ON THE CAUCHY PROBLEM FOR FOCUSING AND DEFOCUSING GROSS-PITAEVSKII HIERARCHIES

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Abstract. We consider the dynamical Gross-Pitaevskii (GP) hierarchy on $\mathbb{R}^d$, $d \geq 1$, for cubic, quintic, focusing and defocusing interactions. For both the focusing and defocusing case, and any $d \geq 1$, we prove local wellposedness of the Cauchy problem in weighted Sobolev spaces $\mathcal{H}^\alpha_\xi$ of sequences of marginal density matrices, for

$$\alpha \begin{cases} > & \frac{1}{2} & \text{if } d = 1 \\ > & \frac{d}{2} - \frac{1}{2(p-1)} & \text{if } d \geq 2 \text{ and } (d,p) \neq (3,2) \\ \geq & 1 & \text{if } (d,p) = (3,2)\,, \end{cases}$$

where $p = 2$ for the cubic, and $p = 4$ for the quintic GP hierarchy; the parameter $\xi > 0$ is arbitrary and determines the energy scale of the problem. This result includes the proof of an a priori spacetime bound conjectured by Klainerman and Machedon for the cubic GP hierarchy in $d = 3$. In the defocusing case, we prove global wellposedness in $\mathcal{H}^1_\xi$ of the cubic GP hierarchy for $1 \leq d \leq 3$, and of the quintic GP hierarchy for $1 \leq d \leq 2$. For the focusing GP hierarchies, we prove lower bounds on the blowup rate, and pseudoconformal invariance in the cases corresponding to L^2 criticality. All of these results hold without the assumption of factorized initial conditions.

1. Introduction

The derivation of the nonlinear Schrödinger equation as the dynamical mean field limit of the manybody quantum dynamics of interacting Bose gases is a research area that is recently experiencing remarkable progress, see [7, 8, 9, 15, 14, 21] and the references therein, and also [1, 6, 10, 11, 12, 13, 23]. A main motivation to investigate this problem is to understand the dynamical behavior of Bose-Einstein condensates. For recent developments in the mathematical analysis of Bose gases and their condensation, we refer to the fundamental work of Lieb, Seiringer, Yngvason, et al.; see [2, 16, 17, 18] and the references therein.

The procedure developed in the landmark works of Erdös, Schlein, and Yau, [7, 8, 9], to obtain the dynamical mean field limit of an interacting Bose gas, comprises the following main ingredients. One determines the BBGKY hierarchy of marginal density matrices for particle number N, and derives the Gross-Pitaevskii (GP) hierarchy in the limit $N \to \infty$, for a scaling where the particle interaction potential tends to a delta distribution; see also [15, 22]. For factorized initial data, the solutions of the GP hierarchy are governed by a cubic NLS for systems with 2-body interactions, [7, 8, 9, 15], and quintic NLS for systems with 3-body interactions, [5]. The proof of the uniqueness of solutions of the GP hierarchy is the most difficult part of this analysis, and is obtained in [7, 8, 9] by use of highly sophisticated Feynman graph expansion methods inspired by quantum field theory.

Recently, an alternative method to prove the uniqueness of solutions in the $d = 3$ case has been developed by Klainerman and Machedon in [14], using spacetime bounds on the density matrices in the GP hierarchy; this result makes the assumption of a particular a priori spacetime bound on the density matrices which has so far remained conjectural. In the work [15] of Kirkpatrick, Schlein, and Staffilani, the corresponding problem in $d = 2$ is solved, and the assumption made in [14] is replaced by a spatial a priori bound which is proven in [15]. Alternative methods to obtain dynamical mean field limits of interacting Bose gases using operator-theoretic methods are developed by Fröhlich et al in [10, 11, 12].

All of the above mentioned works discuss Bose gases with *repulsive* interactions; it is currently not known how to obtain a GP hierarchy from the $N \to \infty$ limit of a BBGKY hierarchy with attractive interactions. In the work at hand, we have nothing to add to this issue. Instead, we start here directly from the level of the GP hierarchy, and are thus free to also consider *attractive* interactions within this context. Accordingly, we will refer to the corresponding GP hierarchies as *cubic*, *quintic*, *focusing*, or *defocusing GP hierarchies*, depending on the type of the NLS governing the solutions one would obtain if one chooses factorized initial conditions.

In the present work, we investigate the Cauchy problem for the cubic and quintic GP hierarchy with focusing and defocusing interactions. Our results do not assume any factorization of the initial data. The main results proven in this paper are:

(1) We prove local wellposedness of the Cauchy problem for the cubic and quintic GP hierarchy with focusing or defocusing interactions, in a generalized weighted Sobolev space $\mathcal{H}^\alpha_\xi$ of sequences of marginal density matrices, in dimensions $d \geq 1$. The parameter α determines the regularity of the solution, and our results hold for $\alpha \in \mathfrak{A}(d,p)$ where

$$\mathfrak{A}(d,p) := \begin{cases} (\frac{1}{2}, \infty) & \text{if } d = 1 \\ (\frac{d}{2} - \frac{1}{2(p-1)}, \infty) & \text{if } d \geq 2 \text{ and } (d,p) \neq (3,2) \\ \left[1, \infty\right) & \text{if } (d,p) = (3,2)\,, \end{cases} \tag{1.1}$$

in dimensions $d \geq 1$, and where $p = 2$ for the cubic, and $p = 4$ for the quintic GP hierarchy. The parameter $\xi > 0$ determines the energy scale of the problem.

(2) As part of the local wellposedness result, we prove the a priori spacetime bound conjectured by Klainerman and Machedon in [14].

(3) For the defocusing case, we prove global wellposedness in $\mathcal{H}^1_\xi$ of the cubic GP hierarchy for $1 \leq d \leq 3$, and of the quintic GP hierarchy for $1 \leq d \leq 2$.

(4) We indroduce generalized pseudoconformal transformations, and prove the invariance of the cubic GP hierarchy in $d = 2$, and of the quintic GP hierarchy in $d = 1$, under their application. Because the NLS obtained from factorized initial data in these cases are L^2-critical, we will, for brevity,

refer to these GP hierarchies as being L^2-critical.

(5) For the focusing cubic or quintic GP hierarchy, we prove lower bounds on the blowup rate in $\mathcal{H}^\alpha_\xi$ and $\mathcal{L}^r_\xi$, where both spaces are defined in Section 2 below.

The introduction of the Banach spaces $\mathcal{H}^\alpha_\xi = \{ \Gamma \in \mathfrak{G} \,|\, \| \Gamma \|_{\mathcal{H}^\alpha_\xi} < \infty \}$ is an important aspect of our arguments. Here,

$$\mathfrak{G} = \{ \Gamma = (\gamma^{(k)}(x_1, \dots, x_k; x_1', \dots, x_k'))_{k\in\mathbb{N}} \,|\, \mathrm{Tr}\gamma^{(k)} < \infty \} \tag{1.2}$$

is the space of sequences of k-particle density matrices, and

$$\| \Gamma \|_{\mathcal{H}^\alpha_\xi} := \sum_{k\in\mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} . \tag{1.3}$$

The parameter $\xi > 0$ is determined by the initial condition, and it sets the energy scale of a given Cauchy problem. If $\Gamma \in \mathcal{H}^\alpha_\xi$, then ξ^{-1} is the typical H^α-energy per particle. We reformulate the spacetime bound conjectured by Klainerman and Machedon in [14] in the language of the sequence of density matrices Γ. In this context, it corresponds to the statement that Γ satisfies a Strichartz type estimate on the level of the GP hierarchy, with respect to $L^2_{t\in[0,T]}\mathcal{H}^1_\xi$, for some $T > 0$ and $\xi > 0$. We prove this estimate in Section 3 using a Picard-type fixed point argument on the space $L^2_{t\in I}\mathcal{H}^1_\xi$, as a key part of our proof of local wellposedness; it corresponds to the inequality (2.15). See also Remark 2.4 below.

An important ingredient of our proof of local wellposedness is the use of the spacetime bounds established in [14] for the cubic GP hierarchy in $d = 3$ (which were generalized to cubic in $d = 2$ in [15], and to the quintic GP hierarchy in [5]), and the "boardgame estimates" developed in [14] (and generalized to the quintic case in [5]), which were motivated by the Feynman graph expansion techniques of [7, 8]. For our discussion of blowup solutions of the focusing (cubic or quintic) GP hierarchy, we make extensive use of a quantity that controls the average H^α-energy per particle, and, in a different form, the average L^r-norm per particle. It is introduced in Definition 2.6 below, and turns out to be the key observable for our discussion of blowup solutions.

Organization of the paper. In Section 2, we introduce the cubic and quintic GP hierarchy, and state our main theorems. In Section 3, we prove the local wellposedness of the Cauchy problem for the cubic and quintic GP hierarchy, for both focusing and defocusing interactions. In Section 4, the local wellposedness is enhanced to global wellposedness for the cubic and quintic defocusing GP hierarchies, using energy conservation. In Section 5, we prove lower bounds on the blowup rate of blowup solutions in the spaces $\mathcal{H}^\alpha_\xi$ and $\mathcal{L}^r_\xi$ (see below for their definitions). In Section 6, we prove the pseudoconformal invariance of the L^2-critical cubic (in $d = 2$) and quintic (in $d = 1$) GP hierarchies. In the Appendix, we reformulate the Klainerman-Machedon spacetime bounds in a form convenient for our work.

2. Definition of the model and statement of the main results

We introduce the space

$$\mathfrak{G} := \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk}) \tag{2.1}$$

of sequences of density matrices

$$\Gamma := (\gamma^{(k)})_{k\in\mathbb{N}} \tag{2.2}$$

where $\gamma^{(k)} \geq 0$, $\mathrm{Tr}\gamma^{(k)} = 1$, and where every $\gamma^{(k)}(\underline{x}_k, \underline{x}_k')$ is symmetric in all components of $\underline{x}_k$, and in all components of $\underline{x}_k'$, respectively.

We call $\Gamma = (\gamma^{(k)})_{k\in\mathbb{N}}$ admissible if

$$\begin{aligned}&\gamma^{(k)}(\underline{x}_k; \underline{x}_k') \\ &\quad = \int dx_{k+1} \cdots dx_{k+\frac{p}{2}}\, \gamma^{(k+\frac{p}{2})}(\underline{x}_k, x_{k+1}, \dots, x_{k+\frac{p}{2}}; \underline{x}_k', x_{k+1}, \dots, x_{k+\frac{p}{2}})\end{aligned} \tag{2.3}$$

for all $k \in \mathbb{N}$.

Let $0 < \xi < 1$ and $r > 1$. We define

$$\mathcal{L}^r_\xi := \Big\{ \Gamma \in \mathfrak{G} \,\Big|\, \|\Gamma\|_{\mathcal{L}^r_\xi} < \infty \Big\} \tag{2.4}$$

where

$$\|\Gamma\|_{\mathcal{L}^r_\xi} := \sum_{k=1}^{\infty} \xi^k \|\, \gamma^{(k)} \,\|_{L^r(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}\,. \tag{2.5}$$

Furthermore, we define

$$\mathcal{H}^\alpha_\xi := \Big\{ \Gamma \in \mathfrak{G} \,\Big|\, \|\Gamma\|_{\mathcal{H}^\alpha_\xi} < \infty \Big\} \tag{2.6}$$

where

$$\|\Gamma\|_{\mathcal{H}^\alpha_\xi} = \sum_{k=1}^{\infty} \xi^k \|\, \gamma^{(k)} \,\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}\,, \tag{2.7}$$

with

$$\|\, \gamma^{(k)} \,\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} = \|\, S^{(k,\alpha)}\, \gamma^{(k)} \,\|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})}\,, \tag{2.8}$$

and $S^{(k,\alpha)} := \prod_{j=1}^k \langle \nabla_{x_j} \rangle^\alpha \langle \nabla_{x_j'} \rangle^\alpha$.

Clearly, $\mathcal{L}^r_\xi$, $\mathcal{H}^\alpha_\xi$ are Banach spaces.

We note that Banach spaces of integral kernels of a similar type as those introduced above are, for instance, used for operator-theoretic renormalization group methods in the spectral analysis of quantum electrodynamics, [3].

Let $p \in \{2, 4\}$. We consider the p-GP (Gross-Pitaevskii) hierarchy given by

$$i\partial_t \gamma^{(k)} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma^{(k)}] + \mu B_{k+\frac{p}{2}} \gamma^{(k+\frac{p}{2})} \tag{2.9}$$

in d dimensions, for $k \in \mathbb{N}$. Here,

$$\begin{aligned}
&\Big(B_{k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\Big)(t,x_1,\dots,x_k;x_1',\dots,x_k') \qquad (2.10)\\
&\quad := \sum_{j=1}^{k}\Big(B_{j;k+1,\dots,k+\frac{p}{2}}\gamma^{(k+\frac{p}{2})}\Big)(t,x_1,\dots,x_k;x_1',\dots,x_k')\\
&\quad := \sum_{j=1}^{k}\int dx_{k+1}\cdots dx_{k+\frac{p}{2}}dx_{k+1}'\cdots dx_{k+\frac{p}{2}}'\\
&\qquad\qquad \Big[\prod_{\ell=k+1}^{k+\frac{p}{2}}\delta(x_j-x_\ell)\cdots\delta(x_j-x_\ell') - \prod_{\ell=k+1}^{k+\frac{p}{2}}\delta(x_j'-x_\ell)\delta(x_j'-x_\ell')\Big]\\
&\qquad\qquad\quad \gamma^{(k+\frac{p}{2})}(t,x_1,\dots,x_{k+\frac{p}{2}};x_1',\dots,x_{k+\frac{p}{2}}')
\end{aligned}$$

accounts for $\frac{p}{2}+1$-body interactions between the Bose particles.

For a factorized initial condition

$$\gamma^{(k)}(0) \,=\, |\,\phi_0\,\rangle\langle\,\phi_0\,|^{\otimes k} \qquad (2.11)$$

with $\phi_0 \in H^\alpha$, one obtains that

$$\gamma^{(k)}(t) \,=\, |\,\phi(t)\,\rangle\langle\,\phi(t)\,|^{\otimes k} \qquad (2.12)$$

is a solution of (2.9) if ϕ_t satisfies the NLS

$$i\partial_t\phi_t \,+\, \Delta_x\phi_t \,-\, \mu\,|\phi_t|^p\,\phi_t \,=\, 0 \qquad (2.13)$$

with initial condition $\phi(0)=\phi_0$, where $\mu\in\{1,-1\}$. For $p=2$, this is the cubic NLS, and for $p=4$, this is the quintic NLS. The NLS is defocusing for $\mu=1$, and focusing for $\mu=-1$.

Accordingly, we refer to (2.9) as the *cubic GP hierarchy* if $p=2$, and as the *quintic GP hierarchy* if $p=4$. Moreover, for $\mu=1$ or $\mu=-1$ we refer to the GP hierarchies as being defocusing or focusing, respectively.

We recall the definition of the set $\mathfrak{A}(d,p)$, for $p=2,4$ and $d\geq 1$,

$$\mathfrak{A}(d,p) \,=\, \begin{cases} (\frac{1}{2},\infty) & \text{if } d=1\\ (\frac{d}{2}-\frac{1}{2(p-1)},\infty) & \text{if } d\geq 2 \text{ and } (d,p)\neq(3,2)\\ \big[1,\infty) & \text{if } (d,p)=(3,2)\end{cases} \qquad (2.14)$$

Our main result in this paper is the following theorem.

Theorem 2.1. *Let $0<\xi_2=\eta\xi_1\ll\xi_1<1$. Then, the following hold.*

- *Assume that $\alpha\in\mathfrak{A}(d,p)$ where $d\geq 1$ and $p\in\{2,4\}$. The Cauchy problem for the defocusing or focusing p-GP hierarchy, with initial condition $\Gamma(0)=\Gamma_0\in\mathcal{H}^\alpha_{\xi_1}\subset\mathcal{H}^\alpha_{\xi_2}$, is locally wellposed in $\mathcal{H}^\alpha_{\xi_2}$.*

- *Let $I=[0,T]$. Then, in particular, when $p=2$ the spacetime bound*

$$\|\widehat{B}\Gamma\|_{L^2_{t\in I}\mathcal{H}_{\xi_2}} \quad\leq\quad \frac{(cT\xi_1^{-4})^{\frac{1}{2}}}{1-(cT\xi_2^{-2})^{\frac{1}{2}}}\,\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} \qquad (2.15)$$

holds for sufficiently small $T > 0$.

- *The Cauchy problem for the defocusing p-GP hierarchy in $\mathcal{H}^1_{\xi_2}$, with initial condition $\Gamma(0) = \Gamma_0 \in \mathcal{H}^1_{\xi_1}$, is globally wellposed for $p = 2$ (cubic) in dimensions $1 \leq d \leq 3$, and for $p = 4$ (quintic) in dimensions $1 \leq d \leq 2$.*

Remark 2.2. *The role of the parameters ξ_1, ξ_2 is as follows: Given initial data $\Gamma_0 = (\gamma^{(k)})_{k\in\mathbb{N}}$ with $\|\gamma^{(k)}\|_{H^\alpha(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} < \infty$ for all k, we determine $\xi_1 > 0$ sufficiently small such that $\Gamma_0 \in \mathcal{H}^\alpha_{\xi_1}$. This means that the energy per particle in Γ_0 is bounded by ξ_1^{-1}. In cases of physical interest, $\xi_1 > 0$; the notion of an energy per particle will be quantified below. Then, we find a suitable $\xi_2 = \eta\xi_1 \ll \xi_1$ such that the Cauchy problem for the the GP hierarchy can be solved in a big enough space $\mathcal{H}^\alpha_{\xi_2}$. The requirement $\xi_2 \ll \xi_1$ is used to ensure that a solution $\Gamma(t)$ does not drift out of $\mathcal{H}^\alpha_{\xi_2}$ for $t \in I = [0, T]$ with $T = T(\xi_2) > 0$; we thereby impose the assumption that the energy per particle does not exceed ξ_2^{-1} while $t \in I$, but once this assumption is violated, one can easily pick a smaller $\xi_2' < \xi_2$ to continue the solution to $T(\xi_2') > T(\xi_2)$.*

Remark 2.3. *In particular, there is no implication of the size of ξ_2 on the regularity accounted for by α. For factorized initial data, the statement that the solution of the NLS remains in H^α for $t \in I$ is equivalent to the statement that the solution of the GP hierarchy remains in $\mathcal{H}^\alpha_\xi$ for* an arbitrary *nonzero $\xi > 0$.*

Remark 2.4. *We note that the estimate (2.15), for the cubic GP hierarchy with $d = 3$ and $\alpha = 1$, proves the a priori spacetime conjectured in* [14]*. For factorized initial data $\Gamma = (|\phi_0\rangle\langle\phi_0|^{\otimes k})_{k\in\mathbb{N}}$ in the cubic case, so that $\Gamma = (|\phi(t)\rangle\langle\phi(t)|^{\otimes k})_{k\in\mathbb{N}}$ where $i\partial_t\phi + \Delta\phi - \mu|\phi|^2\phi = 0$, it corresponds to the inequality*

$$\| \, |\phi|^2\phi \, \|^{\frac{1}{3}}_{L^2_{t\in I} H^\alpha} \leq C(T) \, \| \, \phi_0 \, \|_{H^\alpha} \tag{2.16}$$

which is of Strichartz type (similar to the case determined by the Strichartz admissible pair $(2, 6)$ in $d = 3$). The example of factorized solutions with $\phi(t) \in H^1$, $t \in I$, is discussed in detail in [14].

Definition 2.5. *We say that a solution $\Gamma(t)$ of the GP hierarchy blows up in finite time with respect to H^α if there exists $T^* < \infty$ such that for every $\xi > 0$ there exists $T^*_{\xi,\Gamma} < T^*$ such that $\|\Gamma(t)\|_{\mathcal{H}^\alpha_\xi} \to \infty$ as $t \nearrow T^*_{\xi,\Gamma}$, and $T^*_{\xi,\Gamma} \nearrow T^*$ as $\xi \to 0$.*

For the study of blowup solutions, it is convenient to introduce the following quantity.

Definition 2.6. *We refer to*

$$\mathrm{Av}_{H^\alpha}(\Gamma) := \Big[\sup \big\{ \xi > 0 \, \big| \, \| \, \Gamma \, \|_{\mathcal{H}^\alpha_\xi} < \infty \big\} \Big]^{-1}, \tag{2.17}$$

$$\mathrm{Av}_{L^r}(\Gamma) := \Big[\sup \big\{ \xi > 0 \, \big| \, \| \, \Gamma \, \|_{\mathcal{L}^r_\xi} < \infty \big\} \Big]^{-1}, \tag{2.18}$$

respectively, as the typical (or average) H^α-energy and the typical L^r-norm per particle.

We note that

$$\Gamma = (\,|\,\phi\,\rangle\langle\,\phi\,|^{\otimes k}\,)_{k\in\mathbb{N}} \quad\Rightarrow\quad \mathrm{Av}_{H^\alpha}(\Gamma) = \|\phi\|_{H^\alpha}^2 \text{ and } \mathrm{Av}_{L^r}(\Gamma) = \|\phi\|_{L^r}^2 \tag{2.19}$$

in the factorized case.

The fact that $\Gamma \in \mathcal{H}_\xi^\alpha$ means that the typical energy per particle is bounded by $\mathrm{Av}_{H^\alpha}(\Gamma) < \xi^{-1}$. Therefore, the parameter ξ determines the H^α-energy scale in the problem. While solutions with a bounded H^α-energy remain in the same $\mathcal{H}_\xi^\alpha$, blowup solutions make transitions $\mathcal{H}_{\xi_1}^\alpha \to \mathcal{H}_{\xi_2}^\alpha \to \mathcal{H}_{\xi_3}^\alpha \to \cdots$ where the sequence $\xi_1 > \xi_2 > \cdots$ converges to zero as $t \to T^*$.

It is easy to see that blowup in finite time of $\Gamma(t)$ with respect to H^α is equivalent to the statement that $\mathrm{Av}_{H^\alpha}(\Gamma(t)) \to \infty$ as $t \nearrow T^*$.

Clearly, $(\mathrm{Av}_N(\Gamma))^{-1}$ is the convergence radius of $\|\Gamma\|_{\mathcal{N}_\xi}$ as a power series in ξ, for the norms $N = H^\alpha, L^r$ and $\mathcal{N}_\xi = \mathcal{H}_\xi^\alpha, \mathcal{L}_\xi^r$, respectively.

Theorem 2.7. *Assume that $\Gamma(t)$ is a solution of the (cubic $p = 2$ or $p = 4$ quintic) p-GP hierarchy with initial condition $\Gamma(t_0) = \Gamma_0 \in \mathcal{H}_\xi^\alpha$, for some $\xi > 0$, which blows up in finite time. Then, the following lower bounds on the blowup rate hold:*

(a) *Assume that $\frac{4}{d} \le p < \frac{4}{d-2\alpha}$. Then,*

$$(\,\mathrm{Av}_{H^\alpha}(\Gamma(t))\,)^{\frac12} > \frac{C}{|T^*-t|^{(2\alpha-d+\frac{4}{p})/4}}\,. \tag{2.20}$$

Thus specifically, for the cubic GP hierarchy in $d = 2$, and for the quintic GP hierarchy in $d = 1$,

$$(\,\mathrm{Av}_{H^1}(\Gamma(t))\,)^{\frac12} \ge \frac{C}{|t-T^*|^{\frac12}}\,, \tag{2.21}$$

with respect to the Sobolev spaces H^α, $\mathcal{H}_\xi^\alpha$.

(b)

$$(\,\mathrm{Av}_{L^r}(\Gamma(t))\,)^{\frac12} \ge \frac{C}{|t-T^*|^{\frac1p-\frac{d}{2r}}}\,,\ \textit{for}\ \frac{pd}{2} < r. \tag{2.22}$$

Remark 2.8. *We note that in the factorized case, the above lower bounds on the blow-up rate coincide with the known lower bounds on the blow-up rate for solutions to the NLS (see, for example, [4]).*

The cubic GP hierarchy in $d = 2$, and the quintic GP hierarchy in $d = 1$ are distinguished by being invariant under a class of generalized pseudoconformal transformations, as present below. Let us first recall pseudoconformal invariance on the level of the NLS (2.13). If the NLS (2.13) is L^2-critical, that is, $p = \frac{4}{d}$, it is invariant under the pseudoconformal transformations

$$\mathcal{P}\phi_t(x) := \frac{1}{(1+bt)^{1/2}}\, e^{-i\frac{bx^2}{1+bt}}\,\phi_{\frac{1}{1+bt}}\Big(\frac{x}{1+bt}\Big)\,, \tag{2.23}$$

for $b \in \mathbb{R}\setminus\{0\}$. That is,

$$i\partial_t\mathcal{P}\phi_t + \Delta\mathcal{P}\phi_t - \mu\,|\mathcal{P}\phi_t|^p\,\mathcal{P}\phi_t = 0\,; \tag{2.24}$$

see for instance [4]. There are two cases of L^2-critical NLS with $p \in \mathbb{N}$: The cubic ($p = 2$) NLS in $d = 2$, and the quintic ($p = 4$) NLS in $d = 1$.

For the GP hierarchy, one can likewise introduce pseudoconformal transformations, and as we prove in this paper, the GP hierarchy is pseudoconformally invariant when $p = 2$ and $d = 2$ (cubic), or $p = 4$ and $d = 1$ (quintic). This property is independent of whether the GP hierarchy is defocusing, $\mu = 1$, or focusing, $\mu = -1$.

Theorem 2.9. *For $d = 2$ and $p = 2$ (cubic), or $d = 1$ and $p = 4$ (quintic), the focusing or defocusing ($\mu \in \{1, -1\}$) GP hierarchy* (2.9) *is invariant under the pseudoconformal transformations*

$$\begin{aligned} &\mathcal{P}\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \\ &\quad := \frac{1}{(1+bt)^{dk}} \, e^{-i\frac{b(|\underline{x}_k|^2 - |\underline{x}'_k|^2)}{1+bt}} \gamma^{(k)}\Big(\frac{1}{1+bt} \, , \, \frac{\underline{x}_k}{1+bt} \, ; \, \frac{\underline{x}'_k}{1+bt} \Big) \, , \end{aligned} \tag{2.25}$$

for $b \in \mathbb{R} \setminus \{0\}$.

That is,

$$i\partial_t \mathcal{P}\gamma^{(k)} \, + \, \Delta_{\pm}^{(k)} \mathcal{P}\gamma^{(k)} \, - \, \mu \, B_{k+\frac{p}{2}} \mathcal{P}\gamma^{(k+2)} = 0 \, , \tag{2.26}$$

for all $k \geq 1$.

The proof is given in Section 6. For a survey of related matters for the NLS, see for instance [4, 20, 24].

Of course, the following is immediately clear.

Theorem 2.10. *Assume that $\alpha \in \mathfrak{A}(d, p)$ where $d \geq 1$ and $p \in \{2, 4\}$. Moreover, assume that $\Gamma(t) \in \mathcal{H}^\alpha_{\xi_2}$ solves the (cubic or quintic) focusing ($\mu = -1$) GP hierarchy with factorized initial condition $\Gamma_0 = (|\phi_0\rangle\langle\phi_0|^{\otimes k})_{k\in\mathbb{N}} \in \mathcal{H}^\alpha_\xi$ for some $\xi > 0$, where $\phi_0 \in H^\alpha$.*

Then, if there exists $T^* < \infty$ such that $\|\phi(t)\|_{H^\alpha} \to \infty$ as $t \nearrow T^*$, it follows that also $\mathrm{Av}_{H^\alpha}(\Gamma(t)) \to \infty$ as $t \nearrow T^*$.

Proof. This follows from $\mathrm{Av}_{H^\alpha}(\Gamma(t)) = \|\phi(t)\|^2_{H^\alpha}$ for product states. □

For various scenarios in which blowup occurs for solutions of the cubic or quintic NLS, we refer to the literature; see for instance [4, 20] for surveys.

3. Local wellposedness for the focusing and defocusing GP hierarchy

In this section, we prove local wellposedness for the cubic and quintic GP hierarchy for both focusing and defocusing interactions.

Theorem 3.1. *Assume that* $\Gamma_0 \in \mathcal{H}^\alpha_{\xi_1}$ *is admissible, and that* $\alpha \in \mathfrak{A}(d,p)$ *where* $d \geq 1$ *and* $p \in \{2,4\}$*.*

Then, both the focusing and defocusing (i.e., $\mu \in \{1,-1\}$*) GP hierarchy have a unique solution* $\Gamma \in L^\infty_{[0,T]}\mathcal{H}^\alpha_{\xi_2}$ *with initial condition* $\Gamma(0) = \Gamma_0 \in \mathcal{H}^\alpha_{\xi_1}$*, provided that* $T > 0$ *is sufficiently small, and* $\xi_2 = \eta\xi_1 \ll \xi_1$*.*

Proof. We formulate everything for the cubic hierarchy ($p = 2$). For the quintic hierarchy ($p = 4$), the generalizations are straightforward. We introduce the notation

$$\Delta^{(k)}_\pm \; = \; \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k} \tag{3.1}$$

with

$$\Delta_{\underline{x}_k} \; = \; \sum_{j=1}^k \Delta_{x_j} \tag{3.2}$$

and

$$\Delta_{\pm,x_j} \; = \; \Delta_{x_j} - \Delta_{x'_j} \, . \tag{3.3}$$

Moreover, we write

$$\widehat{\Delta}_\pm\Gamma \; := \; (\, \Delta^{(k)}_\pm \gamma^{(k)} \,)_{k\in\mathbb{N}} \tag{3.4}$$

and

$$\widehat{B}\Gamma \; := \; (\, B_{k+1}\gamma^{(k+1)} \,)_{k\in\mathbb{N}} \, . \tag{3.5}$$

Hence the p-GP hierarchy (2.9) can be written as

$$i\partial_t\Gamma + \widehat{\Delta}_\pm\Gamma = \mu\widehat{B}\Gamma \, , \tag{3.6}$$

which, in turn, in integral formulation looks as:

$$\Gamma(t) \; = \; e^{it\widehat{\Delta}_\pm}\Gamma_0 \; - \; i\mu\int_0^t ds \, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s) \, . \tag{3.7}$$

In order to prove local wellposedness for the solution $\Gamma(t)$ of the cubic GP hierarchy in $\mathcal{H}^\alpha_{\xi_2}$, we proceed with a similar strategy as in the case of NLS: We first prove a result corresponding to a Strichartz inequality, and then formulate the Picard fixed point principle for $\Gamma(t)$.

As stated in Remark 2.4, the estimate

$$\| \, \widehat{B}\Gamma \, \|_{L^2_{t\in[0,T]}\mathcal{H}^\alpha_{\xi_2}} \; < \; C(T) \, \| \, \Gamma_0 \, \|_{\mathcal{H}^\alpha_\xi} \tag{3.8}$$

is of Strichartz type. A condition equivalent to this bound is assumed to hold in [14], but has not been proven.

In order to prove (3.8), we first consider the sequence $\widehat{B}\Gamma(t)$ which satisfies, in integral form,

$$\widehat{B}\Gamma(t) \,=\, \widehat{B}e^{it\widehat{\Delta}_\pm}\Gamma_0 \,-\, i\mu \int_0^t ds\, \widehat{B}e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s)\,. \tag{3.9}$$

We observe that this is a *fixed point equation* for $\widehat{B}\Gamma$.

We want to prove that (3.9) has a unique solution in $L^2_{t\in[0,T]}\mathcal{H}^\alpha_\xi$. We define, for an admissible sequence of density matrices $\widetilde{\Gamma} = (\widetilde{\gamma}^{(k)})_{k\in\mathbb{N}}$,

$$\begin{aligned}
&\mathrm{Duh}_j(\widetilde{\Gamma})^{(k+1)}(t) \\
&:= \;(-i\mu)^j \int_0^t dt_1 \cdots \int_0^{t_{j-1}} dt_j e^{i(t-t_1)\Delta_\pm^{(k+1)}} B_{k+2} e^{-i(t_1-t_2)\Delta_\pm^{(k+2)}} \\
&\qquad\qquad B_{k+3}\cdots\cdots B_{k+j+1} e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j+1)}} \widetilde{\gamma}^{(k+j+1)}(t_j)\,.
\end{aligned} \tag{3.10}$$

Then, any solution of (3.9) satisfies the fixed point equation (obtained from iterating the Duhamel formula k times for the k-th component of $\widehat{B}\Gamma$)

$$(\widehat{B}\Gamma)^{(k)}(t) \,=\, \sum_{j=1}^{k-1} B_{k+1}\mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \,+\, B_{k+1}\mathrm{Duh}_k(\widehat{B}\Gamma)^{(k+1)}(t)\,. \tag{3.11}$$

To formulate a Picard-type fixed point argument, we define

$$\Phi(\widehat{B}\Gamma) = (\Phi(\widehat{B}\Gamma)^{(k)})_{k\in\mathbb{N}} \tag{3.12}$$

where the k-th component is given by

$$\Phi(\Gamma)^{(k)}(t) \,=\, \sum_{j=1}^{k-1} B_{k+1}\mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \,+\, B_{k+1}\mathrm{Duh}_k(\widehat{B}\Gamma)^{(k+1)}(t)\,. \tag{3.13}$$

We prove local wellposedness of $(\widehat{B}\Gamma)(t)$ for $\alpha \in \mathfrak{A}(d,p)$ for $d \geq 1$. The constraints on α are clarified in the Appendix.

Similarly as in [5], we use different approaches when $d \geq 2$ and when $d = 1$. In dimension $d = 1$, and for both the cubic and quintic GP hierarchy, we use a spatial a priori bound as in [5] where we refer for details.

In dimensions $d \geq 2$, we apply the Klainerman-Machedon spacetime bounds similarly to [14] and [15]. This is explained in detail in the Appendix.

The case $d \geq 2$.

We prove the Strichartz type estimate (3.8) on the level of the GP hierarchy. To this end, we use the fact that

$$\begin{aligned}
& \| \sum_{j=1}^{k-1} B_{k+1} \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \|_{L^2_{t\in I} H^\alpha} \\
& < \quad kC^k \sum_{j=1}^{k-1} (cT)^{\frac{j}{2}} \|\gamma_0^{(k+j+1)}\|_{H^\alpha} && (3.14)\\
& < \quad k(C\xi_1^{-1})^{k+1} \sum_{j=1}^{k-1} (cT\xi_1^{-2})^{\frac{j}{2}} \xi_1^{k+j+1} \|\gamma_0^{(k+j+1)}\|_{H^\alpha} && (3.15)\\
& < \quad (cT\xi_1^{-2})^{\frac{1}{2}} k(C\xi_1^{-1})^{k+1} \sum_{j=1}^{k-1} \xi_1^{k+j+1} \|\gamma_0^{(k+j+1)}\|_{H^\alpha} && (3.16)
\end{aligned}$$

using Propositions A.1 and A.2 in the Appendix. They generalize the $L^2(\mathbb{R}\times\mathbb{R}^{dk}\times\mathbb{R}^{dk})$ spacetime bounds, and the "board game" arguments, developed in [14].

Therefore,

$$\begin{aligned}
& \sum_{k=1}^{\infty} \xi_2^k \| \sum_{j=1}^{k-1} B_{k+1} \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \|_{L^2_{t\in I} H^\alpha} \\
& < \quad (cT\xi_1^{-4})^{\frac{1}{2}} \sum_{k=1}^{\infty} k\Big(C\frac{\xi_2}{\xi_1}\Big)^k \sum_{j=1}^{k-1} \xi_1^{k+j+1} \|\gamma_0^{(k+j+1)}\|_{H^\alpha} && (3.17)\\
& < \quad (cT\xi_1^{-4})^{\frac{1}{2}} \sum_{k=1}^{\infty} k\Big(C\frac{\xi_2}{\xi_1}\Big)^k \sum_{\ell=1}^{2k} \xi_1^{\ell} \|\gamma_0^{(\ell)}\|_{H^\alpha} && (3.18)\\
& < \quad (cT\xi_1^{-4})^{\frac{1}{2}} \sum_{k=1}^{\infty} k\big(C\eta\big)^k \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} && (3.19)\\
& < \quad (cT\xi_1^{-4})^{\frac{1}{2}} \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} && (3.20)
\end{aligned}$$

for $\xi_2 = \eta\xi_1 \ll \xi_1$.

This implies that, for $I=[0,T]$, and any $T>0$,

$$\Big(\sum_{j=1}^{k-1} B_k \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \Big)_{k\in\mathbb{N}} \in L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2} \tag{3.21}$$

if $\Gamma_0 \in \mathcal{H}^\alpha_{\xi_1}$ with $\xi_1 > \xi_2$.

Our next step is to prove that Φ is a contraction on $L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}$. To this end, we use the bound

$$\|\Phi(\widehat{B}\Gamma_1)^{(k)} - \Phi(\widehat{B}\Gamma_2)^{(k)}\|_{L^2_{t\in I}H^\alpha} \ \le\ k\,(CT)^{\frac{k}{2}} \|\widehat{B}\Gamma_1^{(2k)} - \widehat{B}\Gamma_2^{(2k)}\|_{L^2_{t\in I}H^\alpha} \tag{3.22}$$

obtained in the same manner as above, using Propositions A.1 and A.2 in the Appendix.

We obtain

$$\begin{aligned}
&\|\Phi(\widehat{B}\Gamma_1)-\Phi(\widehat{B}\Gamma_2)\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}}\\
&= \sum_{k=1}^\infty \xi_2^k\|\Phi(\widehat{B}\Gamma_1)^{(k)}-\Phi(\widehat{B}\Gamma_2)^{(k)}\|_{L^2_{t\in I}H^\alpha}\\
&\leq \sum_{k=1}^\infty k\,(CT\xi_2^{-2})^{\frac{k}{2}}\xi_2^{2k}\,\|\widehat{B}\Gamma_1^{(2k)}-\widehat{B}\Gamma_2^{(2k)}\|_{L^2_{t\in I}H^\alpha}\\
&\leq \sup_k\{\,k\,(CT\xi_2^{-2})^{\frac{k}{2}}\,\}\sum_{k=1}^\infty \xi_2^{2k}\,\|\widehat{B}\Gamma_1^{(2k)}-\widehat{B}\Gamma_2^{(2k)}\|_{L^2_{t\in I}H^\alpha}\\
&\leq (CT\xi_2^{-2})^{\frac12}\,\|\widehat{B}\Gamma_1-\widehat{B}\Gamma_2\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}}\,, \qquad (3.23)
\end{aligned}$$

for $T>0$ sufficiently small. This allows us to conclude that Φ is a contraction on $L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}$ if T is sufficiently small.

Consequently, there exists a unique solution $\widehat{B}\Gamma$ of (3.9) in $L^2_{t_I}\mathcal{H}^\alpha_{\xi_2}$, for a given initial condition $\Gamma_0\in\mathcal{H}^\alpha_{\xi_1}$ with $\xi_2=\eta\xi_1\ll\xi_1$. For this solution, we find

$$\|\widehat{B}\Gamma\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}} \leq (cT\xi_1^{-4})^{\frac12}\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} + (CT\xi_2^{-2})^{\frac12}\,\|\widehat{B}\Gamma\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}} \qquad (3.24)$$

using the same arguments as above, so that sufficiently small $T>0$,

$$\|\widehat{B}\Gamma\|_{L^2_{t\in I}\mathcal{H}_{\xi_2}} \leq \frac{(cT\xi_1^{-4})^{\frac12}}{1-(cT\xi_2^{-2})^{\frac12}}\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}}\,. \qquad (3.25)$$

We note that if $\Gamma_0=(0)_{k\in\mathbb{N}}$, it follows that $\widehat{B}\Gamma(t)=(0)_{k\in\mathbb{N}}$ for $t<T$, which is equivalent to the uniqueness of the solution. In particular, (3.25) proves (2.15).

Solutions to the original GP hierarchy satisfy

$$\Gamma(t) = e^{it\widehat{\Delta}_\pm}\Gamma_0 - i\mu\int_0^t ds\, e^{i(t-s)\widehat{\Delta}_\pm}\widehat{B}\Gamma(s)\,,$$

and thus,

$$\begin{aligned}
\|\Gamma(t)\|_{\mathcal{H}^\alpha_{\xi_2}} &\leq \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_2}} + (CT)^{\frac12}\,\|\widehat{B}\Gamma\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}}\\
&\leq \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} + (cT)^{\frac12}\,\frac{(cT\xi_1^{-4})^{\frac12}}{1-(cT\xi_2^{-2})^{\frac12}}\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}}\\
&= \Big(1+\frac{cT\xi_1^{-2}}{1-(cT\xi_2^{-2})^{\frac12}}\Big)\,\|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}}\,. \qquad (3.26)
\end{aligned}$$

This implies the existence of a solution locally in time, for $cT<\xi_2^2$.

Uniqueness follows immediately from the fact that $\Gamma_0=(0)_{k\in\mathbb{N}}$ implies that $\Gamma(t)=0$, for all $t<T$.

For the quintic GP hierarchy, all steps of the above proof can be adopted with minor modifications. A key difference is the fact that (3.26) is replaced by

$$\begin{aligned}\|\Gamma(t)\|_{\mathcal{H}^\alpha_{\xi_2}} &\leq \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_2}} + (CT)^{\frac{1}{2}} \, \|\widehat{B}\Gamma\|_{L^2_{t\in I}\mathcal{H}^\alpha_{\xi_2}} \\ &\leq \Big(1 + \frac{cT\xi_1^{-4}}{1-(cT\xi_2^{-4})^{\frac{1}{2}}} \Big) \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} \, . \end{aligned} \tag{3.27}$$

This implies the existence of a unique solution locally in time, for $cT < \xi_2^4$.

<u>The case $d = 1$.</u>

In this case, we can straightforwardly adapt the proof given in [5] of the uniqueness of solutions of the quintic GP hierarchy in $d = 1$. The spacetime bounds of Proposition A.1 is not available in $d = 1$ since it would produce divergent bounds. However, the spatial bounds in $d = 1$ proven in [5] apply for both the cubic and the quintic GP hierarchy, under the assumption that $\alpha > \frac{1}{2}$.

The result is that we get a factor t instead of $t^{\frac{1}{2}}$, in all of the bounds found above for the cubic GP hierarchy that produced a factor $t^{\frac{1}{2}}$. Accordingly, we find

$$\|\Gamma(t)\|_{\mathcal{H}^\alpha_{\xi_2}} \leq \Big(1 + \frac{cT^2\xi_1^{-2}}{1-cT\xi_2^{-1}} \Big) \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} \tag{3.28}$$

for the cubic GP hierarchy instead of (3.26), and

$$\|\Gamma(t)\|_{\mathcal{H}^\alpha_{\xi_2}} \leq \Big(1 + \frac{cT^2\xi_1^{-4}}{1-cT\xi_2^{-2}} \Big) \|\Gamma_0\|_{\mathcal{H}^\alpha_{\xi_1}} \tag{3.29}$$

for the quintic GP hierarchy instead of (3.27), respectively. Again, we obtain local wellposedness for sufficiently small $T > 0$. □

4. Global wellposedness for the defocusing GP hierarchy

In this section, we establish global wellposedness in $\mathcal{H}^1_\xi$ for the defocusing cubic GP hierarchy in dimensions $1 \leq d \leq 3$, and for the defocusing quintic GP hierarchy in dimensions $1 \leq d \leq 2$.

Theorem 4.1. *Assume that $1 \leq d \leq 3$ for $p = 2$, and in $1 \leq d \leq 2$ for $p = 4$ such that $\{1\} \in \mathfrak{A}(d,p)$. Moreover, assume that $\Gamma_0 \in \mathcal{H}^\alpha_{\xi_1}$, with $\alpha \geq 1$, is admissible. Then, the defocusing ($\mu = +1$) GP hierarchy has a unique solution $\Gamma(t) \in \mathcal{H}^1_{\xi_2}$ with initial condition $\Gamma(0) = \Gamma_0$, for all $t \in \mathbb{R}$, provided that $\xi_2 = \eta\xi_1 \ll \xi_1$.*

Proof. It is proved for the $d = 2,3$ cubic case in [7, 8, 15], and for the $d = 1,2$ quintic case in [5], that whenever $\Gamma(t) = (\gamma^{(k)}(t))_{k\in\mathbb{N}}$ is a solution of the GP hierarchy with initial condition satisfying

$$\mathrm{Tr}\big(S^{(k,1)} \, \gamma_0^{(k)} \big) \, < \, C^k \tag{4.1}$$

for some constant C, then

$$\mathrm{Tr}\big(\, S^{(k,1)}\,\gamma^{(k)}(t)\,\big) \,<\, C_0^k \tag{4.2}$$

with C_0 independent of $t \geq 0$.

This follows from energy conservation in the N-particle system of which the GP hierarchy is the $N \to \infty$ limit. The proofs given in [7, 8, 15] and [5] can be straightforwardly generalized to any dimension $d \geq 1$.

We consider a fixed k. Let $\gamma^{(k)}$ be non-negative, normalized trace class, $\mathrm{Tr}(\gamma^{(k)}) = 1$, and hermitean. Then, we have that

$$\gamma^{(k)}(\underline{x}_k;\underline{x}_k') \,=\, \sum_j \lambda_j \,|\psi_j(\underline{x}_k')\rangle\langle\psi_j(\underline{x}_k)| \tag{4.3}$$

for an orthogonal basis ψ_j of $L^2(\mathbb{R}^{dk})$ with $\lambda_j \geq 0$ and $\sum \lambda_j = 1$. Then,

$$\begin{aligned} \|\,\gamma^{(k)}\,\|_{H^1} \;&=\; \sum_{j,j'} \lambda_j\,\lambda_{j'} \big|\big\langle\,\langle\nabla_{\underline{x}_k}\rangle\psi_j\,\big|\,\langle\nabla_{\underline{x}_k}\rangle\psi_{j'}\,\big\rangle\big|^2 && (4.4)\\ &\leq\; \Big(\sum_j \lambda_j \,\big\|\,\langle\nabla_{\underline{x}_k}\rangle\psi_j\,\big\|^2\Big)^2 && (4.5)\\ &=\; \Big(\,\mathrm{Tr}\big(\,S^{(k,1)}\,\gamma^{(k)}\,\big)\,\Big)^2 . && (4.6) \end{aligned}$$

Thus, for a solution $\gamma^{(k)}(t)$ of the (cubic or quintic) GP hierarchy with initial condition satisfying (4.1), we have that

$$\|\,\gamma^{(k)}(t)\,\|_{H^1} \,<\, C_0^k \tag{4.7}$$

with C_0 independent of t.

Thus, for ξ sufficiently small,

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \,\leq\, \Big(\,\sum_{k=1}^\infty (C_0\xi)^{2k}\,\Big)^{\frac12} \,<\, \infty\,. \tag{4.8}$$

This establishes the claim. □

5. Lower bound on the blowup rates

In this section, we establish Theorem 2.7. We adapt a standard proof given for L^2-critical focusing NLS to the GP hierarchy; see for instance [20]. Let $p \in \{2,4\}$. Similarly as in (6.1), one finds that the p-GP hierarchy is invariant under the rescaling

$$\begin{aligned} &\mathcal{R}_{\lambda,t}\gamma^{(k)}(\tau,\underline{x}_k;\underline{x}_k') \\ &\quad:= \frac{1}{(\lambda(t))^{4k/p}}\,\gamma^{(k)}\Big(\,t+(\lambda(t))^{-2}\tau\,,\,(\lambda(t))^{-1}\underline{x}_k\,;\,(\lambda(t))^{-1}\underline{x}_k'\,\Big) \end{aligned} \tag{5.1}$$

If $\Gamma(t)=(\gamma^{(k)}(t))_{k\in\mathbb{N}}$ solves the p-GP hierarchy, then $\mathcal{R}_{\lambda,t}\Gamma=(\mathcal{R}_{\lambda,t}\gamma^{(k)})_{k\in\mathbb{N}}$ is also a solution of the p-GP hierarchy. The proof can be straightforwardly adapted from the one given in Section 6.

Proof of statement (a).

Blowup in finite time means that there exists $T^*<\infty$ such that $\mathrm{Av}_{H^\alpha}(\Gamma(t))\to\infty$ as $t\to T^*$. To prove a lower bound on the blowup rate, we may assume that $1<\mathrm{Av}_{H^\alpha}(\Gamma(t))<\infty$ at a fixed time t, and choose

$$\lambda(t)\;=\;(\mathrm{Av}_{H^\alpha}(\Gamma(t)))^{\frac{1}{2\alpha-d+\frac{4}{p}}}\;>\;1\,. \tag{5.2}$$

We note that $\frac{4}{d}<p<\frac{4}{d-2\alpha}$ implies that $2\alpha-d+\frac{4}{p}>0$. Let

$$S^{(k,\alpha)}_{\lambda(t)}\;:=\;\prod_{j=1}^{k}\langle\,(\lambda(t))^{-1}\,\nabla_{x_j}\rangle^{\alpha}\,\langle\,(\lambda(t))^{-1}\,\nabla_{x_j'}\,\rangle^{\alpha} \tag{5.3}$$

where $\langle b\nabla_x\rangle=\sqrt{1-b^2\Delta_x}$ for any $b\in\mathbb{R}$. Clearly,

$$\begin{aligned}&\Big\|\,S^{(k,\alpha)}\mathcal{R}_{\lambda,t}\gamma^{(k)}(\tau)\,\Big\|_{L^2_{\underline{x}_k,\underline{x}_k'}}\\ &=\;\;(\lambda(t))^{k(d-\frac{4}{p})}\,\|\,(\,S^{(k,\alpha)}_{\lambda(t)}\,\gamma^{(k)}\,)(t+(\lambda(t))^{-2}\tau)\,\|_{L^2_{\underline{x}_k,\underline{x}_k'}}\,,\end{aligned} \tag{5.4}$$

and

$$(\lambda(t))^{-2\alpha k}S^{(k,\alpha)}\;\le\;S^{(k,\alpha)}_{\lambda(t)}\;\le\;S^{(k,\alpha)} \tag{5.5}$$

since we are assuming that $\lambda(t)>1$.

We define

$$\xi_<(\xi,t,\lambda)\;:=\;\xi\,(\lambda(t))^{-\frac{4}{p}+d-2\alpha}\;=\;\xi\,(\mathrm{Av}_{H^\alpha}(\Gamma(t)))^{-1} \tag{5.6}$$

and

$$\xi_>(\xi,t,\lambda)\;:=\;\xi(\lambda(t))^{-\frac{4}{p}+d}\;=\;\xi(\mathrm{Av}_{H^\alpha}(\Gamma(t)))^{\frac{d-\frac{4}{p}}{2\alpha-d+\frac{4}{p}}}\,. \tag{5.7}$$

Clearly, (5.4) and (5.5) imply that

$$\begin{aligned}\|\,\Gamma(t+(\lambda(t))^{-2}\tau)\,\|_{\mathcal{H}^\alpha_{\xi_<(\xi,t,\lambda)}}\;&\le\;\;\|\,\mathcal{R}_{\lambda,t}\Gamma(\tau)\,\|_{\mathcal{H}^\alpha_\xi}\\ &\le\;\;\|\,\Gamma(t+(\lambda(t))^{-2}\tau)\,\|_{\mathcal{H}^\alpha_{\xi_>(\xi,t,\lambda)}}\,.\end{aligned} \tag{5.8}$$

As a consequence of the definition of $\mathrm{Av}_{H^\alpha}(\Gamma(t))$, it follows that for $\tau=0$,

$$0\;<\;\|\,\Gamma(t)\,\|_{\mathcal{H}^\alpha_{\xi_<(\xi,t,\lambda)}}\;<\;c \tag{5.9}$$

for any $0<\xi<1$.

To ensure that $\|\Gamma(t+(\lambda(t))^{-2}\tau)\|_{\mathcal{H}^\alpha_{\xi_>(\xi,t,\lambda)}}<c$, we use the fact that according to (5.7), $\xi_>(\xi,t,\lambda)$ can be made arbitrarily small by choosing ξ small.

We note that our assumption $\frac{4}{d}<p<\frac{4}{d-2\alpha}$ implies that $2\alpha-d+\frac{4}{p}>0$ and $d-\frac{4}{p}>0$, so that the exponent on the rhs of (5.7) is positive. If blowup occurs, such that $\mathrm{Av}_{H^\alpha}(\Gamma(t))\to\infty$ as $t\nearrow T^*$, the above considerations necessitate the choice of values of ξ (whose reciprocal determines the energy scale) tending to zero as $t \nearrow T^*$.

Thus, for $\xi_1 > 0$ sufficiently small,

$$\| \mathcal{R}_{\lambda,t}\Gamma(0) \|_{\mathcal{H}^\alpha_{\xi_1}} \leq \| \Gamma(t) \|_{\mathcal{H}^\alpha_{\xi_>(\xi_1,t,\lambda)}} \tag{5.10}$$

Due to Theorem 3.1, we may pick $0 < \xi_2 = \eta\xi_1 \ll \xi_1 < 1$, such that there exists a solution $\mathcal{R}_{\lambda,t}\gamma^{(k)}(\tau) \in \mathcal{H}^\alpha_{\xi_2}$ if $\tau \in [0, \tau_{\max}]$, for $\tau_{\max} > 0$ sufficiently small.

But this implies that

$$\| \Gamma(t + (\lambda(t))^{-2}\tau) \|_{\mathcal{H}^\alpha_{\xi_<(\xi_2,t,\lambda)}} \leq \| \mathcal{R}_{\lambda,t}\Gamma(\tau) \|_{\mathcal{H}^\alpha_{\xi_2}} < \infty \tag{5.11}$$

for $\tau \in [0, \tau_{\max}]$ so that there is no blowup if τ lies in that interval. Therefore, the blowup time T^* is bounded from below by

$$T^* > t + (\lambda(t))^{-2}\tau_{\max} , \tag{5.12}$$

and hence,

$$(\mathrm{Av}_{H^\alpha}(\Gamma(t)))^{\frac{1}{2}} = \lambda(t)^{(\alpha - \frac{d}{2} + \frac{2}{p})} > \frac{C}{|T^* - t|^{(2\alpha - d + \frac{4}{p})/4}} . \tag{5.13}$$

This proves (a).

<u>*Proof of statement (b).*</u>

It is easy to see that

$$\left\| \mathcal{R}_{\lambda,t}\gamma^{(k)}(\tau) \right\|_{L^r_{\underline{x}_k,\underline{x}'_k}} = (\lambda(t))^{-2k(\frac{2}{p} - \frac{d}{r})} \| \gamma^{(k)}(t + (\lambda(t))^{-2}\tau) \|_{L^r_{\underline{x}_k,\underline{x}'_k}} , \tag{5.14}$$

which, in turn, implies that

$$\begin{aligned} \| \mathcal{R}_{\lambda,t}\gamma^{(k)}(0) \|_{\mathcal{L}^r_\xi} &= \sum_{k\geq 1} \xi^k \left\| \mathcal{R}_{\lambda,t}\gamma^{(k)}(0) \right\|_{L^r_{\underline{x}_k,\underline{x}'_k}} \\ &= \sum_{k\geq 1} \xi^k \, (\lambda(t))^{-2k(\frac{2}{p} - \frac{d}{r})} \| \gamma^{(k)}(t) \|_{L^r_{\underline{x}_k,\underline{x}'_k}} \\ &= \sum_{k\geq 1} \left(\frac{\xi}{(\lambda(t))^{(\frac{4}{p} - \frac{2d}{r})}} \right)^k \| \gamma^{(k)}(t) \|_{L^r_{\underline{x}_k,\underline{x}'_k}} . \end{aligned} \tag{5.15}$$

However (5.15) is bounded for every $\xi < 1$, if we choose

$$\lambda(t) = (\mathrm{Av}_{L^r}(\Gamma(t)))^{\frac{1}{\frac{4}{p} - \frac{2d}{r}}} . \tag{5.16}$$

Now we argue as in the proof of the part (a) by using the local well-posedness Theorem 3.1 to conclude that the H^α blowup time T^* is bounded from below by

$$T^* > t + (\lambda(t))^{-2}\tau_{\max} . \tag{5.17}$$

Therefore

$$(\mathrm{Av}_{L^r}(\Gamma(t)))^{\frac{1}{2}} = \lambda(t)^{(\frac{2}{p} - \frac{d}{r})} > \frac{C}{|T^* - t|^{\frac{1}{p} - \frac{d}{2r}}} . \tag{5.18}$$

Hence (b) is proved. □

6. Proof of pseudoconformal invariance

In this section, we prove Theorem 2.9. We recall the pseudoconformal transformations

$$\begin{aligned}&\mathcal{P}\gamma^{(k)}(t,\underline{x}_k;\underline{x}'_k)\\ &:= \frac{1}{(1+bt)^{dk}}\, e^{-i\frac{b(|\underline{x}_k|^2-|\underline{x}'_k|^2)}{1+bt}}\gamma^{(k)}\Big(\frac{1}{1+bt}\,,\,\frac{\underline{x}_k}{1+bt}\,;\,\frac{\underline{x}'_k}{1+bt}\Big)\,, \end{aligned} \tag{6.1}$$

for any $b \in \mathbb{R}\setminus\{0\}$. Similarly as in the case of NLS, one can verify that

$$\begin{aligned}&(i\partial_t+\Delta_{\underline{x}_k}-\Delta_{\underline{x}'_k})\mathcal{P}\gamma^{(k)}(t,\underline{x}_k;\underline{x}'_k)\\ &= \frac{1}{(1+bt)^2}\frac{1}{(1+bt)^{dk}}\, e^{-i\frac{b(|\underline{x}_k|^2-|\underline{x}'_k|^2)}{1+bt}}\\ &\qquad ((i\partial_t+\Delta_{\underline{x}_k}-\Delta_{\underline{x}'_k})\gamma^{(k)})\Big(\frac{1}{1+bt}\,,\,\frac{\underline{x}_k}{1+bt}\,;\,\frac{\underline{x}'_k}{1+bt}\Big)\,. \end{aligned} \tag{6.2}$$

Now we shall prove the pseudoconformal invariance of the quintic GP hieararchy when $d=1$. In particular, we find that

$$\begin{aligned}&B^1_{j;k+1,k+2}\mathcal{P}\gamma^{(k+2)}(t,\underline{x}_k;\underline{x}'_k)\\ &= \frac{1}{(1+bt)^{d(k+2)}}\, e^{-i\frac{b(|\underline{x}_k|^2-|\underline{x}'_k|^2)}{1+bt}}\int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}\\ &\qquad \delta(x_j-x_{k+1})\delta(x_j-x'_{k+1})\delta(x_j-x_{k+2})\delta(x_j-x'_{k+2})\\ &\qquad \gamma^{(k)}\Big(\frac{1}{1+bt}\,,\,\frac{(\underline{x}_k,x_{k+1},x_{k+2})}{1+bt}\,;\,\frac{(\underline{x}'_k,x'_{k+1},x'_{k+2})}{1+bt}\Big) \end{aligned} \tag{6.3}$$

$$\begin{aligned}&= \frac{1}{(1+bt)^{2d}}\frac{1}{(1+bt)^{dk}}\, e^{-i\frac{b(|\underline{x}_k|^2-|\underline{x}'_k|^2)}{1+bt}}\\ &\qquad \gamma^{(k)}\Big(\frac{1}{1+bt}\,,\,\frac{(\underline{x}_k,x_j,x_j)}{1+bt}\,;\,\frac{(\underline{x}'_k,x_j,x_j)}{1+bt}\Big)\,, \end{aligned} \tag{6.4}$$

$$\begin{aligned}&= \frac{1}{(1+bt)^{2d}}\frac{1}{(1+bt)^{dk}}\, e^{-i\frac{b(|\underline{x}_k|^2-|\underline{x}'_k|^2)}{1+bt}}\\ &\qquad B^1_{j;k+1,k+2}\gamma^{(k+2)}\Big(\frac{1}{1+bt}\,,\,\frac{\underline{x}_k}{1+bt}\,;\,\frac{\underline{x}'_k}{1+bt}\Big) \end{aligned} \tag{6.5}$$

with $B_{j;k+1,k+2}=B^1_{j;k+1,k+2}-B^2_{j;k+1,k+2}$; in $B^2_{j;k+1,k+2}$, the variable x_j in $B^1_{j;k+1,k+2}$ is replaced by x'_j. Notably, we have used that

$$e^{-i\frac{b((x^2_{k+1}+x^2_{k+2})-(x'^2_{k+1}+x'^2_{k+2}))}{1+bt}}\Big|_{x_{k+1}=x_{k+2}=x'_{k+1}=x'_{k+2}=x_j} = 1\,. \tag{6.6}$$

Thus, when $d = 1$, we obtain

$$\begin{aligned}
&\Big((i\partial_t + \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\mathcal{P}\gamma^{(k)} - \mu \sum_{j=1}^{k} B_{j;k+1,k+2}\mathcal{P}\gamma^{(k+2)} \Big)(t, \underline{x}_k; \underline{x}'_k) \\
&= \quad \frac{1}{(1+bt)^2} \frac{1}{(1+bt)^{dk}} \, e^{-i\frac{b(|\underline{x}_k|^2 - |\underline{x}'_k|^2)}{1+bt}} \\
&\qquad \Big((i\partial_t + \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)} - \mu \sum_{j=1}^{k} B_{j;k+1,k+2}\gamma^{(k+2)} \Big)\Big(\frac{1}{1+bt} \, , \, \frac{\underline{x}_k}{1+bt} \, ; \, \frac{\underline{x}'_k}{1+bt} \Big) \\
&= \quad 0
\end{aligned} \tag{6.7}$$

if $d = 1$. This proves pseudoconformal invariance of the quintic GP hierarchy in dimension $d = 1$.

For the cubic GP hierarchy, the operators $B_{j;k+1,k+2}$ are replaced by operators $B_{j;k+1}$ which contract x_j, x'_j only with x_{k+1} and x'_{k+1},

$$(i\partial_t + \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)} - \mu \sum_{j=1}^{k} B_{j;k+1}\gamma^{(k+1)} \, = \, 0 \, . \tag{6.8}$$

The same considerations as above then produce

$$\begin{aligned}
&\Big((i\partial_t + \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\mathcal{P}\gamma^{(k)} - \mu \sum_{j=1}^{k} B_{j;k+1}\mathcal{P}\gamma^{(k+1)} \Big)(t, \underline{x}_k; \underline{x}'_k) \\
&= \quad \frac{1}{(1+bt)^2} \frac{1}{(1+bt)^{dk}} \, e^{-i\frac{b(|\underline{x}_k|^2 - |\underline{x}'_k|^2)}{1+bt}} \\
&\qquad \Big((i\partial_t + \Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)} - \mu \sum_{j=1}^{k} B_{j;k+1}\gamma^{(k+1)} \Big)\Big(\frac{1}{1+bt} \, , \, \frac{\underline{x}_k}{1+bt} \, ; \, \frac{\underline{x}'_k}{1+bt} \Big) \\
&= \quad 0
\end{aligned} \tag{6.9}$$

if $d = 2$. This proves Theorem 2.9. □

Appendix A. The Klainerman-Machedon spacetime bounds

We present the Klainerman-Machedon spacetime bounds in dimensions $d \geq 2$ in the form required for this paper, with $\alpha \in \mathfrak{A}(d,p)$; see (2.14). In the regime $\alpha > \frac{d}{2} - \frac{1}{2(p-1)}$, we present a simple argument to prove the result. In the endpoint case $(d,p) = (3,2)$ and $\alpha = 1$, we invoke a result of [14].

Proposition A.1. *Let $p = 2, 4$ account for the cubic and quintic GP hierarchy, respectively, and assume that $\alpha \in \mathfrak{A}(d,p)$. Let $\gamma^{(k+\frac{p}{2})}$ be the solution of*

$$i\partial_t \gamma^{(k+\frac{p}{2})}(t, \underline{x}_{k+\frac{p}{2}}; \underline{x}'_{k+\frac{p}{2}}) + (\Delta_{\underline{x}_{k+\frac{p}{2}}} - \Delta_{\underline{x}'_{k+\frac{p}{2}}}) \gamma^{(k+\frac{p}{2})}(t, \underline{x}_{k+\frac{p}{2}}; \underline{x}'_{k+\frac{p}{2}}) = 0 \tag{A.1}$$

with initial condition

$$\gamma^{(k+\frac{p}{2})}(0, \cdot) = \gamma_0^{(k+\frac{p}{2})} \in \mathcal{H}^\alpha . \tag{A.2}$$

Then, there exists a constant C such that

$$\begin{aligned} & \Big\| S^{(k,\alpha)} B_{j;k+1,\dots,k+\frac{p}{2}} \gamma^{(k+\frac{p}{2})} \Big\|_{L^2(\mathbb{R}\times\mathbb{R}^{2(k+\frac{p}{2})}\times\mathbb{R}^{2(k+\frac{p}{2})})} \\ & \qquad \leq C \Big\| S^{(k+\frac{p}{2},\alpha)} \gamma_0^{(k+\frac{p}{2})} \Big\|_{L^2(\mathbb{R}^{2(k+\frac{p}{2})}\times\mathbb{R}^{2(k+\frac{p}{2})})} \end{aligned} \tag{A.3}$$

holds.

Proof. For notational convenience, we discuss the proof for the quintic GP hierarchy where $p = 4$.

Let $(\tau, \underline{u}_k, \underline{u}'_k)$, $\underline{q} := (q_1, q_2)$, and $\underline{q}' := (q'_1, q'_2)$ denote the Fourier conjugate variables corresponding to $(t, \underline{x}_k, \underline{x}'_k)$, (x_{k+1}, x_{k+2}), and (x'_{k+1}, x'_{k+2}), respectively.

Without any loss of generality, we may assume that $j = 1$ in $B_{j;k+1,k+2}$. Then, abbreviating

$$\delta(\cdots) := \delta\big(\, \tau + (u_1 + q_1 + q_2 - q'_1 - q'_2)^2 + \sum_{j=2}^{k} u_j^2 + |\underline{q}|^2 - |\underline{u}'_k|^2 - |\underline{q}'|^2 \,\big) \tag{A.4}$$

we find

$$\begin{aligned} & \Big\| S^{(k,\alpha)} B_{1;k+1,k+2} \gamma^{(k+2)} \Big\|^2_{L^2(\mathbb{R}\times\mathbb{R}^{d(k+2)}\times\mathbb{R}^{d(k+2)})} \\ & = \int_{\mathbb{R}} d\tau \int d\underline{u}_k d\underline{u}'_k \prod_{j=1}^{k} \langle u_j \rangle^{2\alpha} \langle u'_j \rangle^{2\alpha} \\ & \qquad \Big(\int d\underline{q} d\underline{q}' \, \delta(\cdots) \widehat{\gamma}^{(k+2)}(\tau, u_1 + q_1 + q_2 - q'_1 - q'_2, u_2, \dots, u_k, \underline{q}; \underline{u}'_k, \underline{q}') \Big)^2 . \end{aligned} \tag{A.5}$$

Using the Schwarz estimate, this is bounded by

$$\begin{aligned} \leq \; & \int_{\mathbb{R}} d\tau \int d\underline{u}_k d\underline{u}'_k \, I_\alpha(\tau, \underline{u}_k, \underline{u}'_k) \int d\underline{q} d\underline{q}' \, \delta(\cdots) \\ & \langle u_1 + q_1 + q_2 - q'_1 - q'_2 \rangle^{2\alpha} \langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q'_1 \rangle^{2\alpha} \langle q'_2 \rangle^{2\alpha} \prod_{j=2}^{k} \langle u_j \rangle^{2\alpha} \prod_{j'=1}^{k} \langle u'_{j'} \rangle^{2\alpha} \\ & \qquad \Big| \widehat{\gamma}^{(k+2)}(\tau, u_1 + q_1 + q_2 - q'_1 - q'_2, u_2, \dots, u_k, \underline{q}; \underline{u}'_k, \underline{q}') \Big|^2 \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} I_\alpha(\tau, \underline{u}_k, \underline{u}_k') & \\ := \int d\underline{q}\, d\underline{q}' \, \frac{\delta(\cdots)\, \langle u_1 \rangle^{2\alpha}}{\langle u_1 + q_1 + q_2 - q_1' - q_2' \rangle^{2\alpha} \langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q_1' \rangle^{2\alpha} \langle q_2' \rangle^{2\alpha}} \, . \end{aligned} \tag{A.7}$$

Similarly as in [14, 15], we observe that

$$\langle u_1 \rangle^{2\alpha} \leq C \Big[\langle u_1 + q_1 + q_2 - q_1' - q_2' \rangle^{2\alpha} + \langle q_1 \rangle^{2\alpha} + \langle q_2 \rangle^{2\alpha} + \langle q_1' \rangle^{2\alpha} + \langle q_2' \rangle^{2\alpha} \Big] , \tag{A.8}$$

so that

$$I_\alpha(\tau, \underline{u}_k, \underline{u}_k') \leq \sum_{\ell=1}^{5} J_\ell \tag{A.9}$$

where J_ℓ is obtained from bounding the numerator of (A.7) using (A.8), and from canceling the ℓ-th term on the rhs of (A.8) with the corresponding term in the denominator of (A.7). Thus, for instance,

$$J_1 < \int d\underline{q}\, d\underline{q}' \, \frac{\delta(\cdots)}{\langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q_1' \rangle^{2\alpha} \langle q_2' \rangle^{2\alpha}} \, , \tag{A.10}$$

and each of the terms J_ℓ with $\ell = 2, \dots, 5$ can be brought into a similar form by appropriately translating one of the momenta q_i, q_j'.

Further following [14, 15], we observe that the argument of the delta distribution equals

$$\tau + (u_1 + q_1 + q_2 - q_1')^2 + \sum_{j=2}^{k} u_j^2 + |\underline{q}|^2 - |\underline{u}_k'|^2 - (q_1')^2 - 2(u_1 + q_1 + q_2 - q_1') \cdot q_2' \, ,$$

and we integrate out the delta distribution using the component of q_2' parallel to $(u_1 + q_1 + q_2 - q_1')$. This leads to the bound

$$J_1 \quad < \quad C_\alpha C \int d\underline{q} dq_1' \, \frac{1}{|u_1 + q_1 + q_2 - q_1'| \langle q_1 \rangle^{2\alpha} \langle q_2 \rangle^{2\alpha} \langle q_1' \rangle^{2\alpha}} \tag{A.11}$$

where

$$C_\alpha := \int_{\mathbb{R}} \frac{d\zeta}{\langle \zeta \rangle^{2\alpha}} \, . \tag{A.12}$$

Clearly, C_α is finite for any $\alpha > \frac{1}{2}$. Moreover, it is clear that J_1 is monotonically decreasing in α.

For the cubic GP hierarchy, the above arguments lead to the condition that instead of (A.11), the integral

$$\int dq_1 \, \frac{1}{|u_1 + q_1| \langle q_1 \rangle^{2\alpha}} \tag{A.13}$$

must be bounded.

<u>Proof for $\alpha > \frac{d}{2} - \frac{1}{2(p-1)}$.</u>

We first consider the case $p = 4$ corresponding to the quintic GP hierarchy, and argue as follows. To bound (A.11), we pick a spherically symmetric function $h \geq 0$

with rapid decay away from the unit ball in $\mathbb{R}^d$, such that $h^\vee(x) \geq 0$ decays rapidly outside of the unit ball in $\mathbb{R}^d$, and

$$\frac{1}{\langle q \rangle^{2\alpha}} \;<\; h * \Big(\chi_{B_1} \,+\, \frac{\chi_{B_1}^c}{|\,\cdot\,|^{2\alpha}} \Big)(q)\,. \tag{A.14}$$

(for example, $h(u) = c_1 e^{-c_2 u^2}$, for suitable constants c_1, c_2), where $\chi_{B_1} + \chi_{B_1}^c = 1$ is a smooth partition of unity with χ_{B_1} supported on the unit ball, with $\chi_{B_1}(u) = 1$ for $|u| \leq \frac{1}{2}$, and $\chi_{B_1}(u) = 0$ for $|u| > \frac{1}{2}$. Clearly, $h * \chi_{B_1}$ and $h * \frac{\chi_{B_1}^c}{|\cdot|^{2\alpha}}$ are both in L^∞, for any $\alpha > 0$.

Then, assuming that $\alpha < \frac{d}{2}$, inserting this into (A.11), the most singular part is given by

$$\begin{aligned} & C_\alpha C \Big\langle \Big(\frac{1}{|\,\cdot\,|} * (h * \frac{\chi_{B_1}^c}{|\,\cdot\,|^{2\alpha}}) \Big) * (h * \frac{\chi_{B_1}^c}{|\,\cdot\,|^{2\alpha}})\,,\, (h * \frac{\chi_{B_1}^c}{|\,\cdot\,|^{2\alpha}}) \Big\rangle_{L^2(\mathbb{R}^d)} \\ &= \; C_\alpha C \int dx \, \Big(\frac{\chi_{B_1}^c}{|\,\cdot\,|} \Big)^\vee (x) \Big((h * \frac{\chi_{B_1}^c}{|\,\cdot\,|^{2\alpha}})^\vee (x) \Big)^3 \\ &= \; C_\alpha C' \int dx \, \frac{1}{|\,x\,|^{d-1}} (h^\vee(x))^3 \Big(\Big((\chi_{B_1}^c)^\vee * \frac{1}{|\,\cdot\,|^{d-2\alpha}} \Big)(x) \Big)^3 \\ &< \; C_\alpha C' \int dx \, \frac{1}{|\,x\,|^{d-1}} (h^\vee(x))^3 \Big(\frac{1}{|\,x\,|^{d-2\alpha}} \Big)^3 . \end{aligned} \tag{A.15}$$

For sufficiently large C', this is an upper bound on all of the remaining terms that are obtained from substituting the bound (A.14) into (A.11). We have here used that $(\chi_{B_1}^c)^\vee = 1^\vee - \chi_{B_1}^\vee = \delta - \chi_{B_1}^\vee$, so that $|((\chi_{B_1}^c)^\vee * \frac{1}{|\cdot|^{d-2\alpha}})(x)| \leq C \frac{1}{|x|^{d-2\alpha}}$ holds for $\alpha < \frac{d}{2}$.

We conclude that (A.15) is finite provided that the singularity at $x = 0$ is integrable, since $h^\vee(x)$ falls off rapidly as $|x| \to \infty$. In dimension d, this is the case if the exponents in the denominator satisfy

$$d - 1 \,+\, 3d \,-\, 6\alpha \,<\, d\,, \tag{A.16}$$

such that

$$\alpha \,>\, \frac{d}{2} - \frac{1}{6}\,. \tag{A.17}$$

This proves the claim for the quintic GP hierarchy, i.e., for $p = 4$. In order to prove the lower bound (A.17) on α, we have assumed that $\alpha < \frac{d}{2}$, which is consistent with it. Now, since J_1 is monotonically decreasing in α, we arrive at the asserted result.

For the cubic GP hierarchy, the same considerations lead to the condition that $(A.11) < \infty$ if $\alpha > \frac{d}{2} - \frac{1}{2}$. For a general p-GP hierarchy, one obtains the condition $\alpha > \frac{d}{2} - \frac{1}{2(p-1)}$.

The case $\alpha = 1$ for the cubic GP hierarchy in $d = 3$.

In the situation $d = 3$ and $p = 2$ of the cubic GP hierarchy in 3 dimensions, we have the endpoint case $\frac{d}{2} - \frac{1}{2(p-1)} = 1$. Klainerman and Machedon have proven in [14] that (A.7) is bounded in this case. □

Next, we prove the iterated spacetime estimates for the cubic GP hierarchy used in Section 3, which involve the "boardgame estimates" of [14], which are motivated by the Feynman graph techniques in [7, 8, 9]. The corresponding results for the quintic GP hierarchy are obtained in an analogous manner, and we refer to [5] for details.

Proposition A.2. *Assume α as in Proposition A.1, for the cubic GP hierarchy ($p = 2$). Then, for $k \geq 1$ and $j \leq k$, and $t \in I = [0, T]$,*

$$\| B_k \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \|_{L^2_{t\in I} H^\alpha} \; < \; k C^k (cT)^{\frac{j}{2}} \| \gamma_0^{(k+j+1)} \|_{H^\alpha} \, . \tag{A.18}$$

Moreover,

$$\| B_{k+1} \mathrm{Duh}_k(\widehat{B}\Gamma)^{(k+1)} \|_{L^2_{t\in I} H^\alpha} \; < \; k C^k (cT)^{\frac{k}{2}} \| \, (\widehat{B}\gamma)^{(2k)} \, \|_{H^\alpha} \, . \tag{A.19}$$

Proof. Let $I = [0, T]$. Using an argument presented as a "board game", it is proven in [14] that the following holds.

Let $\mathcal{E}_{j,k+1}$ denote the space of sequences $\underline{\mu}_s = (\mu_s(1), \dots, \mu_s(j))$ where $\mu(i) \in \{1, \dots, k+i\}$, where for every $i \in \{1, \dots, j\}$, one has $\mu(i) \geq \mu(i')$ for all $i' > i$. The elements of $\mathcal{E}_{j,k+1}$ parametrize $(k+j) \times j$ matrices in so-called "special upper echelon form" (see [14] for definitions). The cardinality of this set satisfies $|\mathcal{E}_{j,k+1}| \leq C^{j+k}$.

For every $\underline{\mu}_s \in \mathcal{E}_{j,k+1}$, one associates the term

$$\begin{aligned} &(\, \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \,)_{\underline{\mu}_s} \\ &:= \int_{I^j} dt_1 \cdots dt_j e^{i(t-t_1)\Delta_\pm^{(k+1)}} B_{\mu(1),k+2} e^{i(t_1-t_2)\Delta_\pm^{(k+2)}} \\ &\qquad \cdots B_{\mu_s(j-1),k+j} e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} B_{\mu_s(j),k+j+1} \gamma_0^{(k+j+1)} \, . \end{aligned} \tag{A.20}$$

Notably, the integration domain I^j for the variables t_i is not a simplex, in contrast to what is found in $\mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t)$. Then, it is proven in [14] that

$$\begin{aligned} &\| B_{k+1} \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \|_{L^2_{t\in I} H^\alpha} \\ &\quad \leq \sum_{\underline{\mu}_s \in \mathcal{E}_{j,k+1}} \| B_{k+1} (\, \mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \,)_{\underline{\mu}_s} \|_{L^2_{t\in I} H^\alpha} \, . \end{aligned} \tag{A.21}$$

For the proof of (A.21) in the case of the cubic GP hierarchy, we refer to [14]. For the case of the quintic GP hierarchy, we refer to [5].

Using Proposition A.1, we have, under the given assumptions on α,

$$\begin{aligned}
&\| B_{k+1}(\mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t))_{\underline{\mu}_s} \|_{L^2_{t\in I}H^\alpha} \\
&\leq \sum_{\ell=1}^{k} \Big\| B_{\ell,k+1} e^{it\Delta_\pm^{(k+1)}} \int_{I^j} dt_1 \cdots dt_j e^{-it_1\Delta_\pm^{(k+1)}} B_{\mu_s(1),k+2} e^{i(t_1-t_2)\Delta_\pm^{(k+2)}} \\
&\qquad \cdots B_{\mu_s(j-1),k+j} e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} B_{\mu_s(j),k+j+1} \gamma_0^{(k+j+1)} \Big\|_{L^2_{t\in I}H^\alpha} \\
&\leq k \int_{I^j} dt_1 \cdots dt_j \Big\| B_{\mu_s(1),k+2} e^{i(t_1-t_2)\Delta_\pm^{(k+2)}} \\
&\qquad \cdots B_{\mu_s(j-1),k+j} e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} B_{\mu_s(j),k+j+1} \gamma_0^{(k+j+1)} \Big\|_{H^\alpha} \qquad (A.22) \\
&\leq k\,(cT)^{\frac{1}{2}} \int_{I^{j-1}} dt_2 \cdots dt_j \Big\| B_{\mu_s(1),k+2} e^{i(t_1-t_2)\Delta_\pm^{(k+2)}} \cdots \\
&\qquad \cdots B_{\mu_s(j-1),k+j} e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} B_{\mu_s(j),k+j+1} \gamma_0^{(k+j+1)} \Big\|_{L^2_{t_1\in I}H^\alpha} \qquad (A.23) \\
&\leq \cdots \\
&\leq k(cT)^{\frac{j-1}{2}} \int_I dt_j \Big\| B_{\mu_s(j-1),k+j} e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} B_{\mu_s(j),k+j+1} \gamma_0^{(k+j+1)} \Big\|_{L^2_{t_{j-1}\in I}H^\alpha} \qquad (A.24) \\
&\leq k(cT)^{\frac{j-1}{2}} \int_I dt_j \Big\| B_{\mu_s(j),k+j+1} e^{it_j\Delta_\pm^{(k+j+1)}} e^{-it_j\Delta_\pm^{(k+j+1)}} \gamma_0^{(k+j+1)} \Big\|_{H^\alpha} \\
&\leq k(cT)^{\frac{j}{2}} \Big\| \gamma_0^{(k+j+1)} \Big\|_{H^\alpha}
\end{aligned} \qquad (A.25)$$

where to obtain (A.22), we used Proposition A.1, and to obtain (A.23), we used the Hölder estimate. Then, we iterated the above steps to obtain (A.24), and finally obtained (A.25) by using Hölder, Proposition A.1, and the unitarity of $e^{-it_j\Delta_\pm^{(k+j+1)}}$.

Then, estimating by C^{j+k} the number of terms in the sum over $\underline{\mu}_s \in \mathcal{E}_{j,k+1}$,

$$\| B_{k+1}\mathrm{Duh}_j(\Gamma_0)^{(k+1)}(t) \|_{L^2_{t\in I}H^\alpha} \leq kC^k(cT)^{\frac{j}{2}} \|\gamma_0^{(k+j+1)}\|_{H^\alpha} , \qquad (A.26)$$

as claimed.

In the same manner, we prove (A.19) where we refer to [5] for details. □

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