = \hat{A}_j$. By Proposition A.4-(iii) and Corollary A.5, there exists a dense subsapce $\mathcal{D}_0$ such that $\mathcal{D}_0 \subset \cap_{\ell_j \geq 0, j=1,\cdots,2n} D(\hat{A}_1^{\ell_1}\hat{A}_2^{\ell_2}\cdots\hat{A}_{2n}^{\ell_{2n}})$ and

$$[\hat{A}_j, \hat{A}_k] = i\alpha_{jk} \tag{5.11}$$

on $\mathcal{D}_0$. This implies (1.1)–(1.3) on $\mathcal{D}_0$. Thus the first half of the proposition holds. The second half follows from Proposition A.1. □

Remark 5.3 The converse of Proposition 5.2 does not hold. As we shall show later, there exists a self-adjoint representation of $\mathrm{QPS}_n(\Lambda)$ which is not a Weyl one.

Proposition 5.4 *The set* $\{e^{it\hat{Q}_j}, e^{it\hat{P}_j} | t \in \mathbb{R}, j = 1, \cdots, n\}$ *is irreducible if and only if so is* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$.

Proof. A simple application of Corollary A.8. □

5.2 A review of Weyl representations of CCR's

We say that two self-adjoint operators A and B on a Hilbert space strongly commute if their spectral measures commute. A set $\{S_1, \cdots, S_n\}$ of self-adjoint operators on a Hilbert space is said to be strongly commuting if, for all $j, k = 1, \cdots, n$ with $j \neq k$, S_j and S_k strongly commute.

For an n-tuple $\mathbf{L} = (L_1, \cdots, L_n)$ of linear operators L_j $(j = 1, \cdots, n)$ on a Hilbert sapce and $\mathbf{a} \in \mathbb{R}^n$, we define

$$\mathbf{a} \cdot \mathbf{L} := \sum_{j=1}^n a_j L_j.$$

Let $\{Q_j, P_j\}_{j=1}^n$ be a Weyl representation of the CCR's with n degrees of freedom on a Hilbert space $\mathcal{H}$ and define operators T_j as follows:

$$T_j := \begin{cases} Q_j \ ; & j = 1, \cdots, n \\ P_{j-n} \ ; & j = n+1, \cdots, 2n \end{cases} \tag{5.12}$$

and

$$\Delta_{jk} := \begin{cases} 0 \ ; & j, k = 1, \cdots, n \\ 0 \ ; & j, k = n+1, \cdots, 2n \\ \delta_{j(k-n)} \ ; & j = 1, \cdots, n; k = n+1, \cdots, 2n \\ -\delta_{k(j-n)} \ ; & j = n+1, \cdots, 2n; k = 1, \cdots, n \end{cases} \tag{5.13}$$

Then (5.1)–(5.3) are equivalent to the following relations:

$$e^{itT_j} e^{isT_k} = e^{-ist\Delta_{jk}} e^{isT_k} e^{itT_j}, \quad j, k = 1, \cdots, 2n. \tag{5.14}$$

Hence we can apply the facts proved in Appendix of the present paper to prove the following lemma:

Lemma 5.5 *For all* $\mathbf{a} \in \mathbb{R}^n$ *and* $\mathbf{b} \in \mathbb{R}^n$, *the operator*

$$\phi_{\mathbf{a},\mathbf{b}} := \overline{\mathbf{a} \cdot \mathbf{Q} + \mathbf{b} \cdot \mathbf{P}}. \tag{5.15}$$

is self-adjoint and

$$e^{i\phi_{\mathbf{a},\mathbf{b}}} = e^{i\mathbf{a}\cdot\mathbf{b}/2} e^{i\overline{\mathbf{a}\cdot\mathbf{Q}}} e^{i\overline{\mathbf{b}\cdot\mathbf{P}}} = e^{i\langle\mathbf{a},\mathbf{b}\rangle/2} \left(\prod_{j=1}^n e^{ia_j Q_j}\right) \left(\prod_{j=1}^n e^{ib_j P_j}\right). \tag{5.16}$$

Proof. We need only to apply Theorem A.6 with $N = 2n$ and $A_j = T_j$; for (5.16), we use also the strong commutativity of $\{Q_j\}_{j=1}^n$ (resp. $\{P_j\}_{j=1}^n$) which follows from(5.2) (resp. (5.3))[2]. □

Lemma 5.6 *For all* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$,

$$e^{i\phi_{\mathbf{a},\mathbf{b}}} e^{i\phi_{\mathbf{c},\mathbf{d}}} = e^{-i(\mathbf{a}\cdot\mathbf{d} - \mathbf{b}\cdot\mathbf{c})} e^{i\phi_{\mathbf{c},\mathbf{d}}} e^{i\phi_{\mathbf{a},\mathbf{b}}}. \tag{5.17}$$

Proof. One needs only to use (5.16) and (5.1)–(5.3). □

5.3 Construction of a Weyl representation of $\mathrm{QPS}_n(\Lambda)$ from a Weyl representation of the CCR's with n degrees of freedom

Theorem 5.7 *Let $\{Q_j, P_j\}_{j=1}^n$ be a Weyl representation of the CCR's with n degrees of freedom. Let Λ be normal with a generating matrix G of the form (2.7) and and let $\hat{Q}_j$ and $\hat{P}_j$ be defined by (3.4). Then $\{\overline{\hat{Q}}_j, \overline{\hat{P}}_j\}_{j=1}^n$ is a Weyl representation of* $\mathrm{QPS}_n(\Lambda)$.

Proof. By Lemma 5.6, we have

$$e^{it\bar{\hat{Q}}_j} e^{is\bar{\hat{Q}}_k} = e^{-ist((A^{\mathrm{t}}B)_{jk} - (B^{\mathrm{t}}A)_{jk})} e^{is\bar{\hat{Q}}_k} e^{it\bar{\hat{Q}}_j}.$$

By (2.5), we have

$$(A^{\mathrm{t}}B)_{jk} - (B^{\mathrm{t}}A)_{jk} = \theta_{jk}.$$

Hence (5.5) holds. Similarly one can prove (5.4) and (5.5). □

It is well known that the Schrödinger representation $\{q_j, p_j\}_{j=1}^n$ of the CCR's with n degrees of freedom is an irreducible Weyl representation. Hence Theorem 5.7 immediately leads us to the following fact:

Corollary 5.8 *Let Λ be normal with a generating matrix G of the form (2.7). Then the representation $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ of* $\mathrm{QPS}_n(\Lambda)$ *is an irreducible Weyl representation of* $\mathrm{QPS}_n(\Lambda)$.

6 Uniqueness Theorems on Weyl Representations of $\mathrm{QPS}_n(\Lambda)$

In this section we prove that, for each regular parameter Λ, every Weyl representation of $\mathrm{QPS}_n(\Lambda)$ on a *separable* Hilbert space is unitarily equivalent to a direct sum of a quasi-Schrödinger representation $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$.

[2] An application of a criterion for strong commutativity of self-adjoint operators (e.g., [13, Theorem VIII.13]).

Theorem 6.1 *Assume (A). Let* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ *be a Weyl representation of* $\mathrm{QPS}_n(\Lambda)$ *on a separable Hilbert space* $\mathcal{H}$. *Then there exist closed subspaces* $\mathcal{H}_\ell$ *such that the following (i)–(iii) hold:*

(i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$ *(N is a positive integer or* ∞*).*

(ii) For each $j = 1, \cdots, n$, $\hat{Q}_j$ *and* $\hat{P}_j$ *are reduced by each* $\mathcal{H}_\ell, \ell = 1, \cdots, N$. *We denote by* $\hat{Q}_j^{(\ell)}$ *(resp.* $\hat{P}_j^{(\ell)}$*) the reduced part of* $\hat{Q}_j$ *(resp.* $\hat{P}_j$*) to* $\mathcal{H}_\ell$.

(iii) For each ℓ, *there exists a unitary operator* $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n)$ *such that*

$$U_\ell \hat{Q}_j^{(\ell)} U_\ell^{-1} = \bar{\hat{q}}_j, \quad U_\ell \hat{P}_j^{(\ell)} U_\ell^{-1} = \bar{\hat{p}}_j, \quad j = 1, \cdots, n, \tag{6.1}$$

where $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ *is the quasi-Schrödinger representation of* $\mathrm{QPS}_n(\Lambda)$ *defined by (2.11).*

Proof. We define $Q_j(\Lambda)$ and $P_j(\Lambda)$ by (3.7) and (3.8) respectively. For simplicity, we put $Q_j := Q_j(\Lambda)$ and $P_j := P_j(\Lambda)$ throughout the proof. Note that Q_j and P_j can be written as follows:

$$Q_j = \sum_{k=1}^{2n} c_{jk} \hat{A}_k, \quad P_j = \sum_{k=1}^{2n} d_{jk} \hat{A}_k, \quad j = 1, \cdots, n,$$

where $\hat{A}_j$ is defined by (5.7) and

$$c_{jk} := \begin{cases} (F_1)_{jk} & ;\ k = 1, \cdots, n \\ (F_2)_{j(k-n)} & ;\ k = n+1, \cdots, 2n \end{cases}, \tag{6.2}$$

$$d_{jk} := \begin{cases} (F_3)_{jk} & ;\ k = 1, \cdots, n \\ (F_4)_{j(k-n)} & ;\ k = n+1, \cdots, 2n \end{cases} \tag{6.3}$$

Hence, by an application of Theorem A.6, Q_j and P_j are essentially self-adjoint and

$$e^{it\bar{Q}_j} = e^{it^2 \sum_{k<\ell}^{2n} \alpha_{k\ell} c_{jk} c_{j\ell}/2} e^{itc_{j1}\hat{A}_1} \cdots e^{itc_{j(2n)}\hat{A}_{2n}}, \tag{6.4}$$

$$e^{it\bar{P}_j} = e^{it^2 \sum_{k<\ell}^{2n} \alpha_{k\ell} d_{jk} d_{j\ell}/2} e^{itd_{j1}\hat{A}_1} \cdots e^{itd_{j(2n)}\hat{A}_{2n}}, \quad t \in \mathbb{R}. \tag{6.5}$$

Using (5.9), we have for all $t, s \in \mathbb{R}$

$$e^{it\bar{Q}_j} e^{is\bar{Q}_k} = e^{-its \sum_{h,g=1}^{2n} \alpha_{hg} c_{jh} c_{kg}} e^{is\bar{Q}_k} e^{it\bar{Q}_j}.$$

But, by the anti-symmetry of α_{hg} in h and g, $\sum_{h,g=1}^{2n} \alpha_{hg} c_{jh} c_{kg} = 0$. Hence

$$e^{it\bar{Q}_j} e^{is\bar{Q}_k} = e^{is\bar{Q}_k} e^{it\bar{Q}_j}.$$

Similarly we can show that

$$e^{it\bar{P}_j} e^{is\bar{P}_k} = e^{is\bar{P}_k} e^{it\bar{P}_j}.$$

As for $e^{it\bar{Q}_j}e^{is\bar{P}_k}$, we have

$$e^{it\bar{Q}_j}e^{is\bar{P}_k} = e^{-itsM_{jk}}e^{is\bar{P}_k}e^{it\bar{Q}_j},$$

where

$$M := F_1\theta^{\mathrm{t}}F_3 + F_1{}^{\mathrm{t}}F_4 - F_2{}^{\mathrm{t}}F_3 + F_2\eta^{\mathrm{t}}F_4.$$

By (2.26), $M = I_n$. Hence

$$e^{it\bar{Q}_j}e^{is\bar{P}_k} = e^{-its\delta_{jk}}e^{is\bar{P}_k}e^{it\bar{Q}_j}.$$

Thus $e^{it\bar{Q}_j}$ and $e^{is\bar{P}_k}$ $(s, t \in \mathbb{R}, j, k = 1, \cdots, n)$ obey the Weyl relations with n degrees of freedom. Namely $\{\bar{Q}_j, \bar{P}_j\}_{j=1}^n$ is a Weyl representation of the CCR's with n degrees of freedom. Hence, by the von Neumann uniqueness theorem (e.g.,[13, Theorem VIII.14]), there exist closed subspaces $\mathcal{H}_\ell$ such that the following (i)–(iii) hold:

(i) $\mathcal{H} = \oplus_{\ell=1}^N \mathcal{H}_\ell$ (N is a positive integer or ∞).

(ii) For each $j = 1, \cdots, n$ and all $t \in \mathbb{R}$, $e^{it\bar{Q}_j}$ and $e^{it\bar{P}_j}$ leave each $\mathcal{H}_\ell$ invariant $(\ell = 1, \cdots, N)$.

(iii) For each ℓ, there exists a unitary operator $U_\ell : \mathcal{H}_\ell \to L^2(\mathbb{R}^n)$ such that

$$U_\ell e^{it\bar{Q}_j}U_\ell^{-1} = e^{itq_j}, \quad U_\ell e^{it\bar{P}_j}U_\ell^{-1} = e^{itp_j}, \quad t \in \mathbb{R}, j = 1, \cdots, n, \tag{6.6}$$

By (3.9) and (5.16), we have

$$e^{it\hat{Q}_j} = e^{it^2\sum_{h=1}^n A_{jh}B_{jh}/2}\left(\prod_{h=1}^n e^{itA_{jh}\bar{Q}_h}\right)\left(\prod_{h=1}^n e^{itB_{jh}\bar{P}_h}\right), \tag{6.7}$$

$$e^{it\hat{P}_j} = e^{it^2\sum_{h=1}^n C_{jh}D_{jh}/2}\left(\prod_{h=1}^n e^{itC_{jh}\bar{Q}_h}\right)\left(\prod_{h=1}^n e^{itD_{jh}\bar{P}_h}\right), \quad t \in \mathbb{R}. \tag{6.8}$$

Hence $e^{it\hat{Q}_j}$ and $e^{it\hat{P}_j}$ leave $\mathcal{H}_\ell$ invariant $(\ell = 1, \cdots, n)$. Therefore $\hat{Q}_j$ and $\hat{P}_j$ are reduced by each $\mathcal{H}_\ell$. We denote the reduced part of $\hat{Q}_j$ (resp. $\hat{P}_j$) to $\mathcal{H}_\ell$ by $\hat{Q}_j^{(\ell)}$ (resp. $\hat{P}_j^{(\ell)}$). Then, by (6.6)–(6.8), we have

$$U_\ell e^{it\hat{Q}_j^{(\ell)}}U_\ell^{-1} = e^{it\bar{\hat{q}}_j}, \quad U_\ell e^{it\hat{P}_j^{(\ell)}}U_\ell^{-1} = e^{it\bar{\hat{p}}_j}.$$

Thus (6.1) follows. □

Theorem 6.1 tells us that, under the assumption there, every Weyl representation $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$ is unitarily equivalent to a direct sum of the quasi-Schrödinger representation $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$, because the operator

$$U := \oplus_{\ell=1}^N U_\ell : \mathcal{H} \to \oplus^N L^2(\mathbb{R}^n),$$

is unitary and

$$U\hat{Q}_jU^{-1} = \oplus^N \bar{\hat{q}}_j, \quad U\hat{P}_jU^{-1} = \oplus^N \bar{\hat{p}}_j.$$

Theorem 6.1 and the irreducibility of the representation $\{\bar{\hat{q}}_j, \bar{\hat{p}}_j\}_{j=1}^n$ (Corollary 5.8) immediately lead us to the following fact:

Corollary 6.2 *Assume (A). Let* $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ *be an irreducible Weyl representation of* $\mathrm{QPS}_n(\Lambda)$ *on a separable Hilbert space* $\mathcal{H}$. *Then there exists a unitary operator* $W : \mathcal{H} \to L^2(\mathbb{R}^n)$ *such that*

$$W\hat{Q}_jW^{-1} = \bar{\hat{q}}_j, \quad W\hat{P}_jW^{-1} = \bar{\hat{p}}_j, \quad j = 1, \cdots, n.$$

Applying this corollary to the case where $\{\hat{Q}_j, \hat{P}_j\}_{j=1}^n$ is a quasi-Schrödinger representation of $\mathrm{QPS}_n(\Lambda)$, we obtain the following result:

Corollary 6.3 *Let* Λ *be regular. Let* G *and* G' *be two generating matrices of* Λ*:* G *is given by (2.7) and*

$$G' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},$$

where A', B', C' *and* D' *are* $n \times n$ *real matrices. Let* $\{\bar{\hat{q}}'_j, \bar{\hat{p}}'_j\}_{j=1}^n$ *be the quasi-Schrödinegr representation of* $\mathrm{QPS}_n(\Lambda)$ *with generating matrix* G'*:*

$$\hat{\mathbf{q}}' := A'\mathbf{q} + B'\mathbf{p}, \quad \hat{\mathbf{p}}' = C'\mathbf{q} + D'\mathbf{p}.$$

Then there exists a unitary operator $V : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ *such that*

$$V\bar{\hat{q}}'_jV^{-1} = \bar{\hat{q}}_j, \quad V\bar{\hat{p}}'_jV^{-1} = \bar{\hat{p}}_j, \quad j = 1, \cdots, n. \tag{6.9}$$

Corollary 6.3 shows that, for each regular parameter Λ, quasi-Schrödinger representations of $\mathrm{QPS}_n(\Lambda)$ are unique up to unitary equivalences.

7 Non-Quasi-Schrödinger Representations of QPS

From representation theoretic points of view, it is interesting to investigate if there exists a self-adjoint representation of $\mathrm{QPS}_n(\Lambda)$ which is not unitarily equivalent to any direct sum of a quasi-Schrödinger representation $\{\hat{q}_j, \hat{p}_j\}_{j=1}^n$ of $\mathrm{QPS}_n(\Lambda)$. In this section, we show that there exist such representations of $\mathrm{QPS}_n(\Lambda)$.

We say that a representation of $\mathrm{QPS}_n(\Lambda)$ is *non-quasi-Schrödinger* (resp. *non-Schrödinger*) if it is not unitarily equivalent to any direct sum of a quasi-Schrödinegr (resp. the Schrödinegr) representation $\{\hat{q}_j, \hat{p}_j\}_{j=1}^n$ (resp. $\{\hat{q}_j^{(\mathrm{S})}, \hat{p}_j^{(\mathrm{S})}\}_{j=1}^n$) of $\mathrm{QPS}_n(\Lambda)$ (resp. $\mathrm{QPS}_n(\Lambda_{\mathrm{S}})$).

7.1 A general case

Let $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^n)$ be a self-adjoint representation of the CCR's with n degrees of freedom on a Hilbert space $\mathcal{H}$ such that $\mathcal{D}$ is dense in $\mathcal{H}$ and a common core of Q_j and P_j $(j = 1, \cdots, n)$. Let $\hat{Q}_j$ and $\hat{P}_j$ be defined by (3.4) $(j = 1, \cdots, n)$.

Theorem 7.1 *Assume (A). Suppose that* $\{Q_j, P_j\}_{j=1}^n$ *is not unitarily equivalent to any direct sum of the Schrödinger representation* $\{q_j, p_j\}_{j=1}^n$. *Then the representation* $\{\overline{\hat{Q}}_j, \overline{\hat{P}}_j\}_{j=1}^n$ *of* $\mathrm{QPS}_n(\Lambda)$ *is non-quasi-Schrödinger.*

Proof. We have (3.5) and (3.6) on $\mathcal{D}$. Since $\mathcal{D}$ is a core of Q_j and P_j by the present assumption, we have

$$Q_j = \overline{\sum_{k=1}^n \Big((F_1)_{jk}\hat{Q}_k + (F_2)_{jk}\hat{P}_k \Big)}, \tag{7.1}$$

$$P_j = \overline{\sum_{k=1}^n \Big((F_3)_{jk}\hat{Q}_k + (F_4)_{jk}\hat{P}_k \Big)}. \tag{7.2}$$

Now suppose that there exists a unitary operator $U : \mathcal{H} \to \oplus^N L^2(\mathbb{R}^n)$ ($N \in \mathbb{N}$ or $N = \infty$) such that

$$U\overline{\hat{Q}}_j U^{-1} = \oplus^N \hat{q}_j, \quad U\overline{\hat{P}}_j U^{-1} = \oplus^N \hat{p}_j.$$

Then, by (7.1) and (7.2), we have

$$UQ_jU^{-1} = \bigoplus^N \overline{\sum_{k=1}^n ((F_1)_{jk}\hat{q}_k + (F_2)_{jk}\hat{p}_k)} = \oplus^N q_j,$$

$$UP_jU^{-1} = \bigoplus^N \overline{\sum_{k=1}^n ((F_3)_{jk}\hat{q}_k + (F_4)_{jk}\hat{p}_k)} = \oplus^N p_j.$$

But this contradicts the present assumption. □

7.2 Non-Schrödinger representations of QPS

Examples of non-Schrödinger representations of $\mathrm{QPS}_n(\Lambda_\mathrm{S})$ can be constructed from those of CCR's with n degrees of freedom. For simplicity, we consider the case $n = 2$ here and we take Λ_S as

$$\Lambda_\mathrm{S} = (\xi^2 a\epsilon, \xi^2 b\epsilon),$$

where ϵ is given by (1.5), $a > 0, b > 0$ and ξ is defined by (2.15). Let $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j\}_{j=1}^2)$ be a self-adjoint representation of the CCR's with two degrees of freedom on a Hilbert space $\mathcal{H}$ with $\mathcal{D}$ dense in $\mathcal{H}$. Suppose that $\mathcal{D}$ is a common core of Q_j and P_j, $j = 1, 2$ such that Q_1 (resp. Q_2) strongly commutes with P_2 (resp. P_1). Then, by functional calculus of strongly commuting self-adjoint operators (e.g., [16, Theorem 9.1.2]), the operators

$$\hat{Q}_1 := \xi\left(Q_1 - \frac{1}{2}aP_2\right), \quad \hat{Q}_2 := \xi\left(Q_2 + \frac{1}{2}aP_1\right),$$
$$\hat{P}_1 := \xi\left(P_1 + \frac{1}{2}bQ_2\right), \quad \hat{P}_2 := \xi\left(P_2 - \frac{1}{2}bQ_1\right).$$

are essentially self-adjoint. Hence $\{\bar{\hat{Q}}_j, \bar{\hat{P}}_j\}_{j=1,2}$ is a self-adjoint representation of $\mathrm{QPS}_2(\Lambda_\mathrm{S})$.

Corollary 7.2 *Suppose that one of the following conditions holds:*

(i) (Q_1, P_1) *is not a Weyl representation of the CCR with one degree of freedom.*

(ii) (Q_2, P_2) *is not a Weyl representation of the CCR with one degree of freedom.*

(iii) The operators Q_1 *and* Q_2 *are not strongly commuting.*

(iv) The operators P_1 *and* P_2 *are not strongly commuting.*

Then $\{\bar{\hat{Q}}_j, \bar{\hat{P}}_j\}_{j=1}^2$ *is a non-Schrödinger representation of* $\mathrm{QPS}_2(\Lambda_\mathrm{S})$.

Proof. In each case of (i)–(iv), $\{Q_j, P_j\}_{j=1}^2$ is not a Weyl representation of the CCR'S with two degrees of freedom. Thus, by Theorem 7.1, the desired result follows. □

Example 7.3 We consider the case where $\mathcal{H} = L^2(\mathbb{R}^2)$ and

$$Q_1 := \overline{q_1 + \exp(-\sqrt{2\pi}p_1)}, \quad P_1 := \overline{p_1 + \exp(-\sqrt{2\pi}q_1)},$$
$$Q_2 := q_2, \quad P_2 := p_2.$$

For $n \in \{0\} \cup \mathbb{N}, r > 0$ and $c \in \mathbb{C}$, we define a function $f_{n,r,c}$ on $\mathbb{R}$ by $f_{n,r,c}(x_1) := x_1^n e^{-rx_1^2 + cx_1}, x_1 \in \mathbb{R}$. Let $\mathcal{D}$ be the linear span of $\{f_{n,r,c} \otimes g | n \in \{0\} \cup \mathbb{N}, r > 0, c \in \mathbb{C}, g \in C_0^\infty(\mathbb{R})\}$. Then Q_j and P_j $(j = 1, 2)$ are essentially self-adjoint on $\mathcal{D}$ and $(L^2(\mathbb{R}^2), \mathcal{D}, \{Q_j, P_j\}_{j=1}^2)$ is a self-adjoint representation of the CCR's with two degrees of freedom [6]. It is obvious that Q_1 (resp. Q_2) strongly commutes with P_2 (resp. P_1). Fuglede [6] proved that $\{Q_1, P_1\}$ is not a Weyl representation. Hence condition (i) in Corollary 7.2 holds. Thus the corresponding representation $\{\bar{\hat{Q}}_j, \bar{\hat{P}}_j\}_{j=1}^2$ of $\mathrm{QPS}_2(\Lambda_\mathrm{S})$ is non-Schrödinger.

Example 7.4 Let $\alpha_1, \cdots, \alpha_N$ $(N \in \mathbb{N})$ be mutually distinct points in the complex plane $\mathbb{C}$ and $f(z)$ be a holomorphic function on $\mathbb{C} \setminus \{\alpha_n | n = 1, \cdots, N\}$ with possible poles at

$\alpha_n, n = 1, \cdots, N$. Let a_n be the point in $\mathbb{R}^2$ corresponding to α_n and $S := \{a_n | n = 1, \cdots, N\}$. Then one can define functions $A_1(x)$ and $A_2(x)$ on $M := \mathbb{R}^2 \setminus S$ by

$$A_1(x) := \operatorname{Im} f(x_1 + ix_2), \quad A_2(x) := \operatorname{Re} f(x_1 + ix_2), \quad x = (x_1, x_2) \in M,$$

where, for $z \in \mathbb{C}$, $\operatorname{Re} z$ (resp. $\operatorname{Im} z$) denotes the real (resp. imaginary) part of z. By the Cauchy–Riemann equation, we have

$$B(x) := \partial_1 A_2(x) - \partial_2 A_1(x) = 0, \quad x \in M, \tag{7.3}$$

where $\partial_j := \partial/\partial x_j, j = 1, 2$. Since the Lebesgue measure of S is zero, each function A_j defines a self-adjoint multiplication operator on $L^2(\mathbb{R}^2)$; we denote it by the same symbol A_j. We can prove that, for all $\lambda \in \mathbb{R} \setminus \{0\}$, the operators

$$P_1 := p_1 - \lambda A_1, \quad P_2 := p_2 - \lambda A_2$$

are essentially self-adjoint on $C_0^\infty(M)$ ([1, Proposition 2.1]).

Let

$$Q_1 = q_1, \quad Q_2 := q_2$$

acting in $L^2(\mathbb{R}^2)$. Then $(L^2(\mathbb{R}^2), C_0^\infty(M), \{Q_j, \bar{P}_j\}_{j=1}^2)$ is a self-adjoint representation of the CCR's with two degrees of freedom. It is easy to see that Q_1 (resp. Q_2) strongly commutes with $\bar{P}_2$ (resp. $\bar{P}_1$).

By (7.3), the line integral

$$\gamma_n := \int_{|x-a_n|=\varepsilon} (A_1(x)dx_1 + A_2(x)dx_2)$$

along the circle $|x - a_n| = \varepsilon$ with center a_n and radius $\varepsilon > 0$ (the orientation is taken to be anticlockwise) is independent of ε sufficiently small. It can be shown that, if there exists an n such that $\gamma_n \notin 2\pi\mathbb{Z}/\lambda$ ($\mathbb{Z}$ is the set of integers), then $\bar{P}_1$ and $\bar{P}_2$ are not strongly commuting [1, Theorem 5.4]. Hence condition (iv) in Corollary 7.2 holds in the present case. Thus the corresponding representation $\{\hat{\bar{Q}}_j, \hat{\bar{P}}_j\}_{j=1}^2$ of $\mathrm{QPS}_2(\Lambda_\mathrm{S})$ is non-Schrödinger.

Physically this example appears in a two dimensional quantum system with perpendicular magnetic field B concentrated on the set S in the distribution sense. In this context, (A_1, A_2) represents a vector potential of B. The condition $\gamma_n \notin 2\pi\mathbb{Z}/\lambda$ for some n corresponds to the occurrence of the so-called *Aharonov-Bohm effect*. Therefore the non-Schrödinger representation of $\mathrm{QPS}_2(\Lambda_\mathrm{S})$ is connected with a physically interesting and important situation.

In a series of papers ([1] and references therein), the present author showed that there appear self-adjoint representations of the CCR's with two degrees of freedom in two-dimensional quantum systems with singular magnetic fields (the example discussed above is one of them) and that, in each case, there is a correspondence between the occurrence of the Aharonov-Bohm effect and a non-Schrödingerness of the representation under consideration. The result derived above can be extended to a more general case.

A Some Properties of Self-Adjoint Operators Satisfying Relations of Weyl Type

Let $N \geq 2$ be an integer and A_j $(j = 1, \cdots, N)$ be self-adjoint operators on a Hilbert space $\mathcal{H}$ satisfying relations of Weyl type:

$$e^{itA_j} e^{isA_k} = e^{-itsa_{jk}} e^{isA_k} e^{itA_j}, \quad t, s \in \mathbb{R},\ j, k = 1, \cdots, N, \tag{A.1}$$

where a_{jk}'s are real constants. It follows that

$$a_{jk} = -a_{kj}, \quad j, k = 1, \cdots, N. \tag{A.2}$$

The unitarity of e^{itA_j} and functional calculus imply that

$$\exp(ise^{itA_j} A_k e^{-itA_j}) = \exp(is(A_k - ta_{jk})), \quad s, t \in \mathbb{R}.$$

Hence we have the operator equality

$$e^{itA_j} A_k e^{-itA_j} = A_k - ta_{jk}, \quad t \in \mathbb{R}, j, k = 1, \cdots, N. \tag{A.3}$$

For a linear operator A on a Hilbert space, we denote the spectrum of A by $\sigma(A)$.

Proposition A.1 *Suppose that there exists a pair (j, k) such that $a_{jk} \neq 0$ (hence $j \neq k$). Then*

$$\sigma(A_j) = \mathbb{R}, \quad \sigma(A_k) = \mathbb{R}. \tag{A.4}$$

Moreover, A_j and A_k are purely absolutely continuous.

Proof. By (A.3) and the unitary invariance of spectrum, we have $\sigma(A_k) = \sigma(A_k - ta_{jk})$ for all $t \in \mathbb{R}$. Since $a_{jk} \neq 0$, this implies the second equation of (A.4). By (A.2), we have $a_{kj} \neq 0$. Hence, by considering the case of (j, k) replaced by (k, j), we obtain the first equation of (A.4).

Relation (A.3) means that (A_k, A_j) is a weak Weyl representation of the CCR with one degree of freedom [2]. Hence A_j is purely absolutely continuous [2, 12, 17]. Similarly we can show that A_k is purely absolutely continuous. □

Proposition A.2 *Let j and k be fixed. Then, for all $\psi \in D(A_j) \cap D(A_j A_k)$, ψ is in $D(A_k A_j)$ and*

$$[A_j, A_k]\psi = i a_{jk} \psi. \tag{A.5}$$

Proof. By (A.3), we have for all $\psi \in D(A_k)$

$$A_k e^{-itA_j} \psi = e^{-itA_j}(A_k \psi - t a_{jk} \psi). \tag{A.6}$$

Let $\psi \in D(A_j A_k) \cap D(A_j)$. Then the right hand side of (A.6) is strongly differentiable in t with

$$\frac{d}{dt} e^{-itA_j}(A_k \psi - t a_{jk} \psi) = -i e^{-itA_j} A_j (A_k \psi - t a_{jk} \psi) - e^{-itA_j} a_{jk} \psi.$$

Hence so is the left hand side of (A.6). This implies that $A_j \psi$ is in $D(A_k)$ and

$$\frac{d}{dt} A_k e^{-itA_j} \psi = -i A_k A_j e^{-itA_j} \psi.$$

Hence, considering the case $t = 0$, we obtain

$$-i A_k A_j \psi = -i A_j A_k \psi - a_{jk} \psi.$$

Thus the desired result follows. □

For each function $f \in C_0^\infty(\mathbb{R}^N)$ and each vector $\psi \in \mathcal{H}$, we define a vector ψ_f by

$$\psi_f := \int_{\mathbb{R}^N} f(t_1, \cdots, t_N) e^{it_1 A_1} \cdots e^{it_N A_N} \psi dt_1 \cdots dt_N, \tag{A.7}$$

where the integral on the right hand side is taken in the strong sense. We introduce

$$\mathcal{D}_0 := \mathrm{Span}\{\psi_f | \psi \in \mathcal{H}, f \in C_0^\infty(\mathbb{R}^N)\}, \tag{A.8}$$

where Span$\{\cdots\}$ denotes the subspace algebraically spanned by the vectors in the set $\{\cdots\}$. It is easy to see that $\mathcal{D}_0$ is dense in $\mathcal{H}$.

For $f : \mathbb{R}^N \to \mathbb{C}$, we set

$$\|f\|_1 := \int_{\mathbb{R}^N} |f(t_1, \cdots, t_N)| dt_1 \cdots dt_N.$$

Lemma A.3 *Let $f_n, f \in C_0^\infty(\mathbb{R}^N)$ such that $\|f_n - f\|_1 \to 0\,(n \to \infty)$. Then $\|\psi_{f_n} - \psi_f\| \to 0\,(n \to \infty)$.*

Proof. Since $e^{it_j A_j}$ is unitary, we have

$$\|\psi_{f_n} - \psi_f\| \le \|f_n - f\|_1 \|\psi\|.$$

Thus the desired result follows. □

Proposition A.4

(i) For all $t \in \mathbb{R}$ and $j = 1, \cdots, N$, e^{itA_j} leaves $\mathcal{D}_0$ invariant.

(ii) For each $j = 1, \cdots, N$ and all $\ell \in \mathbb{N}$, $\mathcal{D}_0 \subset D(A_j^\ell)$ with

$$A_j^\ell \psi_f = (-i)^\ell \psi_{F_j^\ell(f)}, \quad f \in C_0^\infty(\mathbb{R}^N), \tag{A.9}$$

where $F_j : C_0^\infty(\mathbb{R}^N) \to C_0^\infty(\mathbb{R}^N)$ is defined by

$$F_j(f) := -\partial_j f - i \sum_{k=1}^{j-1} a_{jk} t_k f, \quad f \in C_0^\infty(\mathbb{R}^N) \tag{A.10}$$

and F_j^ℓ is the ℓ times composition of F_j

(iii) For all $\ell_1, \cdots, \ell_N \in \mathbb{N} \cup \{0\}$, $\mathcal{D}_0 \subset D(A_1^{\ell_1} A_2^{\ell_2} \cdots A_N^{\ell_N})$ and

$$A_1^{\ell_1} A_2^{\ell_2} \cdots A_N^{\ell_N} \psi_f = \psi_{F_1^{\ell_1} \cdots F_N^{\ell_N}(f)}, \quad f \in C_0^\infty(\mathbb{R}^N). \tag{A.11}$$

Proof. (i) Let ψ_f be as above. Then we have

$$e^{itA_j} \psi_f = \int_{\mathbb{R}^N} f(t_1, \cdots, t_N) e^{itA_j} e^{it_1 A_1} \cdots e^{it_N A_N} \psi dt_1 \cdots dt_N.$$

By (A.1), we have

$$e^{itA_j} e^{it_1 A_1} \cdots e^{it_N A_N} = e^{-it \sum_{k=1}^{j-1} a_{jk} t_k} e^{it_1 A_1} \cdots e^{it_{j-1} A_{j-1}} e^{i(t_j + t) A_j} e^{it_{j+1} A_{j+1}} \cdots e^{it_N A_N}.$$

Hence

$$e^{itA_j} \psi_f = \int_{\mathbb{R}^N} f(t_1, \cdots, t_{j-1}, t_j - t, t_{j+1}, \cdots, t_N) e^{-it \sum_{k=1}^{j-1} a_{jk} t_k} e^{it_1 A_1} \cdots e^{it_N A_N} \psi dt_1 \cdots dt_N.$$

We define $f_j^{(t)} : \mathbb{R}^N \to \mathbb{C}$ by

$$f_j^{(t)}(t_1, \cdots, t_N) := f(t_1, \cdots, t_{j-1}, t_j - t, t_{j+1}, \cdots, t_N) e^{-it \sum_{k=1}^{j-1} a_{jk} t_k}. \tag{A.12}$$

It is easy to see that $f_j^{(t)}$ is in $C_0^\infty(\mathbb{R}^N)$ and

$$e^{itA_j} \psi_f = \psi_{f_j^{(t)}} \in \mathcal{D}_0. \tag{A.13}$$

Thus e^{itA_j} leaves $\mathcal{D}_0$ invariant.

(ii) By (A.13), we have for $t \in \mathbb{R} \setminus \{0\}$

$$\frac{(e^{itA_j} - 1)\psi_f}{t} = \psi_{(f_j^{(t)} - f)/t}.$$

It is easy to see that $\|(f_j^{(t)} - f)/t - F_j(f)\|_1 \to 0 (t \to 0)$. Hence, by Lemma A.3,

$$\lim_{t \to 0} \frac{(e^{itA_j} - 1)\psi_f}{t} = \psi_{F_j(f)}.$$

Therefore ψ_f is in $D(A_j)$ and $iA_j\psi_f = \psi_{F_j(f)}$. Hence (A.9) with $\ell = 1$ holds. Then one can prove (A.9) by induction.

(iii) This easily follows from (ii). □

Corollary A.5 *We have*

$$[A_j, A_k] = ia_{jk}, \quad j, k = 1, \cdots, N, \tag{A.14}$$

on $\mathcal{D}_0$.

Proof. This follows from Proposition A.2 and Proposition A.4. □

Theorem A.6 *For all* $c_j \in \mathbb{R}, j = 1, \cdots, N$, $\sum_{j=1}^N c_j A_j$ *is essentially self-adjoint on* $\mathcal{D}_0$ *and*

$$e^{it\overline{\sum_{j=1}^N c_j A_j}} = e^{it^2 \sum_{j<k}^N a_{jk} c_j c_k/2} e^{itc_1 A_1} e^{itc_2 A_2} \cdots e^{itc_N A_N}. \tag{A.15}$$

Proof. For each $t \in \mathbb{R}$, we define an operator $U(t)$ by

$$U(t) := e^{it^2 \sum_{j<k}^N a_{jk} c_j c_k/2} e^{itc_1 A_1} e^{itc_2 A_2} \cdots e^{itc_N A_N}.$$

By using (A.1), one can show that $\{U(t)\}_{t \in \mathbb{R}}$ is a strongly continuous one-parameter unitary group. Hence, by the Stone theorem, there exists a unique self-adjoint operator A on $\mathcal{H}$ such that

$$U(t) = e^{itA}, \quad t \in \mathbb{R}.$$

By Proposition A.4-(i), $U(t)$ leaves $\mathcal{D}_0$ invariant. In the same manner as in the proof of Proposition A.4-(ii), (iii), one can show that, for all $\psi \in \mathcal{D}_0$, $U(t)\psi$ is strongly differentiable in t and

$$\left.\frac{dU(t)\psi}{dt}\right|_{t=0} = i \sum_{j=1}^N c_j A_j \psi.$$

Hence $\mathcal{D}_0$ is a core of A (e.g., [13, Theorem VIII.10])and $A\psi = \sum_{j=1}^N c_j A_j \psi$. Thus the desired result follows. □

Finally we give a remark on irreducibility of the set $\{e^{itA_j} | t \in \mathbb{R}, j = 1, \cdots, N\}$. There is a general fact on irreducibility of a set consisting of strongly continuous one-parameter unitary groups:

Proposition A.7 *Let $S_1, \cdots, S_N$ be self-adjoint operators on a Hilbert space. Then the set $\{e^{itS_j} | t \in \mathbb{R}, j = 1, \cdots, N\}$ is irreducible if and only if so is $\{S_j | j = 1, \cdots, N\}$.*

Proof. Suppose that $\{e^{itS_j} | t \in \mathbb{R}, j = 1, \cdots, N\}$ is irreducible. Let $B \in \mathsf{B}(\mathcal{H})$ be an operator such that $BS_j \subset S_j B, j = 1, \cdots, N$. Then, by Lemma 4.1, we have $e^{itS_j}B = Be^{itS_j}$ for all $t \in \mathbb{R}$ and $j = 1, \cdots, N$. Hence $B = cI$ with some $c \in \mathbb{C}$. Thus $\{S_j | j = 1, \cdots, N\}$ is irreducible.

Conversely, suppose that $\{S_j | j = 1, \cdots, N\}$ is irreducible. Let $B \in \mathsf{B}(\mathcal{H})$ be an operator such that $e^{itS_j}B = Be^{itS_j}$ for all $t \in \mathbb{R}$ and $j = 1, \cdots, N$. For each $\psi \in \mathcal{H}$, we put $f_\psi(t) := e^{itS_j}B\psi, g_\psi(t) := Be^{itS_j}\psi$. Then we have $f_\psi(t) = g_\psi(t)$. Let ψ be in $D(S_j)$. Then $g_\psi(t)$ is strongly differentiable in t with $dg_\psi(t)/dt = iBS_j e^{itS_j}\psi$. Hence $f_\psi(t)$ also is strongly differentiable in t, which implies that $B\psi \in D(S_j)$ and $df_\psi(t)/dt = ie^{itS_j}S_j B\psi$. Considering the case $t = 0$, we obtain $BS_j \subset S_j B$, $j = 1, \cdots, N$. Hence $B = cI$ with some $c \in \mathbb{C}$. Thus $\{e^{itS_j} | t \in \mathbb{R}, j = 1, \cdots, N\}$ is irreducible. □

As a corollary of Proposition A.7, we have the following fact:

Corollary A.8 *The set $\{e^{itA_j} | t \in \mathbb{R}, j = 1, \cdots, N\}$ is irreducible if and only if so is $\{A_j | j = 1, \cdots, N\}$.*

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References

[1] A. Arai, Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: canonical commutation relations, quantum algebras, and reduction to lattice quantum systems, *J. Math. Phys.* **39** (1998), 2476–2498.

[2] A. Arai, Generalized weak Weyl relation and decay in quantum dynamics, *Rev. Math. Phys.* **17** (2005), 1-39.

[3] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, Field theory on noncommutative spacetimes: Quasiplanar Wick products, *Phys. Rev. Lett. D* **71** (2005), 025022(1–12).

[4] S. Dulat and K. Li, The Aharonov–Casher effect for spin-1 particles in noncommutative quantum mechanics, *Eur. Phys. J* **C 54** (2008), 333–337.

[5] S. Doplicher, K. Fredenhagen and J. E. Roberts, The quantum structure of spacetime at the Planck scale and quantum fields, *Commun. Math. Phys.* **172** (1995), 187–220.

[6] B. Fuglede, On the relation $PQ - QP = -iI$, *Math. Scand.* **20** (1967), 79–88.

[7] H. Grosse and M. Wohlgenannt, Induced gauge theory on a noncommutative space, *Eur. Phys. J.* **C 52** (2007), 435–450.

[8] Y. Habara, A new approach to scalar field theory on noncommutative space, *Prog. Theor. Phys.* **107** (2002), 211–230.

[9] L. Jonke and S. Meljanac, Representations of non-commutative quantum mechanics and symmetries, *Eur. Phys. J* **C 29** (2003), 433–439.

[10] K. Li and J. Wang, The topological AC effect on non-commutative phase space, *Eur. Phys. J.* **C 50** (2007), 1007–1011.

[11] Y.-G. Miao, H. J. W. Müller-Kirsten and D. K. Park, Chiral bosons in noncommutative spacetime, *J. High Energy Phys.* **08** (2003), 038.

[12] M. Miyamoto, A generalized Weyl relation approach to the time operator and its connection to the survival probability, *J. Math. Phys.* **42** (2001), 1038–1052.

[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York, 1972.

[14] M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.

[15] L. R. Riberio, E. Passos, C. Furtado and J. R. Nascimento, Landau analog levels for dipoles in non-commutative space and phase space, *Eur. Phys. J.* **C 56** (2008), 597–606.

[16] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Birkhäuser, Basel, 1990.

[17] K. Schmüdgen, On the Heisenberg commutation relation. I, *J. Funct. Anal.* **50** (1983), 8–49.

[18] J. von Neumann, Die Eindeutigkeit der Schrödingerschen Operatoren, *Math. Ann.* **104** (1931), 570–578.

[19] J.-Z. Zhang, Consistent deformed bosonic algebra in noncommutative quantum mechanics, *Int. J. Mod. Phys.* A **23**(2008), 1393–1403.