Trotter-Kato product formula for unitary groups

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Abstract: Let A and B be non-negative self-adjoint operators in a separable Hilbert space such that its form sum C is densely defined. It is shown that the Trotter product formula holds for imaginary times in the L^2-norm, that is, one has

$$\lim_{n\to+\infty}\int_0^T \left\|\left(e^{-itA/n}e^{-itB/n}\right)^n h - e^{-itC}h\right\|^2 dt = 0$$

for any element h of the Hilbert space and any $T>0$. The result remains true for the Trotter-Kato product formula

$$\lim_{n\to+\infty}\int_0^T \left\|\left(f(itA/n)g(itB/n)\right)^n h - e^{-itC}h\right\|^2 dt = 0$$

where $f(\cdot)$ and $g(\cdot)$ are so-called holomorphic Kato functions; we also derive a canonical representation for any function of this class.

1 Introduction

The aim of this paper is to prove a Trotter-Kato-type formula for unitary groups. Apart of a pure mathematical interest such a product formula can be related to physical problems. In particular, Trotter formula provides us with a way to define Feynman path integrals [6, 13] and extending it beyond the essentially self-adjoint case would allow us to treat in this way Schrödinger operators with a much wider class of potentials.

In order to put our investigation into a proper context let us describe first the existing related results. Let $-A$ and $-B$ be two generators of contraction semigroups in the Banach space $\mathfrak{X}$. In the seminal paper [23] Trotter proved that if the operator $-C$,

$$C := \overline{A + B},$$

is the generator of a contraction semigroup in $\mathfrak{X}$, then the formula

$$e^{-tC} = \operatorname{s-lim}_{n\to\infty} \left(e^{-tA/n} e^{-tB/n}\right)^n \tag{1.1}$$

holds in $t \in [0, T]$ for any $T > 0$. Formula (1.1) is usually called the Trotter or Lie-Trotter product formula. The result was generalized by Chernoff in [2] as follows: Let $F(\cdot) : \mathbb{R}_+ \longrightarrow \mathfrak{B}(\mathfrak{X})$ be a strongly continuous contraction valued function such that $F(0) = I$ and the strong derivative $F'(0)$ exists and is densely defined. If $-C$, $C := \overline{F'(0)}$, is the generator of a C_0-contraction semigroup, then the generalized Lie-Trotter product formula

$$e^{-tC} = \operatorname{s-lim}_{n\to\infty} F(t/n)^n \tag{1.2}$$

holds for $t \geq 0$. In [3, Theorem 3.1] it is shown that in fact the convergence in the last formula is uniform in $t \in [0, T]$ for any $T > 0$. Furthermore, in [3, Theorem 1.1] this result was generalized as follows: Let $F(\cdot) : \mathbb{R}_+ \longrightarrow \mathfrak{B}(\mathfrak{X})$ a family of linear contractions on a Banach space $\mathfrak{X}$. Then the generalized Lie-Trotter product formula (1.2) holds uniformly in $t \in [0, T]$ for any $T > 0$ if and only if there is a $\lambda > 0$ such that

$$(\lambda + C)^{-1} = \operatorname{s-lim}_{\tau\to+0} (\lambda + S_\tau)^{-1}$$

where

$$S_\tau := \frac{I - F(\tau)}{\tau}, \quad \tau > 0.$$

Using the results of Chernoff, Kato was able to prove in [14] the following theorem: Let A and B be two non-negative self-adjoint operators in a separable Hilbert space $\mathfrak{H}$. Let us assume that the intersection $\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})$ is dense in $\mathfrak{H}$. If $C := A \dotplus B$ is the form sum of the operators A and B, then Lie-Trotter product formula

$$e^{-tC} = \mathrm{s}\text{-}\lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n \tag{1.3}$$

holds true uniformly in $t \in [0, T]$ for any $T > 0$. In addition, it was proven that a symmetrized Lie-Trotter product formula,

$$e^{-tC} = \mathrm{s}\text{-}\lim_{n\to\infty} \left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n, \tag{1.4}$$

is valid. In fact, the Lie-Trotter formula was extended to more general products of the form $(f(tA/n)g(tB/n))^n$ or $\left(f(tA/n)^{1/2}g(tB/n)f(tA/n)^{1/2}\right)^n$ where f (and similarly g) is a real valued function $f(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ obeying $0 \le f(t) \le 1$, $f(0) = 1$ and $f'(0) = -1$ which are called *Kato functions* in the following. Usually product formulæ of that type are labeled as Lie-Trotter-Kato.

It is a longstanding open question in linear operator theory to indicate assumptions under which the Lie-Trotter product formulæ (1.3) and (1.4) remain to hold for imaginary times, that is, under which assumptions the formulæ

$$e^{-itC} = \mathrm{s}\text{-}\lim_{n\to\infty} \left(e^{-itA/n}e^{-itB/n}\right)^n, \qquad C = A \dotplus B, \tag{1.5}$$

or

$$e^{-itC} = \mathrm{s}\text{-}\lim_{n\to\infty} \left(e^{-itA/2n}e^{-itB/n}e^{-itA/2n}\right)^n, \qquad C = A \dotplus B, \tag{1.6}$$

are valid, see [3, Remark p. 91], [9], [12] and [21]. We note that if A and B be non-negative selfadjoint operators in $\mathfrak{H}$ and the limit

$$U(t) := \mathrm{s}\text{-}\lim_{n\to\infty} \left(e^{-itA/n}e^{-itB/n}\right)^n$$

exists for all $t \in \mathbb{R}$, then $\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})$ is dense in $\mathfrak{H}$ and it holds $U(t) = e^{-itC}$, $t \in \mathbb{R}$, where $C := A \dotplus B$, see [13, Proposition 11.7.3]. Hence it makes sense to assume that $\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})$ is dense in $\mathfrak{H}$. Furthermore, applying Trotter's result [23] one immediately gets that formulæ (1.5) and (1.6) are valid if $C := \overline{A + B}$ is self-adjoint. Modifying

Lie-Trotter product formula to a kind of Lie-Trotter-Kato product formula Lapidus was able to show in [16], see also [17], that one has

$$e^{-itC} = \operatorname*{s-lim}_{n\to\infty} \left((I + itA/n)^{-1}(I + itB/n)^{-1}\right)^n$$

uniformly in t on bounded subsets of $\mathbb{R}$. In [1] Cachia extended the Lapidus result as follows. Let $f(\cdot)$ be a Kato function which admits a holomorphic continuation to the right complex plane $\mathbb{C}_{\text{right}} := \{z \in \mathbb{C} : \Re\mathrm{e}\,(z) > 0\}$ such that $|f(z)| \le 1$, $z \in \mathbb{C}_{\text{right}}$. Such functions we call holomorphic Kato functions in the following. We note that functions from this class admit limits $f(it) = \lim_{\epsilon\to+0} f(\epsilon + it)$ for a.e. $t \in \mathbb{R}$, see Section 5. In [1] it was in fact shown that if f and g holomorphic Kato functions, then

$$\lim_{n\to\infty} \int_0^T \left\| \left(\frac{f(2itA/n) + g(2itB/n)}{2} \right)^n h - e^{-itC}h \right\|^2 dt = 0.$$

for any $h \in \mathfrak{H}$ and $T > 0$. Since $f(t) = e^{-t}$, $t \in \mathbb{R}_+$, belongs to the holomorphic Kato class we find

$$\lim_{n\to\infty} \int_0^T \left\| \left(\frac{e^{-2itA/n} + e^{-2itB/n}}{2} \right)^n h - e^{-itC}h \right\|^2 dt = 0.$$

for any $h \in \mathfrak{H}$ and $T > 0$.

Before we close this introductory survey, let us mention one more family of related results. The paper [1] was inspired by a work of Ichinose and one of us [7] devoted to the so-called Zeno product formula which can be regarded as a kind of degenerated symmetric Lie-Trotter product formula. Specifically, in this formula one replaces the unitary factor $e^{-itA/2}$ by an orthogonal projection onto some closed subspace $\mathfrak{h} \subseteq \mathfrak{H}$ and defines the operator C as the self-adjoint operator which corresponds to the quadratic form $\mathfrak{k}(h,k) := \left(\sqrt{B}h, \sqrt{B}k\right)$, $h,k \in \mathrm{dom}(\mathfrak{k}) := \mathrm{dom}(\sqrt{B}) \cap \mathfrak{h}$ where it is assumed that $\mathrm{dom}(\mathfrak{k})$ is dense in $\mathfrak{h}$. In the paper [7] it was proved that

$$\lim_{n\to\infty} \int_0^T \left\| \left(Pe^{-itB/n}P\right)^n h - e^{-itC}h \right\| dt = 0$$

holds for any $h \in \mathfrak{h}$ and $T > 0$ where P is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{h}$. Subsequently, an attempt was made in [8] to replace the strong L^2-topology of [7] by the usual strong topology of $\mathfrak{H}$. To this end a class of admissible functions was introduced which consisted of Borel measurable functions $\phi(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{C}$ obeying $|\phi(x)| \leq 1$, $x \in \mathbb{R}_+$, $\phi(0) = 1$ and $\phi'(0) = -i$. It was shown in [8] that if ϕ is an admissible function such that $\Im\mathrm{m}\,(\phi(x)) \leq 0$, $x \in \mathbb{R}_+$, then

$$e^{-itC} = \operatorname*{s-lim}_{n\to\infty} (P\phi(tB/n)P)^n = e^{-itC}$$

holds uniformly in $t \in [0,T]$ for any $T > 0$. We stress that the function $\phi(x) = e^{-ix}$, $x \in \mathbb{R}_+$, is admissible but does not satisfy the condition $\Im\mathrm{m}\,(e^{-ix}) \leq 0$ for $x \in \mathbb{R}_+$, and the question about convergence of the Zeno product formula in the strong topology of $\mathfrak{H}$ remains open.

The paper is organized as follows: In Section 2 we formulate our main result and relate it to the Feynman integral. In Section 3 is devoted to the proof of the main result. The main result is generalized to Trotter-Kato product formulas for holomorphic Kato function in Section 4. Finally, in Section 5 we try to characterize holomorphic Kato functions.

2 The main result

With the above preliminaries, we can pass to our main result which can be stated as follows:

Theorem 2.1 *Let A and B two non-negative self-adjoint operators on the Hilbert space $\mathfrak{H}$. If their form sum $C := A \dotplus B$ is densely defined, then*

$$\lim_{n\to\infty} \int_0^T \left\| \left(e^{-itA/n}e^{-itB/n}\right)^n h - e^{-itC}h \right\|^2 dt = 0 \tag{2.1}$$

and

$$\lim_{n\to\infty} \int_0^T \left\| \left(e^{-itA/2n}e^{-itB/n}e^{-itA/2n}\right)^n h - e^{-itC}h \right\|^2 dt = 0 \tag{2.2}$$

holds for any $h \in \mathfrak{H}$ and $T > 0$.

We note that Theorem 2.1 partially solves [13, Problem 11.3.9] by changing slightly the topology.

Remark 2.2 From the viewpoint of physical applications, the formula (2.1) allows us to extend the Trotter-type definition of Feynman integrals to

Schrödinger operators with a wider class of potentials. Following [13, Definition 11.2.21] the Feynman integral $\mathcal{F}^t_{\rm TP}(V)$ associated with the potential V is the strong operator limit

$$\mathcal{F}^t_{\rm TP}(V) := \text{s-}\lim_{n\to\infty} \left(e^{-itH_0/n}e^{-itV/n}\right)^n$$

where $H_0 := -\frac{1}{2}\Delta$ and $-\Delta$ is the usually defined Laplacian operator in $L^2(\mathbb{R}^d)$. From [13, Corollary 11.2.22] one gets that the Feynman integral exists if $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is Lebesgue measurable and non-negative as well as $V \in L^2_{\rm loc}(\mathbb{R}^d)$.

Taking into account Theorem 2.1 it is possible to extend the Trotter-type definition of Feynman integrals if one replaces the $L^2(\mathbb{R}^d)$-topology by the $L^2([0,T]\times\mathbb{R}^d)$-topology. Indeed, let us define the generalized Feynman integral $\mathcal{F}^t_{\rm gTP}(V)$ by

$$\lim_{n\to\infty}\int_0^T \left\|\left(e^{-itH/n}e^{-itV/n}\right)^n h - \mathcal{F}^t_{\rm gTP}(V)h\right\|^2 dt = 0$$

for $h \in L^2(\mathbb{R}^d)$ and $T > 0$. Obviously, the existence of $\mathcal{F}^t_{\rm TP}(V)$ yields the existence of $\mathcal{F}^t_{\rm gTP}(V)$ where the converse is in general not true. By Theorem 2.1 one immediately gets that the generalized Feynman integral exists if $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is Lebesgue measurable and non-negative as well as $V \in L^1_{\rm loc}(\mathbb{R}^d)$. This essentially extends the class of admissible potentials. The same class of potentials is covered by the so-called modified Feynman integral $\mathcal{F}^t_M(V)$ defined by

$$\mathcal{F}^t_M(V) := \text{s-}\lim_{n\to\infty}\left([I + i(t/n)H_0]^{-1}[I + i(t/n)V]^{-1}\right)^n,$$

see [13, Definition 11.4.4] and [13, Corollary 11.4.5]. However, in this case the exponents are replaced by resolvents which leads to the loss of the typical structure of Feynman integrals.

Remark 2.3

(i) Formula (2.1) holds if and only if convergence in measure takes place, that is, for any $\eta > 0$, $h \in \mathfrak{H}$ and $T > 0$ one has

$$\lim_{n\to\infty}\left|\left\{t\in[0,T] : \left\|\left(e^{-itA/n}e^{-itB/n}\right)^n h - e^{-itC}h\right\| \geq \eta\right\}\right| = 0. \qquad (2.3)$$

where $|\cdot|$ denotes the Lebesgue measure.

(ii) We note that the relation (1.3) can be rewritten as follows: for any $\eta > 0$, $h \in \mathfrak{H}$ and $T > 0$ one has

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left\| \left(e^{-tA/n}e^{-tB/n}\right)^n h - e^{-tC}h \right\| = 0.$$

This shows that passing to imaginary times one effectively switches from a uniform convergence to a convergence in measure.

(iii) Theorem 2.1 immediately implies the existence of a non-decreasing subsequence $n_k \in \mathbb{N}$, $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \left\| \left(e^{-itA/n_k}e^{-itB/n_k}\right)^{n_k} h - e^{-itC}h \right\| = 0$$

holds for any $h \in \mathfrak{H}$ and a.e. $t \in [0, T]$.

3 Proof of Theorem 2.1

The argument is based on the following lemma.

Lemma 3.1 *Let $\{S_\tau(\cdot)\}_{\tau>0}$ be a family of bounded holomorphic operator-valued functions defined in $\mathbb{C}_{\rm right}$ such that $\Re e\,(S_\tau(z)) \geq 0$ for $z \in \mathbb{C}_{\rm right}$. Let $R_\tau(z) := (I + S_\tau(z))^{-1}$, $z \in \mathbb{C}_{\rm right}$. If the limit*

$$s\text{-}\lim_{\tau\to+0} R_\tau(t)$$

exists for all $t > 0$, then the following claims are valid:

(i) *The limit*

$$R(z) := s\text{-}\lim_{\tau\to+0} R_\tau(z)$$

exists everywhere in $\mathbb{C}_{\rm right}$, the convergence is uniform with respect to z in any compact subset of $\mathbb{C}_{\rm right}$, and the limit function $R(z)$ is holomorphic in $\mathbb{C}_{\rm right}$.

(ii) *The limits*

$$R_\tau(it) := s\text{-}\lim_{\epsilon\to+0} R_\tau(\epsilon + it)$$

and

$$R(it) := s\text{-}\lim_{\epsilon\to+0} R(\epsilon + it)$$

exist for a.e. $t \in \mathbb{R}$.

(iii) *If, in addition, there is a non-negative self-adjoint operator C such that the representation $R(t) = (I+tC)^{-1}$ is valid for $t > 0$, then $R(z) = (I+zC)^{-1}$ for $z \in \overline{\mathbb{C}_{\rm right}}$ and*

$$\lim_{\tau\to+0} \int_0^T \left\|R_\tau(it)h - (I+itC)^{-1}h\right\|^2 dt = 0 \tag{3.1}$$

holds for any $h \in \mathfrak{H}$ and $T > 0$.

Proof. The claims (i) and (ii) are obtained easily; the first one is a consequence of [11, Theorem 3.14.1], the second follows from [22, Section 5.2]. It remains to check the third claim. To prove $R(z) = (I + zC)^{-1}$ we note that $(I + tC)^{-1}$, $t > 0$, admits an analytic continuation to $\mathbb{C}_{\rm right}$ which is equal to $(I + zC)^{-1}$, $z \in \mathbb{C}_{\rm right}$. Since $R(z)$ is an analytic function in $\mathbb{C}_{\rm right}$, by (i) one immediately proves $R(z) = (I + zC)^{-1}$ for $z \in \mathbb{C}_{\rm right}$. In particular, we get the representation

$$R(it) = (I + itC)^{-1}$$

for a.e. $t \in \mathbb{R}$. Furthermore, by [1, Lemma 2] one has

$$\lim_{\tau\to+0} \int_{\mathbb{R}} (R_\tau(it)h, v(t))\, dt = \int_{\mathbb{R}} (R(it)h, v(t))\, dt \tag{3.2}$$

for any $h \in \mathfrak{H}$ and $v \in L^1(\mathbb{R}, \mathfrak{H})$. Let $p(\cdot) \in L^1(\mathbb{R})$ be real and non-negative, i.e. $p(t) \geq 0$ a.e. in $\mathbb{R}$. In particular, if $v(t) := p(t)h$ we find

$$\lim_{\tau\to+0} \int_{\mathbb{R}} p(t)\,(R_\tau(it)h, h)\, dt = \int_{\mathbb{R}} p(t)\,(R(it)h, h)\, dt$$

which yields

$$\lim_{\tau\to+0} \int_{\mathbb{R}} p(t)\Re\mathrm{e}\,\{(R_\tau(it)h, h)\}\, dt = \int_{\mathbb{R}} p(t)\Re\mathrm{e}\,\{(R(it)h, h)\}\, dt. \tag{3.3}$$

Since for each $\tau > 0$ the function $S_\tau(z)$ is bounded in $\mathbb{C}_{\rm right}$ the limit $S_\tau(it) :=$ s-$\lim_{\epsilon\to+0} S_\tau(\epsilon + it)$ exists for a.e. $t \in \mathbb{R}$, see [22, Section 5.2], and we have $\Re\mathrm{e}\,(S_\tau(it)) \geq 0$. Furthermore, from (3.3) we get

$$\begin{aligned}\lim_{\tau\to+0} \int_{\mathbb{R}} p(t)\,((I + \Re\mathrm{e}\,\{S_\tau(it)\})R_\tau(it)h, R_\tau(it)h)\, dt \qquad\qquad (3.4)\\ = \int_{\mathbb{R}} p(t)\Re\mathrm{e}\,\{(R_\tau(it)h, h)\}\, dt = \int_{\mathbb{R}} p(t)\,\|R(it)h\|^2\, dt.\end{aligned}$$

Obviously, we have

$$\int_{\mathbb{R}} p(t)\left\|R_\tau(it)h - R(it)h\right\|^2 dt = \int_{\mathbb{R}} p(t)\left\|R_\tau(it)h\right\|^2 dt$$
$$+ \int_{\mathbb{R}} p(t)\left\|R(it)h\right\|^2 dt - 2\Re\mathrm{e}\left\{\int_{\mathbb{R}} p(t)\left(R_\tau(it)h, R(it)h\right) dt\right\}.$$

If $p(t) \geq 0$ for a.e. $t \in \mathbb{R}$, then

$$\int_{\mathbb{R}} p(t)\left\|R_\tau(it)h - R(it)h\right\|^2 dt$$
$$\leq \int_{\mathbb{R}} p(t)\left((I + \Re\mathrm{e}\{S_\tau(it)\})R_\tau(it)h, R_\tau(it)h\right) dt$$
$$+ \int_{\mathbb{R}} p(t)\left\|R(it)h\right\|^2 dt - 2\Re\mathrm{e}\left\{\int_{\mathbb{R}} p(t)\left(R_\tau(it)h, R(it)h\right) dt\right\}.$$

Choosing $v(t) = p(t)R(it)h$ we obtain from (3.2) that

$$\lim_{\tau\to+0}\int_{\mathbb{R}} p(t)\left(R_\tau(it)h, R(it)h\right) dt = \int_{\mathbb{R}} p(t)\left\|R(it)h\right\|^2 dt. \tag{3.5}$$

Taking then into account (3.4) and (3.5) we find

$$\lim_{\tau\to+0}\int_{\mathbb{R}} p(t)\left\|R_\tau(it)h - R(it)h\right\|^2 dt = 0$$

and choosing finally $p(t) := \chi_{[0,T]}(t)$, $T > 0$, we arrive at the formula (3.1) for any $h \in \mathfrak{H}$ and $T > 0$. $\square$

Now we are in position to prove Theorem 2.1. We set

$$F_\tau(z) := e^{-\tau z A/2}e^{-\tau z B}e^{-\tau z A/2}, \quad \tau \geq 0,$$

and

$$S_\tau(z) := \frac{I - F_\tau(z)}{\tau}, \quad \tau > 0,$$

for $z \in \overline{\mathbb{C}_{\mathrm{right}}}$. Obviously, the family $\{S_\tau(\cdot)\}_{\tau>0}$ consists of bounded holomorphic operator-valued functions defined in $\mathbb{C}_{\mathrm{right}}$. Since $\|F_\tau(z)\| \leq 1$ for $z \in \mathbb{C}_{\mathrm{right}}$ we get that $\Re\mathrm{e}\{S_\tau(z)\} \geq 0$ for $z \in \mathbb{C}_{\mathrm{right}}$ and $\tau > 0$. Using formula (2.2) of [14] we find

$$\operatorname*{s-lim}_{\tau\to+0}(I + S_\tau(t))^{-1} = (I + tC)^{-1}$$

for $t \in \mathbb{R}$. Obviously, we have

$$R_\tau(it) = (I + S_\tau(it))^{-1}$$

for a.e $t \in \mathbb{R}$ where

$$S_\tau(it) = \frac{I - e^{-i\tau t A/2} e^{-i\tau t B} e^{-i\tau t A/2}}{\tau}$$

for $t \in \mathbb{R}$ and $\tau > 0$. Applying Lemma 3.1 we obtain

$$\lim_{\tau \to +0} \int_0^T \left\| (I + S_\tau(it))^{-1} h - (I + itC)^{-1} h \right\|^2 dt = 0 \tag{3.6}$$

for any $h \in \mathfrak{H}$ and $T > 0$.

Now we pass to $\mathfrak{H}$-valued functions introducing $\widehat{\mathfrak{H}} := L^2([0,T], \mathfrak{H})$. We set

$$(\widehat{A} f)(t) = tAf(t), \quad f \in \mathrm{dom}(\widehat{A}) = \{f \in \widehat{\mathfrak{H}} : tAf(t) \in \widehat{\mathfrak{H}}\}$$

and in the same way we define $\widehat{B}$ and $\widehat{C}$ associated with the operators B and C, respectively. It is obvious that the operators $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$ are non-negative. Setting

$$\widehat{F}_\tau := e^{-i\tau \widehat{A}/2} e^{-i\tau \widehat{B}} e^{-i\tau \widehat{A}/2}, \quad \tau > 0,$$

and

$$\widehat{S}_\tau := \frac{\widehat{I} - \widehat{F}_\tau}{\tau}, \quad \tau > 0,$$

we have

$$(\widehat{F}_\tau \widehat{h})(t) = F_\tau(it) \widehat{h}(t) \quad \text{and} \quad (\widehat{S}_\tau \widehat{h})(t) = \frac{I - F_\tau(it)}{\tau} \widehat{h}(t),$$

where $\widehat{h} \in \widehat{\mathfrak{H}}$. From Lemma 3.1 one immediately gets that

$$\lim_{\tau \to +0} \|(\widehat{I} + \widehat{S}_\tau)^{-1} \widehat{h} - (\widehat{I} + \widehat{C})^{-1} \widehat{h}\|_{\widehat{\mathfrak{H}}} = 0$$

for any $\widehat{h} \in \widehat{\mathfrak{H}}$. Applying now [3, Theorem 1.1] we find

$$\operatorname{s-lim}_{n \to \infty} \widehat{F}^{\,n}_{s/n} = e^{-is\widehat{C}}$$

uniformly in $s \in [0, \widehat{T}\,]$ for any $\widehat{T} > 0$ which yields

$$\lim_{n\to\infty} \int_0^T \left\| \left(e^{-istA/2n} e^{-istB/n} e^{-istA/2n}\right)^n \widehat{h}\,(t) - e^{-istC}\,\widehat{h}\,(t) \right\|^2 dt = 0$$

for any $\widehat{h} \in \widehat{\mathfrak{H}}$ and $s \in [0, \widehat{T}\,]$, $\widehat{T} > 0$. Setting finally $\widehat{h}\,(t) = \chi_{[0,T]}(t)h$, $h \in \mathfrak{H}$, and $s = 1$ we arrive at the symmetrized form (2.2) of the product formula. To get the other one, we take into account the relation

$$\left(e^{-istA/2n} e^{-itB/n} e^{-itA/2n}\right)^n = e^{itA/2n} \left(e^{-itA/n} e^{-itB/n}\right)^n e^{-itA/2n}$$

which yields

$$\left\| \left(e^{-itA/2n} e^{-itB/n} e^{-itA/2n}\right)^n h - e^{-itC} h \right\|^2 = \\ \left\| \left(e^{-itA/n} e^{-itB/n}\right)^n e^{-itA/2n} h - e^{-itA/2n} e^{-itC} h \right\|^2$$

and through that the sought formula (2.1).

4 A generalization

Let $f(\cdot)$ be a holomorphic Kato function. In general, one cannot expect that for any non-negative operator A the formula

$$\operatorname*{s-lim}_{\epsilon\to+0} f((\epsilon + it)A) = f(itA)$$

would be valid for all $t \in \mathbb{R}$. This is due to the fact that the limit $f(iy)$ does not exist for each $y \in \mathbb{R}_+$, see Section 5. In order to avoid difficulties we assume in the following that the limit $f(iy)$ exist for all $y \in \mathbb{R}$ and indicate in Section 5 conditions which guarantee this property.

Theorem 4.1 *Let A and B two non-negative self-adjoint operators on the Hilbert space $\mathfrak{H}$. Assume that $C := A \dot{+} B$ is densely defined. If f and g be holomorphic Kato functions such that the limit $f(iy) = \lim_{x\to+0} f(x + iy)$ exist for all $y \in \mathbb{R}$, then*

$$\lim_{n\to\infty} \int_0^T \left\| (f(itA/n)g(itB/n))^n h - e^{-itC} h \right\|^2 dt = 0$$

for any $h \in \mathfrak{H}$ and $T > 0$.

Proof. We set

$$F_\tau(z) := f(\tau z A) g(\tau z B), \quad z \in \mathbb{C}_{\rm right}, \quad \tau \ge 0,$$

and

$$S_\tau(z) := \frac{I - F_\tau(z)}{\tau}, \quad z \in \mathbb{C}_{\rm right}, \quad \tau > 0.$$

We note that $\{S_\tau(z)\}_{\tau>0}$ is a family of bounded holomorphic operator-valued functions defined in $\mathbb{C}_{\rm right}$ obeying $\Re\mathrm{e}\,\{S_\tau(z)\} \ge 0$. We set $R_\tau(z) := (I + S_\tau(z))^{-1}$, $z \in \mathbb{C}_{\rm right}$, $\tau > 0$. By [14] we know that

$$\operatorname*{s-lim}_{n\to\infty} \left(f(tA/n)g(tB/n)\right)^n = e^{-tC}$$

uniformly in $t \in [0, T]$ for any $T > 0$. Applying Theorem 1.1 of [3] we find

$$\operatorname*{s-lim}_{\tau\to+0} R_\tau(t) = (I + tC)^{-1}$$

for $t \in \mathbb{R}_+$. Since $S_\tau(z)$, $z \in \mathbb{C}_{\rm right}$, is a holomorphic continuation of $S_\tau(t)$, $t \in \mathbb{R}_+$, one gets that $R_\tau(z)$, $z \in \mathbb{C}_{\rm right}$, is in turn a holomorphic continuation of $R_\tau(t)$, $t \in \mathbb{R}_+$. Since

$$F_\tau(it) := \operatorname*{s-lim}_{\epsilon\to+0} F_\tau(\epsilon + it) = f(i\tau tA)g(i\tau tB), \quad \tau > 0,$$

for $t \in \mathbb{R}$ we find that

$$S_\tau(it) := \operatorname*{s-lim}_{\epsilon\to+0} S_\tau(\epsilon + it) = \frac{I - f(i\tau tA)g(i\tau tB)}{\tau}, \quad \tau > 0,$$

holds for $t \in \mathbb{R}$, which further yields

$$R_\tau(it) := \operatorname*{s-lim}_{\epsilon\to+0} R_\tau(\epsilon + it) = (I + S_\tau(it))^{-1}, \quad \tau > 0,$$

for $t \in \mathbb{R}$. Applying Lemma 3.1 we prove (3.6). Following now the line of reasoning used after formula (3.6) we complete the proof. $\square$

Obviously, the Kato functions $f_k(x) := (1 + x/k)^{-k}$, $x \in \mathbb{R}_+$, are holomorphic Kato functions. Indeed, each function f_k admits a holomorphic continuation, $f(z) = (1 + z/k)^{-k}$ on $z \in \mathbb{C}_{\rm right}$and, moreover, the limit

$$f_k(it) := \lim_{\epsilon\to+0} f(\epsilon + it) = (1 + it/k)^{-k}$$

exists for any $t \in \mathbb{R}$. This yields

$$\lim_{n\to+\infty} \int_0^T \left\| \left((I+itA/kn)^{-k}(I+itB/kn)^{-k}\right)^n h - e^{-itC}h \right\| dt = 0$$

for any $h \in \mathfrak{H}$ and $T > 0$. We note that for the particular case $k = 1$ Lapidus demonstrated in [16] that

$$\text{s-}\lim_{n\to+\infty} \left((I+itA/n)^{-1}(I+itB/n)^{-1}\right)^n = e^{-itC} \tag{4.1}$$

holds uniformly in $t \in [0,T]$ for any $T > 0$. By Theorem 4.1 one gets that formula (4.1) is valid in a weaker topology as in [16]. This discrepancy will be clarified in a forthcoming paper.

5 Holomorphic Kato functions

5.1 Representation

To make use of the results of the previous section one should know properties of holomorphic Kato functions. To this purpose we will try in the following to find a canonical representation for this function class.

Theorem 5.1 *If f is a holomorphic Kato function, then*

(i) *there is an at most countable set of complex numbers $\{\xi_k\}_k$, $\xi_k \in \mathbb{C}_{\rm right}$ with $\Im\mathrm{m}\,(\xi_k) \geq 0$ satisfying the condition*

$$\varkappa := 4\sum_k \frac{\Re\mathrm{e}\,(\xi_k)}{|\xi_k|^2} \leq 1 \tag{5.1}$$

(ii) *there is a Borel measure ν defined on $\overline{\mathbb{R}}_+ = [0,\infty)$ obeying $\nu(\{0\}) = 0$ and*

$$\int_{\mathbb{R}_+} \frac{1}{1+t^2}\, d\nu(t) < \infty$$

such that the limit $\beta := \lim_{x\to+0} \frac{2}{\pi}\int_{\mathbb{R}_+} \frac{1}{x^2+t^2}\, d\nu(t)$ exists and satisfies the condition $\beta \leq 1-\varkappa$;

(iii) *the Kato function f admits the representation*

$$f(x) = D(x)\exp\left\{-\frac{2x}{\pi}\int_{\mathbb{R}_+} \frac{1}{x^2+t^2} d\nu(t)\right\} e^{-\alpha x}, \quad x \in \mathbb{R}_+, \tag{5.2}$$

where $\alpha := 1 - \varkappa - \beta$ *and* $D(x)$ *is a Blaschke-type product given by*

$$D(x) := \prod_k \frac{x^2 - 2x\Re\mathrm{e}\,(\xi_k) + |\xi_k|^2}{x^2 + 2x\Re\mathrm{e}\,(\xi_k) + |\xi_k|^2}, \quad x \in \mathbb{R}_+. \tag{5.3}$$

The factor $D(x)$ *is absent if the set* $\{\xi_k\}_k$ *is empty; in that case we set* $\varkappa := 0$.

Conversely, if a real function f *admits the representation* (5.2) *such that the assumptions* (i) *and* (ii) *are satisfied as well as* $\alpha + \varkappa + \beta = 1$ *holds, then* f *is a holomorphic Kato function and its holomorphic extension to* $\mathbb{C}_{\mathrm{right}}$ *is given by*

$$f(z) = D(z) \exp\left\{-\frac{2z}{\pi}\int_{\mathbb{R}_+} \frac{1}{z^2 + t^2} d\nu(t)\right\} e^{-\alpha z}, \quad z \in \mathbb{C}_{\mathrm{right}}.$$

Proof. If f is a holomorphic Kato function, then $G(z) := f(-iz)$, $z \in \mathbb{C}_+$, belongs to $H^\infty(\mathbb{C}_+)$. We have $f(z) = G(iz)$, $z \in \mathbb{C}_{\mathrm{right}}$, and taking into account Section C of [15] we find that if $G(\cdot) \in H^\infty(\mathbb{C}_+)$, then there is a real number $\gamma \in [0, 2\pi)$, a sequence of complex numbers $\{z_k\}_k$, $z_k \in \mathbb{C}_+$, satisfying

$$\sum_{k=1}^{n} \frac{\Im\mathrm{m}\,(z_k)}{|i + z_k|^2} < \infty, \tag{5.4}$$

a Borel measure ν defined on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} \frac{1}{1 + t^2}\, d\nu(t) < \infty,$$

and a real number $\alpha \geq 0$ such that $G(\cdot)$ admits the factorization

$$G(z) = e^{i\gamma} B(z) \exp\left\{-\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{1 + t^2}\right) d\nu(t)\right\} e^{iaz}, \quad z \in \mathbb{C}_+,$$

where $B(z)$ is the Blaschke product given by

$$B(z) := \prod_k \left(e^{i\alpha_k}\frac{z - z_k}{z - \overline{z_k}}\right), \quad z \in \mathbb{C}_+,$$

and $\{\alpha_k\}_k$ is a sequence of real numbers $\alpha_k \in [0, 2\pi)$ determined by the requirement

$$e^{i\alpha_k}\frac{i - z_k}{i - \overline{z_k}} \geq 0.$$

The sequence $\{z_k\}_k$ coincides with the zeros of $G(z)$ counting multiplicities. The quantities γ, $\{z_k\}_k$, ν, a are uniquely determined by $G(\cdot)$.

Using the relation $f(z) = G(iz)$, $z \in \mathbb{C}_{\rm right}$, one gets from here a factorization of the holomorphic Kato function,

$$f(z) = e^{i\gamma} B(iz) \exp\left\{-\frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right) d\nu(t)\right\} e^{-\alpha z}, \tag{5.5}$$

$z \in \mathbb{C}_{\rm right}$. Setting next $\xi_k = -iz_k \in \mathbb{C}_{\rm right}$ the condition (5.4) takes the form

$$\sum_{k=1}^{n} \frac{\Re\mathrm{e}\,(\xi_k)}{|1+\xi_k|^2} < \infty$$

and the Blaschke product can be written as

$$D(z) := B(iz) = \prod_k \left(e^{i\alpha_k}\frac{z-\xi_k}{z+\overline{\xi_k}}\right), \quad z \in \mathbb{C}_{\rm right}, \tag{5.6}$$

where the sequence of real numbers $\{\alpha_k\}_k$ is determined now by

$$e^{i\alpha_k}\frac{1-\xi_k}{1+\overline{\xi_k}} \geq 0. \tag{5.7}$$

The complex numbers ξ_k are the zeros of $f(\cdot)$.

Since the Kato function has to be real on $\mathbb{R}_+$ we easily find that the condition $f(z) = \overline{f(\overline{z})}$, $z \in \mathbb{C}_{\rm right}$, has to be satisfied. Hence ξ_k and $\overline{\xi_k}$ are simultaneously zeros of $f(z)$ and the Blaschke-type product $D(z)$ always contains the factors $e^{i\alpha_k}\frac{z-\xi_k}{z-\overline{\xi_k}}$ and $e^{-i\alpha_k}\frac{z-\overline{\xi_k}}{z-\xi_k}$ simultaneously. This allows us to put $D(z)$ into the form

$$D(z) = \prod_k \frac{z^2 - 2z\Re\mathrm{e}\,(\xi_k) + |\xi_k|^2}{z^2 + 2z\Re\mathrm{e}\,(\xi_k) + |\xi_k|^2} \prod_l \frac{z-\eta_l}{z+\eta_l}, \quad z \in \mathbb{C}_{\rm right}, \tag{5.8}$$

where $\Re\mathrm{e}\,(\xi_k) > 0$, $\Im\mathrm{m}\,(\xi_k) > 0$ for complex conjugated pairs and $\eta_l > 0$ for the remaining real zeros. Hence we have $D(z) = \overline{D(\overline{z})}$ for $z \in \mathbb{C}_{\rm right}$. Using this relation we find that

$$e^{i\gamma - g(z)} = e^{-i\gamma - \widetilde{g}(z)}, \quad z \in \mathbb{C}_{\rm right},$$

for $z \in \mathbb{C}_{\text{right}}$ where

$$g(z) := \frac{i}{\pi} \int_{\mathbb{R}} \frac{1+izt}{iz-t}\, d\mu(t) \quad \text{and} \quad \widetilde{g}(z) := \overline{g(\overline{z})} = \frac{i}{\pi} \int_{\mathbb{R}} \frac{1-izt}{iz+t}\, d\mu(t)$$

and $d\mu(t) = (1+t^2)^{-1} d\nu(t)$. Since $g(1) = \widetilde{g}(1)$ we find $e^{2i\gamma} = 1$ which yields $\gamma = 0$ or $\gamma = \pi$. In both cases we have

$$e^{-g(z)} = e^{-\widetilde{g}(z)}, \quad z \in \mathbb{C}_{\text{right}}.$$

By $g(1) = \widetilde{g}(1)$ we find that $g(z) = \widetilde{g}(z)$, $z \in \mathbb{C}_{\text{right}}$. Setting $\widetilde{\mu}(X) := \mu(-X)$ for any Borel set X of $\mathbb{R}$ we find

$$\int_{\mathbb{R}} \frac{1+izt}{iz-t}\, d\mu(t) = \int_{\mathbb{R}} \frac{1+izt}{iz-t}\, d\widetilde{\mu}(t), \quad z \in \mathbb{C}_{\text{right}}.$$

Using

$$\int_{\mathbb{R}} \frac{1+izt}{iz-t}\, d\mu(t) = (1-z^2) \int_{\mathbb{R}} \frac{1}{iz-t}\, d\mu(t) - \int_{\mathbb{R}} d\mu(t)$$

and

$$\int_{\mathbb{R}} \frac{1+izt}{iz-t}\, d\widetilde{\mu}(t) = (1-z^2) \int_{\mathbb{R}} \frac{1}{iz-t}\, d\widetilde{\mu}(t) - \int_{\mathbb{R}} d\widetilde{\mu}(t)$$

as well as the relation $\int_{\mathbb{R}} d\mu(t) = \int_{\mathbb{R}} d\widetilde{\mu}(t)$ we find

$$\int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t) = \int_{\mathbb{R}} \frac{1}{z-t}\, d\widetilde{\mu}(t), \quad z \in \mathbb{C}_{\text{right}},$$

which yields $\mu = \widetilde{\mu}$. Hence the Borel measure obeys $\mu(X) = \mu(-X)$ for any Borel set $X \subseteq \mathbb{R}$ and this in turn implies $\nu(X) = \nu(-X)$ for any Borel set. Using this property we get

$$\begin{aligned}
\int_{\mathbb{R}} \left(\frac{1}{iz-t} + \frac{t}{1+t^2} \right) d\nu(t) &= \int_{\mathbb{R}} \frac{1+izt}{iz-t}\, d\mu(t) \\
&= \frac{1}{iz}\mu(\{0\}) + \int_{\mathbb{R}_+} \left(\frac{1+izt}{iz-t} + \frac{1-izt}{iz+t} \right) d\mu(t), \quad z \in \mathbb{C}_{\text{right}},
\end{aligned}$$

where $\mathbb{R}_+ = (0, \infty)$. In this way we find

$$\int_{\mathbb{R}} \left(\frac{1}{iz-t} + \frac{t}{1+t^2} \right) d\nu(t) = \frac{1}{iz}\nu(\{0\}) - 2iz \int_{\mathbb{R}_+} \frac{1}{z^2+t^2} d\nu(t)$$

for $z \in \mathbb{C}_{\rm right}$. Summing up we find that a holomorphic Kato function admits the representation

$$f(x) = e^{i\gamma} D(x) \exp\left\{-\frac{1}{\pi x}\nu(\{0\}) - \frac{2x}{\pi}\int_{\mathbb{R}_+} \frac{1}{x^2+t^2}\,d\nu(t)\right\} e^{-\alpha x},$$

$x \in \mathbb{R}_+$, where $D(z)$ is given by (5.8). Since $f(x) \geq 0$, $x \in \mathbb{R}_+$, one gets that $\gamma = 0$ and $D(x) \geq 0$, $x \in \mathbb{R}_+$, which means that the real zeros of $f(z)$ are of even multiplicity. Consequently, the Blaschke-type product $D(z)$ is of the form

$$D(z) = \prod_k \frac{z^2 - 2z\Re\mathrm{e}\,(\xi_k) + |\xi_k|^2}{z^2 + 2z\Re\mathrm{e}\,(\xi_k) + |\xi_k|^2}, \quad z \in \mathbb{C}_{\rm right}.$$

We note that the inequality $0 \leq f(x) \leq 1$, $x \in \mathbb{R}_+$, is valid.

Next we have to satisfy the conditions $f(0) := \lim_{x\to+0} f(x) = 1$ and $f'(0) = \lim_{x\to+0}\frac{f(x)-1}{x} = -1$. Firstly we note that

$$f(x) \leq \exp\left\{-\frac{\nu(\{0\})}{\pi x}\right\}, \quad x \in \mathbb{R}_+.$$

If $\nu(\{0\}) \neq 0$, then it follows that $f(0) = 0$ which contradicts the assumption $f(0) = 1$, hence $\nu(\{0\}) = 0$. Next we set $D_k(x) := \frac{x^2-2x\Re\mathrm{e}\,(\xi_k)+|\xi_k|^2}{x^2+2x\Re\mathrm{e}\,(\xi_k)+|\xi_k|^2}$, $x \in \mathbb{R}_+$. Since $0 \leq D_k(x) \leq 1$, $x \in \mathbb{R}_+$, we get

$$\begin{aligned}1 - f(x) \geq 1 - D_1(x) + D_1(1 - D_2(x)) + D_1(x)D_2(x)(1 - D_3(x)) + \cdots\\ + \prod_{k=1}^n D_k(x)\left(1 - \prod_{k=n+1} D_k(x)\exp\left\{-\frac{2x}{\pi}\int_{\mathbb{R}_+}\frac{1}{x^2+t^2}\,d\nu(t)\right\}e^{-\alpha x}\right)\end{aligned}$$

for $x \in \mathbb{R}_+$ and $n = 1, 2, \ldots$. In this way we find the estimate

$$\begin{aligned}1 - f(x) \geq 1 - D_1(x) + D_1(x)(1 - D_2(x)) +\\ D_1(x)D_2(x)(1 - D_3(x)) + \cdots + \prod_{k=1}^{n-1} D_k(x)(1 - D_n(x))\end{aligned}$$

for $x \in \mathbb{R}_+$ and $n = 1, 2 \ldots$. This yields

$$\begin{aligned}\frac{1 - f(x)}{x} \geq \frac{1 - D_1(x)}{x} + D_1(x)\frac{1 - D_2(x)}{x} +\\ D_1(x)D_2(x)\frac{1 - D_3(x)}{x} + \cdots + \prod_{k=1}^{n-1} D_k(x)\frac{1 - D_n(x)}{x}\end{aligned}$$

for $x \in \mathbb{R}_+$ and $n = 1, 2 \ldots$, and since $\lim_{x \to +0} D_k(x) = 1$ and

$$\lim_{x \to +0} \frac{1 - D_k(x)}{x} = 4 \frac{\Re\mathrm{e}\,(\xi_k)}{|\xi_k|^2}$$

for $k = 1, 2, \ldots$, we immediately obtain (5.1). In particular, we infer that the limit $D'(0) := \lim_{x \to +0} \frac{D(x)-1}{x} = -\varkappa$ exists. Furthermore, we note that condition (5.1) implies (5.6). Furthermore, we have

$$1 - f(x) \geq 1 - \exp\left\{-\frac{2x}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2 + t^2}\, d\nu(t)\right\}, \quad x \in \mathbb{R}_+,$$

which yields

$$\lim_{x \to +0} \exp\left\{-\frac{2x}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2 + t^2}\, d\nu(t)\right\} = 1\,,$$

or

$$\lim_{x \to +0} \frac{2x}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2 + t^2}\, d\nu(t) = 0.$$

Moreover, we have

$$\begin{aligned} &\frac{1 - f(x)}{x} \\ &\geq \exp\left\{-\frac{2x}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2 + t^2}\, d\nu(t)\right\} \frac{\exp\left\{\frac{2x}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2+t^2}\, d\nu(t)\right\} - 1}{x} \\ &\geq \exp\left\{-\frac{2x}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2 + t^2}\, d\nu(t)\right\} \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2 + t^2}\, d\nu(t) \end{aligned}$$

which yields $1 \geq \limsup_{x \to +0} \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2+t^2}\, d\nu(t)$. However, the function $p(x) := \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2+t^2}\, d\nu(t)$, $x \in \mathbb{R}_+$, is decreasing which implies the existence of $\beta := \lim_{x \to +0} \frac{2}{\pi} \int_{\mathbb{R}_+} \frac{1}{x^2+t^2}\, d\nu(t)$. Summing up these considerations we have found

$$f'(0) = \lim_{x \to +0} \frac{f(x) - 1}{x} = -\varkappa - \beta - \alpha = -1 \tag{5.9}$$

which completes the proof of the necessity of the conditions. The converse is obvious. □

5.2 On the existence of $f(iy)$ everywhere

Besides the fact that $f(x)$ has to be a holomorphic Kato function one needs that the limit $f(iy) := \lim_{x\to+0} f(x+iy)$ exist for all $y \in \mathbb{R}$. First we note that the limit $f(iy)$ exists for a.e. $y \in \mathbb{R}$. This is a simple consequence of the fact that the function $G(z) := f(-iz)$, $z \in \mathbb{C}_{\rm right}$, belongs to $H^\infty(\mathbb{C}_+)$: for such functions the limit $G(x) := \lim_{\epsilon\to+0} G(x+i\epsilon)$ exists for a.e. $x \in \mathbb{R}$ which immediately yields that $f(iy)$ exists for a.e. $y \in \mathbb{R}$. To begin with, let us ask about the existence of the limit $|f|(iy) := \lim_{x\to+0} |f(x+iy)|$. For this purpose we note that the measure ν of Theorem 5.1 admits the unique decomposition $\nu = \nu_s + \nu_{ac}$ where ν_s is singular and ν_{ac} is absolutely continuous, and furthermore, the measure $\nu_{ac}(\cdot)$ can be represented as

$$d\nu_{ac}(t) = h(t)dt$$

where the function $h(t)$ is non-negative and obeys

$$\int_{\mathbb{R}_+} h(t)\,\frac{dt}{1+t^2} < \infty.$$

Proposition 5.2 *Let $f(\cdot)$ be a holomorphic Kato function and let Δ be an open interval of $\mathbb{R}$. The limit $|f|(iy) = \lim_{x\to+0}|f(x+iy)|$ exists for every $y \in \Delta$, is continuous and different from zero on Δ if and only if the limit*

$$\lim_{x\to+0} |D(x+iy)| = 1 \tag{5.10}$$

exist for every $y \in \Delta$, $\nu_s(\Delta) = 0$ and the extended weight function $\widetilde{h}(t) := h(|t|)$, $t \in \mathbb{R}$, is continuous on Δ.

In particular, the limit $|f|(iy)$ exists for every $y \in \mathbb{R}$, is continuous and different from zero on $\mathbb{R}$ if and only if the limit (5.10) exists for every $y \in \mathbb{R}$, $\nu_s \equiv 0$ and the extended function $\widetilde{h}(\cdot)$ is continuous on $\mathbb{R}$.

Proof. The measure ν of Theorem 5.1 is given on $[0,\infty)$. We extend it to the real axis $\mathbb{R}$ setting $\nu(X) := \nu(-X)$ for any Borel set $X \subseteq (-\infty,0)$. Using $\nu(X) := \nu(-X)$ we obtain from (5.5) and (5.6) the representation

$$|f(x+iy)| = |D(x+iy)| \exp\left\{-\frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y+t)^2}\,d\nu(t)\right\} e^{-ax},$$

$z = x+iy \in \mathbb{C}_{\rm right}$, or

$$|f(x+iy)| = |D(x+iy)| \exp\left\{-\frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\,d\nu(t)\right\} e^{-ax},$$

$z = x + iy \in \mathbb{C}_{\rm right}$; in this way we find

$$-\log(|f(x+iy)|) = -\log(|D(x+iy)|) + \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\,d\nu(t) + \alpha x$$

for $z = x + iy \in \mathbb{C}_{\rm rigth}$. Since one has $\lim_{x\to+0}|D(x+iy)| = 1$ for a.e. $y \in \mathbb{R}$ we infer that

$$-\lim_{x\to+0}\log(|f(x+iy)|) = \lim_{x\to+0}\frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+(y-t)^2}\,d\nu(t)$$

for a.e. $y \in \mathbb{R}$. Since

$$\lim_{x\to+0}\frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+(y-t)^2}\,d\nu(t) = \widetilde{h}(y)$$

holds for almost all $y \in \mathbb{R}$ we obtain $-\log(|f|(iy)) = \widetilde{h}(y)$ for a.e. $y \in \mathbb{R}$. By assumption $|f|(iy)$ is continuous and different from zero on Δ. Hence the extended weight function $\widetilde{h}(y)$ can be assumed to be continuous on Δ. However, if $\widetilde{h}(\cdot)$ is continuous on Δ, then one has

$$\lim_{x\to+0}\frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+(y-t)^2}\,\widetilde{h}(t)\,dt = \widetilde{h}(y)$$

for each $y \in \Delta$ which means that

$$\lim_{x\to+0}\left\{-\log(|D(x+iy)|) + \frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+(y-t)^2}\,d\nu_s(t)\right\} = 0$$

for each $y \in \Delta$. Since $-\log(|D(x+iy)|) \geq 0$ we find $\lim_{x\to+0}\log(|D(x+iy)|) = 0$ and

$$\lim_{x\to+0}\frac{1}{\pi}\int_{\mathbb{R}}\frac{x}{x^2+(y-t)^2}\,d\nu_s(t) = 0$$

for each $y \in \Delta$. Taking into account [19] one can conclude that the symmetric derivative $\nu_s'(y)$,

$$\nu_s'(y) := \lim_{\epsilon}\frac{\nu_s((y-\epsilon, y+\epsilon))}{2\epsilon}$$

exists and obeys $\nu_s'(y) = 0$ for every $y \in \Delta$. If $\nu_s(\{y_0\}) > 0$ for $y_0 \in \Delta$, then

$$0 = \lim_{\epsilon\to+0}\frac{\nu_s((y_0-\epsilon, y_0+\epsilon))}{2\epsilon} \geq \lim_{\epsilon\to+0}\frac{\nu_s(\{y_0\})}{2\epsilon}$$

which yields $\nu_s(\{y_0\}) = 0$, hence $\nu(\{y\}) = 0$ for any $y \in \Delta$. This means that ν_s has to be singular continuous. Let us introduce the function $\theta(t) := \nu_s([0,t))$, $t \in [0,t)$. The function $\nu_s(t)$ is continuous and monotone. From $\nu_s'(y) = 0$ we get that the derivative of $\theta'(y)$ exists and $\theta'(y) = 0$ for each $y \in \Delta$. Hence the function is constant which yields that $\nu_s(\Delta) = 0$.

Conversely, let us assume that $\widetilde{h}(\cdot)$ is continuous on Δ, $\nu_s(\Delta) = 0$, and condition (5.10) holds. Then we have the representation

$$\begin{aligned} |f(x+iy)| &= |D(x+iy)| \\ &\times \exp\left\{-\frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\, d\nu_s(t) - \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\,\widetilde{h}(t)\, dt\right\} e^{-ax} \end{aligned}$$

If $y \in \Delta$, then $\lim_{x\to+0} \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\, d\nu_s(t) = 0$. Since $\widetilde{h}(\cdot)$ is continuous on the interval Δ we have $\lim_{x\to+0} \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\,\widetilde{h}(t)\, dy = \widetilde{h}(y)$ for each $y \in \Delta$. Thus we find $\lim_{x\to+0} |f(x+iy)| = e^{-\widetilde{h}(y)}$ for each $y \in \Delta$ and the limit $|f|(iy)$ is continuous on Δ. Since $\widetilde{h}(y)$ is finite for each $y \in \Delta$ the limit $|f|(iy)$ is different from zero for each $y \in \Delta$. $\square$

Conditions of the type appearing in the proposition were discussed in [20]. In particular, it turns out that the condition (5.10) is satisfied if and only if

$$\lim_{x\to+0} \frac{\tau(iy,x)}{x} = 0 \tag{5.11}$$

holds for every $y \in \Delta$ where

$$\tau(iy,t) := \sum_{|iy-\xi_k|\le t} \Re\mathrm{e}\,(\xi_k), \quad y \in \mathbb{R}_+, \quad t > 0. \tag{5.12}$$

It is clear that the validity of the condition (5.11) is related to the distribution of zeros in $\mathbb{C}_{\rm right}$. Of course, if there is only a finite number of zeros ξ_k, then condition (5.11) is satisfied.

Theorem 5.3 *Let $f(\cdot)$ is a holomorphic Kato function and let Δ be an open interval of $\mathbb{R}$. The limit $f(iy) = \lim_{x\to+0} f(x+iy)$ exists for every $y \in \Delta$, is locally Hölder continuous and different from zero on Δ if and only if the zeros of $f(\cdot)$ do not accumulate to any point of $i\Delta := \{iy : y \in \Delta\}$, $\nu_s(\Delta) = 0$ and the extended weight function $\widetilde{h} := h(|t|)$, $t \in \mathbb{R}$, is locally Hölder continuous on Δ.*

In particular, the limit $f(iy)$ exists for every $y \in \mathbb{R}$, is locally Hölder continuous and different from zero on $\mathbb{R}$ if and only if $f(\cdot)$ has only a finite number of zeros in every bounded open set of $\mathbb{C}_{\rm right}$, $\nu_s \equiv 0$ and the extended weight function $\widetilde{h}(\cdot)$ is locally Hölder continuous on $\mathbb{R}$.

Proof. We note that the existence of the limit $f(iy) = \lim_{x\to+0} f(x+iy)$ for each $y \in \Delta$ yields the existence of $|f|(iy) = \lim_{x\to+0} |f(x+iy)|$ and the relation $|f(iy)| = |f|(iy)$ for each $y \in \Delta$. Hence $|f|(\cdot)$ is continuous. Applying Proposition 5.2 we get that condition (5.10) is satisfied, $\nu_s(\Delta) = 0$ and $\widetilde{h}(\cdot)$ is continuous. In fact, one has $h(y) = -\log(|f|(iy))$, $y \in \Delta$. This yields that the function $\widetilde{h}(\cdot)$ is locally Hölder continuous on Δ as well. If $\widetilde{h}(\cdot)$ is locally Hölder continuous on Δ, then the limit

$$\begin{aligned}\varphi(y) &:= \lim_{x\to+0} \frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right) d\nu(t)\\ &= \lim_{x\to+0}\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right) d\nu_s(t) + \frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right)\widetilde{h}(t)dt\right\},\end{aligned}$$

$z = x + iy \in \mathbb{C}_{\rm right}$, exist for every $y \in \Delta$. Indeed, we have

$$\begin{aligned}&\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right) d\nu_s(t) \qquad (5.13)\\ &\quad = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\, d\nu_s(t) - \frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{y-t}{x^2+(y-t)^2} + \frac{t}{1+t^2}\right) d\nu_s(t)\end{aligned}$$

where we have used $\nu_s(-X) = \nu_s(X)$. Taking into account that $\nu_s(\Delta) = 0$ we immediately get from the representation (5.13) that the limit

$$\varphi_s(y) := \lim_{x\to+0} \frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right) d\nu_s(t) = -\frac{i}{\pi}\int_{\mathbb{R}} \frac{1+yt}{y-t}\frac{d\nu_s(t)}{1+t^2},$$

$z = x + iy \in \mathbb{C}_{\rm right}$, exist for each $y \in \Delta$. Since

$$\begin{aligned}&\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right)\widetilde{h}(t)\,dt\\ &\quad = \frac{1}{\pi}\int_{\mathbb{R}} \frac{x}{x^2+(y-t)^2}\widetilde{h}(t)\,dt - \frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{y-t}{x^2+(y-t)^2} + \frac{t}{1+t^2}\right)\widetilde{h}(t)\,dt\end{aligned}$$

we infer that

$$\begin{aligned}\varphi_{ac}(y) &:= \lim_{x\to+0}\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t}+\frac{t}{1+t^2}\right)\widetilde{h}(t)\,dt\\ &= \widetilde{h}(y)+\lim_{x\to+0}\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{y-t}{x^2+(y-t)^2}+\frac{t}{1+t^2}\right)\widetilde{h}(t)\,dt,\end{aligned}$$

$z=x+iy\in\mathbb{C}_{\rm right}$. If $\widetilde{h}(\cdot)$ is locally Hölder continuous on Δ, then the limit

$$\widetilde{\varphi}_{ac}(y) := \lim_{x\to+0}\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{y-t}{x^2+(y-t)^2}+\frac{t}{1+t^2}\right)\widetilde{h}(t)dt$$

exists for each $y\in\Delta$, and consequently, the limit $\varphi(y)=\varphi_s(y)+\varphi_{ac}(y)$ exist for every $y\in\Delta$. Using the representation

$$\exp\left\{\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t}+\frac{t}{1+t^2}\right)\widetilde{h}(t)dt\right\}f(x+iy)e^{\alpha z}=D(x+iy) \tag{5.14}$$

for $z=x+iy\in\mathbb{C}_{\rm right}$ we find the existence of the limit

$$D(iy) := \lim_{x\to+0}D(x+iy) \tag{5.15}$$

for every $y\in\Delta$. Taking into account (5.14) we find that $D(iy)$ is continuous on Δ. Using the conformal mapping $\mathbb{C}_{\rm right}\ni z\longrightarrow\frac{1-z}{1+z}\in\mathbb{D}:=\{z\in\mathbb{C}: |z|<1\}$ which maps $\mathbb{C}_{\rm right}$ onto $\mathbb{D}$ and setting

$$B(z) := D((1-z)(1+z)^{-1}),\quad z\in\mathbb{D},$$

one defines a Blaschke product in $\mathbb{D}$. The open set Δ transforms into an open set δ of $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$. By the Lindelöf sectorial theorem [18] we get that $B(z)$ admits radial boundary values for each point of δ. The boundary function $B(e^{i\theta}):=\lim_{r\to1}B(re^{i\theta})$ admits the representation

$$B(e^{i\theta})=D(-i\tan(\theta/2)),\quad e^{i\theta}\in\delta. \tag{5.16}$$

Since $D(iy)$ is continuous on Δ the Blaschke product $B(e^{i\theta})$ is continuous on δ. If $e^{i\theta_0}\in\delta$ is an accumulation point of zeros of, then for every $\epsilon>0$ the set $\{B(e^{i\theta}:|\theta-\theta_0|<\epsilon\}$ contains $\mathbb{T}$, see [4, Chapter 5] or [5, Remark 4.A.3]. Since $B(e^{i\theta})$ is continuous on δ, this is impossible which shows that $e^{i\theta_0}$ is not an accumulation point of zeros of $B(z)$. Hence no point of δ is an accumulation point which yields that no point of Δ is an accumulation point of zeros of $f(\cdot)$.

Conversely, let us assume that no point of $i\Delta$ is an accumulation point of zeros of $f(\cdot)$. This yields that no point of δ is an accumulation point of zeros of $B(z)$. Since $\inf_{k\in\mathbb{N}} |e^{i\theta} - z_k| > 0$ for any $e^{i\theta} \in \delta$ by a result of Frostman [10] one gets that the radial boundary values $B(e^{i\theta}) = \lim_{r\to 1} B(re^{i\theta})$ exist for each $e^{i\theta} \in \delta$. Using [5, Remark 4.A.2] we get that $B(e^{i\theta})$ is continuous on δ. Applying again the Lindelöf sectorial theorem [18] we find that $D(iy)$ exists for each $y \in \Delta$ and is continuous.

Since $\nu_s(\Delta) = 0$ the limit $\varphi_s(\cdot)$ exists for every $y \in \Delta$. Because $\widetilde{h}(\cdot)$ is locally Hölder continuous on Δ we conclude that the limit $\varphi_{ac}(y)$ exist for every $y \in \Delta$. Hence the limit $\varphi(y)$ exists for every $y \in \Delta$ and

$$S(iy) := \lim_{x\to+0} \exp\left\{-\frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{iz-t} + \frac{t}{1+t^2}\right) d\nu(t)\right\} e^{-\alpha z},$$

$z = x+iy \in \mathbb{C}_{\rm right}$, exists for every $y \in \Delta$. In this way we have demonstrated the existence of $f(iy)$ and the representation $f(iy) = D(iy)S(iy)e^{-iay}$ for each $y \in \Delta$. Using this representation we get that $f(iy)$ is locally Hölder continuous on Δ and different from zero.

If the limit $f(iy)$ exist for each $y \in \mathbb{R}$, is locally Hölder continuous and different from zero, then in view of the first part no point of the imaginary axis is an accumulation point of zeros of $f(\cdot)$. Therefore, any rectangle of the form $\mathcal{O} := \{z \in \mathbb{C}_{\rm right} : |\Im{\rm m}\,(z)| < y_0, \quad 0 < \Re{\rm e}\,(z) < x_0\}$ contains only a finite number of zeros. Otherwise, it would be exists an imaginary accumulation point. Hence any bounded open sets contains only a finite number of zeros. From the first part it follows that $\widetilde{h}(\cdot)$ is locally Hölder continuous on $\mathbb{R}$.

Conversely, if any open set contains only a finite number of zeros, then, in particular, the rectangle of the form $\mathcal{O}$ contains only a finite number of zeros. Hence imaginary accumulation points do not exists. By the first part it immediately follows that $f(\cdot)$ is locally Hölder continuous and different from from zero on $\mathbb{R}$. $\square$

5.3 Examples

1. If the holomorphic Kato function $f(\cdot)$ has no zeros in $\mathbb{C}_{\rm right}$ and $\nu \equiv 0$, then $f(z) = e^{-z}$, $z \in \mathbb{C}_{\rm right}$, where $\alpha = 1$ follows from condition (5.9).

2. If the holomorphic Kato function $f(\cdot)$ has zeros and the measure $\nu \equiv 0$, then $f(\cdot)$ is of the form $f(z) = D(z)e^{-\alpha z}$, where the Blaschke-type product $D(z)$ is given by (5.3). In particular, if $n = 1$ we find the representation

$$f(z) = \frac{z^2 - 2z\Re e\,(\xi) + |\xi|^2}{z^2 + 2z\Re e\,(\xi) + |\xi|^2}\, e^{-\alpha z}, \quad z \in \mathbb{C}_{\text{right}}\,,$$

where $\xi \in \mathbb{C}_{\text{right}}$ such that

$$\alpha + 4\frac{\Re e\,(\xi)}{|\xi|^2} = 1.$$

This gives the representation

$$f(z) = \frac{z^2 - 2\eta\left(z - \frac{2}{1-\alpha}\right)}{z^2 + 2\eta\left(z + \frac{2}{1-\alpha}\right)}\, e^{-\alpha z}, \quad z \in \mathbb{C}_{\text{right}}, \tag{5.17}$$

$0 < \eta \le \frac{4}{1-\alpha}$, $0 \le \alpha \le 1$, where we have denoted $\xi = \eta + i\tau$, $\eta > 0$, and $\tau = \sqrt{\frac{4}{(1-\alpha)^2} - \left(\eta - \frac{2}{1-\alpha}\right)^2}$. The limit $f(iy) := \lim_{\epsilon \to +0} f(\epsilon + iy)$, $y \in \mathbb{R}$, exists for each $y \ge 0$ and is given by

$$f(iy) = \frac{y^2 + 4\eta\frac{1}{1-\alpha} + 2i\eta y}{y^2 - 4\eta\frac{1}{1-\alpha} + 2i\eta y} e^{-i\alpha y} =: \phi(y), \quad y \in \mathbb{R}.$$

We note that $\phi(\cdot)$ is admissible.

3. If the holomorphic Kato function $f(z)$ has no zeros and the measure ν is atomar, then $f(z)$ admits the representation

$$f(z) = \exp\left\{-\frac{2z}{\pi}\sum_l \frac{1}{z^2 + s_l^2}\nu(\{s_l\})\right\} e^{-\alpha z}, \quad z \in \mathbb{C}_{\text{right}},$$

where $\{s_l\}_l$ the point where $\nu(\{s_l\}) \ne 0$. In the particular case when $d\nu(t) = c\delta(t-s)dt$, $s > 0$, we have

$$f(z) = \exp\left\{-\frac{2zc}{\pi}\frac{1}{z^2 + s^2}\right\} e^{-\alpha z},$$

and $\alpha + \frac{2c}{\pi}\frac{1}{s^2} = 1$ which yields $c = \frac{1}{2}(1-\alpha)\pi s^2$ and

$$f(z) := \exp\left\{-z(1-\alpha)\frac{s^2}{z^2+s^2}\right\} e^{-\alpha z}$$

The limit $f(iy) := \lim_{\epsilon\to+0} f(\epsilon+iy)$, $y \in \mathbb{R}$, exists for all $y \in \mathbb{R}\setminus\{-s,s\}$ and is given by

$$f(iy) = \exp\left\{iy(1-\alpha)\frac{s^2}{y^2-s^2}\right\} e^{-i\alpha y} := \phi(y), \quad y \in \mathbb{R}\setminus\{-s,s\}.$$

The function $\phi(y)$ is admissible.

4. If the holomorphic Kato function $f(z)$ has no zeros and the measure ν is absolutely continuous, that is, $d\nu(t) = h(t)dt$, $h(t)(1+t^2)^{-1} \in L^1(\mathbb{R}_+)$, then $f(z)$ admits the representation

$$f(z) = \exp\left\{-\frac{2z}{\pi}\int_0^\infty \frac{h(t)}{z^2+t^2}dt\right\} e^{-\alpha z}, \quad z \in \mathbb{C}_{\text{right}}$$

such that

$$\alpha + \lim_{x\to+0}\frac{2}{\pi}\int_0^\infty \frac{h(t)}{x^2+t^2}dt = 1.$$

In particular, if $f(x) = (1+\frac{x}{k})^{-k}$, $x \in \mathbb{R}_+$, then the holomorphic continuation $f(z) = (1+\frac{z}{k})^{-k}$ has no zeros which means that in the representation (5.2) the Blaschke-type product $D(x)$ is absent. Moreover, the limit $f(iy) = (1+\frac{iy}{k})^{-k}$ exists for all $y \in \mathbb{R}_+$, $|f(iy)|$ is locally Hölder continuous and different from zero on $\mathbb{R}_+$. Taking into account Theorem 5.3 this yields the representation

$$f(z) = \exp\left\{-\frac{kz}{\pi}\int_{\mathbb{R}_+}\frac{1}{z^2+t^2}\ln\left(1+\frac{t^2}{k^2}\right)dt\right\} e^{-\alpha z}, \quad z \in \mathbb{C}_{\text{right}}.$$

A straightforward computation shows that

$$\lim_{x\to+0}\frac{k}{\pi}\int_{\mathbb{R}_+}\frac{1}{x^2+t^2}\ln\left(1+\frac{t^2}{k^2}\right)dt = 1$$

which yields $\alpha = 0$, and consequently, we have

$$f(z) = \exp\left\{-\frac{kz}{\pi}\int_{\mathbb{R}_+}\frac{1}{z^2+t^2}\ln\left(1+\frac{t^2}{k^2}\right)dt\right\}$$

for $z \in \mathbb{C}_{\text{right}}$.

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