On Spectral Gap, U(1) Symmetry and Split Property in Quantum Spin Chains

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Abstract: In this article, we consider a class of ground states for quantum spin chains on an integer lattice. First we show that presence of the spectral gap between the ground state energy and the rest of spectrum implies the split property of certain subsystems. As a corollary, we show that gapless excitation exists for spinless Fermion if the pure gauge invariant ground state is non-trivial and translationally invariant.

Keywords: quantum spin chain, spectral gap, $U(1)$ gauge symmetry, Haag duality, Bell's inequality,Lieb-Robinson bound.
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1 Introduction.

In what follows, we show presence of the spectral gap between the ground state energy and the rest of spectrum implies statistical independence of left and right semi-infinite subsystems. This independence is called *split property* in the context of local quantum field theory (local QFT) in th sense of R. Haag. (c.f. [11]) The split property is known to hold in a number of situations of local QFT while, as far as we are aware, this condition for quantum spin chains was never proved in the situation we consider here. As a corollary we will see that a gapless excitation exists in certain $U(1)$ symmetric quantum lattice models on $\mathbf{Z}$.

Now we will exhibit typical examples of Hamiltonians we have in our mind. These are $U(1)$ gauge invariant finite range Hamiltonians for quantum spin chains such as the Heisenberg Hamiltonians H_{XXX} on the one-dimensional integer lattice $\mathbf{Z}$ or fermionic systems on $\mathbf{Z}$, H_F as described as follows:

$$H_{XXX} = \sum_{i,j\in\mathbf{Z}} \sum_{\alpha=x,y,z} J_{ij}\sigma_\alpha^{(i)}\sigma_\alpha^{(j)}, \tag{1.1}$$

$$H_F = \sum_{i,j\in\mathbf{Z}} t_{ij}c^* c_j + \sum_{k\in\mathbf{Z}} V_k(n), \tag{1.2}$$

where $\sigma_\alpha^{(j)}$ in (1.1) is the spin operator at the site j in which the direction of the spin is denoted by α. c_j^* ,c_i are Fermion creation - annihilation operators satisfying the anti-commutation relations and J_{ij} and t_{ij} are coupling constants satisfying the conditions (finite rangeness and translational invariance);

$$J_{ij} = J_{i-j0},\ t_{ij} = t_{i-j0}, \quad J_{i0} = 0,\ t_{i0} = 0$$

if $|i| > r$. $V_k(n)$ is a polynomial of the local number operators $n_i = c_i^* c_i$ at the site i . A simple example of $V_k(n)$ is $V_k(n) = Kn_kn_{k+1}$ (K: a constant).

Both H_{XXX} and H_F are invariant under the global $U(1)$ gauge transformation where, for the spin chain, the gauge transformation is defined via the rotation around the z axis and, for Fermion system, it is defined via the formula $c_j \to e^{i\theta}c_j$. Then, one of our results is expressed as follows.

Theorem 1.1
(i) Consider the quantum spin chain on $\mathbf{Z}$ *and the spin at each site is* $1/2$. *Let* H_S *be a translationally invariant ,* $U(1)$ *gauge invariant finite range Hamiltonian. Suppose that* φ *is a* $U(1)$ *gauge invariant , translationally invariant pure ground state of* H_S. *If* φ *is not a product state, a gapless excitation exists between the ground state energy and the rest of the spectrum of the effective Hamiltonian on the GNS representation space.*
(ii) Consider the spinless Fermion lattice system on $\mathbf{Z}$. *Let* H_F *be a translationally invariant ,* $U(1)$ *gauge invariant finite range Hamiltonian. Suppose that* φ *is a* $U(1)$ *gauge invariant , translationally invariant pure ground state of* H_F. *Suppose further that* φ *is not neither the standard Fock state* ψ_F *nor the*

standard anti-Fock state, a gapless excitation exists between the ground state energy and the rest of the spectrum of the effective Hamiltonian on the GNS representation space .

In the above theorem, by the standard Fock state we mean the state ψ_F specified by the identity $\psi_F(c_j^* c_j) = 0$ for any j and the standard anti-Fock state is the state ψ_{AF} specified by the identity $\psi_{AF}(c_j c_j^*) = 1$ for any j.

The infinite volume ground state we consider here is defined in [7]. φ is a ground state for an infinite volume Hamiltonian H if φ is a normalized positive functional on the C^*-algebra of quasi-local observables satisfying

$$\varphi(Q^*[H, Q]) \geq 0 \tag{1.3}$$

for any quasi-local observable Q. The infinite volume limit of ground states for finite volume Hamiltonians with any boundary conditions gives rise to a state satisfying (1.3). More precisely let H_n be a sequence of finite volume Hamiltonians satisfying $\lim[H_n, Q] = [H, Q]$ for any local observable Q and Ω_n be a unit eigenvector for the least eigenvalue of H_n. Set

$$\varphi(Q) = \lim_n (\Omega_n, Q\Omega_n).$$

Then, the state φ satisfies the inequality (1.3).

Results similar to Theorem1.1 were obtained before for several cases. For example, for antiferromagnetic Heisenberg model, presence of gapless excitation was proved by I.Affleck and E.Lieb in [5] . Results on Fermion models were obtained by T.Koma in [15] . The difference between previous results and ours lies in the two points. First our result is on ground state for arbitrary boundary conditions. The second point is that our proof is new and the argument is based on three mathematical ingredient:
(1) Results on Bell's inequality for infinite quantum systems due to S.Summers and R.F.Werner ([26]) ,
(2) Haag duality of quantum spin chain recently proved by us [14] and
(3) Improved Lieb-Robinson bound due to R.Simms and B.Nachteregaele [25].

We will see that these three results imply that any translationally invariant pure ground states have the split property if there is a spectral gap between the ground state energy and the rest of spectrum. More precisely, we will prove the following theorem.

Theorem 1.2
(i) Consider a quantum spin chain on $\mathbf{Z}$*. Let* H_S *be a translationally invariant finite range Hamiltonian and let* φ *be a translationally invariant pure ground state of* H_S*. Suppose that there is a gap between the ground state energy and the rest of the spectrum of the effective Hamiltonian on the GNS representation space associated with* φ *.*

Then, φ is equivalent to a product state $\psi_L \otimes \psi_R$ where ψ_L is a state of the algebra of observables localized in $(-\infty, 0]$ and ψ_R is a state of the algebra of observables localized in $[1, \infty)$
(ii) Consider a Fermion lattice system on $\mathbf{Z}$ with a finite number of components at each lattice site. Let H_F be a translationally invariant finite range Hamiltonian and let φ be a translationally invariant pure ground state. Suppose that there is a gap between the ground state energy and the rest of the spectrum of the effective Hamiltonian on the GNS representation space associated with φ . Then, φ is equivalent to a product state $\psi_L \otimes_{Z_2} \psi_R$ where ψ_L is a even state of the algebra of observables localized in $(-\infty, 0]$ and ψ_R is a even state of the algebra of observables localized in $[1, \infty)$ and by $\otimes_{Z_2}$ we denote the graded tensor product.

The notion of *graded tensor product* is fermion analogue of the tensor product which we introduce in Section 4.

We employ the C^*-algebraic method to prove Theorem 1.1 and Theorem 1.2. The method is an abstract functional analysis which can be applied to Hamiltonians with a form more general than in (1.1) and (1.2) . The standard references for the framework and basic notions of the C^*-algebraic method are [6] and [7]. In Section 2, we introduce several notions to describe our results precisely. We introduce Lieb-Robinson bound and uniform exponential clustering. In Section 3, we prove twisted Haag duality for the fermionic system. In section 4, we explain the reason why that the spectral gap implies split property with the aid of Bell's inequality in infinite quantum systems. We present our proof of Theorem 1.1 and Theorem 1.2 in the final section.

2 Infinite Volume Ground States and Spectral Gap.

First we introduce several notations and notions of quantum spin chain on $\mathbf{Z}$ and then later we mention the case of Fermions. We denote the C^*-algebra of (quasi)local observables by $\mathfrak{A}$. This means that $\mathfrak{A}$ is the UHF C^*-algebra n^∞ (the C^*-algebraic completion of the infinite tensor product of n by n matrix algebras):

$$\mathcal{A} = \overline{\bigotimes_{\mathbf{Z}} M_n(\mathbf{C})}^{C^*} ,$$

where $M_n(\mathbf{C})$ is the set of all n by n complex matirces. Each component of the tensor product is specified with a lattice site $j \in \mathbf{Z}$. $\mathcal{A}$ is the totality of quasi-local observables. We denote by $Q^{(j)}$ the element of $\mathcal{A}$ with Q in the j th component of the tensor product and the identity in any other components :

$$Q^{(j)} = \cdots \otimes 1 \otimes 1 \otimes \underbrace{Q}_{j\text{th component}} \otimes 1 \otimes 1 \otimes \cdots$$

For a subset Λ of $\mathbf{Z}$, $\mathfrak{A}_\Lambda$ is defined as the C^*-subalgebra of $\mathfrak{A}$ generated by elements $Q^{(j)}$ with all j in Λ. We set

$$\mathfrak{A}_{loc} = \cup_{\Lambda \subset \mathbf{Z}:|\Lambda|<\infty} \ \mathfrak{A}_\Lambda$$

where the cardinality of Λ is denoted by $|\Lambda|$. We call an element of $\mathfrak{A}_{loc}$ a local observable or a strictly local observable.

By a state φ of a quantum spin chain, we mean a normalized positive linear functional on $\mathfrak{A}$ which gives rise to the expectation value of a quantum mechanical state.

When φ is a state of $\mathfrak{A}$, the restriction of φ to $\mathfrak{A}_\Lambda$ will be denoted by φ_Λ :

$$\varphi_\Lambda = \varphi|_{\mathfrak{A}_\Lambda}.$$

We set

$$\mathfrak{A}_R = \mathfrak{A}_{[1,\infty)} \ , \ \mathfrak{A}_L = \mathfrak{A}_{(-\infty,0]} \ , \ \varphi_R = \varphi_{[1,\infty)} \ , \ \varphi_L = \varphi_{(-\infty,0]} \ .$$

By τ_j, we denote the automorphism of $\mathfrak{A}$ determined by

$$\tau_j(Q^{(k)}) = Q^{(j+k)}$$

for any j and k in $\mathbf{Z}$. τ_j is referred to as the lattice translation of $\mathfrak{A}$.

Given a state φ of $\mathfrak{A}$, we denote the GNS representation of $\mathfrak{A}$ associated with φ by $\{\pi_\varphi(\mathfrak{A}), \Omega_\varphi, \mathfrak{H}_\varphi\}$ where $\pi_\varphi(\cdot)$ is the representation of $\mathfrak{A}$ on the GNS Hilbert space $\mathfrak{H}_\varphi$ and Ω_φ is the GNS cyclic vector satisfying

$$\varphi(Q) = (\Omega_\varphi, \pi_\varphi(Q)\Omega_\varphi) \qquad Q \in \mathfrak{A}.$$

Let π be a representation of $\mathfrak{A}$ on a Hilbert space. The von Neumann algebra generated by $\pi(\mathfrak{A}_\Lambda)$ is denoted by $\mathfrak{M}_\Lambda$. We set

$$\mathfrak{M}_R = \mathfrak{M}_{[1,\infty)} = \pi(\mathfrak{A}_R)'', \quad \mathfrak{M}_L = \mathfrak{M}_{(-\infty,0]} = \pi(\mathfrak{A}_L)''.$$

In terms of the above definitions, we introduce the notion of the dynamics (the Heisenberg time evolution) and the ground state for infinite volume systems. By *Interaction* we mean an assignment $\{\Psi(X)\}$ of each finite subset X of $\mathbf{Z}$ to a selfadjoint operator $\Psi(X)$ in $\mathfrak{A}_X$. We say that an interaction is of finite range if there exists a positive number r such that $\Psi(X) = 0$ if that the diameter of X is larger than r. An interaction is translationally invariant if and only if $\tau_j(\Psi(X)) = \Psi(X+j)$ for any $X \subset \mathbf{Z}$ and for any $j \in \mathbf{Z}$. For a translationally invariant finite range interaction, we consider the formal infinite volume Hamiltonian H which is an infinite sum of local observables.

$$H = \sum_{X \subset \mathbf{Z}} \Psi(X).$$

This sum does not converge in the norm topology, however the following commutator and the limit make sense:

$$[H,Q] = \lim_{n\to\infty}[H_n,Q] = \sum_{X\subset \mathbf{Z}}[\Psi(X),Q], \quad \lim_{n\to\infty} e^{itH_n}Qe^{-itH_n} \quad Q\in\mathfrak{A}_{loc}$$

where $H_n = \sum_{X\subset[-n,n]}\Psi(X)$.

More generally, for any finite subset Λ the finite volume Hamiltonian H_Λ is defined by $H_\Lambda = \sum_{X\subset\Lambda}\Psi(X)$. Then if $\{\Psi(X)\}$ is a translationally invariant interaction, and if we assume that

$$\sum_{X\ni 0}|X|||\Psi(X)|| < \infty, \tag{2.1}$$

the following limit exists

$$\alpha_t(Q) = \lim_{\Lambda\to\mathbf{Z}} e^{itH_\Lambda}Qe^{-itH_\Lambda}$$

for any element Q of $\mathfrak{A}$ in the C^* norm topology. We call $\alpha_t(Q)$ the time evolution of Q.

Definition 2.1 *Suppose the time evolution $\alpha_t(Q)$ associated with an interaction satisfying (2.1) is given. Let φ be a state of $\mathfrak{A}$. φ is a ground state of H if and only if*

$$\varphi(Q^*[H,Q]) = \frac{1}{i}\frac{d}{dt}\varphi(Q^*\alpha_t(Q)) \geq 0 \tag{2.2}$$

for any Q in $\mathfrak{A}_{loc}$.

Suppose that φ is a ground state for α_t . In the GNS representation of $\{\pi_\varphi(\mathfrak{A}), \Omega_\varphi, \mathfrak{H}_\varphi\}$, there exists a positive selfadjoint operator $H_\varphi \geq 0$ such that

$$e^{itH_\varphi}\pi_\varphi(Q)e^{-itH_\varphi} = \pi_\varphi(\alpha_t(Q)), \quad e^{itH_\varphi}\Omega_\varphi = \Omega_\varphi$$

for any Q in $\mathfrak{A}$. Roughly speaking, the operator H_φ is the effective Hamiltonian on the physical Hilbert space $\mathfrak{H}_\varphi$ obtained after regularization via subtraction of the vacuum energy.

The spectral gap we are interested in is that of H_φ. Note that, in principle, a different choice of a ground state gives rise to a different spectrum.

Definition 2.2 *We say that H_φ has the spectral gap if 0 is a non-degenerate eigenvalue of H_φ and for a positive $M > 0$, H_φ has no spectrum in $(0,M)$,i.e. $Spec(H_\varphi)\cap(0,M) = \emptyset$.*

It is easy to see that H_φ has the spectral gap if and only if there exists a positive constant M such that

$$\varphi(Q^*[H,Q]) \geq M(\varphi(Q^*Q) - |\varphi(Q)|^2).$$

In the quantum field theory with locality it is known that presence of the spectral gap implies exponential decay of (spacial) correlation. (c.f.[4], [10]) The most general result is obtained by K.Fredenhagen and the result is referred to as Fredenhagen's cluster theorem.

The nature of locality in quantum spin chains is quite different from the relativistic quantum field theory as we do not have speed of light. Nevertheless, there is a control of propagation of quasi locality which is due to E.Lieb and D.Robinson. The bound of this kind is called *the Lieb-Robinson bound.* ([16]) The inequality is described as follows.

$$||[\alpha_t(Q), R]|| \leq C(Q,R)e^{-ad(X,Y)-v|t|} \tag{2.3}$$

where Q(resp. R) is an element of $\mathfrak{A}_X$ (resp. $\mathfrak{A}_Y$ and $C(Q,R)$ is a constant positive depending on Q and R and v is another positive constant called "group velocity". Though not straightforward, once the above quasi-locality estimate is established we may expect a lattice model analogue of Fredenhagen's cluster theorem. This was achieved relatively recently. (See [22], [12] and see also [8], [15], [21], [24] for extension and application of the results.) The estimate of spacial decay of correlation we need for our purpose is due to B.Nachtergaele and R.Sims. Now we present this version in [25] .

Theorem 2.3 (B.Nachtergaele and R.Sims 2007) *We consider the quantum spin chain on* $\mathbf{Z}$*. Assume that the interaction* $\{\Psi(X)\}$ *is translationally invariant and of finite range. Let* φ *be a translationally invariant pure ground state of the Hamiltonian* H*. Assume further that the effective Hamiltonian* H_φ *has the spectral gap.*
Then, there exists positive constants C *and* K *such that for any* Q *in* $\mathfrak{A}_L$*,* R *in* $\mathfrak{A}_R$ *and any positive integer* j*, the following estimate is valid.*

$$|\varphi(Q\tau_j(R)) - \varphi(Q)\varphi(R)| \leq C||Q|| \cdot ||R||e^{-Kj} \tag{2.4}$$

C *and* K *are independent of* Q *and* R*.*

That C is independent on the size of support of observables Q and R is crucial to our argument below and it seems that this independence was never obtained before [25].

Next we introduce the $U(1)$ gauge action to describe our main result (Theorem 1.1) more precisely. We consider the spin 1/2 system , thus $\mathfrak{A}$ is the infinite tensor product of 2 by 2 matrix algebras. We set

$$S_z = \frac{1}{2}\sum_{j=-\infty}^{\infty} \sigma_z^{(j)}, \;\; \gamma_\theta(Q) = e^{i\theta S_z}Qe^{-i\theta S_z} \tag{2.5}$$

Then $\gamma_{2\pi}(Q) = Q$ for any Q in $\mathfrak{A}$ and γ_θ gives rise a $U(1)$ action on $\mathfrak{A}$. Instead of proving Theorem 1.1, we will show an equivalent result stated as follows.

Theorem 2.4 *Suppose that the spin S is one half, hence the one-site observable algebra is $M_2(\mathbf{C})$. Let φ be a translationally invariant pure ground state of a finite range translationally invariant Hamiltonian. Suppose further that the effective Hamiltonian H_φ has the spectral gap and φ is γ_θ invariant for any θ. Then, φ is a product state.*

Next we consider fermionic systems. The results discussed above are extended in a natural way. Let $\mathfrak{A}^{CAR}$ be the CAR (canonical anti-commutation relations) algebra generated by Fermion creation-annihilation operators c_j and c_k^* ($j, k \in \mathbf{Z}$) satisfying

$$\{c_j, c_k\} = 0, \ \{c_j^*, c_k^*\} = 0, \ \{c_j, c_k^*\} = \delta_{jk}1 \tag{2.6}$$

For each subset Λ of $\mathbf{Z}$, $\mathfrak{A}_\Lambda^{CAR}$ is the C^*-subalgebra generated by c_j and c_k^* for $j, k \in \Lambda$. $\mathfrak{A}_{loc}^{CAR}$, $\mathfrak{A}_L^{CAR}$, $\mathfrak{A}_R^{CAR}$ are defined as before. The *paritiy* Θ is an automorphism of $\mathfrak{A}^{CAR}$ defined by $\Theta(c_j) = -c_j$, $\Theta(c_j^*) = -c_j^*$ for any j. We set

$$(\mathfrak{A}^{CAR})_\pm = \{Q \in \mathfrak{A}^{CAR} \mid \Theta(Q) = \pm Q\},$$
$$(\mathfrak{A}_\Lambda^{CAR})_\pm = (\mathfrak{A}^{CAR})_\pm \cap \mathfrak{A}_\Lambda^{CAR}, \quad (\mathfrak{A}_{loc}^{CAR})_\pm = (\mathfrak{A}^{CAR})_\pm \cap \mathfrak{A}_{loc}^{CAR}.$$

τ_j is the shift automorphisms defined by

$$\tau_j(c_k) = c_{k+j}, \quad \tau_j(c_k^*) = c_{k+j}^*.$$

We introduce the $U(1)$ gauge action γ_θ via the following equation:

$$\gamma_\theta(c_j) = e^{-i\theta}c_j, \quad \gamma_\theta(c_j^*) = e^{i\theta}c_j^*$$

For fermionic systems, an interaction is an assignment $\{\Psi(X)\}$ of each finite subset X of $\mathbf{Z}$ to a selfadjoint operator $\Psi(X)$ in $(\mathfrak{A}_X^{CAR})_+$. If we assume finite rangeness and translational invariance of interactions and the formal infinite volume Hamiltonian $H = \sum_{X\subset\mathbf{Z}} \Psi(X)$ gives rise to the time evolution α_t of the system via the formula,

$$\frac{d}{dt}\alpha_t(Q)|_{t=0} = [H, Q].$$

The notions of the effective Hamiltonian and the spectral gap are formulated as before and the Lieb-Robinson bound is valid for Θ-twisted commutators.

$$||\{\alpha_t(Q), R\}|| \leq C(Q, R)e^{-ad(X,Y)-v|t|} \tag{2.7}$$

where Q(resp. R) is an element of $(\mathfrak{A}_X^{CAR})_-$ (resp. $(\mathfrak{A}_Y^{CAR})_-$).

Theorem 2.5 *We consider the spinless Fermion on $\mathbf{Z}$. Assume that the interaction $\{\Psi(X)\}$ is translationally invariant and of finite range. Let φ be a translationally invariant pure ground state of the Hamiltonian H. Assume further that the effective Hamiltonian H_φ has the spectral gap.*

Then, there exists positive constants C and K such that for any Q in $\mathfrak{A}_L^{CAR}$, R in $\mathfrak{A}_R^{CAR}$ and any positive integer j, the following estimate is valid.

$$|\varphi(Q\tau_j(R)) - \varphi(Q)\varphi(R)| \leq C||Q|| \cdot ||R||e^{-Kj} \tag{2.8}$$

C and K are independent of Q and R.

The standard Fock state ψ_F is a state of Fermion which is determined uniquely by the formula

$$\psi_F(c_j^* c_j) = 0 \tag{2.9}$$

for any j. Similarly, the standard anti-Fock state ψ_{AF} is a state of Fermion which is determined uniquely by the formula

$$\psi_{AF}(c_j c_j^*) = 0 \tag{2.10}$$

for any j.

Theorem 2.6 *Consider the spinless Fermion on* $\mathbf{Z}$ *and let φ be a translationally invariant pure ground state for a finite range translationally invariant Hamiltonian. Suppose further that the effective Hamiltonian H_φ has the spectral gap and φ is γ_θ invariant for any θ. Then, φ is either ψ_F or ψ_{AF} .*

3 Twisted Haag duality

In this section, we show twisted Haag duality for translationally invariant pure states of Fermion systems. First, let us recall the definition of the Haag duality for quantum spin chains. Consider an irreducible representation $\pi(\mathfrak{A})$ of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$. We say Haag duality holds for a subset Λ in $\mathbf{Z}$ if $\pi(\mathfrak{A}_\Lambda)'' = \pi(\mathfrak{A}_{\Lambda^c})'$ where Λ^c is the complement of Λ in $\mathbf{Z}$. At first sight, this duality may be expected. However, if one recalls other examples of infinite quantum systems such as positive energy representation of loop groups, the duality turns out to be highly non-trivial. (c.f. [32]) In the loop group case when we choose the upper semi-circle as Λ, the duality does not hold for non-vacuum sectors of positive energy representations.

If $\pi(\mathfrak{A}_\Lambda)''$ is a type I von Neumann algebra, it is easy to show Haag duality for any Λ in $\mathbf{Z}$. If Λ is the semi-interval $[1,\infty)$ and the representation is associated with a (gapless) ground state $\mathfrak{M}_R = \pi(\mathfrak{A}_\Lambda)''$ can be of non type I. Nevertheless in [14], we succeeded in proving the Haag duality for $\mathfrak{M}_R$ in GNS representations of translationally invariant pure states

Theorem 3.1 *Let φ be a translationally invariant pure state of the UHF algebra $\mathfrak{A}$, and let $\{\pi_\varphi(\mathfrak{A}), \Omega_\varphi, \mathfrak{H}_\varphi\}$ be the GNS triple for φ. Then, the Haag duality holds:*

$$\mathfrak{M}_R = \mathfrak{M}_L' \tag{3.1}$$

Next we consider the GNS representation of $\mathfrak{A}^{CAR}$ associated with a translationally invariant pure state ψ and we show the fermionic version of Haag duality. In general, any translationally invariant factor state ψ of $\mathfrak{A}^{CAR}$ is Θ invariant. (See [3] for proof.) Suppose that a state ψ of $\mathfrak{A}^{CAR}$ is Θ invariant and let $\{\pi_\psi(\mathfrak{A}_{CAR}), \Omega_\psi, \mathfrak{H}_\psi\}$ be the GNS triple associated with ψ. There exists a (unique) selfadjoint unitary Γ on $\mathfrak{H}_\psi$ satisfying

$$\Gamma\pi_\psi(Q)\Gamma^{-1} = \pi_\psi(\Theta(Q)),\ \ \Gamma^2 = 1,\ \ \Gamma = \Gamma^*,\ \ \Gamma\Omega_\psi = \Omega_\psi. \tag{3.2}$$

With aid of Γ, we introduce another representation $\overline{\pi}_\psi$ of $\mathfrak{A}^{CAR}$ via the following equation:

$$\overline{\pi}_\psi(c_j) = \pi_\psi(c_j)\Gamma,\ \ \overline{\pi}_\psi(c_j^*) = \Gamma\pi_\psi(c_j^*) \tag{3.3}$$

for any integer j.

Let Λ be a subset of $\mathbf{Z}$ and ψ be a state of $\mathfrak{A}^{CAR}$ which is Θ invariant. By definition,$\pi_\psi(\mathfrak{A}^{CAR}_\Lambda)'' \subset \overline{\pi}_\psi(\mathfrak{A}^{CAR}_{\Lambda^c})'$. We say the twisted Haag duality is valid for Λ if and only if

$$\pi_\psi(\mathfrak{A}^{CAR}_\Lambda)'' = \overline{\pi}_\psi(\mathfrak{A}^{CAR}_{\Lambda^c})' \tag{3.4}$$

holds.

Theorem 3.2 *Let ψ be a translationally invariant pure state of the CAR algebra $\mathfrak{A}^{CAR}$. and let $\{\pi_\psi(\mathfrak{A}^{CAR}), \Omega_\psi, \mathfrak{H}_\psi\}$ be the GNS triple for ψ. Then, the twisted Haag duality holds for $\Lambda = [1,\infty)$.*

$$\pi_\psi((\mathfrak{A}^{CAR})_L)'' = \overline{\pi}_\psi((\mathfrak{A}^{CAR})_R)' \tag{3.5}$$

Now we prove this twisted duality using results of [1], [2] and [14]. Fermion systems and quantum spin chains are formally equivalent via the Jordan-Wigner transformation.However this is not mathematically precise as the Jordan-Wigner transformation contains an infinite product of Pauli spin matrices which may not converge in the GNS spaces. We follow the idea of [1]. First we introduce an automorphism Θ_- of $\mathfrak{A}_{CAR}$ by the following equations:

$$\Theta_-(c_j^*) = -c_j^*, \ \Theta_-(c_j) = -c_j \ (j \leq 0),$$
$$\Theta_-(c_k^*) = c_k^*, \ \Theta_-(c_k) = c_k \ (k > 0).$$

Let $\tilde{\mathfrak{A}}$ be the crossed product of $\mathfrak{A}_{CAR}$ by the $\mathbf{Z_2}$ action Θ_- . $\tilde{\mathfrak{A}}$ is the C^*-algebra generated by $\mathfrak{A}_{CAR}$ and a unitary T satisfying

$$T = T^*, \ T^2 = 1, \ \ TQT = \Theta_-(Q) \quad (Q \in \mathfrak{A}_{CAR}).$$

Via the following formulae, we regard $\mathfrak{A}$ as a subalgebra of $\tilde{\mathfrak{A}}$:

$$\begin{aligned} \sigma_z^{(j)} &= 2c_j^* c_j - 1 \\ \sigma_x^{(j)} &= TS_j(c_j + c_j^*) \\ \sigma_y^{(j)} &= iTS_j(c_j - c_j^*). \end{aligned} \tag{3.6}$$

where

$$S_n = \begin{cases} \sigma_z^{(1)} \cdots \sigma_z^{(n-1)} & n > 1 \\ 1 & n = 1 \\ \sigma_z^{(0)} \cdots \sigma_z^{(n)} & n < 1. \end{cases}$$

We extend the automorphism Θ of $\mathfrak{A}_{CAR}$ to $\tilde{\mathfrak{A}}$ via the following equations:

$$\Theta(T) = T, \ \Theta(\sigma_x^{(j)}) = -\sigma_x^{(j)}, \ \Theta(\sigma_y^{(j)}) = -\sigma_y^{(j)}, \ \Theta(\sigma_z^{(j)}) = \sigma_z^{(j)}.$$

As is the case of the CAR algebra, we set

$$(\mathfrak{A})_\pm = \{Q \in \mathfrak{A} | \Theta(Q) = \pm Q\}, \ (\mathfrak{A}_\Lambda)_\pm = (\mathfrak{A})_\pm \cap \mathfrak{A}_\Lambda, \ (\mathfrak{A}_{loc})_\pm = (\mathfrak{A})_\pm \cap \mathfrak{A}_{loc}.$$

Then, it is easy to see that

$$(\mathfrak{A})_+ = (\mathfrak{A}^{CAR})_+, \ (\mathfrak{A}_\Lambda)_+ = (\mathfrak{A}_\Lambda^{CAR})_+, \ (\mathfrak{A}_{loc})_+ = (\mathfrak{A}_{loc}^{CAR})_+.$$

Let ψ be a pure state of $\mathfrak{A}_{CAR}$ and assume that ψ is Θ invariant. Let ψ_+ be the restriction of ψ to $(\mathfrak{A}^{CAR})_+ = (\mathfrak{A})_+$. ψ_+ is extendible to a Θ invariant state φ_0 of $\mathfrak{A}$ via the following formula:

$$\varphi_0(Q) = \psi_+(Q_+), \quad Q_\pm = \frac{1}{2}(Q \pm \Theta(Q)) \in (\mathfrak{A})_\pm. \tag{3.7}$$

In general, φ_0 may not be a pure state but if φ is a pure state extension of ψ_+ to $\mathfrak{A}$, the relation between φ_0 and φ is written as $\varphi_0(Q) = \varphi(Q_+)$. That φ_0 and φ are identical or not depends on existence of a unitary implementing Θ_- on $\mathfrak{H}_\psi$.

Proposition 3.3 *Let ψ be a Θ invariant pure state of $\mathfrak{A}^{CAR}$ and ψ_+ be the restriction of ψ to $(\mathfrak{A}^{CAR})_+$.*

(i) *Suppose that ψ and $\psi \circ \Theta_-$ are not unitarily equivalent. The unique Θ invariant extension φ of ψ_+ to $\mathfrak{A}$ is a pure state. If ψ is translationally invariant, φ is translationally invariant as well.*

(ii) *Suppose that ψ and $\psi \circ \Theta_-$ are unitarily equivalent and that ψ_+ and $\psi_+ \circ \Theta_-$ are unitarily equivalent as states of $(\mathfrak{A}^{CAR})_+$. The unique Θ invariant extension φ of ψ_+ to $\mathfrak{A}$ is a pure state. If ψ is translationally invariant, φ is translationally invariant as well.*

(iii) *Suppose that ψ and $\psi \circ \Theta_-$ are unitarily equivalent and that ψ_+ and $\psi_+ \circ \Theta_-$ are not unitarily equivalent as states of $(\mathfrak{A}^{CAR})_+$. There exists a pure state extension φ of ψ_+ to $\mathfrak{A}$ which is not Θ invariant. Furthermore, we can identify the GNS Hilbert spaces $\mathfrak{H}_{\psi_+}$ and $\mathfrak{H}_\varphi$ and*

$$\pi_\varphi(\mathfrak{A})'' = \pi_\varphi((\mathfrak{A})_+)''. \tag{3.8}$$

If ψ is translationally invariant, φ is a periodic state with period 2, $\varphi \circ \tau_2 = \varphi$ and

$$\pi_\varphi(\mathfrak{A}_L)'' = \pi_\varphi((\mathfrak{A}_L)_+)'', \quad \pi_\varphi(\mathfrak{A}_R)'' = \pi_\varphi((\mathfrak{A}_R)_+)'' \tag{3.9}$$

where we set $(\mathfrak{A}_{L,R})_+ = (\mathfrak{A}_{L,R}) \cap (\mathfrak{A})_+$.

Proposition 3.4 **(i)** *Let ψ be a Θ invariant pure state of $\mathfrak{A}^{CAR}$ and Λ be a subset of $\mathbf{Z}$. Then, the twisted Haag duality (3.4) holds for Λ if and only if*

$$\pi_{\psi_+}((\mathfrak{A}^{CAR}_\Lambda)_+)'' = \pi_{\psi_+}((\mathfrak{A}^{CAR}_{\Lambda^c})_+)' \tag{3.10}$$

on the GNS space $\mathfrak{H}_{\psi_+}$ associated with the state ψ_+ of $(\mathfrak{A}^{CAR})_+$

(ii) *Let φ be a Θ invariant pure state of $\mathfrak{A}$ and Λ be a subset of $\mathbf{Z}$. Then, the Haag duality holds for Λ if and only if*

$$\pi_{\varphi_+}((\mathfrak{A}_\Lambda)_+)'' = \pi_{\varphi_+}((\mathfrak{A}_{\Lambda^c})_+)' \tag{3.11}$$

on the GNS space $\mathfrak{H}_{\varphi_+}$ associated with the restriction φ_+ of φ to $(\mathfrak{A})_+$.

Theorem 3.2 follows from the above Proposition 3.3 , Proposition 3.4 and the Haag duality for spin systems.

Proof of Proposition 3.3
Set $X_j = c_j + c_j^*$. As ψ is Θ invariant, the GNS space $\mathfrak{H}_\psi$ is a direct sum of $\mathfrak{H}^{(\pm)}_{(\psi)}$ where

$$\mathfrak{H}^{(+)}_\psi = \overline{\pi_\psi((\mathfrak{A})_+)\Omega}, \quad \mathfrak{H}^{(-)}_\psi = \overline{\pi_\psi((\mathfrak{A})_+ X_j)\Omega}.$$

The representation $\pi_\psi((\mathfrak{A})_+)$ of $(\mathfrak{A})_+$ on $\mathfrak{H}_\psi$ is decomposed into mutually disjoint irreducible representations on $\mathfrak{H}_\psi^{(\pm)}$.

Let ψ and $\tilde{\psi}$ be Θ invariant states of $\mathfrak{A}^{CAR}$. The argument in 2.8 of [28] shows that if ψ_+ and $\tilde{\psi}_+$ of $(\mathfrak{A})_+$ are equivalent, ψ and $\tilde{\psi}$ are equivalent. Now we show (i). If pure states ψ and $\psi \circ \Theta_-$ are not equivalent, $\psi_+ = \varphi_+$ is not equivalent to $(\varphi \circ \Theta_-)_+$ and $(\varphi \circ \Theta_- \circ Ad(X_j))_+$. Consider the GNS representation $\{\pi_\varphi(\mathfrak{A}), \Omega_\varphi, \mathfrak{H}_\varphi\}$ of $\mathfrak{A}$. If we restrict π_φ to $(\mathfrak{A})_+$ it is the direct sum of two irreducible GNS representations associated with $\psi_+ = \varphi_+$ and $(\varphi \circ \Theta_- \circ Ad(X_j))_+$. So we set

$$\mathfrak{H} = \mathfrak{H}_\varphi, \; \mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2, \; \mathfrak{H}_1 = \mathfrak{H}_{\varphi_+}, \; \mathfrak{H}_2 = \mathfrak{H}_{(\varphi\circ\Theta_-\circ Ad(X_j))_+}.$$

Any bounded operator A on $\mathfrak{H}$ is written in a matrix form,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{3.12}$$

where a_{11}(resp. a_{22}) is a bounded operator on $\mathfrak{H}_1$ (resp. $\mathfrak{H}_2$) and a_{12}(resp. a_{21}) is a bounded operator from $\mathfrak{H}_2$ to $\mathfrak{H}_1$ (resp. a bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$. As $\psi_+ = \varphi_+$ is not equivalent to $(\varphi \circ \Theta_- \circ Ad(X_j))_+$,

$$P = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \tag{3.13}$$

is an element of $\pi_\varphi((\mathfrak{A})_+)''$ and $\pi_\varphi(\sigma_x^{(j)})$ looks like

$$\pi_\varphi(\sigma_x^{(j)}) = \begin{pmatrix} 0 & d \\ d^* & 0 \end{pmatrix} \tag{3.14}$$

A direct computation shows that an operator A of the matrix form (3.12) commuting with (3.13) and (3.14) is trivial. This shows that the state φ is pure. The translational invariance of φ follows from translational invariance of ψ and $\varphi(Q) = \psi(Q_+)$.

(ii) of Proposition 3.3 can be proved by constructing the representation of $\mathfrak{A}$ on the GNS space of Fermion. By our assumption, $\pi_{\psi_+}((\mathfrak{A})_+)$ is not equivalent to $\pi_{\psi_+}(Ad(X_j)(\mathfrak{A})_+)$. Hence $\pi_{\psi_+}((\mathfrak{A})_+)$ is equivalent to $\pi_{\psi_+}(\Theta_-(\mathfrak{A})_+))$ and $\pi_{\psi_+}(Ad(X_j)(\mathfrak{A})_+))$ is equivalent to $\pi_{\psi_+}(\Theta_-(Ad(X_j)\mathfrak{A})_+))$. It turns out that there exists a selfadjoint unitary $U(\Theta_-)$ ($U(\Theta_-)^* = U(\Theta_-)$, $U(\Theta_-)^2 = 1$) on $\mathfrak{H}_\psi$ such that

$$U(\Theta_-)\pi_\psi(Q)U(\Theta_-)^*, \quad U(\Theta_-) \in \pi_\psi((\mathfrak{A})_+)'' \tag{3.15}$$

for any Q in $\mathfrak{A}^{CAR}$. Any element R of $\mathfrak{A}$ is writttten in terms of fermion operators and T as follows:

$$R = R_+ + TR_-, \tag{3.16}$$

where

$$R_+ = \frac{1}{2}(R + \Theta(R)) \in (\mathfrak{A}^{CAR})_+, \quad R_- = \frac{1}{2}(TR - T\Theta(R)) \in (\mathfrak{A}^{CAR})_-.$$

Using this formula, for any R in $\mathfrak{A}$, we set

$$\pi(R) = \pi_\psi(R_+) + U(\Theta_-)\pi_\psi(R_-) \tag{3.17}$$

$\pi(R)$ gives rise to a representation of $\mathfrak{A}$ on $\mathfrak{H}_\psi$ and we set

$$\varphi(R) = (\Omega_\psi, \pi(R)\Omega_\psi)\,. \tag{3.18}$$

The representation $\pi(\mathfrak{A})$ is irreducible because $\pi((\mathfrak{A})_+)''$ contains $U(\Theta_-)$ and hence $\pi(\mathfrak{A})''$ contains $\pi((\mathfrak{A}^{CAR})_-)$ and $\pi(\mathfrak{A})'' = \mathfrak{B}(\mathfrak{H}_\varphi)$.
As in (i), the translational invariance of φ follows from Θ invariance of φ (by construction) and translational invariance of ψ .

To show (iii), we construct an irreducible representation of $\mathfrak{A}$ on the GNS space $\mathfrak{H}_+ = \overline{\pi_{\psi_+}((\mathfrak{A}^{CAR})_+)\Omega_\psi}$. Now under our assumption there exists a self-adjoint unitary $V(\Theta_-)$ satisfying

$$V(\Theta_-)\pi_\psi(Q)V(\Theta_-)^* = \pi_\psi(\Theta_(Q)), \quad V(\Theta_-) \in \overline{\pi_\psi((\mathfrak{A})_-)}^w \tag{3.19}$$

for any Q in $\mathfrak{A}^{CAR}$. For R written in the form (3.16), we set

$$\pi(R) = \pi_\psi(R_+) + V(\Theta_-)\pi_\psi(R_-) \tag{3.20}$$

for R in $\mathfrak{A}$ and $\pi(R)$ belongs to the even part $\pi_\psi((\mathfrak{A}^{CAR})_+)''$. and $\pi(\mathfrak{A})$ acts irreducibly on $\mathfrak{H}_+$.
To show periodicity of the state φ, we introduce a unitary W satisfying

$$W\Omega_\psi = \Omega_\psi, \quad W\pi_\psi(Q)W^* = \pi_\psi(\tau_1(Q)), \quad Q \in \mathfrak{A}^{CAR}$$

The adjoint action of both unitaries $WV(\Theta_-)W^*$ and $V(\Theta_-)\pi_\psi(\sigma_z^{(1)})$ gives rise to the same automorphism on $\pi_\psi(\mathfrak{A}^{CAR})$. By irreducibility of the representation $\pi_\psi(\mathfrak{A}^{CAR})$, $WV(\Theta_-)W^*$ and $V(\Theta_-)\pi_\psi(\sigma_z^{(1)})$ differ in a phase factor.

$$WV(\Theta_-)W^* = cV(\Theta_-)\pi_\psi(\sigma_z^{(1)}) \tag{3.21}$$

where c is a complex number with $|c| = 1$. As both sides in (3.21) are selfadjoint , $c = \pm 1$. Then,

$$W^2V(\Theta_-)(W^2)^* = V(\Theta_-)\pi_\psi(\sigma_z^{(1)}\sigma_z^{(2)}) \tag{3.22}$$

This implies that the state φ is periodic, for example,

$$\begin{aligned}
\varphi(\tau_2(\sigma_x^{(1)})) &= \big(\Omega_\psi, (W^2V(\Theta_-)\pi_\psi(c_1 + c_1^*)(W^2)^*\Omega_\psi\big) \\
&= \Big(\Omega_\psi, V(\Theta_-)\pi_\psi((\sigma_z^{(1)}\sigma_z^{(2)})(c_3 + c_3^*))\Omega_\psi\Big) \\
&= \varphi(\tau_2(\sigma_x^{(3)})).
\end{aligned}$$

End of Proof of Proposition 3.3

Remark 3.5 *In [19], using expansion technique(but not the exact solution) we have shown the XXZ Hamiltonian H_{XXZ} with large Ising type anisotorpy $\Delta >> 1$*

$$H_{XXZ} = \sum_{j=-\infty}^{\infty} \{\Delta\sigma_z^{(j)}\sigma_z^{(j+1)} + \sigma_x^{(j)}\sigma_x^{(j+1)} + \sigma_y^{(j)}\sigma_y^{(j+1)}\}$$

has exactly two pure ground states φ and

$$\varphi \circ \Theta = \varphi \circ \tau_1 \neq \varphi.$$

The unique Θ invariant ground state $(1/2\varphi + \varphi \circ \tau_1)$ is a pure state of $(\mathfrak{A})_+$ In this example, the phase factor c of (3.21) is -1.

Proof of Proposition 3.4

We now prove (i). Suppose that ψ is a Θ invariant pure state of $\mathfrak{A}$.
Let $\{\pi_\psi(\mathfrak{A}^{CAR}), \Omega_\psi, \mathfrak{H}_\psi\}$ be the GNS triple associated with ψ and U be the selfadjoint unitary satisfying

$$U\pi_\psi(Q)U^* = \pi_\psi(\Theta(Q)), \quad U\Omega_\psi = \Omega_\psi.$$

We set

$$\mathfrak{H}_\pm = \{\xi \in \mathfrak{H}_\psi \; | U\xi = \pm\xi\}$$

and let $P_\pm$ be the projection to $\mathfrak{H}_\pm$.

First we assume (3.10) and fix k in Λ and l in Λ^c. Any element Q in the commutant of $\pi_\psi(\mathfrak{A}^{CAR})$ is written as

$$Q = Q_1 + Q_2 Z_l, \quad Z_l = U\pi_\psi((c_l + c_l^*)) \tag{3.23}$$

where

$$Q_1 = \frac{1}{2}(Q + UQU^*), \quad Q_2 = \frac{1}{2}(Q - UQU^*)Z_l^*.$$

It is easy to see that $UQ_1U^* = Q_1$, $UQ_2U^* = Q_2$, and that Q_1, Q_2 is in $\pi_\psi(\mathfrak{A}_\Lambda^{CAR})'$. It turns out that, to prove our claim, it suffices to show that an operator Q commuting with U and $\pi_\psi(\mathfrak{A}_\Lambda^{CAR})$ is in the weak closure of $\pi_\psi((\mathfrak{A}_{\Lambda^c}^{CAR})_+)$.

Now let Q be an operator satisfying $[Q, U] = 0, \quad [Q, \pi_\psi(R)] = 0$ for any R in $\pi_\psi(\mathfrak{A}_\Lambda^{CAR})$. Set

$$Q_\pm = P_\pm Q P_\pm.$$

Due to our assumption (3.10), we obtain

$$Q_+ = w - \lim_\alpha P_+\pi_\psi(Q_\alpha)P_+ \tag{3.24}$$

for a sequence Q_α in $(\mathfrak{A}_{\Lambda^c}^{CAR})_+$. As Q commutes with the selfadjoint unitary $\overline{X}_k = \pi_\psi(c_k + c_k^*)$, we get

$$Q_- = P_-\overline{X}_k Q_+ \overline{X}_k P_- = P_-\overline{X}_k P_+ Q P_+ \overline{X}_k P_- = P_-\overline{X}_k Q \overline{X}_k P_-. \tag{3.25}$$

Inserting (3.24) in (3.25) we arrive at

$$\begin{aligned} Q_- &= w - \lim_\alpha P_- \overline{X}_k \pi_\psi(Q_\alpha) \overline{X}_k P_- \\ &= w - \lim_\alpha P_- \pi_\psi((c_k + c_k^*) Q_\alpha (c_k + c_k^*)) P_- \\ &= w - \lim_\alpha P_- \pi_\psi(Q_\alpha) P_- \end{aligned} \tag{3.26}$$

where we used the conditions that $(c_k + c_k^*) \in (\mathfrak{A}_\Lambda^{CAR})$ and that $Q_\alpha^+ \in (\mathfrak{A}_{\Lambda^c}^{CAR})_+$. (3.24) and (3.25) imply that

$$Q = w - \lim_\alpha \pi_\psi(Q_\alpha) \in \pi_\psi(\mathfrak{A}_{\Lambda^c}^{CAR})_+)'' \tag{3.27}$$

(3.27) is the property we claimed.

Next we show (3.11) assuming twisted Haag duality (3.11) . We use the same notation as above.

The representation π_ψ restricted to $(\mathfrak{A}_\Lambda^{CAR})$ is a direct sum of representations $\pi^\pm$ where

$$\pi^\pm((\mathfrak{A}_\Lambda^{CAR})_+)) = P_\pm \pi_\psi((\mathfrak{A}_\Lambda^{CAR})_+)) P_\pm$$

on $\mathfrak{H}_\pm$. We denote $\tilde{\pi}^\pm$ by the representation of $(\mathfrak{A}_{\Lambda^c}^{CAR})_+$ on $\mathfrak{H}_\pm$. $\pi^\pm$ of $(\mathfrak{A}_\Lambda^{CAR})_+$ are mutually unitarily equivalent because the operator Z_l interwtines these representations. The same is true for $\tilde{\pi}^\pm$ for $(\mathfrak{A}_{\Lambda^c}^{CAR})_+$. Let $\mathfrak{M}_\pm$ be the von Neumann algebra on $\mathfrak{H}_\pm$ generated by $\pi^\pm((\mathfrak{A}_\Lambda^{CAR})_+)$ As $\pi^\pm$ are unitarily equivalent, $\Xi = Ad(\pi_\psi((c_k + c_k^*)))$ gives rise to an automorphism of $\mathfrak{M}_\pm$. Thus $Ad(\pi_\psi((c_k + c_k^*))$ is an automorphim of the commutant $\mathfrak{M}_\pm{}'$ on $\mathfrak{H}_\pm$.

Now suppose Q_+ is an element of $\mathfrak{M}_+{}'$ on $\mathfrak{H}_+$ and we have to show that Q_+ is in $\tilde{\pi}^+((\mathfrak{A}_{\Lambda^c}^{CAR})_+)''$.

Set $X_k = (c_k + c_k^*$ and

$$Q = P_+ Q_+ P_+ + P_- \pi_\psi(X_k) Q_+ \pi_\psi(X_k) P_-. \tag{3.28}$$

Then, we claim that Q commutes with $(\pi_\psi(\mathfrak{A}_\Lambda^{CAR})$. To see this, first take R from $(\mathfrak{A}_\Lambda^{CAR})_+$ and we obtain

$$\begin{aligned} & Q\pi_\psi(R) = P_+ Q_+ \pi^+(R) P_+ + P_- \pi_\psi(X_k) Q_+ \pi_\psi(X_k R X_k) \pi_\psi(X_k) P_- \\ & = P_+ \pi^+(R) Q_+ P_+ + P_- \pi_\psi(X_k) P_+ Q_+ P_+ \pi_\psi(X_k R X_k) P_+ \pi_\psi(X_k) P_- \\ & = P_+ \pi^+(R) Q_+ P_+ + P_-(\pi_\psi(X_k) P_+ \pi_\psi(X_k R X_k) P_+ Q_+ P_+ \pi_\psi(X_k) P_- \\ & = \pi_\psi(R) Q. \end{aligned} \tag{3.29}$$

On the other hand,

$$\begin{aligned} & \pi_\psi(X_k) Q \pi_\psi(X_k) = P_+ \pi_\psi(X_k) Q \pi_\psi(X_k) P_+ + P_- \pi_\psi(X_k) Q \pi_\psi(X_k) P_- \\ & = P_+ \pi_\psi(X_k) P_- Q P_- \pi_\psi(X_k) P_+ + P_- \pi_\psi(X_k) P_+ Q P_+ \pi_\psi(X_k) P_- \\ & = P_+ \pi_\psi(X_k) \pi_\psi(X_k) Q_+ \pi_\psi(X_k) \pi_\psi(X_k) P_+ + P_- \pi_\psi(X_k) Q_+ \pi_\psi(X_k) P_- \\ & = P_+ Q_+) P_+ + P_- \pi_\psi(X_k) Q_+ \pi_\psi(X_k) P_- \\ & = Q. \end{aligned} \tag{3.30}$$

As a consequence,

$$Q \in \pi_\psi(\mathfrak{A}_{\Lambda}^{CAR})' = \pi_\psi(\mathfrak{A}_{\Lambda^c}^{CAR})'', \quad Q_+ \in \pi^+((\mathfrak{A}_{\Lambda^c}^{CAR})_+)''$$

As (ii) can be shown in the same manner, we omit the detail.
End of Proof of Proposition 3.4

4 Split Property and Spectral Gap

Once Haag duality is proven, it is possible to show that the presence of the spectral gap implies split property in the sense of S.Doplicher and R.Longo. (cf.[9]) This result is known in case of the relativistic QFT case. We explain the proof rather briefly. In our proof we use results on maximal violation of Bell's inequality due to Stephen J.Summers and Reinhard Werner in [26] .

First let us recall the definition of split property or split inclusion. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be a commuting pair of factors acting on a Hilbert space $\mathfrak{H}$, $\mathfrak{M}_1 \subset \mathfrak{M}_2'$. We say the inclusion is split if there exists an intermediate type I factor $\mathcal{N}$ such that

$$\mathfrak{M}_1 \subset \mathcal{N} \subset \mathfrak{M}_2' \subset \mathfrak{B}(\mathfrak{H}) \tag{4.1}$$

The split inclusion is used for analysis of local QFT and of von Neumann algebras and some general feature of this concept is investigated for abstract von Neumann alegebras. by J.von Neumann and later by S.Doplicher and R.Longo in [9] . R.Longo used this notion of splitting for his solution to the factorial Stone-Weierstrass conjecture in [17].

If (4.1) is valid, the inclusion of the type I factors $\mathcal{N} = \mathfrak{B}(\mathfrak{H}_1) \subset \mathfrak{B}(\mathfrak{H})$ implies factorization of the underlying Hilbert spaces and we obtain $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ and tensor product

$$\mathfrak{M}_1 = \tilde{\mathfrak{M}}_1 \otimes 1_{\mathfrak{H}_2} \subset \mathfrak{B}(\mathfrak{H}_1) \otimes 1_{\mathfrak{H}_2}, \ \mathfrak{M}_2 = 1_{\mathfrak{H}_1} \otimes \tilde{\mathfrak{M}}_2 \subset 1_{\mathfrak{H}_1} \otimes \mathfrak{B}(\mathfrak{H}_2). \tag{4.2}$$

In this sense the split inclusion is statistical independence of two algebras $\mathfrak{M}_1$ and $\mathfrak{M}_2$.

When $\mathfrak{M}_2$ is the commutant $\mathfrak{M}_1'$.of $\mathfrak{M}_1$, the split property of the inclusion $\mathfrak{M}_1 \subset \mathfrak{M}_2'$ is nothing but the condition that $\mathfrak{M}_1$ and hence $\mathfrak{M}_2$ are type I von Neumann algebras . In our case of quantum spin chains, we set $\mathfrak{M}_1 = \mathfrak{M}_R = \pi_\varphi(\mathfrak{A}_R)''$, and $\mathfrak{M}_2 = \mathfrak{M}_L = \pi_\varphi(\mathfrak{A}_L)''$. When the state φ is translationally invariant and pure, $\mathfrak{M}_2$ is the commutant of $\mathfrak{M}_1$ due to Haag dualtiy.

In 1987, Stephen J.Summers and Reinhard Werner found the characterization of split property in terms of violation of Bell's inequality. We now explain their results in [26] . Fix a commuting pair of factors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and let $\mathfrak{M}$ be the von Neumann algebra generated by $\mathfrak{M}_1$ and $\mathfrak{M}_2$, $\mathfrak{M} = \mathfrak{M}_1 \vee \mathfrak{M}_2$ and letφ be a normal state of $\mathfrak{M}$.

By an admissible quadraple $I = \{X_1, X_2, Y_1, Y_2\}$, we mean a quartet of operators X_1, X_2 in $\mathfrak{M}_1$ and Y_1, Y_2 in $\mathfrak{M}_2$ satisfying

$$-1 \leq X_1 \leq 1,\ -1 \leq X_2 \leq 1, \quad -1 \leq Y_1 \leq 1,\ -1 \leq Y_2 \leq 1.$$

We set

$$\beta(\varphi, \mathfrak{M}_1, \mathfrak{M}_2) = \frac{1}{2} \sup_I \varphi(X_1(Y_1 + Y_2) + X_2(Y_1 - Y_2)) \tag{4.3}$$

where the supremun is taken in all admissible quadraple $I = \{X_1, X_2, Y_1, Y_2\}$. We call $\beta(\varphi, \mathfrak{M}_1, \mathfrak{M}_2)$ *the Bell's constant.*

The following results are known. (cf. [27]):

(i) $1 \leq \beta(\varphi, \mathfrak{M}_1, \mathfrak{M}_2) \leq \sqrt{2}$

(ii) If either $\mathfrak{M}_1$ or,$\mathfrak{M}_2$ is commutative, $\beta(\varphi, \mathfrak{M}_1, \mathfrak{M}_2) = 1$.

(iii) If the normal state φ of $\mathfrak{M}$ is a convex combination of product states, then $\varphi = \sum_i \psi_1^{(i)} \otimes \psi_2^{(i)}$, $\beta(\varphi, \mathfrak{M}_1, \mathfrak{M}_2) = 1$.

(iii) If X_1 and X_2 attain the maximum value $\sqrt{2}$ of the Bell's constant , $\beta(\varphi, \mathfrak{M}_1, \mathfrak{M}_2) = \sqrt{2}$, then,

$$\varphi(X_i^2 Q) = \varphi(Q X_i^2) = \varphi(Q), \quad \varphi((X_1 X_2 + X_2 X_1)Q) = 0 \tag{4.4}$$

for $i = 1, 2$ and for any Q in $\mathfrak{M}_1$.

When the state φ is faithful on $\mathfrak{M}_1$, the equation (4.4) means that $\sigma_x = X_1$, $\sigma_y = X_2$, and $\sigma_z = iX_1X_2$ satisfy the relation of Pauli matrices and that the state φ restricted to these Pauli spin matrices is the tracial state. If the maximum value $\sqrt{2}$ of the Bell's constant is not attained by some elements, it is possible to find a sequence of operators asymptotically satisfying the relation of Pauli matrices in the ultra product of $\mathfrak{M}_1$. Then, by applying a result of strong stability of von Neumann algebras due to A.Connes, we are led to the followin relation between split property, strong sability of von Neumann algebras and Bell's constant. (See [26] for proof.)

Theorem 4.1 (S. J.Summers and Reinhard Werner) *Let $\mathfrak{M}$ be a von Neumann algebra in a separable Hilbert space $\mathfrak{H}$ with cyclic separating vector. The following conditions are equivalent.*
(i) $\mathfrak{M}$ is strongly stable, i.e.

$$\mathfrak{M} \cong \mathfrak{M} \otimes \mathfrak{R}_1$$

where $\mathfrak{R}_1$ is the hyperfinite II_1 factor.
(ii) For every normal state φ of $\mathfrak{B}(\mathfrak{H})$, $\beta(\varphi, \mathfrak{M}, \mathfrak{M}') = \sqrt{2}$.

Corollary 4.2 *(i) Let φ be a translationally invariant pure state of $\mathfrak{A}$ and set*

$$C_j = \sup |\varphi(Q\tau_j(R)) - \varphi(Q)\varphi(R)| \tag{4.5}$$

where the supremum is taken for $Q \in \mathfrak{A}_L$, and $R \in \mathfrak{A}_R$ satisfying $\|Q\| \leq 1, \|R\| \leq 1$.

Suppose that the following uniform decay of correlation is valid.

$$\lim_{j\to\infty} C_j = 0 \tag{4.6}$$

Then, $\mathfrak{M}_L$ and $\mathfrak{M}_R$ are of type I.
(ii) Let ψ be a translationally invariant pure state of $\mathfrak{A}^{CAR}$ and set

$$C_j = \sup |\varphi(Q\tau_j(R)) - \varphi(Q)\varphi(R)| \tag{4.7}$$

where the supremum is taken for $Q \in \mathfrak{A}_L^{CAR}$, and $R \in \mathfrak{A}_R^{CAR}$ satisfying $\|Q\| \leq 1, \|R\| \leq 1$.

Suppose that the following uniform decay of correlation is valid.

$$\lim_{j\to\infty} C_j = 0 \tag{4.8}$$

Then, $\mathfrak{M}_L^{CAR} = \pi_\psi(\mathfrak{A}_L^{CAR})''$ and $\mathfrak{M}_R^{CAR} = \pi_\psi(\mathfrak{A}_R^{CAR})''$ are of type I.

Proof of Corollary 4.2. To show the above corollary 4.2 (i), first take j large such that $C_j < \epsilon$ and we have

$$|\,(\Omega_\varphi, QR\Omega_\varphi) - (\Omega_\varphi, Q\Omega_\varphi)\,(\Omega_\varphi, R\Omega_\varphi)\,| < \epsilon||Q|| \cdot ||R|| \tag{4.9}$$

for any Q in $\mathfrak{M}_{(-\infty,0]}$ and any R in $\mathfrak{M}_{[j,\infty)}$. Let $\tilde{\varphi}$ be the vector state associated with Ω_φ and restrict it to $\mathfrak{M}_{(-\infty,0]\cup[j,\infty)}$. Then, $\tilde{\varphi}_{(-\infty,0]\cup[j,\infty)}$ is close to a product state due to (4.9)

$$\beta(\tilde{\varphi}_{(-\infty,0]\cup[j,\infty)}, \mathfrak{M}_{[j,\infty)}, \mathfrak{M}_{(-\infty,0]}) \leq 1 + 2\epsilon \tag{4.10}$$

As the state φ is pure, the von Neumann algebra $\mathfrak{M}_{(-\infty,0]\cup[j,\infty)}$ is type I and by Haag duality explained in the previous section, we have

$$\mathfrak{M}_{(-\infty,0]\cup[j,\infty)} \cap \mathfrak{M}'_{(-\infty,0]} = \mathfrak{M}_{[j,\infty)}.$$

The state $\tilde{\varphi}_{(-\infty,0]\cup[j,\infty)}$ may not be faithful. We reduce $\mathfrak{M}_{[j,\infty)}$ by the support projection P for $\tilde{\varphi}_{(-\infty,0]\cup[j,\infty)}$. We set $\mathfrak{M} = P\mathfrak{M}_{[j,\infty)}P$ and we apply Theorem 4.1 of S.J.Summes and R.Werner. As a result, $\mathfrak{M}$ is not strong stable. By construction, $\mathfrak{M}$ is hyperfinite, so $\mathfrak{M}$ and $\mathfrak{M}_{[j,\infty)}$ are type I von Neumann algebra. (See Section 4 and Appendix of [13].) As $\mathfrak{M}_{1j,\infty)}$ is the tensor product of a matrix algbera and $\mathfrak{M}_{[j,\infty)}$, it is of type I as well.

The case of the corollary 4.2 (ii) can be handle in the same way. Then, instead of (4.9) , we obtain

$$|\,(\Omega_\psi, QR\Omega_\psi) - (\Omega_\psi, Q\Omega_\psi)\,(\Omega_\psi, R\Omega_\psi)\,| < \epsilon||Q|| \cdot ||R|| \tag{4.11}$$

for for any Q in $\tilde{\pi}(\mathfrak{A}_{(-\infty,0]}^{CAR})''$ and any R in $\pi(\mathfrak{A}_{[j,\infty)}^{CAR})''$.

As before, we express any element Q of $\mathfrak{A}^{CAR}$ as a sum of even and odd elements.

$$Q = Q_+ + Q_-, \;\; Q_\pm \in \mathfrak{A}^{CAR})_\pm, \;\; ||Q_\pm|| \leq ||Q||.$$

The state ψ is Θ invariant, and we see

$$\psi(QR) - \psi(Q)\psi(R) = \psi(Q_+R_+) - \psi(Q_+)\psi(R_+) + \psi(Q_-R_-).$$

For Q in $\mathfrak{A}^{CAR}_{[1,\infty)}$ and R in $\mathfrak{A}^{CAR}_{(-\infty,0]}$

$$\begin{aligned} & \mid (\Omega_\psi, \tilde{\pi}_\psi(R)\pi_\psi(Q)\Omega_\psi) - (\Omega_\psi, \tilde{\pi}_\psi(R)\Omega_\psi)\,(\Omega_\psi, \pi_\psi(Q)\Omega_\psi) \mid \\ \leq \;\; & \mid (\Omega_\psi, \pi_\psi(R_+)\pi_\psi(Q_+)\Omega_\psi) - (\Omega_\psi, \pi_\psi(R_+)\Omega_\psi)\,(\Omega_\psi, \pi_\psi(Q_+)\Omega_\psi) \mid \\ + \;\; & \mid (\Omega_\psi, \pi_\psi(R_-)\pi_\psi(Q_-)\Omega_\psi) \end{aligned} \tag{4.12}$$

Thus we obtain the following estimate of the Bell's constant

$$\beta(\tilde{\psi}_{(-\infty,0]\cup[j,\infty)}, \pi(\mathfrak{A}^{CAR}_{[j,\infty)})'', \tilde{\pi}(\mathfrak{A}^{CAR}_{(-\infty,0]})'') \leq 1 + 4\epsilon. \tag{4.13}$$

(4.13) shows that $\pi(\mathfrak{A}^{CAR}_{[1,\infty)})''$ is of type I. *End of Proof of Corollary 4.2.*

By setting $C_j = C_0 e^{-M|j|}$ the above corollary 4.2 implies Theorem 1.2 (i).

We consider fermionic systems. A state ψ of $\mathfrak{A}^{CAR}$ or $\mathfrak{A}^{CAR}_\Lambda$ is called even if $\psi \circ \Theta = \psi$. Suppose that states ψ_1 of $\mathfrak{A}^{CAR}_\Lambda$ and ψ_2 of $\mathfrak{A}^{CAR}_{\Lambda^c}$ are given and that ψ_1 is even. We construct the graded tensor product state $\psi_1 \otimes_{\mathbf{Z_2}} \psi_2$ in the following manner. Let $\{\pi_k(\cdot), \Omega_k, \mathfrak{H}_k\}$ $(k = 1, 2)$ be the GNS representation associated with ψ_k. As ψ_1 is even, there exists a selfadjoint unitary Γ on $\mathfrak{H}_1$ implementing Θ on $\mathfrak{A}^{CAR}_\Lambda$:

$$\Gamma\pi_1(Q)\Gamma^* = \pi_1(\Theta(Q)), \quad Q \in \mathfrak{A}^{CAR}_\Lambda$$

We introduce a representation π of $\mathfrak{A}^{CAR}$ on $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ via the following identity:

$$\pi(c_j) = \pi_1(c_j) \otimes 1, \;\; \pi(c_j) = \pi_2\Gamma \otimes (c_k)$$

for j in Λ and k in Λ^c. We define $\psi_1 \otimes_{\mathbf{Z_2}} \psi_2$ as the vector state for $\Omega_1 \otimes \Omega_2$.

$$\psi_1 \otimes_{\mathbf{Z_2}} \psi_2(Q) = (\Omega_1 \otimes \Omega_2, \pi(Q)\Omega_1 \otimes \Omega_2)\,.$$

If ψ is an even state of $\mathfrak{A}^{CAR}$ and if the restriction of ψ to $\mathfrak{A}^{CAR}_\Lambda$ gives rise to a type I representation, ψ is equivalent to $\psi_1 \otimes_{\mathbf{Z_2}} \psi_2$ where ψ_1 is a even state of $\mathfrak{A}^{CAR}_\Lambda$ and ψ_2 is a state of $\mathfrak{A}^{CAR}_{\Lambda^c}$.

Noticing these facts we see that the corollary 4.2 implies Theorem 1.2 (ii).

5 $U(1)$ Gauge Symmetry

To complete our proof of Theorem 1.1, we use the main theorem of [18]. and the proposition below.

Theorem 5.1 *Suppose that the spin S of one site algebra $M_{2S+1}(n = 2S+1)$ for $\mathfrak{A}$ is $1/2$. Let φ be a translationally invariant pure state of $\mathfrak{A}$ such that φ_R gives rise to a type I representation of $\mathfrak{A}_R$. Suppose further that φ is $U(1)$ gauge invariant , $\varphi \circ \gamma_\theta = \varphi$. Then, φ is a product state.*

Proposition 5.2 *Let ψ be a translationally invariant pure state of $\mathfrak{A}^{CAR}$.*
(i) Suppose further that ψ is $U(1)$ gauge invariant, $\psi \circ \gamma_\theta = \psi$. The Θ invariant extension of ψ_+ to $\mathfrak{A}$ is a translationally invariant pure state.
(ii) Suppose further that ψ is $U(1)$ gauge invariant and the von Neumann algebra $\pi_\psi(\mathfrak{A}_L^{CAR})''$ associated with the GNS representation of ψ_L is of type I. Then, either $\psi = \psi_F$ or $\psi = \psi_{AF}$ holds.

Next we present a proof for Theorem 5.1 partly different from the one in [18]. Let $\{\pi(\mathfrak{A}), \Omega, \mathfrak{H}\}$ be the GNS triple for φ. Suppose that $\mathfrak{M}_R$ is of type I. As φ_R is γ_θ invariant, γ_θ is extendible to an $U(1)$ action on the type I factor $\mathfrak{M}_R$. As any automorphism of an type I factor is inner, there exists a projective unitary representation $U_R(\theta)$ in $\mathfrak{M}_R$ satisfying

$$U_R(\theta)\pi(Q)U_R(\theta)^* = \pi(\gamma_\theta(Q)), \quad Q \in \mathfrak{A}_R. \tag{5.1}$$

For $U(1)$ the cocycle is trivial and we may assume that $U_R(\theta)$ is a representation of $U(1)$. Similarly we obtain a representation $U_L(\theta)$ of $U(1)$ in $\mathfrak{M}_L$ satisfying

$$U_L(\theta)\pi(Q)U_L(\theta)^* = \pi(\gamma_\theta(Q)), \quad Q \in \mathfrak{A}_L. \tag{5.2}$$

Furthermore by suitably choosing phase factors and setting $U(\theta) = U_R(\theta)U_L(\theta)$, we obtain

$$U(\theta)\Omega = \Omega, \quad U_R(\theta)\Omega = U_L(-\theta)\Omega \tag{5.3}$$

We write the Fourrier series for $U_R(\theta)$ and $U_L(\theta)$ as follows:

$$U_R(\theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta}P_R(k), \quad U_L(\theta) = \sum_{k=-\infty}^{\infty} e^{ik\theta}P_L(k)$$

Due to (5.3) we have $P_R(k)\Omega = P_L(-k)\Omega$.

The state φ is translationally invariant, τ_1 restricted to $\mathfrak{A}_R$ is extendible to the von Neumann algebra $\mathfrak{M}_R$ as an endomorphism denoted by Ξ_R.

$$\Xi_R(\pi(Q)) = \pi(\tau_1(Q)), \quad Q \in \mathfrak{A}_R.$$

This endomorphism Ξ_R is a shift of the type I von Neumann algebra $\mathfrak{M}_R$, namely, $\cap_{k=0}^{\infty}\Xi^k(\mathfrak{M}_R) = \mathbf{C}1$. Then, there exists a representation of O_2 in $\mathfrak{M}_R$ implementing Ξ_R .

$$T_k^*T_l \;=\; \delta_{kl}1, \quad \Xi_R(Q) = T_1QT_1^* + T_2QT_2^*, \quad Q \in \mathfrak{M}_R \tag{5.4}$$

We can introduce a backward shift Ξ_L on $\mathfrak{M}_L$ satisfying

$$\Xi_L(\pi(Q)) = \pi(\tau_{-1}(Q)), \quad Q \in \mathfrak{A}_L$$

and another representation of O_2 in $\mathfrak{M}_L$ implementing Ξ_L .

$$S_k^* S_l \;=\; \delta_{kl} 1, \quad \Xi_L(Q) = S_1 Q S_1^* + S_2 Q S_2^*, \quad Q \in \mathfrak{M}_L \tag{5.5}$$

The representations of O_2 satisfying (5.4) and (5.5) is not unique because we have freedom of the $U(2)$ gauge action (or choice of base of the 2 dimensional space) but we may assume that

$$T_2 T_2^* - T_1 T_1^* = \sigma_z^{(1)}, \quad S_2 S_2^* - S_1 S_1^* = \sigma_z^{(0)}.$$

Still we have freedom to choose the phase factor corresponding to the U(1) gauge action. If we set $V = S_1^* T_1 + S_2^* T_2$, a direct computation shows that V is a unitary and

$$V \pi(Q) V^* = \pi(\tau_1(Q)) \tag{5.6}$$

for any Q in $\mathfrak{A}$. As the state φ is translationally invariant we may assume that

$$V\Omega = \Omega, \quad S_k^* \Omega = T_k^* \Omega. \tag{5.7}$$

Next turn to $U_R(\theta) T_k U_R(\theta)^*$. These operators satisfy the relation of the generators of O_2. On the other hand, the adjoint action of $U_R(\theta)$ is same as γ_θ restricted on $\mathfrak{M}_R$. By this fact we conclude

$$U_R(\theta) T_1 U_R(\theta)^* = e^{il\theta} T_1, \quad U_R(\theta) T_2 U_R(\theta)^* = e^{i(l+1)\theta} T_2 \tag{5.8}$$

By the same reason,

$$U_L(\theta) S_1 U_R(\theta)^* = e^{il'\theta} S_1, \quad U_R(\theta) S_2 U_R(\theta)^* = e^{i(l'+1)\theta} S_2 \tag{5.9}$$

We claim that $l = l'$. As τ_1 commutes with γ_θ V commutes with $U(\theta)$ where we used $V\Omega = \Omega, U(\theta)\Omega = \Omega$. By definition,

$$U(\theta) V = e^{i(l-l')\theta} V U(\theta)$$

so we conclude $l = l'$.
(5.8) and (5.9) tell us

$$\begin{aligned} &P_R(k) T_1 = T_1 P_R(k-l), \;\; P_R(k) T_2 = T_2 P_R(k-l-1), \\ &P_L(k) S_1 = S_1 P_L(k-l), \;\; P_L(k) S_2 = S_2 P_L(k-l-1). \end{aligned} \tag{5.10}$$

Setting $S_1 S_1^* = e_1^{(0)}$, $S_2 S_2^* = e_2^{(0)}$ $T_1 T_1^* = e_1^{(1)}$, $T_2 T_2^* = e_2^{(1)}$, we have

$$(\Omega, e_1^{(0)} P_R(k) \Omega) = (S_1^* \Omega, P_R(k) S_1^* \Omega) = (\Omega, T_1 P_R(k) T_1^* \Omega) = (\Omega, P_R(k+l) e_1^{(1)} \Omega)$$

and

$$(\Omega, e_2^{(0)} P_R \Omega) = (\Omega, P_R(k+l+1) e_2^{(1)} \Omega)$$

where we used 5.7. As $e_1^{(0)} + e_2^{(0)} = 1 = e_1^{(1)} + e_2^{(1)}$

$$\begin{aligned}(\Omega, P_R(k)\Omega) &= (\Omega, (e_1^{(1)} + e_2^{(1)})P_R(k)\Omega) \\ &= (\Omega, P_R(k+l)e_1^{(1)} + P_R(k+l+1)e_2^{(1)})\Omega)\end{aligned} \tag{5.11}$$

Suppose that $l = 0$. Then,

$$(\Omega, e_2^{(1)} P_R(k)\Omega) = (\Omega, P_R(k+1)e_2^{(1)}\Omega) = \alpha$$

for any k. Thus, for any m, we obtain

$$(\Omega, e_2^{(1)}\Omega) \geq \sum_{k=n}^{n+m} (\Omega, P_R(k+1)e_2^{(1)}\Omega) = m\alpha.$$

This shows that $\alpha = 0$ and

$$(\Omega, e_2^{(1)}\Omega) = \sum_{k=-\infty}^{\infty} (\Omega, P_R(k+1)e_2^{(1)}\Omega) = 0$$

Thus , φ is a translational invariant pure state satisfying $\varphi(e_1^{(1)}) = 0$ which is a product state.
Suppose that $l = -1$. Then,

$$(\Omega, e_1^{(1)} P_R(k)\Omega) = (\Omega, P_R(k-1)e_1^{(1)}\Omega) = \alpha$$

for any k. By the same line of reasoning

$$(\Omega, e_1^{(1)}\Omega) = 0$$

Thus , φ is a translational invariant pure state satisfying $\varphi(e_2^{(1)}) = 0$ which is a product state.
Suppose that $l \geq 1$. Take sum of k in (5.11)

$$\sum_{k=n}^{\infty} (\Omega, (e_1^{(1)} + e_2^{(1)})P_R(k)\Omega) = \sum_{k=n}^{\infty} (\Omega, P_R(k+l)e_1^{(1)} + P_R(k+l+1)e_2^{(1)})\Omega)$$

It turns out

$$\sum_{k=n}^{l-1} (\Omega, e_1^{(1)} P_R(k)\Omega) + \sum_{k=n}^{l} (\Omega, e_2^{(1)} P_R(k)\Omega) = 0 \tag{5.12}$$

Each summand is positive in (5.12) and we see

$$(\Omega, e_1^{(1)} P_R(k)\Omega) = (\Omega, e_2^{(1)} P_R(k)\Omega) = 0$$

This shows $(\Omega, e_1^{(1)}\Omega) = 0$ $(\Omega, e_2^{(1)}\Omega) = 0$ and we arrive at a contradiction. So $l \geq 1$ is not possible. Similarly $l \leq -2$ is impossible. *End of Proof*

Proof of Proposition 5.2
To prove Proposition 5.2 (i), we show the case (iii) in Proposition 3.3 is impossible due to assumption of γ_θ invariance. There exists $U(\theta)$ implementing γ_θ on the GNS space of ψ. Then

$$U(\theta)V(\Theta_-)U(\theta)^* = c(\theta)V(\Theta_-)$$

as the adjoint action of both unitaries are identical. Moreover these are self-adjoint so $c(\theta) = \pm 1$. Due to continuity in θ we conclude that $c(\theta) = 1$ and $V(\Theta_-)$ is an even element.

Finally, we consider Proposition 5.2 (ii). Due to (i) of Proposition 5.2 (i), the Fermionic state ψ has a translationally invariant pure state extension φ to $\mathfrak{A}$. Then, the split property for Fermion implies that that of the Pauli spin system. It turns out that either $\psi(c_j^* c_j) = \varphi(e_1^{(j)}) = 0$ or $\psi(c_j c_{j^*}) = \varphi(e_2^{(j)}) = 0$ holds. This completes our proof of Proposition 5.2 (ii).
End of Proof of Proposition 5.2

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