ABSOLUTELY CONTINUOUS SPECTRUM OF A POLYHARMONIC OPERATOR WITH A LIMIT PERIODIC POTENTIAL IN DIMENSION TWO.

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ABSTRACT. We consider a polyharmonic operator $H = (-\Delta)^l + V(x)$ in dimension two with $l \geq 6$, l being an integer, and a limit-periodic potential $V(x)$. We prove that the spectrum contains a semiaxis of absolutely continuous spectrum.

1. Main Results.

We study an operator

$$H = (-\Delta)^l + V(x) \tag{1}$$

in two dimensions, where $l \geq 6$ is an integer and $V(x)$ is a limit-periodic potential

$$V(x) = \sum_{r=1}^{\infty} V_r(x); \tag{2}$$

here $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms, namely, V_r has orthogonal periods $2^{r-1}\vec{\beta}_1,\ 2^{r-1}\vec{\beta}_2$ and

$$\|V_r\|_\infty < \hat{C}exp(-2^{\eta r}) \tag{3}$$

for some $\eta > 2 + 64/(2l-11)$. Without loss of generality, we assume that $\hat{C} = 1$ and $\int_{Q_r} V_r(x)dx = 0$, Q_r being the elementary cell of periods corresponding to $V_r(x)$.

The one-dimensional analog of (1), (2) with $l = 1$ is already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1 u = -u'' + Vu$ is generically a Cantor type set. It has positive Lebesgue measure [1, 6]. The spectrum is absolutely continuous [1, 2], [5]–[9]. Generalized eigenfunctions can be represented in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]–[14]. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operators are studied in [15]. As to the continuum multidimensional case, it is proved [14] that the integrated density of states for (1) is the limit of densities of states for periodic operators. A particular case of a periodic operator ($V_r = 0$ when $r \geq 2$) for dimensions $d \geq 2$ and different l is already studied well, e.g., see [16] – [30]. Here we prove that the spectrum of (1), (2) contains a semiaxis of absolutely continuous spectrum. This paper is based on [31]. We proved the following results for the case $d = 2$, $l \geq 6$ in [31].

(1) The spectrum of the operator (1), (2) contains a semiaxis. A proof of the analogous result by different means can be found in [32]. The more general case $8l > d + 3$, $d \neq 1(\text{mod}4)$, is considered in [32], however, under the additional restriction on the potential: the lattices of periods of all periodic potentials V_r

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have to contain a nonzero vector $\vec{\gamma}$ in common, i.e., $V(x)$ is periodic in one direction.

(2) There are generalized eigenfunctions $\Psi_\infty(\vec{k},\vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in a subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k},\vec{x})$ of the equation $H\Psi_\infty=\lambda_\infty\Psi_\infty$ which can be described by the formula

$$\Psi_\infty(\vec{k},\vec{x})=e^{i\langle\vec{k},\vec{x}\rangle}\left(1+u_\infty(\vec{k},\vec{x})\right), \tag{4}$$

$$\|u_\infty\|_{L_\infty(\mathbb{R}^2)} \underset{|\vec{k}|\to\infty}{=} O\left(|\vec{k}|^{-\gamma_1}\right), \quad \gamma_1>0, \tag{5}$$

where $u_\infty(\vec{k},\vec{x})$ is a limit-periodic function

$$u_\infty(\vec{k},\vec{x})=\sum_{r=1}^{\infty}u_r(\vec{k},\vec{x}), \tag{6}$$

$u_r(\vec{k},\vec{x})$ being periodic with periods $2^{r-1}\vec{\beta}_1$, $2^{r-1}\vec{\beta}_2$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_\infty(\vec{k}) \underset{|\vec{k}|\to\infty}{=} |\vec{k}|^{2l}+O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2>0. \tag{7}$$

The "non-resonance" set $\mathcal{G}_\infty$ of vectors $\vec{k}$, for which (4) – (7) hold, is a Cantor type set $\mathcal{G}_\infty=\bigcap_{n=1}^{\infty}\mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets in $\mathbb{R}^2$. Each $\mathcal{G}_n$ has a finite number of holes in each bounded region. More and more holes appear as n increases; however, holes added at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate

$$|\mathcal{G}_\infty\cap\mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1+O(R^{-\gamma_3})\big), \quad \gamma_3>0, \tag{8}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin and $|\cdot|$ is Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda)=\left\{\vec{k}\in\mathcal{G}_\infty:\lambda_\infty(\vec{k})=\lambda\right\},$$

is shown to be a slightly distorted circle with an infinite number of holes. It can be described by the formula

$$\mathcal{D}_\infty(\lambda)=\left\{\vec{k}:\vec{k}=\varkappa_\infty(\lambda,\vec{\nu})\vec{\nu},\ \vec{\nu}\in\mathcal{B}_\infty(\lambda)\right\}, \tag{9}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (4). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda\to\infty$:

$$L\big(\mathcal{B}_\infty(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi+O\left(\lambda^{-\gamma_3/2l}\right), \tag{10}$$

here and below $L(\cdot)$ is a length of a curve. The value $\varkappa_\infty(\lambda,\vec{\nu})$ in (9) is the "radius" of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\varkappa_\infty(\lambda,\vec{\nu})-\lambda^{1/2l}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$. It is shown that the deviation is small

$$\varkappa_\infty(\lambda,\vec{\nu}) \underset{\lambda\to\infty}{=} \lambda^{1/2l}+O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4>0. \tag{11}$$

In this paper, we use the technique of [31] to prove absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k},\vec{x})$.

In [31], we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method to prove the results listed above. The paper [31] is inspired by [33, 34, 35], where the method is used for periodic problems. In [33], KAM method is applied to classical Hamiltonian systems. In [34, 35], the technique developed in [33] is applied for semiclassical approximation for multidimensional periodic Schrödinger operators at high energies. In [31], we consider a sequence of operators

$$H_0 = (-\Delta)^l, \qquad H^{(n)} = H_0 + \sum_{r=1}^{M_n} V_r, \quad n \geq 1, \ M_n \to \infty \text{ as } n \to \infty.$$

Obviously, $\|H - H^{(n)}\| \to 0$ as $n \to \infty$ and $H^{(n)} = H^{(n-1)} + W_n$, where $W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r$. We treat each operator $H^{(n)}$, $n \geq 1$, as a perturbation of the previous operator $H^{(n-1)}$. Each operator $H^{(n)}$ is periodic; however, the periods go to infinity as $n \to \infty$. We show that there exists $\lambda_* = \lambda_*(V)$ such that the semiaxis $[\lambda_*, \infty)$ is contained in the spectra of all operators $H^{(n)}$. For every operator $H^{(n)}$, there is a set of eigenfunctions (corresponding to the semiaxis) close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k},\vec{x})$ of the differential equation $H^{(n)}\Psi_n = \lambda^{(n)}\Psi_n$, which can be represented by the formula

$$\Psi_n(\vec{k},\vec{x}) = e^{i\langle\vec{k},\vec{x}\rangle}\left(1 + \tilde{u}_n(\vec{k},\vec{x})\right), \quad \|\tilde{u}_n\| =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{12}$$

where $\tilde{u}_n(\vec{k},\vec{x})$ has periods $2^{M_n-1}\vec{\beta}_1, 2^{M_n-1}\vec{\beta}_2$.[1] The corresponding eigenvalue $\lambda^{(n)}(\vec{k})$ is close to $|\vec{k}|^{2l}$:

$$\lambda^{(n)}(\vec{k}) \underset{|\vec{k}|\to\infty}{=} |\vec{k}|^{2l} + O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2 > 0.$$

The non-resonance set $\mathcal{G}_n$ is shown to be extensive in $\mathbb{R}^2$:

$$|\mathcal{G}_n \cap \mathbf{B_R}| \underset{R\to\infty}{=} |\mathbf{B_R}|\big(1 + O(R^{-\gamma_3})\big). \tag{13}$$

Estimates (12) – (13) are uniform in n. The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for non-resonant eigenvalue $\lambda^{(n)}(\vec{k})$:

$$\mathcal{D}_n(\lambda) = \left\{\vec{k} \in \mathcal{G}_n : \lambda^{(n)}(\vec{k}) = \lambda\right\}. \tag{14}$$

This set is shown to be a slightly distorted circle with a finite number of holes (see Figs. 1, 2), the set $\mathcal{D}_1(\lambda)$ being strictly inside the circle of the radius $\lambda^{1/2l}$ for sufficiently large λ. The set $\mathcal{D}_n(\lambda)$ can be described by the formula

$$\mathcal{D}_n(\lambda) = \left\{\vec{k} : \vec{k} = \varkappa_n(\lambda,\vec{\nu})\vec{\nu}, \ \vec{\nu} \in \mathcal{B}_n(\lambda)\right\}, \tag{15}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (12). It has an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$L\big(\mathcal{B}_n(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi + O\left(\lambda^{-\gamma_3/2l}\right). \tag{16}$$

[1]Obviously, $\tilde{u}_n(\vec{k},\vec{x})$ is simply related to functions $u_r(\vec{k},\vec{x})$ used in (6): $\tilde{u}_n(\vec{k},\vec{x}) = \sum_{r=M_{n-1}+1}^{M_n} u_r(\vec{k},\vec{x})$.

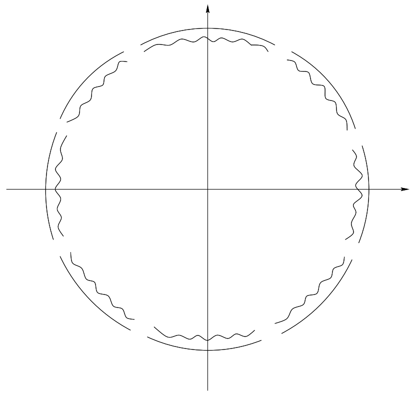

FIGURE 1. Distorted circle with holes, $\mathcal{D}_1(\lambda)$

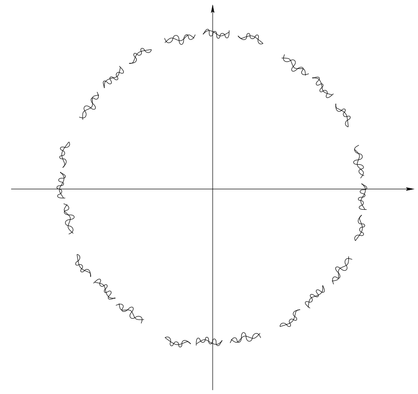

FIGURE 2. Distorted circle with holes, $\mathcal{D}_2(\lambda)$

The set $\mathcal{B}_n(\lambda)$ has only a finite number of holes; however, their number grows with n. More and more holes of a smaller and smaller size are added at each step. The value $\varkappa_n(\lambda,\vec{\nu})-\lambda^{1/2l}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is shown that the deviation is asymptotically small:

$$\varkappa_n(\lambda,\vec{\nu})=\lambda^{1/2l}+O\left(\lambda^{-\gamma_4}\right),\qquad \frac{\partial\varkappa_n(\lambda,\vec{\nu})}{\partial\varphi}=O\left(\lambda^{-\gamma_5}\right),\quad \gamma_4,\gamma_5>0, \tag{17}$$

φ being an angle variable, $\vec{\nu}=(\cos\varphi,\sin\varphi)$. Estimates (16), (17) are uniform in n.

At each step, more and more points are excluded from the non-resonance sets $\mathcal{G}_n$; thus, $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set $\mathcal{G}_\infty=\bigcap_{n=1}^{\infty}\mathcal{G}_n$. It has an infinite number of holes, but nevertheless satisfies the relation (8). For every $\vec{k}\in\mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of $H^{(n)}$ of the type (12). It is shown that the sequence $\Psi_n(\vec{k},\vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ when $\vec{k}\in\mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k},\vec{x})=\lim_{n\to\infty}\Psi_n(\vec{k},\vec{x})$ is a generalized eigenfunction of H. It can be written in the form (4) – (6). Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda^{(n)}(\vec{k})$ as $n\to\infty$.

It is shown that $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets at each step more and more directions being excluded. We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$,

$$\mathcal{B}_\infty(\lambda)=\bigcap_{n=1}^{\infty}\mathcal{B}_n(\lambda).$$

This set has a Cantor type structure on the unit circle. It is shown that $\mathcal{B}_\infty(\lambda)$ has asymptotically full measure on the unit circle (see (10)). We prove that the sequence $\varkappa_n(\lambda,\vec{\nu})$, $n=1,2,...$, describing the isoenergetic curves $\mathcal{D}_n$, converges rapidly (super exponentially) as $n\to\infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense (9), where $\varkappa_\infty(\lambda,\vec{\nu})=\lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu})$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda,\vec{\nu})$ (with respect to the angle variable on the unit circle) have a limit as $n\to\infty$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial\varkappa_\infty(\lambda,\vec{\nu})}{\partial\varphi}$. Using (17), we prove that

$$\frac{\partial\varkappa_\infty(\lambda,\vec{\nu})}{\partial\varphi}=O\left(\lambda^{-\gamma_5}\right). \tag{18}$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite of its Cantor type structure, the tangent vector being the limit of corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n \to \infty$. The curve $\mathcal{D}_\infty(\lambda)$ looks like a slightly distorted circle with infinite number of holes.

The main technical difficulty overcome in [31] is the construction of non-resonance sets $\mathcal{B}_n(\lambda)$ for every fixed sufficiently large λ, $\lambda > \lambda_*(V)$, where λ_* is the same for all n. The set $\mathcal{B}_n(\lambda)$ is obtained by deleting a "resonant" part from $\mathcal{B}_{n-1}(\lambda)$. The definition of $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$ includes Bloch eigenvalues of $H^{(n-1)}$. To describe $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$, one has to use not only non-resonant eigenvalues of the type (7) but also resonant eigenvalues, for which no suitable formulas are known. The absence of formulas causes difficulties in estimating the size of $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$. To deal with this problem, we start by introducing an angle variable $\varphi \in [0, 2\pi)$, $\vec{\nu} = (\cos\varphi, \sin\varphi) \in S_1$ and consider sets $\mathcal{B}_n(\lambda)$ in terms of this variable. Next, we show that the resonant set $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$ can be described as the set of zeros of determinants of the type $\det(I + A_{n-1}(\varphi))$, $A_{n-1}(\varphi)$ being a trace type operator,

$$I + A_{n-1}(\varphi) = \big(H^{(n-1)}\big(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\big) - \lambda - \epsilon\big)\big(H_0\big(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\big) + \lambda\big)^{-1},$$

where $\vec{\varkappa}_{n-1}(\varphi)$ is a vector-function describing $\mathcal{D}_{n-1}(\lambda) : \vec{\varkappa}_{n-1}(\varphi) = \varkappa_{n-1}(\lambda, \vec{\nu})\vec{\nu}$. To obtain $\mathcal{B}_{n-1}(\lambda) \setminus \mathcal{B}_n(\lambda)$, we take all values of ϵ in a small interval and vectors $\vec{b}$ in a finite set, $\vec{b} \neq 0$. Further, we extend our considerations to a complex neighborhood $\varPhi_0$ of $[0, 2\pi)$. We show that the determinants are analytic functions of φ in $\varPhi_0$, and thus reduce the problem of estimating the size of the resonance set to a problem in complex analysis. We use theorems for analytic functions to count the zeros of the determinants and to investigate how far zeros move when ϵ changes. This enables us to estimate the size of the zero set of the determinants and hence the size of the non-resonance set $\varPhi_n \subset \varPhi_0$, which is defined as a non-zero set for the determinants. Proving that the non-resonance set $\varPhi_n$ is sufficiently large, we obtain estimates (13) for $\mathcal{G}_n$ and (16) for $\mathcal{B}_n$, the set $\mathcal{B}_n$ being the intersecton of $\varPhi_n$ with the real line. To obtain $\varPhi_n$ we delete from $\varPhi_0$ more and more holes of smaller and smaller radii at each step. Thus, the non-resonance set $\varPhi_n \subset \varPhi_0$ has the structure of Swiss Cheese (Fig. 7, 8). We call deleting the resonance set from $\varPhi_0$ at each step of the recurrent procedure the "Swiss Cheese Method". The essential difference of our method from those applied earlier in similar situations (see, e.g., [33, 34, 35]) is that we construct a non-resonance set not only in the whole space of a parameter ($\vec{k} \in \mathbb{R}^2$ here) but also on the isoenergetic curves $\mathcal{D}_n(\lambda)$ in the space of parameter when λ is sufficiently large. Estimates for the size of non-resonant sets on a curve require more subtle technical considerations than those sufficient for description of a non-resonant set in the whole space of the parameter.

Here, we use information obtained in [31] to prove absolute continuity of the branch of the spectrum (the semiaxis) corresponding to the functions $\Psi_\infty(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$. Absolute continuity follows from the convergence of the spectral projections corresponding to $\Psi_n(\vec{k}, \vec{x})$, $\vec{k} \in \mathcal{G}_\infty$, to spectral projections of H (in the strong sense uniformly in λ) and properties of the level curves $\mathcal{D}_\infty(\lambda)$, $\lambda > \lambda_*$. Roughly speaking, the area between isoenergetic curves $\mathcal{D}_\infty(\lambda + \epsilon)$ and $\mathcal{D}_\infty(\lambda)$ (integrated density of states) is proportional to ϵ.

Note that generalization of results from the case $l \geq 6$, l being an integer, to the case of rational l satisfying the same inequality is relatively simple; it requires just slightly more careful technical considerations. The restriction $l \geq 6$ is also technical, though it is more difficult to lift. The condition $l \geq 6$ is needed only for the first two steps of the recurrent procedure in [31]. The requirement for super exponential decay of $\|V_r\|$

as $r\to\infty$ is more essential than $l\geq 6$ since it is needed to ensure convergence of the recurrent procedure. It is not essential that potentials V_r have doubling periods; periods of the type $q^{r-1}\vec\beta_1,\ q^{r-1}\vec\beta_2$, $q\in\mathbb{N}$, can be treated in the same way.

The periodic case ($V_r=0$, when $r\geq 2$) is already carefully investigated for dimensions $d\geq 2$ and different l [16]–[30]. For briefness, we mention here only results for dimension two. Absolute continuity of the whole spectrum is proven in [16] for $l=1$, however the proof can be extended for higher integers l. Bethe-Sommerfeld conjecture is first proved for $d=2$, $l=1$ in [17], [18] and for $l\geq 1$ in [21]. The perturbation formulas for eigenvalues are constructed in [20]. The formulas for eigenfunctions and the corresponding isoenergetic surfaces are obtained in [21].

The plan of the paper is the following. In Section 2, we sketch main steps of the recurrent procedure and the "Swiss cheese method" developed in [31]. Section 3 describes eigenfunctions and isoenergetic surfaces of H. The proof of the absolute continuity is in Section 4 using the results in Sections 2 and 3.

2. Recurrent Procedure.

2.1. **The First Approximation.**

2.1.1. *The Main Operator H and the First Operator $H^{(1)}$.* We introduce the first operator $H^{(1)}$, which corresponds to a partial sum in the series (2)

$$H^{(1)}=(-\Delta)^l+W_1,\qquad W_1=\sum_{r=1}^{M_1}V_r, \tag{19}$$

where M_1 is chosen in such a way that $2^{M_1}\approx k^{s_1}$ [2] for a $k>1$, $s_1=(2l-11)/32$. For simplicity, we let the potentials V_r have periods directed along the axes, i.e., the periods of V_r are $2^{r-1}\vec\beta_1=2^{r-1}(\beta_1,0)$ and $2^{r-1}\vec\beta_2=2^{r-1}(0,\beta_2)$. Then, obviously, the periods of W_1 are $(a_1,0)=2^{M_1-1}(\beta_1,0)$ and $(0,a_2)=2^{M_1-1}(0,\beta_2)$, and $a_1\approx k^{s_1}\beta_1/2$, $a_2\approx k^{s_1}\beta_2/2$. Note that

$$\|W_1\|_\infty\leq\sum_{n=1}^{M_1}\|V_n\|_\infty=O(1)\text{ as }k\to\infty.$$

It is well-known (see, e.g., [36]) that spectral analysis of a periodic operator $H^{(1)}$ can be reduced to analysis of a family of operators $H^{(1)}(t)$, $t\in K_1$, where K_1 is the elementary cell of the dual lattice, $K_1=[0,2\pi a_1^{-1})\times[0,2\pi a_2^{-1})$. The vector t is called *quasimomentum*. An operator $H^{(1)}(t)$, $t\in K_1$, acts in $L_2(Q_1)$, Q_1 being the elementary cell of the periods of the potential, $Q_1=[0,a_1]\times[0,a_2]$. The operator $H^{(1)}(t)$ is described by formula (19) and the quasiperiodic boundary conditions for a function and its derivatives:

$$\begin{array}{ll} u(a_1,x_2)=\exp(it_1a_1)u(0,x_2), & u(x_1,a_2)=\exp(it_2a_2)u(x_1,0),\\ u^{(j)}_{x_1}(a_1,x_2)=\exp(it_1a_1)u^{(j)}_{x_1}(0,x_2), & u^{(j)}_{x_2}(x_1,a_2)=\exp(it_2a_2)u^{(j)}_{x_2}(x_1,0), \end{array} \tag{20}$$

$0<j<2l$. Each operator $H^{(1)}(t)$, $t\in K_1$, has a discrete bounded below spectrum $\Lambda^{(1)}(t)$

$$\Lambda^{(1)}(t)=\bigcup_{n=1}^{\infty}\lambda_n^{(1)}(t),\ \lambda_n^{(1)}(t)\to_{n\to\infty}\infty.$$

[2]We write $a(k)\approx b(k)$ when the inequalities $\frac12 b(k)<a(k)<2b(k)$ hold.

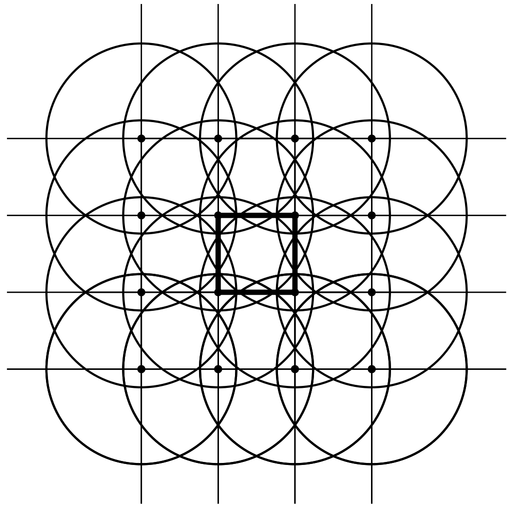

FIGURE 3. The isoenergetic surface $S_0(\lambda)$ of the free operator $H_0^{(1)}$

The spectrum $\Lambda^{(1)}$ of the operator $H^{(1)}$ is the union of the spectra of the operators $H^{(1)}(t)$ over $t \in K_1$, $\Lambda^{(1)} = \cup_{t\in K_1}\Lambda^{(1)}(t) = \bigcup_{n\in\mathbb{N},t\in K_1}\lambda_n^{(1)}(t)$. The functions $\lambda_n^{(1)}(t)$ are continuous in t, so $\Lambda^{(1)}$ has a band structure

$$\Lambda^{(1)} = \bigcup_{n=1}^{\infty}[q_n^{(1)}, Q_n^{(1)}],\ q_n^{(1)} = \min_{t\in K_1}\lambda_n^{(1)}(t),\ Q_n^{(1)} = \max_{t\in K_1}\lambda_n^{(1)}(t). \tag{21}$$

The eigenfunctions of $H^{(1)}(t)$ and $H^{(1)}$ are simply related. Extending all the eigenfunctions of the operators $H^{(1)}(t)$ quasiperiodically (see (20)) to $\mathbb{R}^2$, we obtain a complete system of generalized eigenfunctions of $H^{(1)}$.

Let $H_0^{(1)}$ be the operator (1) corresponding to $V = 0$. We consider that it has periods $(a_1, 0), (0, a_2)$ and that operators $H_0^{(1)}(t)$, $t \in K_1$ are defined in $L_2(Q_1)$. The eigenfunctions of the operator $H_0^{(1)}(t)$, $t \in K_1$, are plane waves satisfying (**??**). They are naturally indexed by points of $\mathbb{Z}^2$

$$\Psi_j^0(t,x) = |Q_1|^{-1/2}\exp i\langle\vec{p}_j(t), x\rangle, \quad j \in \mathbb{Z}^2,$$

the eigenvalue corresponding to $\Psi_j^0(t,x)$ being equal to $p_j^{2l}(t)$, where here and below

$$\vec{p}_j(t) = 2\pi j/a + t,\ \ 2\pi j/a = (2\pi j_1/a_1, 2\pi j_2/a_2),\ j \in \mathbb{Z}^2,\ \ |Q_1| = a_1a_2,\ \ p_j^{2l}(t) = |\vec{p}_j(t)|^{2l}.$$

Next, we introduce an isoenergetic surface[3] $S_0(\lambda)$ of the free operator $H_0^{(1)}$. A point $t \in K_1$ belongs to $S_0(\lambda)$ if and only if $H_0^{(1)}(t)$ has an eigenvalue equal to λ, i.e., there exists $j \in \mathbb{Z}^2$ such that $p_j^{2l}(t) = \lambda$. This surface can be obtained as follows: the circle of radius $k = \lambda^{1/(2l)}$ centered at the origin is divided into pieces by the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$, and then all pieces are translated in a parallel manner into the cell K_1 of the dual lattice. We also can get $S_0(\lambda)$ by drawing sufficiently many circles of radii k centered at the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$ and by looking at the figure in the cell K_1. As

[3] "surface" is a traditional term. In our case, it is a curve.

the result of either of these two procedures we obtain a circle of radius k "packed into the bag K_1" as shown in the Fig. 3. Note that each piece of $S_0(\lambda)$ can be described by an equation $p_j^{2l}(t)=\lambda$ for a fixed j. If $t\in S_0(\lambda)$, then j can be uniquely defined from the last equation, unless t is not a point of self-intersection of the isoenergetic surface. A point t is a self-intersection of $S_0(\lambda)$ if and only if

$$p_q^{2l}(t)=p_j^{2l}(t)=k^{2l} \tag{22}$$

for at least on pair of indices q,j, $q\neq j$.

Note that any vector $\vec{\varkappa}$ in $\mathbb{R}^2$ can be uniquely represented in the form $\vec{\varkappa}=\vec{p}_j(t)$, where $j\in\mathbb{Z}^2$ and $t\in K_1$. Let $\mathcal{K}_1$ be the parallel shift into K_1:

$$\mathcal{K}_1:\mathbb{R}^2\to K_1,\quad \mathcal{K}_1\big(\vec{p}_j(t)\big)=t.$$

Obviously, $\mathcal{K}_1S_k=S_0(\lambda)$ and $L\big(S_0(\lambda)\big)=L\big(S_k\big)=2\pi k$, $k=\lambda^{1/(2l)}$, S_k being the circle of radius k centered at the origin.

The operator $H^{(1)}(t)$, $t\in K_1$, has the following matrix representation in the basis of plane waves $\Psi_j^0(t,x)$:

$$H^{(1)}(t)_{mq}=p_m^{2l}(t)\delta_{mq}+w_{m-q},\quad m,q\in\mathbb{Z}^2,$$

here and below δ_{mq} is the Kronecker symbol, w_{m-q} are Fourier coefficients of W_1, the coefficient w_0 being equal to zero. The matrix $H^{(1)}(t)_{mq}$ also describes an operator in the space l_2 of square summable sequences with indices in $\mathbb{Z}^2$, the operator in l_2 being unitarily equivalent to $H^{(1)}(t)$ in $L_2(Q_1)$. From now on, we denote the operator in l_2 also by $H^{(1)}(t)$. Note that the canonical basis in l_2 does not depend on t, all dependence on t being in the matrix. Thus, the matrix $H^{(1)}(t)_{mq}$ and hence the operator $H^{(1)}(t):l_2\to l_2$ can be analytically extended in t from K_1 to $\mathbb{C}^2$. We consider $H^{(1)}(t):l_2\to l_2$ for real and complex t. Further, when we refer to $H^{(1)}(t)$ for $t\in\mathbb{C}^2$, we mean the operator in l_2.

2.1.2. *Perturbation Formulas.* In this section, we consider the operator $H^{(1)}(t)$ as a perturbation of the free operator $H_0^{(1)}(t)$. We show that for every sufficiently large λ, there is a "non-resonant" subset $\chi_1(\lambda)$ of $S_0(\lambda)$ such that perturbation series for an eigenvalue and a spectral projection of $H^{(1)}(t)$ converge when $t\in\chi_1(\lambda)$. The set $\chi_1(\lambda)$ is obtained by deleting small neighborhoods of self-intersections of $S_0(\lambda)$; see Fig. 4. The self-intersections are described by (22) and correspond to degenerated eigenvalues of $H_0^{(1)}(t)$. The size of the neighborhood is $k^{-1-4s_1-\delta}$, $k=\lambda^{1/(2l)}$, where δ is a small positive number. The set $\chi_1(\lambda)$ is sufficiently large: its relative measure with respect to $S_0(\lambda)$ tends to 1 as $\lambda\to\infty$. The precise formulation of these results is given in the next lemma, proved by elementary geometric considerations in [26] [4].

Lemma 1 (Geometric Lemma)**.** *For an arbitrarily small positive δ, $2\delta<2l-2-4s_1$, and sufficiently large λ, $\lambda>\lambda_0(V,\delta)$, there exists a non-resonance set $\chi_1(\lambda,\delta)\subset S_0(\lambda)$ such that the following hold.*

(1) *For any point $t\in\chi_1(\lambda)$,*
  (a) *there exists a unique $j\in\mathbb{Z}^2$ such that $p_j(t)=k$, $k=\lambda^{1/(2l)}$;*

[4] More precisely, Lemma 1 corresponds to Lemma 2.1 on page 26 in [26]. There is a slight difference between two lemmas. Lemma 2.1 in [26] is proved for the case of fixed periods β_1,β_2. In Lemma 1 here, we consider the periods $a_1\approx k^{s_1}\beta_1/2, a_2\approx k^{s_1}\beta_2/2$. However, the proofs are completely analogous.

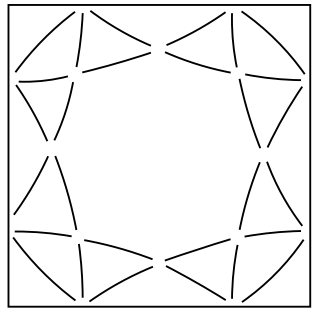

FIGURE 4. The first non-resonance set $\chi_1(\lambda)$

(b)

$$\min_{i\neq j}|p_j^2(t)-p_i^2(t)|>2k^{-4s_1-\delta}. \tag{23}$$

(2) *For any t in the $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, there exists a unique $j\in\mathbb{Z}^2$ such that*

$$|p_j^2(t)-k^2|<5k^{-4s_1-2\delta} \tag{24}$$

and (23) holds.

(3) *The non-resonance set $\chi_1(\lambda)$ has an asymptotically full measure on $S_0(\lambda)$ in the following sense that*

$$\frac{L(S_0(\lambda)\setminus\chi_1(\lambda,\delta))}{L(S_0(\lambda))}\underset{\lambda\to\infty}{=}O(k^{-\delta/2}). \tag{25}$$

Corollary 2. *If t belongs to the $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set $\chi_1(\lambda,\delta)$ in $\mathbb{C}^2$, then, for any $z\in\mathbb{C}^2$ lying on the circle $C_1=\{z:|z-k^{2l}|=k^{2l-2-4s_1-\delta}\}$ and any i in $\mathbb{Z}^2$, the inequality $2|p_i^{2l}(t)-z|>k^{2l-2-4s_1-\delta}$ holds.*

Let $E_j(t)$ be the spectral projection of the free operator, corresponding to the eigenvalue $p_j^{2l}(t):(E_j)_{rm}=\delta_{jr}\delta_{jm}$. In the $(2k^{-1-4s_1-2\delta})$-neighborhood of $\chi_1(\lambda,\delta)$, we define functions $g_r^{(1)}(k,t)$ and operator-valued functions $G_r^{(1)}(k,t)$, $r=1,2,\cdots$ as follows:

$$g_r^{(1)}(k,t)=\frac{(-1)^r}{2\pi ir}\mathrm{Tr}\oint_{C_1}((H_0^{(1)}(t)-z)^{-1}W_1)^r dz, \tag{26}$$

$$G_r^{(1)}(k,t)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_1}((H_0^{(1)}(t)-z)^{-1}W_1)^r(H_0^{(1)}(t)-z)^{-1}dz. \tag{27}$$

To find $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$, it is necessary to compute the residues of a rational function of a simple structure, whose numerator does not depend on z, while the denominator is a product of factors of the type $(p_i^{2l}(t)-z)$. For all t in the non-resonance set within C_1, the integrand has a single pole at the point $z=k^{2l}=p_j^{2l}(t)$. By computing the residue at this point, we obtain explicit expressions for $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$. For example, $g_1^{(1)}(k,t)=0$,

$$g_2^{(1)}(k,t)=\sum_{q\in\mathbb{Z}^2,q\neq0}|w_q|^2(p_j^{2l}(t)-p_{j+q}^{2l}(t))^{-1}=\sum_{q\in\mathbb{Z}^2,q\neq0}\frac{|w_q|^2(2p_j^{2l}(t)-p_{j+q}^{2l}(t)-p_{j-q}^{2l}(t))}{2(p_j^{2l}(t)-p_{j+q}^{2l}(t))(p_j^{2l}(t)-p_{j-q}^{2l}(t))}, \tag{28}$$

$$G_1^{(1)}(k,t)_{rm}=\frac{w_{j-m}}{p_j^{2l}(t)-p_m^{2l}(t)}\delta_{rj}+\frac{w_{r-j}}{p_j^{2l}(t)-p_r^{2l}(t)}\delta_{mj}, \text{if } r\neq m, \quad G_1^{(1)}(k,t)_{jj}=0. \quad (29)$$

It is not difficult to show that $g_2^{(1)}(k,t)>$ for sufficiently large λ. For technical reasons, it is convenient to introduce parameter α in front of the potential W_1. Namely, $H_\alpha^{(1)}=(-\Delta)^l+\alpha W_1$, $0\leq\alpha\leq 1$. We denote the operator $H_\alpha^{(1)}$ with $\alpha=1$ simply by $H^{(1)}$.

Theorem 3. *Suppose t belongs to the $(2k^{-1-4s_1-2\delta})$-neighborhood in K_1 of the non-resonance set $\chi_1(\lambda,\delta)$, $0<2\delta<2l-2-4s_1$. Then for sufficiently large λ, $\lambda>\lambda_0(V,\delta)$, and for all α, $-1\leq\alpha\leq 1$, there exists a single eigenvalue of the operator $H_\alpha^{(1)}(t)$ in the interval $\varepsilon_1(k,\delta):=(k^{2l}-k^{2l-2-4s_1-\delta},k^{2l}+k^{2l-2-4s_1-\delta})$. It is given by the series*

$$\lambda_j^{(1)}(\alpha,t)=p_j^{2l}(t)+\sum_{r=2}^{\infty}\alpha^r g_r^{(1)}(k,t), \quad (30)$$

converging absolutely in the disk $|\alpha|\leq 1$, where the index j is determined according to Parts 1(a) and 2 of Lemma 1. The spectral projection, corresponding to $\lambda_j^{(1)}(\alpha,t)$, is given by the series:

$$E_j^{(1)}(\alpha,t)=E_j+\sum_{r=1}^{\infty}\alpha^r G_r^{(1)}(k,t), \quad (31)$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha|\leq 1$.

For the coefficients $g_r^{(1)}(k,t)$, $G_r^{(1)}(k,t)$ the following estimates hold:

$$|g_r^{(1)}(k,t)|<k^{2l-2-4s_1-\gamma_0 r-\delta}, \|G_r^{(1)}(k,t)\|_1<k^{-\gamma_0 r}, \quad (32)$$

where $\gamma_0=2l-2-4s_1-2\delta$.

Corollary 4. *For the perturbed eigenvalue and its spectral projection, the following estimates hold:*

$$\left|\lambda_j^{(1)}(\alpha,t)-p_j^{2l}(t)\right|\leq 2\alpha^2 k^{2l-2-4s_1-2\gamma_0-\delta}, \quad (33)$$

$$\left\|E_j^{(1)}(\alpha,t)-E_j\right\|_1\leq 2|\alpha|k^{-\gamma_0}. \quad (34)$$

Let us introduce the notations:

$$T(m):=\frac{\partial^{|m|}}{\partial t_1^{m_1}\partial t_2^{m_2}}, \quad m=(m_1,m_2), \quad |m|:=m_1+m_2, \quad m!:=m_1!m_2!, \quad T(0)f:=f.$$

We show, in [26], that the coefficients $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$ can be extended as holomorphic functions of two variables from the real $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set $\chi_1(\lambda,\delta)$ to its complex neighborhood of the same size and the following estimates hold in the complex neighborhood:

$$|T(m)g_r^{(1)}(k,t)|<m!k^{2l-2-4s_1-\delta-\gamma_0 r+|m|(1+4s_1+2\delta)}, \|T(m)G_r^{(1)}(k,t)\|_1<m!k^{-\gamma_0 r+|m|(1+4s_1+2\delta)}.$$

From this, the following theorem follows easily.

Theorem 5. *The series (30), (31) can be extended as holomorphic functions of two variables to the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set χ_1 from its neighborhood in K_1, and the following estimates hold in the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set χ_1:*

$$|T(m)(\lambda_j^{(1)}(\alpha,t)-p_j^{2l}(t))|<2m!\alpha^2 k^{2l-2-4s_1-2\gamma_0-\delta+|m|(1+4s_1+2\delta)}, \quad (35)$$

$$\|T(m)(E_j^{(1)}(\alpha,t)-E_j)\|_1<2m!\alpha k^{-\gamma_0+|m|(1+4s_1+2\delta)}. \quad (36)$$

2.1.3. *Nonresonance Part of Isoenergetic Set of* $H^{(1)}$. Let $S_1(\lambda)$[5] be the isoenergetic set of the operator $H^{(1)}_\alpha$, i.e.,

$$S_1(\lambda)=\{t\in K_1 : \exists n\in\mathbb{N}:\ \lambda^{(1)}_n(\alpha,t)=\lambda\}, \tag{37}$$

where $\{\lambda^{(1)}_n(\alpha,t)\}_{n=1}^{\infty}$ is the complete set of eigenvalues of $H^{(1)}_\alpha(t)$. We construct a "non-resonance" subset $\chi_1^*(\lambda)$ of $S_1(\lambda)$, which corresponds to non-resonance eigenvalues $\lambda^{(1)}_j(\alpha,t)$ given by the perturbation series (30). Note that for every t belonging to the non-resonant set $\chi_1(\lambda,\delta)$ described by 1, there is a single $j\in\mathbb{Z}^2$ such that $p_j(t)=k$, $k=\lambda^{1/(2l)}$. This means that the function $t\to\vec p_j(t)$ maps $\chi_1(\lambda,\delta)$ into the circle S_k. We denote the image of $\chi_1(\lambda,\delta)$ in S_k by $\mathcal{D}_0(\lambda)_{nonres}$. Obviously,

$$\chi_1(\lambda,\delta)=\mathcal{K}_1\mathcal{D}_0(\lambda)_{nonres}, \tag{38}$$

where $\mathcal{K}_1$ establishes a one-to-one relation between two sets. Let $\mathcal{B}_1(\lambda)$ be a set of unit vectors corresponding to $\mathcal{D}_0(\lambda)_{nonres}$,

$$\mathcal{B}_1(\lambda)=\{\vec\nu\in S_1 : k\vec\nu\in\mathcal{D}_0(\lambda)_{nonres}\}.$$

It is easy to see that $\mathcal{B}_1(\lambda)$ is a unit circle with holes, centered at the origin. We denote by $\Theta_1(\lambda)$ the set of angles φ, corresponding to $\mathcal{B}_1(\lambda)$:

$$\Theta_1(\lambda)=\{\varphi\in[0,2\pi):\ (\cos\varphi,\sin\varphi)\in\mathcal{B}_1(\lambda)\}.$$

Let $\vec\varkappa\in\mathcal{D}_0(k)_{nonres}$.[6] Then, there exists $j\in\mathbb{Z}^2$, $t\in\chi_1(\lambda,\delta)$ such that $\vec\varkappa=\vec p_j(t)$. Obviously, $t=\mathcal{K}_1\vec\varkappa$ and, by (38), $t\in\chi_1(\lambda,\delta)$. According to Theorem 3, for sufficiently large k, there exists an eigenvalue of the operator $H^{(1)}_\alpha(t)$, $t=\mathcal{K}_1\vec\varkappa$, $0\le\alpha\le1$, given by (30). It is convenient here to denote $\lambda^{(1)}_j(\alpha,t)$ by $\lambda^{(1)}(\alpha,\vec\varkappa)$; we can do this since there is a one-to-one correspondence between $\vec\varkappa$ and the pair (t,j). We rewrite (30) and (35) in the forms

$$\lambda^{(1)}(\alpha,\vec\varkappa)=\varkappa^{2l}+f_1(\alpha,\vec\varkappa),\qquad \varkappa=|\vec\varkappa|, \tag{39}$$

$$|T(m)f_1(\alpha,\vec\varkappa)|\le 2m!\alpha^2\varkappa^{2l-2-4s_1-2\gamma_0-\delta+|m|(1+4s_1+2\delta)}. \tag{40}$$

where $f_1(\alpha,\vec\varkappa)=\sum_{r=2}^{\infty}\alpha^r g^{(1)}_r(\vec\varkappa)$, $g^{(1)}_r(\vec\varkappa)$ being defined by (26) with j and t such that $\vec p_j(t)=\vec\varkappa$. By Theorem 3, the formulas (39), (40) hold in $(2k^{-1-4s_1-2\delta})$-neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$, i.e., they hold for any $\varkappa\vec\nu$ such that $\vec\nu\in\mathcal{B}_1(\lambda)$, $|\varkappa-k|<2k^{-1-4s_1-2\delta}$. We define $\mathcal{D}_1(\lambda)$ as the level set of the function $\lambda^{(1)}(\alpha,\vec\varkappa)$ in this neighborhood:

$$\mathcal{D}_1(\lambda):=\{\vec\varkappa_1=\varkappa_1\vec\nu:\vec\nu\in\mathcal{B}_1(\lambda),\ |\varkappa_1-k|<2k^{-1-4s_1-2\delta},\ \lambda^{(1)}(\alpha,\vec\varkappa_1)=\lambda\}. \tag{41}$$

Lemma 6. (1) *For sufficiently large λ, the set $\mathcal{D}_1(\lambda)$ is a distorted circle with holes which is strictly inside the circle of the radius k (see Fig. 1); it can be described by the formula*

$$\mathcal{D}_1(\lambda)=\big\{\vec\varkappa_1\in\mathbb{R}^2:\vec\varkappa_1=\varkappa_1(\varphi)\vec\nu,\quad \vec\nu=(\cos\varphi,\sin\varphi)\in\mathcal{B}_1(\lambda)\big\}, \tag{42}$$

where $\varkappa_1(\varphi)=k+h_1(\varphi)$ and $h_1(\varphi)$ obeys the inequalities

$$|h_1|<k^{-1-4s_1-2\gamma_0-\delta},\left|\frac{\partial h_1}{\partial\varphi}\right|\le k^{-2\gamma_0+1+\delta}, \tag{43}$$

$h_1(\varphi)<0$ *when* $W_1\neq0$.

[5] $S_1(\lambda)$ definitely depends on αW_1; however we omit this to keep the notation simple.

[6] Usually the vector $\vec p_j(t)$ is denoted by $\vec k$, the corresponding plane wave being $e^{\langle\vec k,\vec x\rangle}$. We use less common notation $\vec\varkappa$, since we already have other k's in the text.

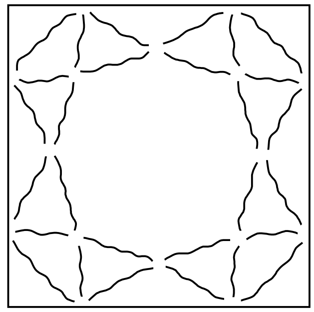

FIGURE 5. The set $\chi_1^*(\lambda)$

(2) *The total length of* $\mathcal{B}_1(\lambda)$ *satisfies the estimate*

$$L(\mathcal{B}_1) = 2\pi(1 + O(k^{-\delta/2})). \tag{44}$$

(3) *The function* $h_1(\varphi)$ *can be extended as a holomorphic function of* φ *to the complex* $(2k^{-2-4s_1-2\delta})$*-neighborhood of each connected component of* $\Theta_1(\lambda)$ *and estimates (43) hold.*

(4) *The curve* $\mathcal{D}_1(\lambda)$ *has a length which is asymptotically close to that of the whole circle in the sense that*

$$L\big(\mathcal{D}_1(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi k\big(1 + O(k^{-\delta/2})\big), \quad \lambda = k^{2l}. \tag{45}$$

Next, we define the non-resonance subset $\chi_1^*(\lambda)$ of isoenergetic set $S_1(\lambda)$ as the parallel shift of $\mathcal{D}_1(\lambda)$ into K_1 (Fig. 5):

$$\chi_1^*(\lambda) := \mathcal{K}_1\mathcal{D}_1(\lambda). \tag{46}$$

Lemma 7. *The set* $\chi_1^*(\lambda)$ *belongs to the* $(k^{-1-4s_1-2\gamma_0-\delta})$*-neighborhood of* $\chi_1(\lambda)$ *in* K_1*. If* $t \in \chi_1^*(\lambda)$*, then the operator* $H_\alpha^{(1)}(t)$ *has a simple eigenvalue* $\lambda_n^{(1)}(\alpha, t)$*,* $n \in \mathbb{N}$*, equal to* λ*, no other eigenvalues being in the interval* $\varepsilon_1(k,\delta)$*,* $\varepsilon_1(k,\delta) := (k^{2l} - k^{2l-2-4s_1-\delta}, k^{2l} + k^{2l-2-4s_1-\delta})$*. This eigenvalue is given by the perturbation series (30), where j is uniquely defined by t from the relation* $p_j^{2l}(t) \in \varepsilon_1(k,\delta)$*.*

Lemma 8. *The formula (46) establishes a one-to-one correspondence between* $\chi_1^*(\lambda)$ *and* $\mathcal{D}_1(\lambda)$*.*

From the geometric point of view, this means that $\chi_1^*(\lambda)$ does not have self-intersections.

2.2. **The Second Step of Approximation.**

2.2.1. *The Operator* $H_\alpha^{(2)}$. Choosing $s_2 = 2s_1$, we define the second operator $H_\alpha^{(2)}$ by the formula

$$H_\alpha^{(2)} = H^{(1)} + \alpha W_2, \quad (0 \le \alpha \le 1), \qquad W_2 = \sum_{r=M_1+1}^{M_2} V_r, \tag{47}$$

where $H^{(1)}$ is defined by (19) and M_2 is chosen in such a way that $2^{M_2} \approx k^{s_2}$. Obviously, the periods of W_2 are $2^{M_2-1}(\beta_1, 0)$ and $2^{M_2-1}(0, \beta_2)$. We write them in the

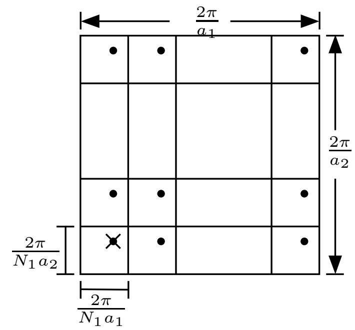

FIGURE 6. Relation between τ and t_p

form $N_1(a_1,0)$ and $N_1(0,a_2)$, where $(a_1,0),\ (0,a_2)$ are the periods of W_1 and $N_1 = 2^{M_2-M_1},\ \frac{1}{4}k^{s_2-s_1} < N_1 < 4k^{s_2-s_1}$. Note that

$$\|W_2\|_\infty \le \sum_{n=M_1+1}^{M_2} \|V_n\|_\infty \le \sum_{n=M_1+1}^{M_2} \exp(-2^{\eta n}) < \exp(-k^{\eta s_1}). \tag{48}$$

2.2.2. *Multiple Periods of $W_1(x)$.* The operator $H^{(1)} = H_0 + W_1(x)$ has the periods $a_1,\ a_2$. The corresponding family of operators, $\{H^{(1)}(t)\}_{t\in K_1}$, acts in $L_2(Q_1)$, where $Q_1 = [0,a_1]\times[0,a_2]$ and $K_1 = [0,2\pi/a_1)\times[0,2\pi/a_2)$. The eigenvalues of $H^{(1)}(t)$ are denoted by $\lambda_n^{(1)}(t)$, $n\in\mathbb{N}$, and its spectrum by $\Lambda^{(1)}(t)$. Now let us consider the same $W_1(x)$ as a periodic function with the periods $N_1a_1,\ N_1a_2$. Obviously, the definition of the operator $H^{(1)}$ does not depend on how we define the periods of W_1. However, the family of operators $\{H^{(1)}(t)\}_{t\in K_1}$ does change when we replace the periods $a_1,\ a_2$ by $N_1a_1,\ N_1a_2$. The family of operators $\{H^{(1)}(t)\}_{t\in K_1}$ has to be replaced by a family of operators $\{\tilde H_1(\tau)\}_{\tau\in K_2}$ acting in $L_2(Q_2)$, where $Q_2 = [0,N_1a_1]\times[0,N_1a_2]$ and $K_2 = [0,2\pi/N_1a_1)\times[0,2\pi/N_1a_2)$. We denote the eigenvalues of $\tilde H_1(\tau)$ by $\tilde\lambda_n^{(1)}(\tau)$, $n\in\mathbb{N}$, and its spectrum by $\tilde\Lambda^{(1)}(\tau)$. The next lemma establishes a connection between spectra of the operators $H^{(1)}(t)$ and $\tilde H_1(\tau)$. It follows easily from Bloch theory (see, e.g., [36]).

Lemma 9. *For any $\tau\in K_2$,*

$$\tilde\Lambda^{(1)}(\tau) = \bigcup_{p\in P} \Lambda^{(1)}(t_p), \tag{49}$$

where

$$P = \{p=(p_1,p_2)\in\mathbb{Z}^2 : 0\le p_1\le N_1-1,\ 0\le p_2\le N_1-1\} \tag{50}$$

and $t_p = (t_{p,1},t_{p,2}) = (\tau_1 + 2\pi p_1/N_1a_1, \tau_2 + 2\pi p_2/N_1a_2)\in K_1$. See Fig. 6.

We defined the isoenergetic set $S_1(\lambda)\subset K_1$ of $H^{(1)}$ by formula (37). Obviously, this definition is directly associated with the family of operators $H^{(1)}(t)$ and, therefore, with the periods a_1,a_2, which we assigned to $W_1(x)$. Now, assuming that the periods are equal to N_1a_1, N_1a_2, we give an analogous definition of the isoenergetic set $\tilde S_1(\lambda)$ in K_2:

$$\tilde S_1(\lambda) := \{\tau\in K_2 : \exists n\in\mathbb{N} :\quad \tilde\lambda_n^{(1)}(\tau)=\lambda\}.$$

By Lemma 9, $\tilde S_1(\lambda)$ can be expressed as follows:

$$\tilde S_1(\lambda) = \big\{\tau\in K_2 : \exists n\in\mathbb{N},\ p\in P :\ \lambda_n^{(1)}\big(\tau+2\pi p/N_1a\big)=\lambda\big\},\quad 2\pi p/N_1a = \left(\frac{2\pi p_1}{N_1a_1},\frac{2\pi p_2}{N_1a_2}\right).$$

The relation between $S_1(\lambda)$ and $\tilde{S}_1(\lambda)$ can be easily understood from the geometric point of view as $\tilde{S}_1 = \mathcal{K}_2 S_1$, where $\mathcal{K}_2$ is the parallel shift into K_2, i.e.,

$$\mathcal{K}_2 : \mathbb{R}^2 \to K_2,\ \mathcal{K}_2(\tau + 2\pi m/N_1 a) = \tau,\ m \in \mathbb{Z}^2,\ \tau \in K_2.$$

Thus, $\tilde{S}_1(\lambda)$ is obtained from $S_1(\lambda)$ by cutting $S_1(\lambda)$ into pieces of the size K_2 and shifting them together in K_2.

Definition 10. We say that τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $n, \hat{n} \in \mathbb{N}$, $n \neq \hat{n}$ such that $\tilde{\lambda}_n^{(1)}(\tau) = \tilde{\lambda}_{\hat{n}}^{(1)}(\tau) = \lambda$.

Remark 11. By Lemma 9, τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $p, \hat{p} \in P$ and a pair $n, \hat{n} \in \mathbb{N}$ such that $|p - \hat{p}| + |n - \hat{n}| \neq 0$ and $\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1 a) = \lambda$.

Now let us recall that the isoenergetic set $S_1(\lambda)$ consists of two parts: $\chi_1^*(\lambda)$ and $S_1(\lambda) \setminus \ \chi_1^*(\lambda)$, where $\chi_1^*(\lambda)$ is the first non-resonance set given by (46). Obviously $\mathcal{K}_2\chi_1^*(\lambda) \subset \mathcal{K}_2 S_1(\lambda) = \tilde{S}_1(\lambda)$ and can be described by the formula:

$$\mathcal{K}_2\chi_1^*(\lambda) = \{\tau \in K_2 :\ \exists p \in P : \tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)\}\,. \tag{51}$$

Let us consider only those self-intersections of $\tilde{S}_1$ which belong to $\mathcal{K}_2\chi_1^*(\lambda)$, i.e., we consider the points of intersection of $\mathcal{K}_2\chi_1^*(\lambda)$ both with itself and with $\tilde{S}_1(\lambda) \setminus \mathcal{K}_2\chi_1^*(\lambda)$.

To obtain a new non-resonance set $\chi_2(\lambda)$, we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ a neighborhood of its self-intersections with$\tilde{S}_1(\lambda)$. More precisely, we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ the set

$$\Omega_1(\lambda) = \big\{\tau \in \mathcal{K}_2\chi_1^*(\lambda) : \exists n, \hat{n} \in \mathbb{N},\ p, \hat{p} \in P,\ p \neq \hat{p} :\ \lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda,$$
$$\tau + 2\pi p/N_1 a \in \chi_1^*(\lambda),\ \Big|\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1 a)\Big| \leq \epsilon_1\big\}, \quad \epsilon_1 = e^{-\frac{1}{4}k^{\eta s_1}}. \tag{52}$$

We define $\chi_2(\lambda)$ by the formula

$$\chi_2(\lambda) = \mathcal{K}_2\chi_1^*(\lambda) \setminus \Omega_1(\lambda). \tag{53}$$

2.2.3. *Perturbation Formulas.*

Lemma 12 (Geometric Lemma)**.** *For an arbitrarily small positive δ, $9\delta < 2l - 11 - 16s_1$ and sufficiently large λ, $\lambda > \lambda_1(V, \delta)$, there exists a non-resonance set $\chi_2(\lambda, \delta) \subset \mathcal{K}_2\chi_1^*$ such that the following hold.*

(1) *For any $\tau \in \chi_2$,*
   (a) *there exists a unique $p \in P$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*$;* [7]
   (b) *$\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) = k^{2l}$, where $\lambda_j^{(1)}(\tau + 2\pi p/N_1 a)$ is given by the perturbation series (30) with $\alpha = 1$, j being uniquely defined by $t = \tau + 2\pi p/N_1 a$ as is described in Part 2 of Lemma 1.*
   (c) *The eigenvalue $\lambda_j^{(1)}(\tau + 2\pi p/N_1 a)$ is a simple eigenvalue of $\tilde{H}^{(1)}(\tau)$, whose distance from all other eigenvalues $\lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1 a)$, $\hat{n} \in \mathbb{N}$, of $\tilde{H}_1(\tau)$ is greater than $\epsilon_1 = e^{-\frac{1}{4}k^{\eta s_1}}$:*

$$|\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1 a)| > \epsilon_1. \tag{54}$$

[7] From the geometric point of view, this means that $\chi_2(\lambda)$ does not have self-intersections.

(2) *For any τ in the $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood in $\mathbb{C}^2$ of χ_2, there exists a unique $p \in P$ such that $\tau + 2\pi p/N_1 a$ is in the $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood in $\mathbb{C}^2$ of χ_1^* and*

$$|\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) - k^{2l}| < \epsilon_1 k^{-\delta}, \tag{55}$$

j being uniquely defined by $\tau + 2\pi p/N_1 a$ as is described in Part 2 of Lemma 1.

(3) *The second non-resonance set χ_2 has asymptotically full measure in χ_1^* in the sense that*

$$\frac{L(\mathcal{K}_2\chi_1^* \setminus \chi_2))}{L(\chi_1^*)} < k^{-2-2s_1}. \tag{56}$$

Remark 13. The dual lattice $2\pi m/N_1 a$ $(m \in \mathbb{Z}^2)$, corresponding to larger periods $N_1a_1,\ N_1a_2$, is finer than the dual lattice $2\pi j/a$ $(j \in \mathbb{Z}^2)$, corresponding to $a_1,\ a_2$. Every point $2\pi m/N_1 a$ of a dual lattice corresponding to the periods $N_1a_1,\ N_1a_2$ can be uniquely represented in the form $2\pi m/N_1a = 2\pi j/a + 2\pi p/N_1 a$, where $m = N_1 j + p$ and $2\pi j/a$ is a point of the dual lattice for periods a_1, a_2, while $p \in P$ is responsible for refining the lattice.

Let us consider a normalized eigenfunction $\psi_n(t,x)$ of $H^{(1)}(t)$ in $L_2(Q_1)$. We extended it quasiperiodically to Q_2, renormalize in $L_2(Q_2)$ and denote the new function by $\tilde\psi_n(\tau,x)$, $\tau = \mathcal{K}_2 t$. The Fourier representations of $\psi_n(t,x)$ in $L_2(Q_1)$ and $\tilde\psi_n(\tau,x)$ in $L_2(Q_2)$ are simply related. If we denote Fourier coefficients of $\psi_n(t,x)$ with respect to the basis of exponential functions $|Q_1|^{-1/2}e^{i\langle t+2\pi j/a,x\rangle}$, $j \in \mathbb{Z}^2$, in $L_2(Q_1)$ by C_{nj}, then, the Fourier coefficients $\tilde C_{nm}$ of $\tilde\psi_n(\tau,x)$ with respect to the basis of exponential functions $|Q_2|^{-1/2}e^{i(\tau+2\pi m/N_1a,x)}$, $m \in \mathbb{Z}^2$, in $L_2(Q_2)$ are given by the formula

$$\tilde C_{nm} = \begin{cases} C_{nj}, & \text{if } m = jN_1 + p; \\ 0, & \text{otherwise,} \end{cases}$$

p being defined from the relation $t = \tau + 2\pi p/N_1 a,\ p \in P$. Hence, the matrices of the projections on $\psi_n(t,x)$ and $\tilde\psi_n(\tau,x)$ with respect to the above bases are simply related by

$$(\tilde E_n)_{j\hat j} = \begin{cases} (E_n)_{m\hat m}, & \text{if } m = jN_1 + p,\ \hat m = \hat j N_1 + p; \\ 0, & \text{otherwise,} \end{cases}$$

$\tilde E_n$ and E_n being the projections in $L_2(Q_2)$ and $L_2(Q_1)$, respectively.

Let us denote by $\tilde E_j^{(1)}\big(\tau + 2\pi p/N_1 a\big)$ the spectral projection $E_j^{(1)}(\alpha,t)$ (see (31)) with $\alpha = 1$ and $t = \tau + 2\pi p/N_1 a$, "extended" from $L_2(Q_1)$ to $L_2(Q_2)$.

By analogy with (26), (27), we define functions $g_r^{(2)}(k,\tau)$ and operator-valued functions $G_r^{(2)}(k,\tau)$, $r = 1, 2, \cdots$, as follows:

$$g_r^{(2)}(k,\tau) = \frac{(-1)^r}{2\pi i r}\mathrm{Tr}\oint_{C_2}\big(\big(\tilde H_1(\tau) - z\big)^{-1}W_2\big)^r dz, \tag{57}$$

$$G_r^{(2)}(k,\tau) = \frac{(-1)^{r+1}}{2\pi i}\oint_{C_2}\big(\big(\tilde H_1(\tau) - z\big)^{-1}W_2\big)^r\big(\tilde H_1(\tau) - z\big)^{-1}dz. \tag{58}$$

We consider the operators $H_\alpha^{(2)} = H^{(1)} + \alpha W_2$ and the family $H_\alpha^{(2)}(\tau)$, $\tau \in K_2$, acting in $L_2(Q_2)$.

Theorem 14. *Suppose τ belongs to the $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood in K_2 of the second non-resonance set $\chi_2(\lambda,\delta)$, $0<9\delta<2l-11-16s_1$, $\epsilon_1=e^{-\frac{1}{4}k^{\eta s_1}}$. Then, for sufficiently large λ, $\lambda>\lambda_1(V,\delta)$ and for all α, $0\le\alpha\le 1$, there exists a single eigenvalue of the operator $H_\alpha^{(2)}(\tau)$ in the interval $\varepsilon_2(k,\delta):=(k^{2l}-\epsilon_1/2,k^{2l}+\epsilon_1/2)$. It is given by the series*

$$\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)=\lambda_j^{(1)}\big(\tau+2\pi p/N_1a\big)+\sum_{r=1}^{\infty}\alpha^r g_r^{(2)}(k,\tau),\qquad \tilde{j}=j+p/N_1, \tag{59}$$

converging absolutely in the disk $|\alpha|\le 1$, where $p\in P$ and $j\in Z^2$ are as in Lemma 12. The spectral projection corresponding to $\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)$ is given by the series

$$E_{\tilde{j}}^{(2)}(\alpha,\tau)=\tilde{E}_j^{(1)}\big(\tau+2\pi p/N_1a\big)+\sum_{r=1}^{\infty}\alpha^r G_r^{(2)}(k,\tau), \tag{60}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha|\le 1$.

The following estimates hold for coefficients $g_r^{(2)}(k,\tau)$, $G_r^{(2)}(k,\tau)$, $r\ge 1$:

$$\big|g_r^{(2)}(k,\tau)\big|<\frac{3\epsilon_1}{2}(4\epsilon_1^3)^r,\qquad \big\|G_r^{(2)}(k,\tau)\big\|_1<6r(4\epsilon_1^3)^r. \tag{61}$$

Corollary 15. *The following estimates hold for the perturbed eigenvalue and its spectral projection:*

$$\big|\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)-\lambda_j^{(1)}\big(\tau+2\pi p/N_1a\big)\big|\le 12\alpha\epsilon_1^4, \tag{62}$$

$$\big\|E_{\tilde{j}}^{(2)}(\alpha,\tau)-\tilde{E}_j^{(1)}\big(\tau+2\pi p/N_1a\big)\big\|_1\le 48\alpha\epsilon_1^3. \tag{63}$$

The proof of Theorem 14 is analogous to that of Theorem 3 and is based on expanding the resolvent $(H_\alpha^{(2)}(\tau)-z)^{-1}$ in a perturbation series for $z\in C_2$, C_2 being the contour around the unperturbed eigenvalue k^{2l}: $C_2=\{z:|z-k^{2l}|=\frac{\epsilon_1}{2}\}$. Integrating the resolvent yields the formulas for an eigenvalue of $H_\alpha^{(2)}$ and its spectral projection.

Theorem 16. *Under the conditions of Theorem 14, the series (59), (60) can be extended as holomorphic functions of τ in the complex $(\frac{1}{2}\epsilon_1 k^{-2l+1-\delta})$-neighborhood of the non-resonance set χ_2 and the following estimates hold in the complex neighborhood:*

$$\left|T(m)\left(\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)-\lambda_j^{(1)}(\tau+2\pi p/N_1a)\right)\right|<\alpha C_m\epsilon_1^{4-|m|}k^{|m|(2l-1+\delta)}, \tag{64}$$

$$\left\|T(m)\left(E_{\tilde{j}}^{(2)}(\alpha,\tau)-\tilde{E}_j^{(1)}(\tau+2\pi p/N_1a)\right)\right\|_1<\alpha C_m\epsilon_1^{3-|m|}k^{|m|(2l-1+\delta)},\quad C_m=48m!2^{|m|}. \tag{65}$$

2.2.4. *Sketch of the Proof of the Geometric Lemma 12.* Parts 1 and 2 of Geometric Lemma 12 easily follow from the definition of the non-resonance set. The main problem is to prove that the non-resonance set exists and is rather extensive, i.e., Part 3. We outline a proof of Part 3 below.

Determinants. Intersections and Quasi-intersections. Description of the set Ω_1 in terms of determinants. We have considered self-intersections of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*$. We describe self-intersections as zeros of determinants of operators of the type $I+A$, $A\in\mathbf{S_1}$. (see, e.g., [36]). Let us represent the operator $(H^{(1)}(t)-\lambda)(H_0(t)+\lambda)^{-1}$ in the form $I+A_1$, $A_1\in\mathbf{S_1}$:

$$(H^{(1)}(t)-\lambda)(H_0(t)+\lambda)^{-1}=I+A_1(t),\quad A_1(t)=(W_1-2\lambda)(H_0(t)+\lambda)^{-1}. \tag{66}$$

Obviously, $A_1(t) \in \mathbf{S_1}$. From properties of determinants and the definition of $S_1(\lambda)$ it follows easily that the isoenergetic set $S_1(\lambda)$ of $H^{(1)}$ is the zero set of $\det\big(I + A_1(t)\big)$ in K_1.

Now recall that the set $\mathcal{D}_1(\lambda)$ can be described in terms of vectors $\vec{\varkappa}_1(\varphi)$, $\varphi \in \Theta_1(\lambda)$; see Lemma 6. By definition, $\chi_1^*(\lambda) = \mathcal{K}_1\mathcal{D}_1(\lambda)$. Lemma 8 shows that $\chi_1^*(\lambda)$ does not have self-intersections (Fig.5), i.e., for every $t \in \chi_1^*(\lambda)$, there is a single $\vec{\varkappa}_1(\varphi) \in \mathcal{D}_1(\lambda)$ such that $t = \mathcal{K}_1\vec{\varkappa}_1(\varphi)$. Next, if $\tau \in \mathcal{K}_2\chi_1^*(\lambda)$, then there is $p \in P$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)$. Note that p is not uniquely defined by τ, since $\mathcal{K}_2\chi_1^*(\lambda)$ may have self-intersections. Hence, every $\tau \in \mathcal{K}_2\chi_1^*(\lambda)$ can be represented as $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$, where $\vec{\varkappa}_1(\varphi)$ is not necessary uniquely defined. The next lemma describes self-intersections of $\tilde{S}_1$ belonging to $\mathcal{K}_2\chi_1^*(\lambda)$ as zeros of a group of determinants.

Lemma 17. *If τ is a point of self-intersection of $\tilde{S}_1$ (Definition 10), belonging to $\mathcal{K}_2\chi_1^*(\lambda)$, then $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$, where $\varphi \in \Theta_1(\lambda)$ and satisfies the equation*

$$\det\big(I + A_1\left(\vec{y}(\varphi)\right)\big) = 0, \quad \vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b}, \quad \vec{b} = 2\pi p/N_1 a, \tag{67}$$

for some $p \in P \setminus \{0\}$. Conversely, if (67) is satisfied for some $p \in P \setminus \{0\}$, then $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$ is a point of self-intersection.

Definition 18. Let $\varPhi_1$ be the complex $\left(k^{-2-4s_1-2\delta}\right)$-neighborhood of Θ_1.

By Lemma 6, $\vec{\varkappa}_1(\varphi)$ is an analytic function in $\varPhi_1$, and hence

$$\det\big(I + A\big(\vec{y}(\varphi)\big)\big), \quad \vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b} \quad \vec{b} \in K_1,$$

is analytic too.

Definition 19. We say that $\varphi \in \varPhi_1$ is a quasi-intersection of $\mathcal{K}_2\chi_1^*$ with $\tilde{S}_1(\lambda)$ if (67) holds for some $p \in P \setminus \{0\}$.

Thus, real intersections correspond to real zeros of the determinant, while quasi-intersections may have a small imaginary part (quasi-intersections include intersections).

Next we describe the resonance set Ω_1 (defined in (52)) in terms of determinants.

Lemma 20. *If $\tau \in \Omega_1$, then $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$ where $\varphi \in \Theta_1$ satisfies the equation*

$$\det\left(\frac{H^{(1)}\big(\vec{y}(\varphi)\big) - k^{2l} - \epsilon}{H_0\big(\vec{y}(\varphi)\big) + k^{2l}}\right) = 0, \quad \vec{y}(\varphi) = \vec{\varkappa}_1(\varphi) + \vec{b}, \quad \vec{b} = 2\pi p/N_1 a, \tag{68}$$

for some $p \in P \setminus \{0\}$ and $|\epsilon| < \epsilon_1$. Conversely, if (68) is satisfied for some $p \in P \setminus \{0\}$ and $|\epsilon| < \epsilon_1$, then $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$ belongs to Ω_1.

We denote by ω_1 the set of $\varphi \in \Theta_1$ corresponding to Ω_1 , i.e., $\omega_1 = \{\varphi \in \Theta_1(\lambda) : \mathcal{K}_2\vec{\varkappa}_1(\varphi) \in \Omega_1\} \subset [0, 2\pi)$.

Complex resonant set. Further we consider a complex resonance set $\omega_1^*(\lambda)$, which is the set of zeros of the determinants (68) in $\varPhi_1$ ($p \in P \setminus \{0\}$, $|\epsilon| < \epsilon_1$). By Lemma 20, $\omega_1 = \omega_1^* \cap \Theta_1$. We prefer to consider quasi-intersections instead of intersections and the complex resonance set instead of just the real one, for the following reason: the determinants (67) and (68), involved in the definitions of quasi-intersections and the complex resonance set ω_1^*, are holomorphic functions of φ in $\varPhi_1$. Thus we can apply theorems of complex analysis to these determinants. Rouché's theorem is particularly important here, since it implies the stability of zeros of a holomorphic function with respect to small perturbations of the function. We take the determinant (67) as a holomorphic function, its zeros being quasi-intersections: the initial determinant corresponds to the case $W_1=0$, the perturbation obtained by "switching on" a potential W_1. Since there is no analogue of Rouché's theorem for real functions on the real axis, introducing the region $\mathit{\Phi}_1$ and analytic extension of the determinants into this region is in the core of our considerations. We also use the well-known inequality for the determinants (see [36])

$$\big|\det(I+A)-\det(I+B)\big|\leq\|A-B\|_1\exp\big(\|A\|_1+\|B\|_1+1\big),\ A,B\in\mathbf{S}_1. \tag{69}$$

Note that $\omega_1^*=\bigcup_{p\in P\setminus\{0\}}\omega_{1,p}^*$, where $\omega_{1,p}^*$ corresponds to a fixed p in (68); and similarly, $\omega_1=\bigcup_{p\in P\setminus\{0\}}\omega_{1,p}$. We fix $p\in P$ and study $\omega_{1,p}^*$ separately. We start by the case $W_1=0$. The corresponding determinant (67) is

$$\det\big(I+A_0\big(\vec{y}_0(\varphi)\big)\big),\ I+A_0\big(\vec{y}_0(\varphi)\big)=\big(H_0\big(\vec{y}_0(\varphi)\big)-\lambda\big)\big(H_0\big(\vec{y}_0(\varphi)\big)+\lambda\big)^{-1}, \tag{70}$$

$\vec{y}_0(\varphi)=k(\cos\varphi,\sin\varphi)+\vec{b}$. This determinant can be investigated by elementary means. We easily check that the number of zeros of the determinant in $\mathit{\Phi}_1$ does not exceed $c_0k^{2+2s_1}$, $c_0=32\beta_1\beta_2$. The resolvent $\big(H_0\big(\vec{y}_0(\varphi)\big)-\lambda\big)^{-1}$ has poles at zeros of the determinant. The resolvent norm at $\varphi\in\mathit{\Phi}_1$ can be easily estimated by the distance from φ to the nearest zero of the determinant. Next, we introduce the union $\mathcal{O}(\vec{b})$ of all disks of radius $r=k^{-4-6s_1-3\delta}$ surrounding zeros of the determinant (70) in $\mathit{\Phi}_1$. Obviously, any $\varphi\in\mathit{\Phi}_1\setminus\mathcal{O}(\vec{b})$ is separated from zeros of the determinant (70) by the distance no less than r. This estimate on the distance yields an estimate for the norm of the resolvent $\big(H_0\big(\vec{y}_0(\varphi)\big)-\lambda\big)^{-1}$, when $\varphi\in\mathit{\Phi}_1\setminus\mathcal{O}(\vec{b})$. Further, we introduce the potential W_1. It is shown in [31] that the number of zeros of each determinant (68) is preserved in each connected component $\Gamma(\vec{b})$ of $\mathcal{O}(\vec{b})$ when we switch from the case $W_1=0, A_1=A_0$ to the case of non-zero W_1 and from $\epsilon=0$ to $|\epsilon|<\epsilon_1$. We also show in [31] that estimates for the resolvent are stable under such change when $\varphi\in\mathit{\Phi}_1\setminus\mathcal{O}(\vec{b})$. We "switch on" the potential W_1 in two steps. First, we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$ and consider $\det\big(I+A_0\big(\vec{y}(\varphi)\big)\big)$ and $\big(H_0\big(\vec{y}(\varphi)\big)-k^{2l}\big)^{-1}$ in $\mathit{\Phi}_1$. We take into account that $\vec{y}(\varphi)-\vec{y}_0(\varphi)$ is small and holomorphic in $\mathit{\Phi}_1$ (Lemma 6), use (69) on the boundary of Γ, and apply Rouché's theorem. This enables us to conclude that the number of zeros of the determinant in $\Gamma(\vec{b})$ is preserved when we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$. Applying Hilbert relation for resolvents, we show that the estimates for the resolvent in $\mathit{\Phi}_1\setminus\mathcal{O}(\vec{b})$ are also stable under such change. In the second step we replace $H_0\big(\vec{y}(\varphi)\big)$ by $H^{(1)}\big(\vec{y}(\varphi)\big)+\epsilon I$ and prove similar results. From this, we see that $\omega_{1,p}^*\subset\mathcal{O}(\vec{b})$, $\vec{b}=2\pi p/N_1a$ and

$$\omega_1^*\subset\mathcal{O}_*:=\bigcup_{p\in P\setminus\{0\}}\mathcal{O}\big(2\pi p/N_1a\big). \tag{71}$$

Considering $\mathcal{O}(\vec{b})$ is formed by no more than $c_0k^{2+2s_1}$ disks and the set P contains no more than $4k^{2s_2-2s_1}$ elements, $s_2=2s_1$, we easily obtain that $\mathcal{O}_*$ contains no more than $4c_0k^{2+4s_1}$ disks. Taking the real parts of the sets, we conclude $\omega_1\subset\mathcal{O}_*\cap\Theta_1(\lambda)$. Noting $\mathcal{O}_*$ is formed by disks of the radius $r=k^{-4-6s_1-3\delta}$ and using the estimate for the number of disks, we obtain that the total length of ω_1 does not exceed $k^{-2-2s_1-3\delta}$ and hence the length of Ω_1 does not exceed $k^{-1-2s_1-3\delta}$.

We introduce the new notation

$$\mathit{\Phi}_2=\mathit{\Phi}_1\setminus\mathcal{O}_*, \tag{72}$$

where $\mathit{\Phi}_1$ is given by Definition 18. Obviously, to obtain $\mathit{\Phi}_2$, we produce round holes in each connected component of $\mathit{\Phi}_1$. The set $\mathit{\Phi}_2$ has a structure of Swiss cheese (Fig. 7); we add more holes of a smaller size at each step of approximation.

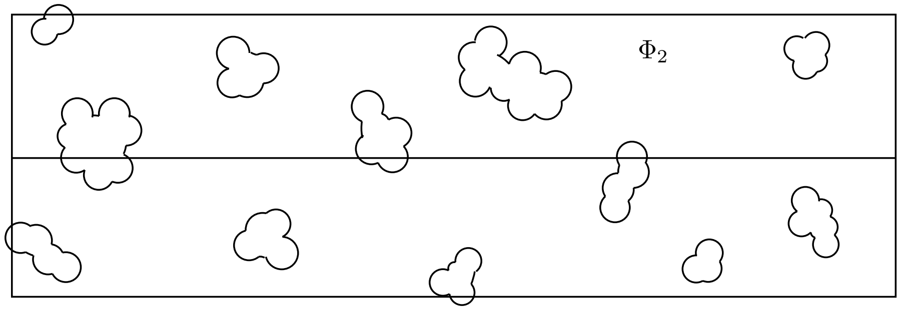

FIGURE 7. The set Φ_2.

Basing on the perturbation formulas (59), (60), we construct $\mathcal{B}_2(\lambda)$, $\mathcal{D}_2(\lambda)$ (see (14), (15) for $n=2$ and Fig. 2) and $\chi_2^*(\lambda)$ in the way analogous to the first step. In particular,

$$|\varkappa_2(\varphi)-\varkappa_1(\varphi)|<2\epsilon_1^4k^{-2l+1} \tag{73}$$

$$\left|\frac{\partial\varkappa_2}{\partial\varphi}(\varphi)-\frac{\partial\varkappa_1}{\partial\varphi}(\varphi)\right|<4\epsilon_1^3k^{1+\delta} \tag{74}$$

for $\vec{\nu}=(\cos\varphi,\sin\varphi)\in\mathcal{B}_2(\lambda)$.

2.3. **Next Steps of Approximation.** On the n-th step, $n\geq 3$, we choose $s_n=2s_{n-1}$ and define the operator $H_\alpha^{(n)}$ by the formula

$$H_\alpha^{(n)}=H^{(n-1)}+\alpha W_n,\quad (0\leq\alpha\leq 1),\qquad W_n=\sum_{r=M_{n-1}+1}^{M_n}V_r,$$

where M_n is chosen in such a way that $2^{M_n}\approx k^{s_n}$. Obviously, the periods of W_n are $2^{M_n-1}(\beta_1,0)$ and $2^{M_n-1}(0,\beta_2)$. We write the periods in the form: $N_{n-1}\cdots N_1(a_1,0)$ and $N_{n-1}\cdots N_1(0,a_2)$, where N_{n-1} is of order of $k^{s_n-s_{n-1}}$, namely, $N_{n-1}=2^{M_n-M_{n-1}}$. Note that $\|W_n\|_\infty\leq\sum_{r=M_{n-1}+1}^{M_n}\|V_r\|_\infty\leq\exp(-k^{\eta s_{n-1}})$.

Let us start by establishing a lower bound for k. Since $\eta s_1>2+2s_1$, there is a number $k_*>e$ such that

$$C_*(1+s_1)k^{2+2s_1}\ln k<k^{\eta s_1},\quad C_*=400l(c_0+1)^2, \tag{75}$$

for any $k>k_*$. Assume also that k_* is sufficiently large to ensure validity of all estimates in the first two steps for any $k>k_*$. Further we consider $\lambda=k^{2l}$, where $k>k_*$.

The geometric lemma for n-th step is the same as that for Step 2 up to shift of indices. Note only that we need an inductive procedure to define the set $\chi_{n-1}^*(\lambda)$, which is defined by (46) for $n=1$ and in the analogous way for $n\geq 2$. The estimate (56) for n-th step takes the form

$$\frac{L\left(\mathcal{K}_n\chi_{n-1}^*\setminus\chi_n)\right)}{L\left(\chi_{n-1}^*\right)}<k^{-\mathcal{S}_n},\qquad \mathcal{S}_n=2\sum_{i=1}^{n-1}(1+s_i). \tag{76}$$

It is easy to see that $\mathcal{S}_n=2(n-1)+(2^n-2)\,s_1$ and $\mathcal{S}_n\approx 2^ns_1\approx s_n$. The formulation of the main results (perturbation formulas) for n-th step is the same as for the second step up to shift of indices. The formula for the resonance set Ω_{n-1} and non-resonance set χ_n are analogous to those for Ω_1, χ_2 (see (53)). The proof of the first and second

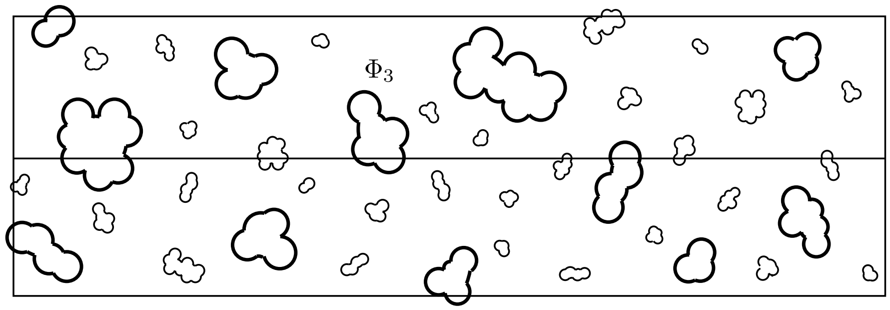

FIGURE 8. The set Φ_3.

statements of Geometric Lemma follows from the definition of the non-resonance set. Now we describe shortly a proof of the third statement.

In the second step, we defined the union $\mathcal{O}(\vec{b})$ of all disks of the radius $r = k^{-4-6s_1-3\delta}$ surrounding zeros of the determinant (70) in Φ_1. Let us change the notation: $\mathcal{O}(\vec{b}) \equiv \mathcal{O}^{(1)}(\vec{b}^{(1)})$. Now we define $\mathcal{O}^{(n-1)}(\vec{b}^{(n-1)})$, $\vec{b}^{(n-1)} \in K_{n-1}$, $n \geq 3$, by the formula

$$\mathcal{O}^{(n-1)}(\vec{b}^{(n-1)}) = \bigcup_{p^{(n-2)} \in P^{(n-2)}} \mathcal{O}_s^{(n-2)}\left(\vec{b}^{(n-1)} + 2\pi p^{(n-2)}/\hat{N}_{n-2}a\right), \tag{77}$$

here and below, $\hat{N}_{n-2} \equiv N_{n-2} \cdots N_1$ and $P^{(m)} = \{p^{(m)} = (p_1^{(m)}, p_2^{(m)}),\ 0 \leq p_1^{(m)} < N_m - 1,\ 0 \leq p_2^{(m)} < N_m - 1\}$. The set $\mathcal{O}_s^{(m)}(\vec{b}^{(m)})$, $m \geq 1$, is a collection of disks of the radius $r^{(m+1)} = r^{(m)} k^{-2-4s_{m+1}-\delta}$, $r^{(1)} = r = k^{-4-6s_1-3\delta}$ around zeros of the determinant $\det\big(I + A_m\left(\vec{y}^{(m)}(\varphi)\right)\big)$ in Φ_m, $\vec{y}^{(m)} = \vec{\varkappa}_m(\varphi) + \vec{b}^{(m)}$, the set Φ_m being defined earlier for $m = 1, 2$ (Definition 18, (72)) and by the formula below for $m \geq 3$.

$$\mathcal{O}_*^{(n-1)} = \bigcup_{p^{(n-1)} \in P^{(n-1)} \setminus \{0\}} \mathcal{O}^{(n-1)}\big(2\pi p^{(n-1)}/\hat{N}_{n-1}a\big), \qquad \Phi_n = \Phi_{n-1} \setminus \mathcal{O}_*^{(n-1)}. \tag{78}$$

If $n = 2$, then (78) gives us $\mathcal{O}_*$, see (71). Note that the complex non-resonance set Φ_n is defined by the recurrent formula analogous to (72).

Lemma 21. *The set $\mathcal{O}_s^{(m)}(\vec{b}^{(m)})$, $\vec{b}^{(m)} \in K_m$ contains no more than $4^{m-1}c_o k^{2+2s_m}$ disks.*

Corollary 22. *The set $\mathcal{O}^{(n-1)}(\vec{b}^{(n-1)})$ contains no more than $4^{n-2}c_o k^{2+2s_{n-1}}$ disks.*

Corollary 23. *The set $\mathcal{O}_*^{(n-1)}$ contains no more than $4^{n-1}c_o k^{2+2s_n}$ disks.*

The lemma is proved by an induction procedure. Corollaries 22 and 23 are based on the fact that $P^{(n-1)}$ contains no more than $4k^{2(s_n - s_{n-1})}$ elements and a similar estimate holds for $P^{(n-2)}$.

Obviously, Φ_n has the structure of Swiss cheese, more and more holes of smaller and smaller radii appear at each step of approximation (Fig. 8). Note that the disks are more and more precisely "targeted" at each step of approximation. At the n-th step the disks of $\mathcal{O}_*^{(n-1)}$ are centered around the zeros of the determinants

$$\det\big(I + A_{n-2}(\vec{\varkappa}_{n-2}(\varphi) + 2\pi p^{(n-2)}/\hat{N}_{n-2}a + 2\pi p^{(n-1)}/\hat{N}_{n-1}a)\big),$$

where $p^{(n-2)} \in P^{(n-2)}$, $p^{(n-1)} \in P^{(n-1)}$, $\vec{\varkappa}_{n-2}(\varphi) \in \mathcal{D}_{n-2}$, the corresponding operator $H^{(n-2)}$ being closer and closer to the operator H. Here, $\lambda^{(n-2)}(\vec{\varkappa}_{n-2}(\varphi)) = \lambda$.

If $W_1 = W_2 = ... = W_{n-2} = 0$, then $\mathcal{O}_*^{(n-1)}$ is just the union of disks centered at quasi-intersections of the "unperturbed" circle $\vec{k} = k(\cos\varphi, \sin\varphi)$, $k = \lambda^{\frac{1}{2l}}$, $\varphi \in [0, 2\pi)$ with circles of the same radius centered at points $2\pi j/a + 2\pi p^{(1)}/N_1 a + ... + 2\pi p^{(n-1)}/\hat{N}_{n-1}a$, these points being nodes of the dual lattice corresponding to the periods $\hat{N}_{n-1}a_1, \hat{N}_{n-1}a_2$. After constructing $\chi_n(\lambda)$ as the real part of Φ_n, we define the non-resonance subset $\chi_n^*(\lambda)$ of the isoenergetic set $S_n(\lambda)$ of $H_\alpha^{(n)}$, $S_n(\lambda) \subset K_n$. It corresponds to the non-resonance eigenvalues given by perturbation series. The sets $\chi_1^*(\lambda)$, $\chi_2^*(\lambda)$ are defined in the previous steps as well as the non-resonance sets $\chi_1(\lambda)$, $\chi_2(\lambda)$. Recall that we started by the definition of $\chi_1(\lambda)$ (Fig. 4) and used it to define $\mathcal{D}_1(\lambda)$ (Fig. 1) and $\chi_1^*(\lambda)$, $\chi_1^* = \mathcal{K}_1\mathcal{D}_1$ (Fig. 5). In the second step, we constructed $\chi_2(\lambda)$, using $\chi_1^*(\lambda)$. Next, we defined $\mathcal{D}_2(\lambda)$ (Fig. 2) and $\chi_2^*(\lambda)$, $\chi_2^* = \mathcal{K}_2\mathcal{D}_2$. Thus, the process looks like $\chi_1 \to \mathcal{D}_1 \to \chi_1^* \to \chi_2 \to \mathcal{D}_2 \to \chi_2^* \to \chi_3 \to \mathcal{D}_3 \to \chi_3^* \to ....$ At Every step, the set χ_n is constructed using χ_{n-1}^* by a formula analogous to (53). Using perturbation formulas, we show that the "radius" $\varkappa_n(\varphi)$ of $\mathcal{D}_n$ satisfies the estimates

$$|\varkappa_n(\varphi) - \varkappa_{n-1}(\varphi)| < 2\epsilon_{n-1}^4 k^{-2l+1}, \quad n \geq 2, \tag{79}$$

$$\left|\frac{\partial \varkappa_n}{\partial \varphi}(\varphi) - \frac{\partial \varkappa_{n-1}}{\partial \varphi}(\varphi)\right| < 4\epsilon_{n-1}^3 k^{1+\delta}, \quad n \geq 2, \tag{80}$$

where

$$\epsilon_n = e^{-\frac{1}{4}k^{\eta s_n}} \tag{81}$$

and η is the parameter in (3). Note that ϵ_n decays super exponentially with n.

3. Limit-Isoenergetic Set and Eigenfunctions

3.1. **Limit-Isoenergetic Set and Proof of Bethe-Sommerfeld Conjecture.** At each step n, we have constructed a set $\mathcal{B}_n(\lambda)$, $\mathcal{B}_n(\lambda) \subset \mathcal{B}_{n-1}(\lambda) \subset S_1(\lambda)$, and a function $\varkappa_n(\lambda, \vec{\nu})$, $\vec{\nu} \in \mathcal{B}_n(\lambda)$, with the following properties. The set $\mathcal{D}_n(\lambda)$ of vectors $\vec{\varkappa} = \varkappa_n(\lambda, \vec{\nu})\vec{\nu}$, $\vec{\nu} \in \mathcal{B}_n(\lambda)$, is a slightly distorted circle with holes; see Figs.1, 2, formula (15) and Lemma 6. For any $\vec{\varkappa}_n(\lambda, \vec{\nu}) \in \mathcal{D}_n(\lambda)$, there is a single eigenvalue of $H^{(n)}(\vec{\varkappa}_n)$ equal to λ and given by a perturbation series analogous to (59). Let $\mathcal{B}_\infty(\lambda) = \bigcap_{n=1}^{\infty} \mathcal{B}_n(\lambda)$. Since $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ for every n, $\mathcal{B}_\infty(\lambda)$ is the unit circle with an infinite number of holes, more and more holes of smaller and smaller size appearing at each step.

Lemma 24. *The length of $\mathcal{B}_\infty(\lambda)$ satisfies estimate (10) with $\gamma_3 = \delta/2$.*

Proof. Using (76) and noting that $S_n \approx 2^n s_1$, we easily conclude that $L(\mathcal{B}_n) = \big(1 + O(k^{-\delta/2})\big)$, $k = \lambda^{1/2l}$, uniformly in n. Since $\mathcal{B}_n$ is a decreasing sequence of sets, (10) holds. □

Let us consider $\varkappa_\infty(\lambda, \vec{\nu}) = \lim_{n\to\infty} \varkappa_n(\lambda, \vec{\nu})$, $\quad \vec{\nu} \in \mathcal{B}_\infty(\lambda)$.

Lemma 25. *The limit $\varkappa_\infty(\lambda, \vec{\nu})$ exists for any $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$. The following estimates hold when $n \geq 1$:*

$$|\varkappa_\infty(\lambda, \vec{\nu}) - \varkappa_n(\lambda, \vec{\nu})| < 4\epsilon_n^4 k^{-2l+1}, \quad \epsilon_n = \exp(-\frac{1}{4}k^{\eta s_n}), \quad s_n = 2^{n-1}s_1. \tag{82}$$

Corollary 26. *For every $\vec{\nu} \in \mathcal{B}_\infty(\lambda)$, estimate (11) holds, where $\gamma_4 = (4l - 3 - 4s_1 - 3\delta)/2l > 0$.*

The lemma follows easily from (79). To obtain the corollary, we use (43) and take into account that $\gamma_0 = 2l - 2 - 4s_1 - 2\delta$.

The estimate (80) justifies convergence of the sequence $\frac{\partial \varkappa_n}{\partial \varphi}$. We denote the limit of this sequence by $\frac{\partial \varkappa_\infty}{\partial \varphi}$.

Lemma 27. *The estimate (18) with* $\gamma_5 = (4l-5-8s_1-4\delta)/2l > 0$ *holds for any* $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$.

We define $\mathcal{D}_\infty(\lambda)$ by (9). Clearly, $\mathcal{D}_\infty(\lambda)$ is a slightly distorted circle of radius k with infinite number of holes. We can assign a tangent vector $\frac{\partial\varkappa}{\partial\varphi}\vec{\nu}+\varkappa\vec{\mu}$, $\vec{\mu}=(-\sin\varphi,\cos\varphi)$ to the curve $\mathcal{D}_\infty(\lambda)$, this tangent vector being the limit of corresponding tangent vectors for curves $\mathcal{D}_n(\lambda)$ at points $\vec{\varkappa}_n(\lambda,\vec{\nu})$ as $n\to\infty$.

Remark 28. We see easily from (82) that any $\vec{\varkappa}\in\mathcal{D}_\infty(\lambda)$ belongs to the $\left(4\epsilon_n^4k^{-2l+1}\right)$-neighborhood of $\mathcal{D}_n(\lambda)$. Applying perturbation formulas for n-th step, we conclude that there is an eigenvalue $\lambda^{(n)}(\vec{\varkappa})$ of $H^{(n)}(\vec{\varkappa})$ satisfying the estimate $\lambda^{(n)}(\vec{\varkappa})=\lambda+\delta_n$, $\delta_n=O\left(\epsilon_n^4\right)$, where the eigenvalue $\lambda^{(n)}(\vec{\varkappa})$ is given by a perturbation series of the type (59). Hence, for every $\vec{\varkappa}\in\mathcal{D}_\infty(\lambda)$, one has the limit

$$\lim_{n\to\infty}\lambda^{(n)}(\vec{\varkappa})=\lambda, \tag{83}$$

$$\left|\lambda^{(n)}(\vec{\varkappa})-\lambda\right|<\delta_n,\quad \delta_n=24\epsilon_n^4. \tag{84}$$

Theorem 29 (Bethe-Sommerfeld Conjecture)**.** *The spectrum of operator* H *contains a semi-axis.*

Proof. By Remark 28, there is a point of the spectrum of H_n in the δ_n-neighborhood of λ for every $\lambda>k_*^{2l}$, where k_* is given by (75). Since $\|H_n-H\|<\epsilon_n^4$, there is a point of the spectrum of H in the δ_n^*-neighborhood of λ, $\delta_n^*=\delta_n+\epsilon_n^4$. Since this is true for every n and the spectrum of H is closed, λ is in the spectrum of H. □

3.2. **Generalized Eigenfunctions of** H**.** A plane wave is usually written by $e^{i\langle\vec{k},x\rangle}$, $\vec{k}\in\mathbb{R}^2$. Here we use $\vec{\varkappa}$ instead of $\vec{k}$ to conform to our previous notations. We show that for every $\vec{\varkappa}$ in the set

$$\mathcal{G}_\infty=\bigcup_{\lambda>\lambda_*}\mathcal{D}_\infty(\lambda),\quad \lambda_*=k_*^{2l}, \tag{85}$$

k_* being given in (75), there is a solution $\Psi_\infty(\vec{\varkappa},x)$ of the equation for eigenfunction equation

$$(-\Delta)^{2l}\Psi_\infty(\vec{\varkappa},x)+V(x)\Psi_\infty(\vec{\varkappa},x)=\lambda_\infty(\vec{\varkappa})\Psi_\infty(\vec{\varkappa},x) \tag{86}$$

which can be represented in the form

$$\Psi_\infty(\vec{\varkappa},x)=e^{i\langle\vec{\varkappa},x\rangle}\big(1+u_\infty(\vec{\varkappa},x)\big), \tag{87}$$

where $u_\infty(\vec{\varkappa},x)$ is a limit-periodic function satisfying the estimate

$$\big\|u_\infty(\vec{\varkappa},x))\big\|_{L_\infty(\mathbb{R}^2)}<10|\vec{\varkappa}|^{-\gamma_1}, \tag{88}$$

$\gamma_1=2l-4-7s_1-2\delta>0$; the eigenvalue $\lambda_\infty(\vec{\varkappa})$ satisfies the asymptotic formula

$$\lambda_\infty(\vec{\varkappa})=|\vec{\varkappa}|^{2l}+O(|\vec{\varkappa}|^{-\gamma_2}),\quad \gamma_2=2l-2-4s_1-3\delta>0. \tag{89}$$

We also show that the set $\mathcal{G}_\infty$ satisfies (8).

In fact, by (82), any $\vec{\varkappa}\in\mathcal{D}_\infty(\lambda)$ belongs to the $(\epsilon_nk^{-2l+1-\delta})$-neighborhood of $\mathcal{D}_n(\lambda)$. Applying the perturbation formulas proved in the previous sections, we obtain the inequalities

$$\big\|E^{(1)}(\vec{\varkappa})-E^{(0)}(\vec{\varkappa})\big\|_1<2k^{-\gamma_0},\quad \gamma_0=2l-2-4s_1-2\delta, \tag{90}$$

$$\big\|E^{(n+1)}(\vec{\varkappa})-\tilde{E}^{(n)}(\vec{\varkappa})\big\|_1<48\epsilon_n^3,\quad n\geq 1, \tag{91}$$

$$\big|\lambda^{(1)}(\vec{\varkappa})-|\vec{\varkappa}|^{2l}\big|<2k^{-\gamma_2}, \tag{92}$$

$$\left|\lambda^{(n+1)}(\vec{\varkappa}) - \lambda^{(n)}(\vec{\varkappa})\right| < 12\epsilon_n^4, \quad n \geq 1, \tag{93}$$

where $E^{(n+1)}$, $\tilde{E}^{(n)}$ are one-dimensional spectral projectors in $L_2(Q_{n+1})$ corresponding to the potentials W_{n+1} and W_n, respectively; $\lambda^{(n+1)}(\vec{\varkappa})$ is the eigenvalue corresponding to $E^{(n+1)}(\vec{\varkappa})$; and $E^{(0)}(\vec{\varkappa})$ corresponds to $V = 0$ and the periods a_1, a_2. The estimate (93) means that for every $\vec{\varkappa} \in \mathcal{G}_\infty$ there is a limit $\lambda_\infty(\vec{\varkappa})$ of $\lambda^{(n)}(\vec{\varkappa})$ as $n \to \infty$:

$$\lambda_\infty(\vec{\varkappa}) = \lim_{n\to\infty} \lambda^{(n)}(\vec{\varkappa}), \tag{94}$$

$$\left|\lambda_\infty(\vec{\varkappa}) - \lambda^{(n)}(\vec{\varkappa})\right| < 24\epsilon_n^4, \quad n \geq 2. \tag{95}$$

The estimates (90), (91) mean that for properly chosen eigenfunctions $\Psi_{n+1}(\vec{\varkappa}, x)$,

$$\|\Psi_1 - \Psi_0\|_{L_2(Q_1)} < 4k^{-\gamma_0}|Q_1|^{1/2}, \quad \Psi_0(x) = e^{i\langle \vec{\varkappa}, x\rangle}, \tag{96}$$

$$\|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_2(Q_{n+1})} < 100\epsilon_n^3 |Q_{n+1}|^{1/2}, \tag{97}$$

where $\tilde{\Psi}_n$ is Ψ_n extended quasi-periodically from Q_n to Q_{n+1}. The eigenfunctions Ψ_n, $n \geq 1$, are chosen to obey two conditions: $\|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}$; [8] and $\mathrm{Im}(\Psi_n, \tilde{\Psi}_{n-1}) = 0$; here $(\cdot,\cdot)$ is an inner product in $L_2(Q_n)$. These two conditions obviously determine a unique choice of each Ψ_n. Noting Ψ_{n+1} and $\tilde{\Psi}_n$ satisfy eigenfunction equations and taking into account (93), (97), we obtain

$$\|\Psi_{n+1} - \tilde{\Psi}_n\|_{W_2^{2l}(Q_{n+1})} < ck^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n \geq 1, \tag{98}$$

and hence $\|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_\infty(Q_{n+1})} < ck^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}$. Since Ψ_{n+1} and $\tilde{\Psi}_n$ obey the same quasiperiodic conditions, the same inequality holds in all of $\mathbb{R}^2$:

$$\|\Psi_{n+1} - \Psi_n\|_{L_\infty(\mathbb{R}^2)} < c_l k^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n \geq 1, \tag{99}$$

where Ψ_{n+1}, Ψ_n are quasiperiodically extended to $\mathbb{R}^2$. Obviously, we have a Cauchy sequence in $L_\infty(\mathbb{R}^2)$. Let

$$\Psi_\infty(\vec{\varkappa}, x) = \lim_{n\to\infty} \Psi_n(\vec{\varkappa}, x). \tag{100}$$

This limit is defined pointwise uniformly in x and in $W_{2,loc}^{2l}(\mathbb{R}^2)$. From the estimate (99), we easily obtain

$$\|\Psi_\infty - \Psi_n\|_{L_\infty(\mathbb{R}^2)} < c)lk^{2l}\epsilon_n^3|Q_{n+1}|^{1/2}, \quad n \geq 2. \tag{101}$$

Theorem 30. *For every sufficiently large λ, $\lambda > \lambda_*(V,\delta)$ and $\vec{\varkappa} \in \mathcal{D}_\infty(\lambda)$, the sequence of functions $\Psi_n(\vec{\varkappa}, x)$ converges in $L_\infty(\mathbb{R}^2)$ and $W_{2,loc}^{2l}(\mathbb{R}^2)$. The limit function $\Psi_\infty(\vec{\varkappa}, x)$ satisfies the equation*

$$(-\Delta)^{2l}\Psi_\infty(\vec{\varkappa}, x) + V(x)\Psi_\infty(\vec{\varkappa}, x) = \lambda\Psi_\infty(\vec{\varkappa}, x). \tag{102}$$

It can be represented in the form (87), where $u_\infty(\vec{\varkappa}, x)$ is the limit-periodic function

$$u_\infty(\vec{\varkappa}, x) = \sum_{n=1}^{\infty} \tilde{u}_n(\vec{\varkappa}, x), \tag{103}$$

and $\tilde{u}_n(\vec{\varkappa}, x)$ are periodic function with the periods $2^{M_n-1}\beta_1, 2^{M_n-1}\beta_2$, $2^{M_n} \approx k^{2^{n-1}s_1}$,

$$\|\tilde{u}_1\|_{L_\infty(\mathbb{R}^2)} < 9k^{-\gamma_1}, \quad \gamma_1 = 2l - 4 - 7s_1 - 2\delta > 0, \tag{104}$$

$$\|\tilde{u}_n\|_{L_\infty(\mathbb{R}^2)} < c_l k^{2l}\epsilon_{n-1}^3|Q_n|^{1/2}, \quad n \geq 2. \tag{105}$$

The eigenvalue λ in (102) is equal to $\lambda_\infty(\varkappa)$, defined by (94), (95), and the estimate (89) holds.

[8] The condition $\|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}$ implies $\|\tilde{\Psi}_n\|_{L_2(Q_{n+1})} = |Q_{n+1}|^{1/2}$.

Corollary 31. *The function $u_\infty(\vec{\varkappa},x)$ satisfies the estimate (88).*

Remark 32. If V is sufficiently smooth, say $V\in C^1(R)$, then estimate (104) and hence (87) can be improved by replacing γ_1 by γ_0.

Proof. Let us show that Ψ_∞ is a limit-periodic function. Obviously, $\Psi_\infty=\Psi_0+\sum_{n=0}^{\infty}(\Psi_{n+1}-\Psi_n)$, the series converging in $L_\infty(\mathbb{R}^2)$ by (99). Writing $u_{n+1}=e^{-i\langle\vec{\varkappa},x\rangle}(\Psi_{n+1}-\Psi_n)$, we arrive at (87), (103). Note that $\tilde{u}_n$ is periodic with the periods $2^{M_n-1}\beta_1, 2^{M_n-1}\beta_2$. Estimate (105) follows from (99). We check (104). Indeed, by (96), the Fourier coefficients $(u_1)_j$, $j\in\mathbb{Z}^2$, satisfy the estimate $|(\tilde{u}_1)_j|<4k^{-\gamma_0}|Q_1|^{1/2}<8k^{-\gamma_0+s_1}$. This estimate is easily improved for j such that $p_j(0)>2k$: $|(\tilde{u}_1)_j|<c|j|^{-2l}$. Summarizing these inequalities and taking into account that the number of $j:p_j(0)\le 2k$ does not exceed $c_0k^{2+2s_1}$, we conclude that (104) holds for sufficiently large k, $k_0(\sum_{r=1}^{\infty}\|V_r\|,l,\delta)$. It remains to prove (102). Indeed, $\Psi_n(\vec{\varkappa},x)$, $n\ge 1$, satisfy the eigenfunction equations: $H^{(n)}\Psi_n=\lambda^{(n)}(\vec{\varkappa})\Psi_n$. Since $\Psi_n(\vec{\varkappa},x)$ converges to $\Psi(\vec{\varkappa},x)$ in $W_{2l,loc}^2$ and relation (83) holds, we arrive at (102). The estimate (89) follows from (92) – (94). □

Remark 33. Theorem 30 holds for $\vec{\varkappa}\in\mathcal{D}_\infty(\lambda)$ and all $\lambda>\lambda_*$. Hence it holds in $\mathcal{G}_\infty=\cup_{\lambda>\lambda*}\mathcal{D}_\infty(\lambda)$.

4. Absolute Continuity of the Spectrum

4.1. **Sets $\mathcal{G}_n$ and Projections $E_n(\mathcal{G}_n')$, $\mathcal{G}_n'\subset\mathcal{G}_n$.** Let us consider the sets $\mathcal{G}_n$ given by

$$\mathcal{G}_n=\bigcup_{\lambda>\lambda_*}\mathcal{D}_n(\lambda), \tag{106}$$

where $\lambda_*=k_*^{2l}$ and k_* is introduced in (75). Since the perturbation formulas hold in a small neighborhood of each point of $\mathcal{G}_n$, we consider, with slightly abused notations, that $\mathcal{G}_n$ is open. The function $\lambda^{(1)}(\vec{\varkappa})$ is differentiable in a neighborhood of each $\vec{\varkappa}\in\mathcal{G}_1$, estimates (39), (40) being valid. Similar results hold for all $\lambda^{(n)}(\vec{\varkappa})$ and $\mathcal{G}_n$, $n=1,2,....$

There is a family of Bloch eigenfunctions $\Psi_n(\vec{\varkappa},x)$, $\vec{\varkappa}\in\mathcal{G}_n$, of the operator $H^{(n)}$, which are described by the perturbation formulas. Let $\mathcal{G}_n'$ be a Lebesgue measurable subset of $\mathcal{G}_n$. We consider the spectral projection $E_n(\mathcal{G}_n')$ of $H^{(n)}$ corresponding to functions $\Psi_n(\vec{\varkappa},x)$, $\vec{\varkappa}\in\mathcal{G}_n'$. Note that, as in [38], $E_n(\mathcal{G}_n'):L_2(\mathbb{R}^2)\to L_2(\mathbb{R}^2)$ can be written as

$$E_n(\mathcal{G}_n')F=\frac{1}{4\pi^2}\int_{\mathcal{G}_n'}\big(F,\Psi_n(\vec{\varkappa})\big)\Psi_n(\vec{\varkappa})d\vec{\varkappa} \tag{107}$$

for any $F\in C_0^\infty(\mathbb{R}^2)$, here and below $\big(\cdot,\cdot\big)$ is the canonical scalar product in $L_2(\mathbb{R}^2)$, i.e.,

$$\big(F,\Psi_n(\vec{\varkappa})\big)=\int_{\mathbb{R}^2}F(x)\overline{\Psi_n(\vec{\varkappa},x)}dx.$$

More precisely, we write

$$E_n(\mathcal{G}_n')=S_n(\mathcal{G}_n')\,T_n(\mathcal{G}_n'), \tag{108}$$

$$T_n:L_2(\mathbb{R}^2)\to L_2(\mathcal{G}_n'),\qquad S_n:L_2(\mathcal{G}_n')\to L_2(\mathbb{R}^2),$$

$$T_nF=\big(F,\Psi_n(\vec{\varkappa})\big)\ \text{ for any } F\in C_0^\infty(\mathbb{R}^2), \tag{109}$$

T_nF being in $L_\infty(\mathcal{G}_n')$, and

$$S_n\varphi=\int_{\mathcal{G}_n'}\varphi(\vec{\varkappa})\Psi_n(\vec{\varkappa},x)d\vec{\varkappa}\ \text{ for any } \varphi\in L_\infty(\mathcal{G}_n'). \tag{110}$$

It is easy to show that $T_nF \in L_\infty(\mathcal{G}_n)$, when $F \in C_0^\infty(\mathbb{R}^2)$. Hence $E_n(\mathcal{G}'_n)$ can be described by formula (107) for $F \in C_0^\infty(\mathbb{R}^2)$. Moreover, as in [38], $\|T_n\| \le 1$ on $C_0^\infty(\mathbb{R}^2)$ and $\|S_n\| \le 1$ on $L_\infty(\mathcal{G}'_n)$ and hence T_n, S_n can be extended by continuity from $C_0^\infty(\mathbb{R}^2)$, $L_\infty(\mathcal{G}'_n)$ to $L_2(\mathbb{R}^2)$ and $L_2(\mathcal{G}'_n)$, respectively. Thus the operator $E_n(\mathcal{G}'_n)$ is described by (108) in the whole space $L_2(\mathbb{R}^2)$.

Let us introduce new coordinates in $\mathcal{G}_n$, (λ_n, φ), $\lambda_n = \lambda^{(n)}(\vec{\varkappa})$, $(\cos\varphi, \sin\varphi) = \frac{\vec{\varkappa}}{|\vec{\varkappa}|}$.

Lemma 34. *Every point $\vec{\varkappa}$ in $\mathcal{G}_n$ is represented by a unique pair (λ_n, φ), $\lambda_n > \lambda_*$, $\varphi \in [0, 2\pi)$, where $\lambda_* = k_*^{2l}$.*

Proof. Obviously, to every $\vec{\varkappa}$ in $\mathcal{G}_n$, there exists a pair (λ_n, φ) such that $\lambda_n = \lambda^{(n)}(\vec{\varkappa})$ and that $(\cos\varphi, \sin\varphi) = \frac{\vec{\varkappa}}{|\vec{\varkappa}|}$. For uniqueness, suppose there are two points $\vec{\varkappa}_1, \vec{\varkappa}_2$ corresponding to (λ_n, φ), i.e., $\lambda^{(n)}(\vec{\varkappa}_1) = \lambda^{(n)}(\vec{\varkappa}_2) = \lambda_n$ and $\frac{\vec{\varkappa}_1}{|\vec{\varkappa}_1|} = \frac{\vec{\varkappa}_2}{|\vec{\varkappa}_2|} = \varphi$. Since both $\vec{\varkappa}_1$ and $\vec{\varkappa}_2$ belong to $\mathcal{D}_n(\lambda_n)$ which is parameterized by φ, $\vec{\varkappa}_1 = \vec{\varkappa}_2$. □

For any function $f(\vec{\varkappa})$ integrable on $\mathcal{G}_n$, we use the new coordinates and write

$$\begin{aligned}\int_{\mathcal{G}_n} f(\vec{\varkappa})d\vec{\varkappa} &= \int_{\mathbb{R}^2} \chi(\mathcal{G}_n, \vec{\varkappa}) f(\vec{\varkappa})d\vec{\varkappa}\\ &= \int_0^{2\pi}\int_{\lambda_*}^\infty \chi(\mathcal{G}_n, \vec{\varkappa}(\lambda_n,\varphi)) f(\vec{\varkappa}(\lambda_n,\varphi)) \frac{\varkappa(\lambda_n,\varphi)}{\frac{\partial\lambda_n}{\partial\varkappa}} d\lambda_n d\varphi,\end{aligned}$$

where $\chi(\mathcal{G}_n, \vec{\varkappa})$ is the characteristic function on $\mathcal{G}_n$.

Let

$$\mathcal{G}_{n,\lambda} = \{\vec{\varkappa} \in \mathcal{G}_n : \lambda_n(\vec{\varkappa}) < \lambda\}. \tag{111}$$

This set is Lebesgue measurable since $\mathcal{G}_n$ is open and $\lambda_n(\vec{\varkappa})$ is continuous on $\mathcal{G}_n$.

Lemma 35. $|\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}| \le 2\pi\lambda^{-(l-1)/l}\varepsilon$ *when* $0 \le \varepsilon \le 1$.

Proof. Considering that $\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda} = \{\vec{\varkappa} \in \mathcal{G}_n : \lambda \le \lambda_n(\vec{\varkappa}) < \lambda + \epsilon\}$, we get

$$|\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}| = \int_{\mathcal{G}_n} \chi(\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}, \vec{\varkappa})\, d\vec{\varkappa} \quad = \int_\lambda^{\lambda+\varepsilon}\int_{\Theta_n(\lambda_n)} \frac{\varkappa(\lambda_n,\varphi)}{\frac{\partial\lambda_n}{\partial\varkappa}} d\varphi d\lambda_n,$$

where $\Theta_n(\lambda_n) \subset [0, 2\pi)$ is the set of φ corresponding to $\mathcal{D}_n(\lambda_n)$. By perturbation formulas (e.g., (40), (64)), we have $\frac{\partial\lambda_n}{\partial\varkappa} = 2l\varkappa^{2l-1}(1 + o(1))$ and easily arrive at the inequality in the lemma. □

By (107), $E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda}) = E_n(\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda})$. Let us obtain an estimate for this projection.

Lemma 36. *For any $F \in C_0^\infty(\mathbb{R}^2)$ and $0 \le \varepsilon \le 1$,*

$$\left\|\big(E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})\big)F\right\|^2_{L_2(\mathbb{R}^2)} \le C(F)\lambda^{-\frac{l-1}{l}}\epsilon, \tag{112}$$

where $C(F)$ is uniform with respect to n and λ.

Proof. Considering formula (107), we easily see that

$$\big(\big(E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})\big)F, F\big) = \int_{\mathcal{G}_{n,\lambda+\varepsilon}\setminus\mathcal{G}_{n,\lambda}} \left|\big(F, \Psi_n(\vec{\varkappa})\big)\right|^2 d\vec{\varkappa}.$$

Using estimates (96), (97) for every cell of periods covering the support of F and summarizing over such cells, we readily obtain

$$\left|\big(F, \Psi_n(\vec{\varkappa})\big)\right|^2 < C(F).$$

Hence, by Lemma 35,

$$\left((E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda}))\, F, F\right) \leq C(F)\, |\mathcal{G}_{n,\lambda+\varepsilon} \setminus \mathcal{G}_{n,\lambda}| \leq C(F)\lambda^{-\frac{l-1}{l}}\varepsilon.$$

Estimate (112) follows since $E_n(\mathcal{G}_{n,\lambda+\varepsilon}) - E_n(\mathcal{G}_{n,\lambda})$ is a projection. □

4.2. **Sets $\mathcal{G}_\infty$ and $\mathcal{G}_{\infty,\lambda}$.** Recall, from (85) and (106), that

$$\mathcal{G}_\infty = \bigcup_{\lambda>\lambda_*} \mathcal{D}_\infty(\lambda) \text{ and } \mathcal{G}_n = \bigcup_{\lambda>\lambda_*} \mathcal{D}_n(\lambda).$$

Lemma 37. *The relation*

$$\mathcal{G}_\infty = \bigcap_{n=1}^{\infty} \mathcal{G}_n \tag{113}$$

holds and $\mathcal{G}_\infty$ satisfies (8) with $\gamma_3 = \delta/2$.

Corollary 38. *The perturbation formulas for $\lambda^{(n)}(\vec{\varkappa})$ and $\Psi_n(\vec{\varkappa})$ hold in $\mathcal{G}_\infty$ for all n. Moreover, Coordinates (λ_n, φ) can be used in $\mathcal{G}_\infty$ for every n.*

Proof. We start by considering a small region $U_n(\lambda_0) = \bigcup_{|\lambda-\lambda_0|<r_n} \mathcal{D}_n(\lambda)$, $r_n = \epsilon_{n-1}k^{-2\delta}$, $k = \lambda_0^{1/2l}$ around the isoenergetic surface $\mathcal{D}_n(\lambda_0)$ for $\lambda_0 > \lambda_*$. Taking into account that the estimate $\nabla\lambda^{(n)}(\vec{\varkappa}) = 2l|\vec{\varkappa}|^{2l-2}\vec{\varkappa} + o(1)$ holds in the $\left(\epsilon_{n-1}k^{-2l+1-2\delta}\right)$-neighborhood of $\mathcal{D}_n(\lambda_0)$, we conclude that $U_n(\lambda_0)$ is an open set (a distorted ring with holes), and the width of the ring is of order $\epsilon_{n-1}k^{-2l+1-2\delta}$. Hence, $|U_n(\lambda_0)| = 2\pi k r_n \left(1 + o(k^{-\delta/2})\right)$. It follows easily from the relations $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ and (79) that $U_{n+1} \subset U_n$. The definition of $\mathcal{D}_\infty(\lambda_0)$ yields $\mathcal{D}_\infty(\lambda_0) = \cap_{n=1}^{\infty} U_n(\lambda_0)$. Hence,

$$\mathcal{G}_\infty \subset \bigcap_{n=1}^{\infty} \mathcal{G}_n^+, \quad \mathcal{G}_n^+ = \bigcup_{\lambda>\lambda_*-\delta_n} \mathcal{D}_n(\lambda).$$

The set $\mathcal{G}_n$ differs from $\mathcal{G}_n^+$ only in the region near $\mathcal{D}_n(\lambda_*)$. Since λ_* is not strictly fixed, this difference is not essential. With a slightly abused notations, we replace $\bigcup_{\lambda>\lambda_*-\delta_n} \mathcal{D}_n(\lambda)$ by $\mathcal{G}_n$. Thus, $\mathcal{G}_\infty \subset \bigcap_{n=1} \mathcal{G}_n$. If $\vec{\varkappa} \in \bigcap_{n=1} \mathcal{G}_n$, then $\lambda_n(\vec{\varkappa})$ exists for every n and satisfies (92), (93). Hence, $\lambda_n(\vec{\varkappa})$ has a limit $\lambda_\infty(\vec{\varkappa}) \equiv \lambda_0$, i.e., $\varkappa \in \mathcal{D}_\infty(\lambda_0)$, $\lambda_0 \geq \lambda_*$. This means $\bigcap_{n=1}^{\infty} \mathcal{G}_n \subset \mathcal{G}_\infty$. The formula (113) is proved.

Now let us estimate the Lebesgue measure of $\mathcal{G}_\infty$. Since $U_{n+1} \subset U_n$ for every $\lambda_0 > \lambda_*$,

$$\mathcal{G}_{n+1} \subset \mathcal{G}_n. \tag{114}$$

Hence $|\mathcal{G}_\infty \cap \mathbf{B_R}| = \lim_{n\to\infty} |\mathcal{G}_n \cap \mathbf{B_R}|$. Summing the volumes of the regions U_n, we conclude that

$$|\mathcal{G}_n \cap \mathbf{B_R}| = |\mathbf{B_R}| \left(1 + O(R^{-\delta/2})\right) \tag{115}$$

uniformly in n. Thus, we have obtained (8) with $\gamma_3 = \delta/2$. □

Let

$$\mathcal{G}_{\infty,\lambda} = \left\{\vec{\varkappa} \in \mathcal{G}_\infty, \lambda_\infty(\vec{\varkappa}) < \lambda\right\}. \tag{116}$$

The function $\lambda_\infty(\vec{\varkappa})$ is a Lebesgue measurable function since it is a limit of the sequence of measurable functions. Hence, the set $\mathcal{G}_{\infty,\lambda}$ is measurable.

Lemma 39. *The measure of the symmetric difference of two sets $\mathcal{G}_{\infty,\lambda}$ and $\mathcal{G}_{n,\lambda}$ converges uniformly in λ to zero as $n \to \infty$:*

$$\lim_{n\to\infty} |\mathcal{G}_{\infty,\lambda} \Delta \mathcal{G}_{n,\lambda}| = 0,$$

where $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Proof. Using the relation $\mathcal{G}_\infty \subset \mathcal{G}_n$ and estimate (95), we readily check that $\mathcal{G}_{\infty,\lambda} \subset \mathcal{G}_{n,\lambda+\delta_n}$, $\delta_n = 24\epsilon_n^4$. Therefore,

$$\mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{n,\lambda} \subset \mathcal{G}_{n,\lambda+\delta_n} \setminus \mathcal{G}_{n,\lambda}.$$

Since $\mathcal{G}_{\infty,\lambda} \supset \mathcal{G}_{n,\lambda-\delta_n} \cap \mathcal{G}_\infty$,

$$\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_{\infty,\lambda} \subset \mathcal{G}_{n,\lambda} \cap \left(\mathcal{G}_{n,\lambda-\delta_n} \cap \mathcal{G}_\infty\right)^c \subset \left(\mathcal{G}_{n,\lambda} \setminus \mathcal{G}_{n,\lambda-\delta_n}\right) \cup \left(\mathcal{G}_n \setminus \mathcal{G}_\infty\right).$$

Combining the two, we get

$$\mathcal{G}_{\infty,\lambda} \Delta \mathcal{G}_{n,\lambda} \subset \left(\mathcal{G}_{n,\lambda+\delta_n} \setminus \mathcal{G}_{n,\lambda-\delta_n}\right) \cup \left(\mathcal{G}_n \setminus \mathcal{G}_\infty\right),$$

hence,

$$|\mathcal{G}_{\infty,\lambda} \Delta \mathcal{G}_{n,\lambda}| \leq |\mathcal{G}_{n,\lambda-\delta_n} \setminus \mathcal{G}_{n,\lambda+\delta_n}| + |\mathcal{G}_n \setminus \mathcal{G}_\infty|.$$

Let us consider the first term of the right hand side. Using Lemma 35 with $\varepsilon = 2\delta_n$, we obtain $|\mathcal{G}_{n,\lambda-\delta_n} \setminus \mathcal{G}_{n,\lambda+\delta_n}| < 48\pi\lambda^{-(l-1)/l}\epsilon_n^4$. By the definition (82) of ϵ_n, we conclude easily that the first term goes to zero uniformly in λ. By (113) and (114), the second term goes to zero too. □

4.3. **Spectral Projections** $E(\mathcal{G}_{\infty,\lambda})$**.** In this section, we show that spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L_2(\mathbb{R}^2)$ as n tends to infinity. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a spectral projection of H. It can be represented in the form $E_\infty(\mathcal{G}_{\infty,\lambda}) = S_\infty T_\infty$, where S_∞ and T_∞ are strong limits of $S_n(\mathcal{G}_{\infty,\lambda})$ and $T_n(\mathcal{G}_{\infty,\lambda})$, respectively. For any $F \in C_0^\infty(\mathbb{R}^2)$, we show

$$E_\infty\left(\mathcal{G}_{\infty,\lambda}\right) F = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda}} \big(F, \Psi_\infty(\vec{\varkappa})\big) \Psi_\infty(\vec{\varkappa}) d\vec{\varkappa}, \tag{117}$$

$$H E_\infty\left(\mathcal{G}_{\infty,\lambda}\right) F = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda}} \lambda_\infty(\vec{\varkappa}) \big(F, \Psi_\infty(\vec{\varkappa})\big) \Psi_\infty(\vec{\varkappa}) d\vec{\varkappa}. \tag{118}$$

Using properties of $E_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$, we prove absolute continuity of the branch of the spectrum corresponding to functions $\Psi_\infty(\vec{\varkappa})$.

Now we consider the sequence of operators $T_n(\mathcal{G}_{\infty,\lambda})$ which are given by (109) with $\mathcal{G}'_n = \mathcal{G}_{\infty,\lambda}$ and act from $L_2(\mathbb{R}^2)$ to $L_2(\mathcal{G}_{\infty,\lambda})$. We prove that the sequence has a strong limit and describe its properties.

Lemma 40. *The sequence $T_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $T_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $T_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|T_\infty\| \leq 1$ and can be described by the formula $T_\infty F = \big(F, \Psi_\infty(\vec{\varkappa})\big)$ for any $F \in C_0^\infty(\mathbb{R}^2)$. The convergence of $T_n(\mathcal{G}_{\infty,\lambda})F$ to $T_\infty(\mathcal{G}_{\infty,\lambda})F$ is uniform in λ for every $F \in L_2(\mathbb{R}^2)$.*

Proof. Let $F \in C_0^\infty(\mathbb{R}^2)$. We consider $T_\infty F = \big(F, \Psi_\infty(\vec{\varkappa})\big)$. It follows from (101) and (109) that

$$\big|(T_\infty - T_n)F(\vec{\varkappa})\big| < C(F) g_n(\vec{\varkappa}), \quad g_n(\vec{\varkappa}) = \varkappa^{2l}\epsilon_n^3 |Q_{n+1}|^{1/2}, \quad \epsilon_n = \exp(-\frac{1}{4}\varkappa^{\eta s_n}).$$

It is easy to see that $g_n(\vec{\varkappa}) \in L_2(\mathcal{G}_\infty)$ for all n and $g_n(\vec{\varkappa})$ tends to zero in $L_2(\mathcal{G}_\infty)$ as $n \to \infty$. Therefore, $g_n(\vec{\varkappa})$ tends to zero in $L_2(\mathcal{G}_{\infty,\lambda})$ uniformly in λ. Hence, $\big\|(T_\infty - T_n)F\big\|_{L_2(\mathcal{G}_{\infty,\lambda})}$ tends to zero uniformly in λ for every $F \in C_0^\infty(\mathbb{R}^2)$ as $n \to \infty$. Considering $\|T_n\| \leq 1$, we obtain that $T_n F$ has a limit for every $F \in L_2(\mathbb{R}^2)$ uniformly in λ. The estimate $\|T_\infty\| \leq 1$ is now obvious. □

Now we consider the sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ which are given by (110) with $\mathcal{G}'_n=\mathcal{G}_{\infty,\lambda}$ and act from $L_2(\mathcal{G}_{\infty,\lambda})$ to $L_2(\mathbb{R}^2)$. We prove that the sequence has a strong limit and describe its properties.

Lemma 41. *The sequence of operators $S_n(\mathcal{G}_{\infty,\lambda})$ has a strong limit $S_\infty(\mathcal{G}_{\infty,\lambda})$. The operator $S_\infty(\mathcal{G}_{\infty,\lambda})$ satisfies $\|S_\infty\|\leq 1$ and can be described by the formula*

$$(S_\infty\varphi)(x)=\int_{\mathcal{G}_{\infty,\lambda}}\varphi(\vec{\varkappa})\Psi_\infty(\vec{\varkappa},x)d\vec{\varkappa} \tag{119}$$

for any $\varphi\in L_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$. The convergence of $S_n(\mathcal{G}_{\infty,\lambda})\varphi$ to $S_\infty(\mathcal{G}_{\infty,\lambda})\varphi$ is uniform in λ for every $\varphi\in L_2\left(\mathcal{G}_\infty\right)$.

Proof. We start by proving that $S_n(\mathcal{G}_{\infty,\lambda})\varphi$ is a Cauchy sequence in $L_2(\mathbb{R}^2)$ for every $\varphi\in L_\infty\left(\mathcal{G}_{\infty,\lambda}\right)$. The function $\Psi_n(\vec{\varkappa},x)$ is quasiperiodic in Q_n and hence can be represented as a combination of plane waves:

$$\Psi_n(\vec{\varkappa},x)=\frac{1}{2\pi}\sum_{r\in\mathbb{Z}^2}c_r^{(n)}(\vec{\varkappa})\exp i\langle\vec{\varkappa}+\vec{p}_r(0)/\hat{N}_{n-1},x\rangle, \tag{120}$$

where $c_r^{(n)}(\vec{\varkappa})$ are Fourier coefficients, $\hat{N}_{n-1}=N_{n-1}\cdots N_1\approx 2^{s_n}$ and $\vec{p}_r(0)=(\frac{2\pi r_1}{a_1},\frac{2\pi r_2}{a_2})$. The Fourier transform of $\widehat{\Psi}_n$ is a combination of δ-functions

$$\widehat{\Psi}_n(\vec{\varkappa},\xi)=\sum_{r\in\mathbb{Z}^2}c_r^{(n)}(\vec{\varkappa})\delta\big(\xi+\vec{\varkappa}+\vec{p}_r(0)/\hat{N}_{n-1}\big).$$

From this, we compute easily the Fourier transform of $S_n\varphi$

$$(\widehat{S_n\varphi})(\xi)=\sum_{r\in\mathbb{Z}^2}c_r^{(n)}\big(-\xi-\vec{p}_r(0)/\hat{N}_{n-1}\big)\varphi\big(-\xi-\vec{p}_r(0)/\hat{N}_{n-1}\big)\chi\big(\mathcal{G}_{\infty,\lambda},-\xi-\vec{p}_r(0)/\hat{N}_{n-1}\big),$$

where $\chi(\mathcal{G}_{\infty,\lambda},\cdot)$ is the characteristic function on $\mathcal{G}_{\infty,\lambda}$. Since $\mathcal{G}_{\infty,\lambda}$ is bounded, the series contains only a finite number of non-zero terms for every ξ. By Parseval's identity, triangle inequality and a parallel shift of the variable,

$$\begin{aligned}&\|S_n\varphi\|_{L_2(\mathbb{R}^2)}=\|\widehat{S_n\varphi}\|_{L_2(\mathbb{R}^2)}\\ &\leq\sum_{r\in\mathbb{Z}^2}\Big\|c_r^{(n)}\big(-\xi-\vec{p}_r(0)/\hat{N}_{n-1}\big)\varphi\big(-\xi-\vec{p}_r(0)/\hat{N}_{n-1}\big)\chi\big(\mathcal{G}_{\infty,\lambda},-\xi-\vec{p}_r(0)/\hat{N}_{n-1}\big)\Big\|_{L_2(\mathbb{R}^2)}=\\ &\sum_{r\in\mathbb{Z}^2}\|c_r^{(n)}\varphi\|_{L_2(\mathcal{G}_{\infty,\lambda})}\leq\|\varphi\|_{L_\infty(\mathcal{G}_{\infty,\lambda})}\sum_{r\in\mathbb{Z}^2}\|c_r^{(n)}\|_{L_2(\mathcal{G}_{\infty,\lambda})}\leq\\ &\|\varphi\|_{L_\infty(\mathcal{G}_{\infty,\lambda})}\left(\sum_{r\in\mathbb{Z}^2}p_r^{2l}(0)\|c_r^{(n)}\|^2_{L_2(\mathcal{G}_{\infty,\lambda})}\right)^{1/2}\left(\sum_{r\in\mathbb{Z}^2}p_r^{-2l}(0)\right)^{1/2}.\end{aligned}$$

By (120), Fourier coefficients $c_r^{(n)}(\vec{\varkappa})$ can be estimated as follows:

$$\sum_{r\in\mathbb{Z}^2}p_r^{2l}(0)|c_r^{(n)}(\vec{\varkappa})|^2\leq\|\Psi_n(\vec{\varkappa},\cdot)\exp -i\langle\vec{\varkappa},\cdot\rangle\|^2_{W_2^{2l}(Q_n)}|Q_n|^{-1}\hat{N}_{n-1}^{2l}<$$

$$2|\vec{\varkappa}|^{2l}\|\Psi_n(\vec{\varkappa},\cdot)\|^2_{W_2^{2l}(Q_n)}|Q_n|^{-1}\hat{N}_{n-1}^{2l}.$$

Integrating the last inequality over $\mathcal{G}_{\infty,\lambda}$, we arrive at

$$\sum_{r\in\mathbb{Z}^2}p_r^{2l}(0)\|c_r^{(n)}\|^2_{L_2(\mathcal{G}_{\infty,\lambda})}\leq 2|\mathcal{G}_{\infty,\lambda}||Q_n|^{-1}\hat{N}_{n-1}^{2l}\sup_{\vec{\varkappa}\in\mathcal{G}_{\infty,\lambda}}|\vec{\varkappa}|^{2l}\|\Psi_n(\vec{\varkappa},\cdot)\|^2_{W_2^{2l}(Q_n)}$$

Considering that $\sum_r p_r^{-2l}(0) < ck^{2s_1}$, we obtain

$$\|S_n\varphi\|_{L_2(\mathbb{R}^2)} < ck^{s_1}|\mathcal{G}_{\infty,\lambda}|^{1/2}\|\varphi\|_{L_\infty(\mathcal{G}_{\infty,\lambda})}|Q_n|^{-1/2}\hat{N}_{n-1}^l \sup_{\vec{\varkappa}\in\mathcal{G}_{\infty,\lambda}} |\vec{\varkappa}|^l\|\Psi_n(\vec{\varkappa},\cdot)\|_{W_2^{2l}(Q_n)}.$$

Similarly,

$$\|(S_{n+1}-S_n)\varphi\|_{L_2(\mathbb{R}^2)}$$
$$< ck^{s_1}|\mathcal{G}_{\infty,\lambda}|^{1/2}\|\varphi\|_{L_\infty(\mathcal{G}_{\infty,\lambda})}|Q_{n+1}|^{-1/2}\hat{N}_n^l \sup_{\vec{\varkappa}\in\mathcal{G}_{\infty,\lambda}} |\vec{\varkappa}|^l\|\big(\Psi_{n+1}(\vec{\varkappa},\cdot)-\tilde{\Psi}_n(\vec{\varkappa},\cdot)\big)\|_{W_2^{2l}(Q_{n+1})}.$$

Now, using (98) and taking into account that $|\vec{\varkappa}|^{2l} < \lambda + o(1)$, $k^{2l} = \lambda$, we obtain

$$\|(S_n - S_{n+1})\varphi\|_{L_2(\mathbb{R}^2)} \leq c|\mathcal{G}_{\infty,\lambda}|^{1/2}\|\varphi\|_{L_\infty(\mathcal{G}_{\infty,\lambda})}\hat{N}_n^l k^{3l+s_1}\epsilon_n^3.$$

Considering that ϵ_n decays super exponentially with n (see (81)) and the estimates $|\mathcal{G}_{\infty,\lambda}| < \pi\lambda(1+o(1))$, $\hat{N}_n \approx k^{s_n}$, we conclude that $S_n\varphi$ is a Cauchy sequence in $L_2(\mathbb{R}^2)$ for every $\varphi \in L_\infty(\mathcal{G}_{\infty,\lambda})$. It is easy to see that convergence is uniform in λ for every $\varphi \in L_\infty(\mathcal{G}_\infty)$. We denote the limit of $S_n(\mathcal{G}_{\infty,\lambda})\varphi$ by $S_\infty(\mathcal{G}_{\infty,\lambda})\varphi$.

We see from formula (110) and estimate (101) that

$$\lim_{n\to\infty}\big(S_n(\mathcal{G}_{\infty,\lambda})\varphi\big)(x) = \int_{\mathcal{G}_{\infty,\lambda}} \varphi(\vec{\varkappa})\Psi_\infty(\vec{\varkappa},x)d\vec{\varkappa},$$

for all $x \in \mathbb{R}^2$ when $\varphi \in L_\infty(\mathcal{G}_{\infty,\lambda})$. Hence, (119) holds.

Since $\|S_n\| \leq 1$, the limit $S_\infty(\mathcal{G}_{\infty,\lambda})\varphi$ exists for all $\varphi \in L_2(\mathcal{G}_{\infty,\lambda})$, the convergence being uniform in λ for every $\varphi \in L_2(\mathcal{G}_\infty)$. It is obvious now that $\|S_\infty\| \leq 1$. □

Lemma 42. *Spectral projections $E_n(\mathcal{G}_{\infty,\lambda})$ have a strong limit $E_\infty(\mathcal{G}_{\infty,\lambda})$ in $L_2(\mathbb{R}^2)$, the convergence being uniform in λ for every element. The operator $E_\infty(\mathcal{G}_{\infty,\lambda})$ is a projection given by the formula (117) for any $F \in C_0^\infty(\mathbb{R}^2)$. The formula (118) holds for $HE_\infty(\mathcal{G}_{\infty,\lambda})$.*

Proof. By (108), $E_n = S_nT_n$. Both S_n and T_n have strong limits S_∞, T_∞ and $\|S_n\| \leq 1$, $\|T_n\| \leq 1$. It follows easily that E_n has the strong limit $E_\infty = S_\infty T_\infty$. Since E_n is a sequence of projections, its strong limit satisfies the relations: $E_\infty = E_\infty^*$, $E_\infty^2 = E_\infty$. Hence E_∞ is a projection [37]. Using last two lemmas and considering that $T_\infty(\mathcal{G}_{\infty,\lambda})F_0 \in L_\infty(\mathcal{G}_{\infty,\lambda})$ for any $F \in C_0^\infty(\mathbb{R}^2)$, we arrive at (117). Applying equation (86) for Ψ_∞, we obtain (118). It remains to prove that convergence of E_nF is uniform in λ for every $F \in L_2(\mathbb{R}^2)$. First, let $F \in C_0^\infty(\mathbb{R}^2)$. By the triangle inequality,

$$\|(E_\infty - E_n)F\| \leq \|(S_\infty - S_n)T_\infty F\| + \|S_n(T_\infty - T_n)F\|.$$

Since T_nF converges to $T_\infty F$ uniformly in λ and $\|S_n\| \leq 1$, the second term goes to zero uniformly in λ. We see easily from (117) that $T_\infty F \in L_\infty(\mathcal{G}_\infty)$. Then, by Lemma 41, $S_nT_\infty F$ converges to $E_\infty(\mathcal{G}_{\infty,\lambda})F$ uniformly in λ. This mean that $E_n(\mathcal{G}_{\infty,\lambda})F$ converges to $E_\infty(\mathcal{G}_{\infty,\lambda})F$ uniformly in λ for $F \in C_0^\infty(\mathbb{R}^2)$. Using $\|E_n\| = 1$, we obtain that uniform convergence holds for all $F \in L_2(\mathbb{R}^2)$.

□

Lemma 43. *There is a strong limit $E_\infty(\mathcal{G}_\infty)$ of the projections $E(\mathcal{G}_{\infty,\lambda})$ as λ goes to infinity.*

Corollary 44. *The operator $E(\mathcal{G}_\infty)$ is a projection.*

Proof. Considering that $\lim_{n\to\infty} E_n(\mathcal{G}_{\infty,\lambda}) = E_\infty(\mathcal{G}_{\infty,\lambda})$ and $E_n(\mathcal{G}_{\infty,\lambda})$ is a monotone in λ, we conclude that $E_\infty(\mathcal{G}_{\infty,\lambda})$ is monotone too. It is well-known that a monotone sequence of projections has a strong limit. □

Lemma 45. *Projections $E_\infty(\mathcal{G}_\infty,\lambda)$, $\lambda\in\mathbb{R}$, and $E_\infty(\mathcal{G}_\infty)$ reduce the operator H.*

Proof. Let us show $E_\infty(\mathcal{G}_\infty,\lambda)$ reduces H, i.e., $E_\infty(\mathcal{G}_{\infty,\lambda})Dom(H)\subset Dom(H)$ and $E_\infty(\mathcal{G}_{\infty,\lambda})H = HE_\infty(\mathcal{G}_{\infty,\lambda})$ on $Dom(H)$ (e.g., see Theorem 40.2 in [39]). For any $F,\ G\in Dom(H)=Dom(H^{(n)})$,

$$\begin{aligned}\big(F,E_\infty(\mathcal{G}_{\infty,\lambda})HG\big) &= \big(E_\infty(\mathcal{G}_{\infty,\lambda})F,HG\big)=\lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\lambda})F,H_nG\big)\\ &=\lim_{n\to\infty}\big(H^{(n)}E_n(\mathcal{G}_{\infty,\lambda})F,G\big)=\lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\lambda})H^{(n)}F,G\big)\\ &=\lim_{n\to\infty}\big(H^{(n)}F,E_n(\mathcal{G}_{\infty,\lambda})G\big)=\big(HF,E_\infty(\mathcal{G}_{\infty,\lambda})G\big)=\big(E_\infty(\mathcal{G}_{\infty,\lambda})HF,G\big)\end{aligned}$$

Hence, $E_\infty(\mathcal{G}_{\infty,\lambda})H$ is symmetric. Since $E_\infty(\mathcal{G}_{\infty,\lambda})$ is bounded, $(E_\infty(\mathcal{G}_{\infty,\lambda})H)^*=HE_\infty(\mathcal{G}_{\infty,\lambda})$ (e.g., see §115 in [37]). Therefore, $E_\infty(\mathcal{G}_{\infty,\lambda})H\subset HE_\infty(\mathcal{G}_{\infty,\lambda})$ which means that for every $F\in Dom(H)$, $E_\infty(\mathcal{G}_{\infty,\lambda})F\in Dom(H)$ and $E_\infty(\mathcal{G}_{\infty,\lambda})HF = HE_\infty(\mathcal{G}_{\infty,\lambda})F$.

Now we show that $E_\infty(\mathcal{G}_\infty)$ reduces H. Noting that $E_\infty(\mathcal{G}_\infty)$ is the strong limit of $E_\infty(\mathcal{G}_{\infty,\lambda})$ as $\lambda\to\infty$, for any $F,\ G\in Dom(H)$,

$$\begin{aligned}\big(F,E_\infty(\mathcal{G}_\infty)HG\big)=\lim_{\lambda\to\infty}\big(F,E_\infty(\mathcal{G}_{\infty,\lambda})HG\big)&=\lim_{\lambda\to\infty}\big(HE_\infty(\mathcal{G}_{\infty,\lambda})F,G\big)\\ &=\lim_{\lambda\to\infty}\big(E_\infty(\mathcal{G}_{\infty,\lambda})HF,G\big)=\big(E_\infty(\mathcal{G}_\infty)HF,G\big),\end{aligned}$$

i.e., $E_\infty(\mathcal{G}_\infty)H$ is symmetric. Considering $(E_\infty(\mathcal{G}_\infty)H)^*=HE_\infty(\mathcal{G}_\infty)$ as before, we obtain $E_\infty(\mathcal{G}_\infty)H\subset HE_\infty(\mathcal{G}_\infty)$ which means that for every $F\in Dom(H)$, $E_\infty(\mathcal{G}_\infty)F\in Dom(H)$ and $E_\infty(\mathcal{G}_\infty)HF = HE(\mathcal{G}_\infty)F$. Thus, $E_\infty(\mathcal{G}_\infty)$ reduces H. □

Lemma 46. *The family of projections $E_\infty(\mathcal{G}_\infty,\lambda)$ is the resolution of identity belonging to the operator $HE_\infty(\mathcal{G}_\infty)$.*

Proof. First, we show that $\lim_{\lambda\to-\infty}E_\infty(\mathcal{G}_{\infty,\lambda})=0$. It is enough to check that $\mathcal{G}_{\infty,\lambda}=\emptyset$ for every $\lambda<\lambda_*$. We see from the definition (106) of $\mathcal{G}_n$ and the definition (111) of $\mathcal{G}_{n,\lambda}$ that $\mathcal{G}_{n,\lambda_*}=\emptyset$. It follows from (95) and (116) that $\mathcal{G}_{\infty,\lambda_*-\delta_n}\subset\mathcal{G}_{n,\lambda_*}$, here $\delta_n=24\epsilon_n^4$, $n\geq 2$. Hence, $\mathcal{G}_{\infty,\lambda}=\emptyset$ for every $\lambda<\lambda_*$.

Second, $\lim_{\lambda\to\infty}E_\infty(\mathcal{G}_{\infty,\lambda})=E_\infty(\mathcal{G}_\infty)$ by Lemma 43.

Third, the family $E_\infty(\mathcal{G}_{\infty,\lambda})$ is left-continuous since each $E_n(\mathcal{G}_{\infty,\lambda})$ is left-continuous and $E_n(\mathcal{G}_{\infty,\lambda})F$ converges to $E_\infty(\mathcal{G}_{\infty,\lambda})F$ uniformly in λ for every F (Lemma 42).

Fourth, let $\lambda>\mu$. Then,

$$\begin{aligned}\big(E_\infty(\mathcal{G}_{\infty,\lambda})E_\infty(\mathcal{G}_{\infty,\mu})F,G\big)&=\big(E_\infty(\mathcal{G}_{\infty,\mu})F,E_\infty(\mathcal{G}_{\infty,\lambda})G\big)=\lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\mu})F,E_n(\mathcal{G}_{\infty,\lambda})G\big)\\ &=\lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\lambda})E_n(\mathcal{G}_{\infty,\mu})F,G\big)=\lim_{n\to\infty}\big(E_n(\mathcal{G}_{\infty,\mu})F,G\big)=\big(E_\infty(\mathcal{G}_{\infty,\mu})F,G\big).\end{aligned}$$

This means that $E_\infty(\mathcal{G}_{\infty,\lambda})E_\infty(\mathcal{G}_{\infty,\mu})=E_\infty(\mathcal{G}_{\infty,\mu})$.

Last, we check that for any $f\in[E_\infty(\mathcal{G}_{\infty,\lambda})-E_\infty(\mathcal{G}_{\infty,\mu})]D(H),\ \lambda>\mu$,

$$\mu\|f\|^2\leq\big(Hf,f\big)\leq\lambda\|f\|^2. \tag{121}$$

In fact, let

$$f=[E_\infty(\mathcal{G}_{\infty,\lambda})-E_\infty(\mathcal{G}_{\infty,\mu})]F,\quad F\in C_0^\infty(\mathbb{R}^2). \tag{122}$$

By (117), (118),

$$f(x) = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{\infty,\mu}} \big(F, \Psi_\infty(\vec{\varkappa})\big) \Psi_\infty(x) d\vec{\varkappa},$$

$$Hf(x) = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{\infty,\mu}} \lambda_\infty(\vec{\varkappa}) \big(F, \Psi_\infty(\vec{\varkappa})\big) \Psi_\infty(x) d\vec{\varkappa},$$

$$\|f\|^2_{L_2(R^2)} = \big(f, F\big) = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{\infty,\mu}} \left|\big(F, \Psi_\infty(\vec{\varkappa})\big)\right|^2 d\vec{\varkappa}, \tag{123}$$

$$\big(Hf, f\big) = \big(Hf, F\big) = \frac{1}{4\pi^2} \int_{\mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{\infty,\mu}} \lambda_\infty(\vec{\varkappa}) \left|\big(F, \Psi_\infty(\vec{\varkappa})\big)\right|^2 d\vec{\varkappa}. \tag{124}$$

By the definitions of $\mathcal{G}_{\infty,\mu}$ and $\mathcal{G}_{\infty,\lambda}$, the inequality $\mu \leq \lambda_\infty(\vec{\varkappa}) < \lambda$ holds, when $\vec{\varkappa} \in \mathcal{G}_{\infty,\lambda} \setminus \mathcal{G}_{\infty,\mu}$. Using the last equality in (124) and considering (123), we obtain (121) for all f given by (122). Since $C_0^\infty(\mathbb{R}^2)$ is dense in $Dom(H)$ with respect to $\|F\|_{L_2(\mathbb{R}^2)} + \|HF\|_{L_2(\mathbb{R}^2)}$ norm, inequality (121) can be extended to all $f = [E_\infty(\mathcal{G}_{\infty,\lambda}) - E_\infty(\mathcal{G}_{\infty,\mu})]F$, $F \in Dom(H)$.

From five properties of $E_\infty(\mathcal{G}_{\infty,\lambda})$ proved above, it follows that $E_\infty(\mathcal{G}_{\infty,\lambda})$ is the resolution of identity belonging to $HE_\infty(\mathcal{G}_\infty)$ [39]. □

4.4. **Proof of Absolute Continuity.** Now we show that the branch of spectrum (semi-axis) corresponding to $\mathcal{G}_\infty$ is absolutely continuous.

Theorem 47. *For any $F \in C_0^\infty(\mathbb{R}^2)$ and $0 \leq \varepsilon \leq 1$,*

$$\left|\big((E_\infty(\mathcal{G}_{\infty,\lambda+\varepsilon}) - E_\infty(\mathcal{G}_{\infty,\lambda}))F, F\big)\right| \leq C_F \varepsilon. \tag{125}$$

Corollary 48. *The spectrum of the operator $HE_\infty(\mathcal{G}_\infty)$ is absolutely continuous.*

Proof. By formula (117),

$$\left|\big((E_\infty(\mathcal{G}_{\infty,\lambda+\varepsilon}) - E(\mathcal{G}_{\infty,\lambda}))F, F\big)\right| \leq C_F \left|\mathcal{G}_{\infty,\lambda+\varepsilon} \setminus \mathcal{G}_{\infty,\lambda}\right|.$$

Applying Lemmas 35 and 39, we immediately get (125).

□

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