WEGNER ESTIMATE FOR DISCRETE ALLOY-TYPE MODELS

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Abstract. We study discrete alloy type random Schrödinger operators on $\ell^2(\mathbb{Z}^d)$. Wegner estimates are bounds on the average number of eigenvalues in an energy interval of finite box restrictions of these types of operators. If the single site potential is compactly supported and the distribution of the coupling constant is of bounded variation a Wegner estimate holds. The bound is polynomial in the volume of the box and thus applicable as an ingredient for a localisation proof via multiscale analysis.

1. Main results

A discrete alloy type model is a family of operators $H_\omega = H_0 + V_\omega$ on $\ell^2(\mathbb{Z}^d)$. Here H_0 denotes an arbitrary symmetric operator. In most applications H_0 is the discrete Laplacian on $\mathbb{Z}^d$. The random part V_ω is a multiplication operator

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} \omega_k \, u(x-k) \tag{1}$$

defined in terms of an i. i. d. sequence $\omega_k \colon \Omega \to \mathbb{R}, k \in \mathbb{Z}^d$ of random variables each having a density f, and a single site potential $u \in \ell^1(\mathbb{Z}^d; \mathbb{R})$. It follows that the mean value $\bar{u} := \sum_{k\in\mathbb{Z}^d} u(k)$ is well defined. We will assume throughout the paper that u does not vanish identically and that $f \in BV$. Here BV denotes the space of functions with finite total bounded and $\|\cdot\|_{BV}$ denotes the corresponding norm. The mathematical expectation w.r.t. the product measure associated with the random variables $\omega_k, k \in \mathbb{Z}^d$ will be denoted by $\mathbb{E}$.

The estimates we want to prove do not concern the operator $H_\omega, \omega \in \Omega$ but rather its finite box restrictions. Thus for the purposes of the present paper domain and selfadjointness properties of H_ω are irrelevant. For $L \in \mathbb{N}$ we denote the subset $[0, L]^d \cap \mathbb{Z}^d$ by Λ_L, its characteristic function by χ_{Λ_L}, the canonical inclusion $\ell^2(\Lambda_L) \to \ell^2(\mathbb{Z}^d)$ by ι_L and the adjoint restriction $\ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda_L)$ by π_L. The finite cube restriction of H_ω is then defined as $H_{\omega,L} := \pi_L H_0 \iota_L + V_\omega \chi_{\Lambda_L} \colon \ell^2(\Lambda_L) \to \ell^2(\Lambda_L)$. For any $\omega \in \Omega$ and $L \in \mathbb{N}$ the restriction $H_{\omega,L}$ is a selfadjoint finite rank operator. In particular its spectrum consists entirely of real eigenvalues $E(\omega, L, 1) \le E(\omega, L, n) \le \cdots \le E(\omega, L, \sharp\Lambda_L)$ counted including multiplicities. Note that if u has compact support, then there exists an $n \in \mathbb{N}$ and an $x \in \mathbb{Z}^d$ such that supp $\subset \Lambda_{-n} + x$, where $\Lambda_{-n} := \{-k \mid k \in \Lambda_n\}$. We

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may assume without loss of generality $x = 0$ without restricting the model (1). The number of points in the support of u is denoted by $\operatorname{rank} u$. Now we are in the position to state our bounds on the expected number of eigenvalues of finite box Hamiltonians $H_{\omega,L}$ in a compact energy interval $[E-\epsilon, E+\epsilon]$.

Theorem 1. *Assume that the single site potential u has support in Λ_{-n}. Then there exists a constant c_u depending only on u such that for any $L \in \mathbb{N}$, $E \in \mathbb{R}$ and $\epsilon > 0$ we have*

$$\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{[E-\epsilon,E-\epsilon]}(H_{\omega,L})\right]\right\} \leq c_u \, \|f\|_{BV} \operatorname{rank} u \; \epsilon \, (L+n)^{d\cdot(n+1)}$$

Remark 2.
1. By the assumption on the support of the single site potential $\operatorname{rank} u \leq n^d$
2. The constant c_u is given in terms of derivatives of a finite array of polynomials constructed in terms of values of the function u.
3. A bound of the type as it is given in Theorem 1 is called Wegner estimate. If such a bound holds one is interested in the dependence of the RHS of the *length of the energy interval* (in our case 2ϵ) and on the *volume of the cube* Λ_L (in our case L^d).
4. Our bound is linear in the energy-interval length and polynomial in the volume of the cube. This implies that the Wegner bound can be used for a localisation proof via multiscale analysis, see e.g. [3, 1, 6]. More precisely, if an appropriate initial scale estimate is available, the multiscale analysis — using as an ingredient the Wegner estimate as given in Theorem 1 — yields Anderson localisation. As the Wegner bound is valid on the whole energy axis one can prove Anderson localisation in any energy region where the initial scale estimate holds.
5. If the single site potential u does not have compact support, one has to use an enhanced version of the multiscale analysis and so-called uniform Wegner estimates to prove localisation, see [8]. However, there exists criteria which allow one to turn a standard Wegner estimate into a uniform one, see Lemma 4.10.2 in [15].
6. The main point of the theorem is that no assumption on u (apart form the compact support) is required. In particular, the sign of u can change arbitrarily. The single site potential may be even degenerate in the sense that $\bar{u} = 0$. Also, note that the result holds on the whole energy axis. These two properties are in contrast to earlier results on Wegner estimates for sign-changing single site potentials. See the discussion of the previous literature at the end of this section.
7. If u does satisfy the assumption $\bar{u} \neq 0$ we obtain an even better bound. This is the content of Theorem 3 below.

The next Theorem applies to single site potentials $u \in \ell^1(\mathbb{Z}^d)$ with non vanishing mean $\bar{u} \neq 0$. Let $m \in \mathbb{N}$ be such that $\sum_{\|k\| \geq m} |u(k)| \leq |\bar{u}/2|$. Here $\|k\| = \|k\|_\infty$ denotes the sup-norm.

Theorem 3. *Assume $\bar{u} \neq 0$ and that f has compact support. Then we have for any $L \in \mathbb{N}$, $E \in \mathbb{R}$ and $\epsilon > 0$*

$$\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{[E-\epsilon,E-\epsilon]}(H_{\omega,L})\right]\right\} \leq \frac{8}{\bar{u}} \, \|f\|_{BV} \, \min\left(L^d, \operatorname{rank} u\right) \; \epsilon \, (L+m)^d$$

In the case that the support of u is compact, we have an important

Corollary 4. *Assume* $\bar{u} \neq 0$ *and* $\operatorname{supp} u \subset \Lambda_{-n}$. *Then we have for any* $L \in \mathbb{N}$, $E \in \mathbb{R}$ *and* $\epsilon > 0$

$$\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{[E-\epsilon,E-\epsilon]}(H_{\omega,L})\right]\right\} \leq \frac{4}{\bar{u}}\,\|f\|_{BV}\ \operatorname{rank} u\ \epsilon\,(L+n)^d$$

Im particular, the function $\mathbb{R} \ni E \to \mathbb{E}\left\{\operatorname{Tr}\left[\chi_{(-\infty,E]}(H_{\omega,L})\right]\right\}$ *is Lipschitz continuous.*

If the operator H_ω has a well defined integrated density of states $N\colon \mathbb{R} \to \mathbb{R}$, meaning that

$$\lim_{L\to\infty} \frac{1}{L^d}\,\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{(-\infty,E]}(H_{\omega,L})\right]\right\} = N(E)$$

at all continuity points of N, then Corollary 4 implies that the integrated density of states is Lipschitz continuous. Consequently its derivative, the density of states, exists for almost all $E \in \mathbb{R}$.

Remark 5. The situation that the two cases $\bar{u} \neq 0$ and $\bar{u} = 0$ have to be distinguished occurs also in other contexts, see for instance the paper [10] on weak disorder localisation.

When looking at Theorems 1 and 3 one might wonder what kind of Wegner bound holds for non-compactly supported single site potentials with vanishing mean. To apply the methods of the present paper in this case it seems that one has to require that u tends to zero exponentially fast. So far only the case of one space dimension is settled:

Theorem 6. *Assume that* f *has compact support and that there exists* $s \in (0,1)$ *and* $C \in (0,\infty)$ *such that* $|u(k)| \leq Cs^{|k|}$ *for all* $k \in \mathbb{Z}$. *Then there exists an integer* $D \in \mathbb{N}_0$ *such that for each* $\beta > D/|\log s|$ *there exists a constant* $K \in (0,\infty)$ *such that for all* $L \in \mathbb{N}$, $E \in \mathbb{R}$ *and* $\epsilon > 0$

$$\mathbb{E}\left\{\operatorname{Tr}\left[\chi_{[E-\epsilon,E-\epsilon]}(H_{\omega,L})\right]\right\} \leq \frac{8}{c}\,\|f\|_{BV}\ \epsilon\,L\,(L+\beta\log L+2K)^{D+1}$$

Let us discuss the relation of the above theorems to previous results [9, 14, 4, 11, 13] on Wegner estimates with single site potentials which are allowed to change sign. The papers [9, 4] concern alloy type Schrödinger operators on $L^2(\mathbb{R}^d)$. The main result is a Wegner estimate for energies in a neighbourhood of the infimum of the spectrum. It applies to arbitrary non-vanishing single site potentials $u \in C_c(\mathbb{R}^d)$ and coupling constants with a picewise absolutely continuous density. The upper bound is linear in the volume of the box and Hölder-continuous in the energy variable.

The results of [14, 11, 13] concern both alloy type Schrödinger operators on $L^2(\mathbb{R}^d)$ and discrete alloy type Schrödinger operators on $\ell^2(\mathbb{Z}^d)$. Since the present paper concerns the latter model we will discuss these results here first. The papers [14, 11, 13] establish results analogous to Corollary 4 above, however only under more restrictive assumptions. For instance in [13] the required hypothesis is that $\operatorname{supp} u$ is compact and that the function

$$\theta \mapsto \sum_{k\in\mathbb{Z}^d} u(k)\mathrm{e}^{-\mathrm{i}k\cdot\theta} \text{ does not vanish on } [0,2\pi)^d. \tag{2}$$

For alloy type Schrödinger operators on $L^2(\mathbb{R}^d)$ [14, 11, 13] derive a Wegner estimate which is linear in the volume of the box and Lipschitz continuous in the energy variable. The bound is valid for all compact intervals along the energy axis. These bounds are valid for single site potentials $u \in L^\infty_c(\mathbb{R}^d)$ which have a generalised step function form and satisfy a condition analogous to (2).

Let us stress that Wegner estimates for sign changing single site potentials are harder to prove for operators on $L^2(\mathbb{R}^d)$ than for ones on $\ell^2(\mathbb{Z}^d)$. The reason is that for discrete models we have in the randomness a degree of freedom for each point in the configuration space $\mathbb{Z}^d$. For the continuum alloy type model the configuration space is $\mathbb{R}^d$ while the degrees of freedom are indexed by a much smaller set, namely $\mathbb{Z}^d$.

Recently a fractional moment bound for the alloy type model on $\ell^2(\mathbb{Z})$ has been proven in [2]. It holds for arbitrary compactly supported single site potentials. The result can be extended to the one-dimensional strip, while the extension to $\mathbb{Z}^d$ is unclear at the moment.

2. An abstract Wegner estimate and the proof of Theorem 3

An important step in the proofs of the Theorems of the last section is an abstract Wegner estimate which we formulate now.

Theorem 7. *Let $L \in \mathbb{N}, E \in \mathbb{R}, \epsilon > 0$ and $I := [E-\epsilon, E+\epsilon]$. Denote by $E(\omega, L, n)$ the n-th eigenvalue of the operator $H_{\omega,L}$. Assume that there exist an $\delta > 0$ and $a \in \ell^1(\mathbb{Z}^d;\mathbb{R})$ such that for all n*

$$\sum_{k\in\mathbb{Z}^d} a(k) \frac{\partial}{\partial\omega_k} E(\omega, L, n) \geq \delta \tag{3}$$

Then

$$\mathbb{E}\left(\operatorname{Tr}\chi_I(H_{\omega,L})\right) \leq \frac{4\epsilon}{\delta} \sum_{k\in\mathbb{Z}^d} a(k)\, \|f\|_{BV}\ \operatorname{rank}(\chi_\Lambda u(\cdot - k))$$

Since $a \in \ell^1$ and the derivatives $\frac{\partial}{\partial\omega_k}E(\omega, L, n)$ are uniformly bounded, the sum (3) is absolutely convergent. Note that one can always replace the sum $\sum_{k\in\mathbb{Z}^d}$ by $\sum_{k\in\Lambda_L^+}$. Here $\Lambda_L^+ = \{k \in \mathbb{Z}^d \mid u(\cdot - k) \cap \Lambda_L \neq \emptyset\}$ denotes the set of lattice points such that the corresponding coupling constant influences the potential in the box Λ_L. In particular, if the support of u is contained in $[-n, \dots, 0]^d$, the sum reduces to $\sum_{k\in\Lambda_{L+n}}$. We give a simple sufficient condition which ensures the hypothesis of Theorem 7.

Corollary 8. *Let $L \in \mathbb{N}, \epsilon > 0$ and $I := [E-\epsilon, E+\epsilon]$. Assume that there exist an $\delta > 0$ and $a \in \ell^1(\mathbb{Z}^d;\mathbb{R})$ such that for all n and all $x \in \Lambda_L$*

$$\sum_{k\in\mathbb{Z}^d} a(k) u(x-k) \geq \delta$$

Then

$$\mathbb{E}\left(\operatorname{Tr}\chi_I(H_{\omega,L})\right) \leq \frac{4\epsilon}{\delta} \sum_{k\in\mathbb{Z}^d} a(k)\, \|f\|_{BV}\ \operatorname{rank}(\chi_\Lambda u(\cdot - k))$$

Proof. By first order perturbation theory, respectively the Helman-Feynman formula we have

$$\frac{\partial}{\partial \omega_k} E(\omega, L, n) = \langle \psi_n, u(\cdot - k)\psi_n \rangle$$

where ψ_n is the normalised eigensolution to $H_{\omega,L}\psi_n = E(\omega, L, n)\psi_n$. Thus

$$\sum_{k \in \mathbb{Z}^d} a(k) \frac{\partial}{\partial \omega_k} E(\omega, L, n) = \sum_{k \in \mathbb{Z}^d} a(k) \langle \psi_n, u(\cdot - k)\psi_n \rangle \geq \delta$$

□

The proof of Theorem 7 relies on quite standard techniques, see e.g. [16, 7, 4, 12]. The main point of the Theorem is that it singles out a relation between properties of linear combinations of single site potentials and a Wegner estimate. In the course of the proof we will need the following estimate.

Lemma 9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $BV \cap L^1(\mathbb{R})$, $\rho \in C^\infty(\mathbb{R})$, $k \in \mathbb{Z}^d$ and $s \in \mathbb{R}$. Then*

$$\sum_{n \in \mathbb{N}} \int d\omega_k f(\omega_k) \frac{\partial}{\partial \omega_k} \rho(E(\omega, L, n) + s) \leq \|f\|_{BV} \operatorname{rank}(\chi_\Lambda u(\cdot - k)) \int |\rho'(x)| dx$$

Note that if $k \notin \Lambda_L^+$ then $\frac{\partial}{\partial \omega_k} E(\omega, L, n) = 0$. Also note that the sum over n is in fact finite since $H_{\omega,L}$ is defined on a finite dimensional vector space.

Proof. We will use that if $g \in C^\infty$ and $f \in BV \cap L^1$ the partial integration bound

$$\int f(x) g'(x) dx \leq \|g\|_\infty \|f\|_{BV}$$

holds. Denote by $E(\omega, \omega_k = 0, L, n)$ the n-th eigenvalue of the operator $H_{\omega,\omega_k=0,L} := H_{\omega,L} - \omega_k u(\cdot - k)$ on $\ell^2(\Lambda_L)$. Partial integration yields

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} \int d\omega_k f(\omega_k) \frac{\partial}{\partial \omega_k} \rho(E(\omega, L, n) + s) \\
&= \int d\omega_k f(\omega_k) \frac{\partial}{\partial \omega_k} \sum_{n \in \mathbb{N}} \Big(\rho(E(\omega, L, n) + s) - \rho(E(\omega, \omega_k = 0, L, n) + s) \Big) \\
&\leq \|f\|_{BV} \sup_{\omega_k \in \operatorname{supp} f} \Big| \sum_{n \in \mathbb{N}} (\rho(E(\omega, L, n) + s) - \rho(E(\omega, \omega_k = 0, L, n) + s)) \Big|
\end{aligned}$$

Here we used that $\omega_k \mapsto E(\omega, L, n)$ is an infinitely differentiable function cf. [5]. Now

$$\begin{aligned}
&\sum_{n \in \mathbb{N}} \rho(E(\omega, L, n) + s) - \rho(E(\omega, \omega_k = 0, L, n) + s) \\
&\qquad = \operatorname{Tr} \Big(\rho((H_{\omega,L} + s) - \rho((H_{\omega,\omega_k=0,L} + s) \Big)
\end{aligned}$$

can be expressed in terms of the spectral shift function $\xi(\cdot, H_{\omega,L}, H_{\omega,\omega_k=0,L})$ of the operator pair $H_{\omega,L}, H_{\omega,\omega_k=0,L}$ as

$$\int \rho'(x) \xi(x, H_{\omega,L}, H_{\omega,\omega_k=0,L}) dx.$$

Since $\|\xi\|_\infty$ is bounded by the rank of the perturbation $\chi_\Lambda u(\cdot - k)$, we obtain

$$\sum_{n\in\mathbb{N}} \rho(E(\omega, L, n) + s) - \rho(E(\omega, \omega_k = 0, L, n) + s) \leq \operatorname{rank}(\chi_\Lambda u(\cdot - k)) \int |\rho'|$$

and the proof of the Lemma is completed. □

Now we turn to the proof of Theorem 7.

Proof of Theorem 7. Let $\rho \in C^\infty(\mathbb{R})$ be a non-decreasing function such that on $(-\infty, -\epsilon]$ it is identically equal to -1, on $[\epsilon, \infty)$ it is identically equal to zero and $\|\rho'\|_\infty \leq 1/\epsilon$. By the chain rule we have

$$\sum_{k\in\mathbb{Z}^d} a(k) \frac{\partial}{\partial \omega_k} \rho(E(\omega, L, n) - E + t)$$
$$= \rho'(E(\omega, L, n) - E + t) \sum_{k\in\mathbb{Z}^d} a(k) \frac{\partial}{\partial \omega_k} E(\omega, L, n)$$

The assumption (3) implies now

$$\rho'(E(\omega, L, n) - E + t) \leq \frac{1}{\delta} \sum_{k\in\mathbb{Z}^d} a(k) \frac{\partial}{\partial \omega_k} \rho(E(\omega, L, n) - E + t)$$

Since $\chi_I \leq \int_{-2\epsilon}^{2\epsilon} dt\, \rho'(x - E + t)$ for $I := [E - \epsilon, E + \epsilon]$ we have

$$\operatorname{Tr} \chi_I(H_{\omega,L}) \leq \frac{1}{\delta} \int_{-2\epsilon}^{2\epsilon} dt \sum_{n\in\mathbb{N}} \sum_{k\in\mathbb{Z}^d} a(k) \frac{\partial}{\partial \omega_k} \rho(E(\omega, L, n) - E + t)$$

Note that for a random variable $F\colon \Omega \to \mathbb{R}$ we have $\mathbb{E}(F) = \mathbb{E}\left(\int f(\omega_k) d\omega_k F(\omega)\right)$ Thus using Lemma 9 and $\int |\rho'(x)| dx = 1$ we obtain

$$\mathbb{E}(\operatorname{Tr} \chi_I(H_{\omega,L})) \leq \frac{4\epsilon}{\delta} \sum_{k\in\mathbb{Z}^d} a(k)\, \|f\|_{BV}\ \operatorname{rank}(u \cdot \chi_\Lambda)$$

□

Now we are in the position to give a

Proof of Theorem 3. Let ψ_n be a normalised eigenfunction associated to $E(\omega, L, n)$ and $Q(L, m) = \bigcup_{k\in\Lambda_L} \left(k + [-m, m]^d \cap \mathbb{Z}^d\right)$. W.l.o.g. we may assume $\bar{u} > 0$. Then $\sum_{k\in Q(L,m)} u(k) \geq \bar{u}/2$. Choose now the coefficients in Corollary 8 in the following way: $a(k) = 1$ for $k \in Q(L, m)$ and $a(k) = 0$ for k in the complement of $Q(L, m)$. Then

$$\sum_{k\in\mathbb{Z}^d} a(k) \langle \psi_n, u(\cdot - k) \psi_n \rangle = \langle \psi_n, \sum_{k\in Q(L,m)} u(\cdot - k) \psi_n \rangle \geq \bar{u}/2.$$

□

Proof of Corollary 4. Set $a(k) = 1$ for $k \in \Lambda_{L+n}$ and $a(k) = 0$ for k in the complement of Λ_{L+n}. Then

$$\sum_{k\in\mathbb{Z}^d} a(k) \langle \psi_n, u(\cdot - k) \psi_n \rangle = \langle \psi_n, \sum_{k\in\Lambda_{L+n}} u(\cdot - k) \psi_n \rangle = \bar{u}$$

An application of Corollary 8 now completes the proof. □

3. Proof of Theorem 1

In this section we give a proof of Theorem 1. In view of Theorem 3 it is sufficient to consider the case that the single site potential $u\colon \mathbb{Z}^d \to \mathbb{R}, u \in \ell^1(\mathbb{Z}^d)$ is degenerate in the sense that $\sum_{x\in\mathbb{Z}^d} u(x) = 0$. We explain how to find in this situation an appropriate linear combination of single site potentials — or, equivalently, an appropriate linear transformation of the random variables — which can be efficiently used for averaging. The aim of the linear transformation is to extract a perturbation potential which is strictly positive on the box Λ.

Let us first consider the case $d = 1$. Then we can assume without loss of generality that $\operatorname{supp} u \subset \{-n, \dots 0\}$. For a given cube $\Lambda_L = \{0, \dots, L\}$ we are looking for an array of numbers $a_k, k \in \Lambda_{L+n}$ such that we have

$$\sum_{k\in\Lambda_{L+n}} a_k u(x-k) = \mathit{constant} > 0 \quad \text{for all } x \in \Lambda_L \tag{4}$$

In fact, we will find a sequence of numbers $a_k, k \in \mathbb{N}$ such that we have

$$\sum_{k\in\mathbb{N}} a_k u(x-k) = \mathit{constant} > 0 \quad \text{for all } x \in \mathbb{N} \tag{5}$$

If we truncate this sequence, we obtain an array of numbers satisfying (4).

For a function $F\colon (1-\epsilon, 1+\epsilon) \to \mathbb{R}$ with $\epsilon > 0$ we say that it has a root of order $m \in \{0, \dots, n\}$ at $t = 1$ iff it is in $C^m(1-\epsilon, 1+\epsilon)$ and

$$\Big(\frac{d^j}{dt^j} F(t)\Big)\Big|_{t=1} = 0 \quad \text{for } j = 0, \dots, m-1 \tag{6}$$

$$c(F) := \Big(\frac{d^m}{dt^m} F(t)\Big)\Big|_{t=1} \neq 0 \tag{7}$$

In particular, $m = 0$ means that $F(1) \neq 0$. If F is a polynomial of degree not exceeding m, if (6) holds and in addition $c(F) = 0$, then $F \equiv 0$. In this case we say that F has a root of infinite order at $t = 1$.

Given a function $w\colon \mathbb{Z} \to \mathbb{R}$ such that $F_w(t) := \sum_{\nu\in\mathbb{Z}} t^\nu w(-\nu)$ converges for $t \in (1-\epsilon, 1+\epsilon)$ we call $(1-\epsilon, 1+\epsilon) \ni t \mapsto F(t) := F_w(t)$ the accompanying (Laurent) series of w. If $\operatorname{supp} w \subset \{-n, \dots, 0\}$ we call $t \mapsto p(t) := p_w(t) := \sum_{\nu=0}^n t^\nu w(-\nu)$ the accompanying polynomial of w.

Lemma 10. *Let $D \in \mathbb{N}_0$ and $a_k = k^D$ for all $k \in \mathbb{N}$. Let m be the order of the root $t = 1$ of the Laurent series f accompanying the function $w\colon \mathbb{Z} \to \mathbb{R}$ with convergent series $\sum_{\nu\in\mathbb{Z}} t^\nu w(-\nu)$ for $t \in (1-\epsilon, 1+\epsilon)$.*
(a) If $m > D$ then $\sum_{k\in\mathbb{Z}} a_k w(x-k) = 0$ for all $x \in \mathbb{N}$.
(b) If $m = D$ then $\sum_{k\in\mathbb{Z}} a_k w(x-k) = c(F)$ for all $x \in \mathbb{N}$.

An important and well known special case is

Corollary 11. *Let $D \in \mathbb{N}_0$ and $a_k = k^D$ for all $k \in \mathbb{N}$. Let m be the order of the root $t = 1$ of the polynomial p accompanying the function $w\colon \mathbb{Z} \to \mathbb{R}$ with* $\operatorname{supp} w \subset \{-n, \dots, 0\}$.
(a) If $m > D$ then $\sum_{k=x}^{x+n} a_k w(x-k) = 0$ for all $x \in \mathbb{N}$.

(b) If $m = D$ then $\sum_{k=x}^{x+n} a_k w(x-k) = c(p)$ for all $x \in \mathbb{N}$.

Due to the support condition $\sum_{k\in\mathbb{N}} a_k w(x-k) = \sum_{k=x}^{x+n} a_k w(x-k)$ for all $x \in \mathbb{N}$.

Proof of Lemma 10. First note that for arbitrary $\nu \in \mathbb{N}$ and $s \in \mathbb{R}$ we have

$$\frac{d^\nu}{ds^\nu} F(e^s) = \sum_{\kappa=1}^{\nu} c_\kappa \, F^{(\kappa)}(e^s)\, e^{\kappa s}$$

with some $c_1, \dots, c_{\nu-1} \in \mathbb{N}_0$ and $c_\nu = 1$. For the value $s = 0$ it follows from (6) that $\frac{d^\nu}{ds^\nu} F(e^s) = 0$ for $\nu = 0, \dots, m-1$ and from (7) that $\frac{d^m}{ds^m} F(e^s) = F^{(m)}(e^s)\, e^{ms} = c(F)$.

We note that $a_k = \frac{d^D}{ds^D} e^{ks}$ for $s = 0$ and insert this into the LHS of (5) to obtain

$$\sum_{k\in\mathbb{Z}} a_k w(x-k) = \sum_{k\in\mathbb{Z}} w(x-k) \frac{d^D}{ds^D} e^{ks} = \sum_{\nu\in\mathbb{Z}} w(-\nu) \frac{d^D}{ds^D} e^{(\nu+x)s}$$

$$= \frac{d^D}{ds^D}\big(e^{xs}\, F(e^s)\big) = \sum_{r=0}^{D} \binom{D}{r} \Big(\frac{d^r}{ds^r} F(e^s)\Big)\Big(\frac{d^{D-r}}{ds^{D-r}} e^{xs}\Big). \tag{8}$$

For $s = 0$, (8) vanishes if $D < m$ and equals $c(F)$ if $D = m$. □

Thus we have found in the case $d = 1$ and $w = u$ a linear combination with the desired property (5). In the multidimensional situation we will reduce the dimension one by one and construct from a non-vanishing single site potential in dimension j a non-vanishing one in dimension $j-1$. In each reduction step we apply Corollary 11.

Let $w^{(j)} : \mathbb{Z}^j \to \mathbb{R}$ be compactly supported and not identically vanishing. W.l.o.g. we assume $\operatorname{supp} w^{(j)} \subset [-n, 0]^j \cap \mathbb{Z}^j$. Next we define a 'projected' single site potential $w^{(j-1)} : \mathbb{Z}^{j-1} \to \mathbb{R}$ as follows. Consider the family of polynomials $p(x_1, \dots, x_{j-1}, \cdot) : \mathbb{R} \to \mathbb{R}$, indexed by $(x_1, \dots, x_{j-1}) \in \{-n, \dots, 0\}^{j-1}$ and defined by

$$p(x_1, \dots, x_{j-1}, t) := \sum_{\nu=0}^{n} t^\nu \, w^{(j)}(x_1, \dots, x_{j-1}, -\nu). \tag{9}$$

Let $m(x_1, \dots, x_{j-1}) \in \{0, \dots, n, \infty\}$ be the order of the root $t = 1$ of the polynomial $p(x_1, \dots, x_{j-1}, \cdot)$ and $M := M_j := \min\big\{m(x_1, \dots, x_{j-1}) \mid x_1, \dots, x_{j-1} \in \{-n, \dots, 0\}\big\}$ the minimal degree occurring in the family. Since $w^{(j)}$ does not vanish identically, $M_j \le n$. Set $I_{j-1} := \{(x_1, \dots, x_{j-1}) \in \{-n, \dots, 0\}^{j-1} \mid m(x_1, \dots, x_{j-1}) = M_j\}$ and $J_{j-1} := \{(x_1, \dots, x_{j-1}) \in \{-n, \dots, 0\}^{j-1} \mid m(x_1, \dots, x_{j-1}) > M_j\}$.

Lemma 12. *For all $(x_1, \dots, x_{j-1}) \in \{-n, \dots, 0\}^{j-1}$ we have the equality*

$$\sum_{k\in\mathbb{N}} k^M \, w^{(j)}(x_1, \dots, x_{j-1}, x_j - k) = \Big(\frac{d^M}{dt^M} p(x_1, \dots, x_{j-1}, t)\Big)\Big|_{t=1}. \tag{10}$$

We denote the function in (10) by $w^{(j-1)} : \mathbb{Z}^j \to \mathbb{R}$. Then $w^{(j-1)}$ is independent of the variable x_j and therefore we call it the single site potential in reduced *dimension and consider it sometimes as a function* $w^{(j-1)}\colon \mathbb{Z}^{j-1} \to \mathbb{R}$. *Its support is contained in* $\{-n,\dots,0\}^{j-1}$.

Moreover, $w^{(j-1)}(x_1,\dots,x_{j-1}) = 0$ *if* $(x_1,\dots,x_{j-1}) \in J_{j-1}$ *and* $w^{(j-1)}(x_1,\dots,x_{j-1}) \neq 0$ *if* $(x_1,\dots,x_{j-1}) \in I_{j-1}$.

Remark 13. The lemma establishes in particular that

- M is an element of $\{0,\dots,n\}$. If we had $M \geq n+1$, then all polynomials $p(x_1,\dots,x_{j-1},\cdot)$ would vanish identically and thus $w^{(j)} \equiv 0$ contrary to our assumption.
- $w^{(j-1)}$ does not vanish identically. In fact $\operatorname{supp} w^{(j-1)} = I_{j-1} \neq \emptyset$ by definition.

Proof. Consider first the case $(x_1,\dots,x_{j-1}) \in J_{j-1}$. Then for any $x_j \in \mathbb{N}$

$$w^{(j-1)}(x_1,\dots,x_{j-1}) = \sum_{k\in\mathbb{N}} k^M\, w^{(j)}(x_1,\dots,x_{j-1},x_j-k) = 0$$

by Lemma 11, part (a), since $t = 1$ is a root of order $M+1$ or higher of the accompanying polynomial $p(x_1,\dots,x_{j-1},\cdot)$.

Now, if $(x_1,\dots,x_{j-1}) \in I_{j-1}$ then the order of the root $t = 1$ of the polynomial $p(x_1,\dots,x_{j-1},\cdot)$ equals M. Thus by part (b) of Lemma 11

$$\begin{aligned} w^{(j-1)}(x_1,\dots,x_{j-1}) &= \sum_{k\in\mathbb{N}} k^M\, w^{(j)}(x_1,\dots,x_{j-1},x_j-k) \\ &= \Big(\frac{d^M}{dt^M} p(x_1,\dots,x_{j-1},t)\Big)\Big|_{t=1} \end{aligned}$$

for all $x_j \in \mathbb{N}$. □

In the last step $j = 1 \to j-1 = 0$ of the induction we obtain a reduced single site potential

$$w^{(0)} = \Big(\frac{d^{M_1}}{dt^{M_1}} p(t)\Big)\Big|_{t=1} = c(p)$$

which is a simply non-zero real.

Now we describe the result which is obtained after the reduction is applied d times. Given a single site potential $u\colon \mathbb{Z}^d \to \mathbb{R}$ with $\operatorname{supp} u \subset [-n,0]^d \cap \mathbb{Z}^d$, set $w^{(d)} = u$ and

$$w^{(0)} = \sum_{k_1\in\mathbb{N}} k_1^{M_1} w^{(1)}(x_1-k_1) \tag{11}$$

$$= \sum_{k_1\in\mathbb{N}} k_1^{M_1} \cdots \sum_{k_d\in\mathbb{N}} k_d^{M_d} w^{(d)}(x_1-k_1,\dots,x_d-k_d) \tag{12}$$

Thus we have produced a linear combination of single site potentials

$$\sum_{k\in\Lambda_{L+n}} b_k w^{(d)}(x_1-k_1,\dots,x_d-k_d) \quad \text{where} \quad b_k := k_1^{M_1}\dots k_d^{M_d}$$

which is a constant, non-vanishing function on the cube Λ_L. Moreover, the coefficients satisfy the bound

$$|b_k| \leq k_1^n \dots k_d^n \leq (L+n)^{d\cdot n} \quad \text{for all} \quad k \in \Lambda_{L+n}$$

Now an application of Corollary 8 with the choice $a_k = b_k$ for $k \in \Lambda_{L+n}$ and $a_k = 0$ for k in the complement of this set completes the *Proof of Theorem 1.*

4. Proof of Theorem 6

The assumption on the exponential decay of u implies that $F(z) = \sum_{\nu\in\mathbb{Z}} z^\nu u(-\nu)$ is an absolutely and uniformly convergent Laurent series on the annulus $\{z \in \mathbb{C} \mid r_1 \leq |z| \leq r_2\}$ for some $0 < r_1 < 1 < r_2 < \infty$ and represents there a holomorphic function. This implies that there exists a $D \in \mathbb{N}_0$ such that $c(F) := \frac{\partial^D}{\partial z^D} F(z) \mid_{z=1} \neq 0$. Otherwise F would be identically vanishing implying that u vanishes identically. Thus the root $z = 1$ of F has a well defined, finite order $D \in \mathbb{N}_0$ and Lemma 10 can be applied.

The problem is now that the series $\sum_{k\in\mathbb{Z}} k^D$ is not absolutely convergent. For this reason we will replace it with an appropriate finite cut-off sum. Assume in the following w.l.o.g. that $c(F) > 0$. A lengthy but easy calculation shows that for all $\beta > D/|\log s|$ there exists a constant $K \in (0,\infty)$ such that

$$\forall\ x \in \Lambda_L : \sum_{k\notin\{-K,\dots,m\}} |k|^D |u(x-k)| \leq \frac{c(F)}{2}$$

where $m = L + \beta \log L + K$. Consequently

$$\forall\ x \in \Lambda_L : \sum_{k\in\{-K,\dots,m\}} k^D u(x-k) \geq \frac{c(F)}{2}$$

Thus we can apply Corollary 8 with the choice $a_k = k^D$ for $k \in \{-k,\dots,m\}$ and $a_k = 0$ for $k \in \{-k_1,\dots,m+1\}$ and obtain

$$\mathbb{E}\left\{\mathrm{Tr}\left[\chi_{[E-\epsilon,E-\epsilon]}(H_{\omega,L})\right]\right\} \leq \frac{8\epsilon}{c(F)} \|f\|_{BV}\ L(L+\beta\log L + 2K)^{D+1}$$

References

[1] H. v. Dreifus and A. Klein. A new proof of localization in the Anderson tight binding model. *Comm. Math. Phys.*, 124(2):285–299, 1989.

[2] A. Elgart, M. Tautenhahn, and I. Veselić. Exponential decay of Green's function for Anderson models on $\mathbb{Z}$ with single-site potentials of finite support. preprint http://arxiv.org/abs/0903.0492.

[3] J. Fröhlich and T. Spencer. Absence of diffusion in the Anderson tight binding model for large disorder or low energy. *Commun. Math. Phys.*, 88:151–184, 1983.

[4] P. D. Hislop and F. Klopp. The integrated density of states for some random operators with nonsign definite potentials. *J. Funct. Anal.*, 195(1):12–47, 2002. www.ma.utexas.edu/mp_arc, preprint no. 01-139 (2001).

[5] T. Kato. *Perturbation Theory of Linear Operators*. Springer, Berlin, 1966.

[6] W. Kirsch. An invitation to random Schroedinger operators. http://wwwarXiv.org/abs/0709.3707.

[7] W. Kirsch. Wegner estimates and Anderson localization for alloy-type potentials. *Math. Z.*, 221:507–512, 1996.

[8] W. Kirsch, P. Stollmann, and G. Stolz. Anderson localization for random Schrödinger operators with long range interactions. *Comm. Math. Phys.*, 195(3):495–507, 1998.

[9] F. Klopp. Localization for some continuous random Schrödinger operators. *Commun. Math. Phys.*, 167:553–569, 1995.

[10] F. Klopp. Weak disorder localization and Lifshitz tails: continuous Hamiltonians. *Ann. Henri Poincaré*, 3(4):711–737, 2002.

[11] V. Kostrykin and I. Veselić. On the Lipschitz continuity of the integrated density of states for sign-indefinite potentials. *Math. Z.*, 252(2):367–392, 2006. http://arXiv.org/math-ph/0408013.

[12] D. Lenz, N. Peyerimhoff, O. Post, and I. Veselić. Continuity properties of the integrated density of states on manifolds. *Jpn. J. Math.*, 3(1):121–161, 2008. http://www.arxiv.org/abs/0705.1079.

[13] I. Veselić. Wegner estimates for sign-changing single site potentials. http://arxiv.org/abs/0806.0482.

[14] I. Veselić. Wegner estimate and the density of states of some indefinite alloy type Schrödinger operators. *Lett. Math. Phys.*, 59(3):199–214, 2002. http://www.ma.utexas.edu/mp_arc/c/00/00-373.ps.gz.

[15] I. Veselić. *Existence and regularity properties of the integrated density of states of random Schrödinger Operators*, volume Vol. 1917 of *Lecture Notes in Mathematics*. Springer-Verlag, 2007.

[16] F. Wegner. Bounds on the DOS in disordered systems. *Z. Phys. B*, 44:9–15, 1981.

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