

# Leaky Quantum Graphs: A Review

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ABSTRACT. The aim of this review is to provide an overview of a recent work concerning “leaky” quantum graphs described by Hamiltonians given formally by the expression  $-\Delta - \alpha\delta(x - \Gamma)$  with a singular attractive interaction supported by a graph-like set in  $\mathbb{R}^\nu$ ,  $\nu = 2, 3$ . We will explain how such singular Schrödinger operators can be properly defined for different codimensions of  $\Gamma$ . Furthermore, we are going to discuss their properties, in particular, the way in which the geometry of  $\Gamma$  influences their spectra and the scattering, strong-coupling asymptotic behavior, and a discrete counterpart to leaky-graph Hamiltonians using point interactions. The subject cannot be regarded as closed at present, and we will add a list of open problems hoping that the reader will take some of them as a challenge.

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## 1. Introduction

In this paper we are going to review results concerning a class of “different” quantum graph models. With this aim in mind, it would be natural to start by recalling briefly the standard quantum graphs, their description, properties, and numerous applications. In this volume, however, this would be clearly to bring

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1991 *Mathematics Subject Classification*. Primary 81Q05; Secondary 35J10, 35P15, 58J05.

*Key words and phrases*. Schrödinger operators, singular interactions, discrete spectrum, geometry, approximations, asymptotic expansions, scattering, point interactions.

The research was supported in part by the Czech Academy of Sciences and Ministry of Education, Youth and Sports within the projects A100480501 and LC06002.

owls to Athens<sup>1</sup> and we refrain from doing that referring to the other articles in these proceedings, or to [BCFK06] as another rich bibliography source.

To motivate a need to look for alternative description of graph-like structures, let us observe that — despite its mathematical simplicity, beauty, and versatility — the standard quantum-graph model has also some drawbacks. In our opinion, the following two are the most important:

- the presence of *ad hoc* parameters in the boundary conditions which describe how the wave functions are matched at the graphs vertices
- the fact that particles are strictly confined to graph edges. While this is often a reasonable approximation when dealing, say, with electrons in semiconductor quantum wires, such a model neglects *quantum tunneling* which can play role once such wires are placed close to each other. One consequence is that in such a description, with the graph embedded in  $\mathbb{R}^\nu$ , spectral properties reflect the topology while the graph geometry enters only through the edge lengths, their shapes being irrelevant

A way to deal with the first problem is to regard a quantum graph as an idealization of a more “realistic” system without such ambiguities; a natural candidate for this role are various “fat graphs”. Limits of such objects when the edge width squeezes to zero were studied extensively, first in the easier Neumann-type case [FW93, KZ01, RS01, Sa01, EP05, Po06, EP07] and quite recently also in the situation with Dirichlet boundaries [Po05, MV07, CE07, Gr07]. These results give a partial answer to the first question<sup>2</sup> while the second problem remains.

Here we are going to discuss a class of quantum graph models which are free of both difficulties; the price we pay is — similarly as for the fat graphs mentioned above — that instead of ordinary differential equations we have to deal with a PDE problem. The idea is to preserve the whole Euclidean space as the configuration space and to suppose that the particle is kept in the vicinity of the graph  $\Gamma \subset \mathbb{R}^\nu$  by an attractive singular interaction. Formally such a Hamiltonian expresses as

$$(1.1) \quad -\Delta - \alpha(x)\delta(x - \Gamma)$$

with  $\alpha(x) > 0$ ; we will consider mostly the situation where the attraction is position independent,  $\alpha(x) = \alpha > 0$ . Before proceeding to a definition of such singular Schrödinger operators and discussion of their properties, let us make a few remarks.

First of all, it is clear that there is no ambiguity related to the graph vertices once  $\Gamma$  and  $\alpha$  are given. It is equally obvious that the confinement in this model takes place at negative energies only. The particle now “lives” in the whole space and can be found even at large distances from  $\Gamma$ , although with a small probability, because the complement  $\mathbb{R}^\nu \setminus \Gamma$  is the classically forbidden region. The presence of the tunneling is the reason why we dub such systems as *leaky quantum graphs*.

Schrödinger operators with interactions supported by curves and other manifolds of a lower dimension were studied already in the early nineties [BT92], and even earlier in examples with a particular symmetry [AGS87, Sha88]. A more systematic investigation motivated by the above considerations was undertaken in a last few years; it is the aim of this review to describe its results. One should

<sup>1</sup>Or, depending on your taste, coal to Newcastle, firewood to the forest, etc. As usual, one can also refer to the Bard: *to throw a perfume on the violet* (The Life and Death of King John).

<sup>2</sup>An alternative approach is to keep the graph fixed and to approximate the vertex coupling through suitably scaled families of regular or singular interactions — see [Ex96, CE04, ETu07].

stress, however, that such mathematical structures can be studied also from other points of view. A prominent example comes from studies of high contrast optical systems used to model photonic crystals — see, e.g. [FKu96, KK98] — which in a suitable approximation yield an analogue of the spectral problem for the operator (1.1); the two differ only by the physical interpretation, the roles of the coupling and spectral parameters being switched. Other problems of this type arise when one deals with contact interactions of several one-dimensional particles [Du07].

The material we are going to review is relatively extensive. We will take care, of course, to explain properly all the notions and the results. On the other hand, proofs will be mostly sketched, however, we will always give references to original papers where the particular complete argument can be found. Let us finally remark that the subject reviewed here cannot be regarded as close, on the contrary, there are many open questions. We devote to them the closing section, and the author of this survey can only hope that his reader will take this problem list as a challenge and a program which will keep him or her busy for some time.

## 2. Leaky graph Hamiltonians

**2.1. Quadratic forms and boundary conditions.** The Hamiltonians we are interested in are generalized Schrödinger operators with a singular interaction supported by a graph-like  $\Gamma$  which is a zero measure set in  $\mathbb{R}^\nu$ . We will use facts about such operators derived, in particular, in [BEKŠ94] specifying them to our present purpose. Let us first suppose that the configuration space dimension  $\nu = 2$  and the coupling “strength” is constant on the interaction support.

To begin with, let us show how such a singular operator can be defined generally through the associated quadratic form. Consider a positive Radon measure  $m$  on  $\mathbb{R}^2$  and a number  $\alpha > 0$  such that

$$(2.1) \quad (1 + \alpha) \int_{\mathbb{R}^2} |\psi(x)|^2 dm(x) \leq a \int_{\mathbb{R}^2} |\nabla\psi(x)|^2 dx + b \int_{\mathbb{R}^2} |\psi(x)|^2 dx$$

holds for all  $\psi \in \mathcal{S}(\mathbb{R}^2)$  and some  $a < 1$  and  $b$ . The map  $I_m$  defined by  $I_m\psi = \psi$  on the Schwartz space  $\mathcal{S}(\mathbb{R}^2)$  extends by density uniquely to

$$(2.2) \quad I_m : W^{1,2}(\mathbb{R}^2) \rightarrow L^2(m) := L^2(\mathbb{R}^2, m) ;$$

for brevity the same symbol is used for a continuous function and the corresponding equivalence classes in both  $L^2(\mathbb{R}^2)$  and  $L^2(m)$ . The inequality (2.1) extends to  $W^{1,2}(\mathbb{R}^2)$  with  $\psi$  replaced by  $I_m\psi$  at the left-hand side. The quadratic form

$$(2.3) \quad \mathcal{E}_{-\alpha m}[\psi] := \int_{\mathbb{R}^2} |\nabla\psi(x)|^2 dx - \alpha \int_{\mathbb{R}^2} |(I_m\psi)(x)|^2 dm(x)$$

is defined on  $W^{1,2}(\mathbb{R}^2)$ ; it is straightforward to check [BEKŠ94] that under the condition (2.1) this form is closed and below bounded, with  $C_0^\infty(\mathbb{R}^2)$  as a core, and consequently, it is associated with a unique self-adjoint operator denoted as  $\hat{H}_{-\alpha m}$ . A sufficient condition for the inequality (2.1) to be valid is that the measure  $m$  belongs to the generalized Kato class, i.e.

$$(2.4) \quad \lim_{\epsilon \rightarrow 0} \sup_{x \in \mathbb{R}^2} \int_{B_\epsilon(x)} |\ln|x-y|| dm(y) = 0,$$

where  $B_\epsilon(x)$  is the ball of radius  $\epsilon$  centered at  $x$ . In such a case, moreover, any positive number can be chosen as  $a$ . So far the construction has been general

and involved also regular Schrödinger operators. Suppose now that  $m$  is the Dirac measure supported by a graph  $\Gamma \subset \mathbb{R}^2$  which has the following properties:

- (g1) **edge smoothness:** each edge  $e_j \in \Gamma$  is a graph of  $C^1$  function  $\gamma_j : I_j \rightarrow \mathbb{R}^2$  where  $I_j$  is an interval (finite, semi-infinite, or the whole  $\mathbb{R}$ ). Moreover, without loss of generality we may suppose that edges are parametrized by the arc length,  $|\dot{\gamma}_j(s)| = 1$ .
- (g2) **cusp absence:** at the vertices of  $\Gamma$  the edges meet at nonzero angles.
- (g3) **local finiteness:** each compact subset of  $\mathbb{R}^2$  contains at most a finite number of edges and vertices of  $\Gamma$ .

The last assumption allows us to extend Theorem 4.1 of [BEKŠ94] applying it to the Dirac measure supported by the graph. More exactly, we consider the measure

$$(2.5) \quad m_\Gamma : m_\Gamma(M) = \ell_1(M \cap \Gamma)$$

for any Borel  $M \subset \mathbb{R}^2$ , where  $\ell_1$  is the one-dimensional Hausdorff measure given in our case by the edge-arc length. Such a straightforward extension implies that  $m \equiv m_\Gamma$  satisfies the condition (2.1) and gives thus rise to the appropriate operator  $\hat{H}_{-\alpha m}$ ; to make the notation explicit we will employ for it in the following the symbol  $H_{\alpha, \Gamma}$ . This is one way how to give meaning to the formal expression (1.1).

An alternative is to use boundary conditions. Consider the operator acting as

$$\left( \dot{H}_{\alpha, \Gamma} \psi \right) (x) = -(\Delta \psi)(x), \quad x \in \mathbb{R}^2 \setminus \Gamma,$$

on any function  $\psi$  which belongs to  $W^{2,2}(\mathbb{R}^2 \setminus \Gamma)$ , is continuous at each edge  $e_j \in \Gamma$  with the normal derivatives having there a jump, namely

$$(2.6) \quad \frac{\partial \psi}{\partial n_+}(x) - \frac{\partial \psi}{\partial n_-}(x) = -\alpha \psi(x), \quad x \in \text{int } e_j;$$

since the edges are smooth by assumption the normal vector exists at each inner point of an edge. In the same way as in [BEKŠ94] one can check that  $\dot{H}_{\alpha, \Gamma}$  is e.s.a., and moreover, by Green's formula it reproduces the form (2.3) on its core, so its closure may be identified with  $H_{\alpha, \Gamma}$  defined above.

REMARKS 2.1. (i) The above definitions easily extend to the situation with the singular interaction strength  $\alpha(s)$  varying along the edges provided the corresponding function  $\alpha : \Gamma \rightarrow \mathbb{R}_+$  is sufficiently regular.

(ii) In a similar way one can define operators corresponding to the formal expression (1.1) for a generalized “graph” whose edges are  $(\nu - 1)$ -dimensional manifolds in  $\mathbb{R}^\nu$  satisfying suitable regularity conditions analogous to (g1)–(g3).

**2.2. Regular potential approximation.** As we shall see below the operators  $H_{\alpha, \Gamma}$  represent a reasonably general class of systems for which various properties can be derived. One can ask nevertheless whether this is not again a too idealized model. Before proceeding further we want to show that  $H_{\alpha, \Gamma}$  can be regarded as weak-coupling approximation to a class of regular Schrödinger operators; for simplicity we restrict ourselves to graphs with a single infinite edge.

Let  $\Gamma$  be a curve described by a  $C^2$  function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ . Then we are able to define the signed curvature  $k(s) := (\dot{\gamma}_1 \ddot{\gamma}_2 - \dot{\gamma}_2 \ddot{\gamma}_1)(s)$ ; we assume that it is bounded,  $|k(s)| < c_+$  for some  $c_+ > 0$  and all  $s \in \mathbb{R}$ . Moreover, we suppose that  $\Gamma$  has neither self-intersections nor “near-intersections”, i.e. that there is a  $c_- > 0$  such that  $|\gamma(s) - \gamma(s')| \geq c_-$  for any  $s, s'$  with  $|s - s'| \geq c_-$ . Then we can define in the vicinity of  $\Gamma$  the standard locally orthogonal system of coordinates [EŠ89],

i.e. the pairs  $(s, u)$  where  $u$  is the (signed) normal distance from  $\Gamma$  and  $s$  is the arc-length coordinate of the point of  $\Gamma$  where the normal  $n(s)$  is taken; the system is unique in the strip neighborhood  $\Sigma_\epsilon := \{x(s, u) : (s, u) \in \Sigma_\epsilon^0\}$ , where

$$(2.7) \quad x(s, u) := \gamma(s) + n(s)u$$

and  $\Sigma_\epsilon^0 := \{(s, u) : s \in \mathbb{R}, |u| < \epsilon\}$  as long as the condition  $2\epsilon < c_-$  is valid.

With these prerequisites we can construct the approximating family. Given  $W \in L^\infty((-1, 1))$ , we define for all  $\epsilon < \frac{1}{2}c_-$  the transversally scaled potential,

$$(2.8) \quad V_\epsilon(x) := \begin{cases} 0 & \dots & x \notin \Sigma_\epsilon \\ -\frac{1}{\epsilon} W\left(\frac{u}{\epsilon}\right) & \dots & x \in \Sigma_\epsilon \end{cases}$$

and put

$$(2.9) \quad H_\epsilon(W, \gamma) := -\Delta + V_\epsilon.$$

The operators  $H_\epsilon(W, \gamma)$  are obviously self-adjoint on  $D(-\Delta) = W^{2,2}(\mathbb{R}^2)$  and we have the following approximation result:

**THEOREM 2.2.** *Under the stated assumptions,  $H_\epsilon(W, \Gamma) \rightarrow H_{\alpha, \Gamma}$  as  $\epsilon \rightarrow 0$  in the norm-resolvent sense, where  $\alpha := \int_{-1}^1 W(t) dt$ .*

**SKETCH OF THE PROOF.** One has to compare the resolvents, that of  $H_{\alpha, \Gamma}$  given below and the Birman-Schwinger expression of  $(H_\epsilon(W, \gamma) - k^2)^{-1}$ . Both are explicit integral operators and their difference can be treated in a way similar to that used in the squeezing approximation of the one-dimensional  $\delta$  interaction — see, e.g., [AGHH04] — a full account of the argument can be found in [EI01].  $\square$

**2.3. The resolvent.** As usual the spectral and scattering properties are encoded in the resolvent and our first task is to find an explicit expression for this operator. We will employ an analogue of the Birman-Schwinger formula for our singular case. If  $k^2$  belongs to the resolvent set of  $H_{\alpha, \Gamma}$  we put  $R_{\alpha, \Gamma}^k := (H_{\alpha, \Gamma} - k^2)^{-1}$ . We look for the difference of this operator and the free resolvent  $R_0^k$  which is for  $\text{Im } k > 0$  an integral operator with the kernel

$$(2.10) \quad G_k(x-y) = \frac{i}{4} H_0^{(1)}(k|x-y|).$$

To this aim we need embedding operators associated with  $R_0^k$ . Let  $\mu, \nu$  be arbitrary positive Radon measures on  $\mathbb{R}^2$  with  $\mu(x) = \nu(x) = 0$  for any  $x \in \mathbb{R}^2$ . By  $R_{\nu, \mu}^k$  we denote the integral operator from  $L^2(\mu) := L^2(\mathbb{R}^2, \mu)$  to  $L^2(\nu)$  with the kernel  $G_k$ , i.e.

$$R_{\nu, \mu}^k \phi = G_k * \phi \mu$$

holds  $\nu$ -a.e. for all  $\phi \in D(R_{\nu, \mu}^k) \subset L^2(\mu)$ . In our case the two measures will be  $m \equiv m_\Gamma$  introduced by (2.5) and the Lebesgue measure  $dx$  on  $\mathbb{R}^2$  in different combinations, which simply means that one or both variables in the kernel (2.10) are restricted to  $\Gamma$ . Using this notation we can state the following result:

**PROPOSITION 2.3.** (i) *There is a  $\kappa_0 > 0$  such that the operator  $I - \alpha R_{m, m}^{i\kappa}$  on  $L^2(m)$  has a bounded inverse for any  $\kappa \geq \kappa_0$ .*

(ii) *Let  $\text{Im } k > 0$ . Suppose that  $I - \alpha R_{m, m}^k$  is invertible and the operator*

$$R^k := R_0^k + \alpha R_{dx, m}^k [I - \alpha R_{m, m}^k]^{-1} R_{m, dx}^k$$

from  $L^2(\mathbb{R}^2)$  to  $L^2(\mathbb{R}^2)$  is everywhere defined. Then  $k^2$  belongs to  $\rho(H_{\alpha,\Gamma})$  and  $(H_{\alpha,\Gamma} - k^2)^{-1} = R^k$ .

(iii)  $\dim \ker(H_{\alpha,\Gamma} - k^2) = \dim \ker(I - \alpha R_{m,m}^k)$  for any  $k$  with  $\text{Im } k > 0$ .

(iv) an eigenfunction of  $H_{\alpha,\Gamma}$  associated with such an eigenvalue  $k^2$  expresses as

$$\psi(x) = \int_0^L R_{dx,m}^k(x,s)\phi(s) ds,$$

where  $\phi$  is the corresponding eigenfunction of  $\alpha R_{m,m}^k$  with the eigenvalue one.

SKETCH OF THE PROOF. The result, which is in fact valid for any operator  $\hat{H}_{-\alpha m}$  of Sec. 2.1, is obtained by verifying the Birman-Schwinger (BS) formula in our singular setting. The procedure requires some care; a full account concerning the claims (i)–(iii) can be found in [BEKŠ94], for (iv) see [Pos04].  $\square$

**2.4. The case of codimension two.** The second one of the above constructions of  $H_{\alpha,\Gamma}$  can also be rephrased in the following way: we first restrict the Laplacian to a symmetric operator defined on function which vanish in the vicinity of  $\Gamma$ , and afterwards we choose a particular self-adjoint extension specified by the condition (2.6). It follows from general properties of partial differential operators [He89] that a similar construction is possible also in higher dimensions as long as  $\text{codim } \Gamma \leq 3$ . Since we want to stick to cases of physical interest, we will mention here only graphs whose edges are curves in  $\mathbb{R}^3$ .

An analogue of the form definition (2.3) does not work in this situation and we have to rely on boundary conditions which are, however, more complicated than (2.6). The difference is of a local character, thus we restrict ourself to the simplest situation when  $\Gamma$  is a single infinite curve described by a  $C^2$  function  $\gamma(s) : \mathbb{R} \rightarrow \mathbb{R}^3$  without self-intersections and such that  $|\dot{\gamma}(s)| = 1$ . In view of the smoothness assumption the curve possesses locally Frenet's frame, i.e. the triple  $(t(s), b(s), n(s))$  of the tangent, binormal and normal vectors; we assume its global existence<sup>3</sup>. For a fixed nonzero  $\rho \in \mathbb{R}^2$  we define the “shifted” curve  $\Gamma_\rho$  as the graph of the function

$$\gamma_\rho(s) := \gamma(s) + \rho_1 b(s) + \rho_2 n(s);$$

the distance between the two is obviously  $r := |\rho|$ . If we suppose in addition that  $\Gamma$  does not have “near-intersections” as in Sec. 2.2 then clearly  $\Gamma_\rho \cap \Gamma = \emptyset$  holds provided  $r$  is small enough. Since any function  $f \in W_{loc}^{2,2}(\mathbb{R}^3 \setminus \Gamma)$  is continuous on  $\mathbb{R}^3 \setminus \Gamma$  its restriction to  $\Gamma_\rho$  is then well defined; we denote it as  $f|_{\Gamma_\rho}$ , in fact, we can regard  $f|_{\Gamma_\rho}$  as a distribution from  $D'(\mathbb{R})$  with the parameter  $\rho$ . We denote by  $\mathcal{D}$  the set of functions  $f \in W_{loc}^{2,2}(\mathbb{R}^3 \setminus \Gamma) \cap L^2(\mathbb{R}^3)$  such that the following limits

$$\begin{aligned} \Xi(f)(s) &:= -\lim_{r \rightarrow 0} \frac{1}{\ln r} f|_{\Gamma_\rho}(s), \\ \Omega(f)(s) &:= \lim_{r \rightarrow 0} f|_{\Gamma_\rho}(s) + \Xi(f)(s) \ln r, \end{aligned}$$

exist a.e. in  $\mathbb{R}$ , are independent of the direction  $\frac{1}{r}\rho$ , and define functions from  $L^2(\mathbb{R})$ ; the limits here are understood in the sense of the  $D'(\mathbb{R})$  topology. Now we

<sup>3</sup>This is true, for instance, if  $\dot{\gamma}$  does not vanish. For curves having isolated straight segments a suitable coordinate system can be obtained by patching local Frenet systems together, possibly with a rotation – cf. a discussion on that point in [EK04a].

are able to define the singular Schrödinger operator in the present case: it acts as

$$(2.11) \quad H_{\alpha,\Gamma} f = -\Delta f \quad \text{for } x \in \mathbb{R}^3 \setminus \Gamma$$

on the domain

$$(2.12) \quad D(H_{\alpha,\Gamma}) := \{g \in \mathcal{D} : 2\pi\alpha\Xi(g)(s) = \Omega(g)(s)\}.$$

In this way we get a well-defined Hamiltonian which we seek **[EK02]**:

**THEOREM 2.4.** *Under the stated assumptions the operator  $H_{\alpha,\Gamma}$  is self-adjoint.*

As in the case of codimension one and the conditions (2.6) the above definition has a natural meaning as a point interaction in the normal plane to  $\Gamma$ .

The proof of Theorem 2.4 is technically more involved and we restrict ourselves to a few remarks referring to **[EK02]** for the full exposition. The argument is in a sense opposite to the previous considerations. It is based on an abstract analogue of Proposition 2.3 proved in **[Pos01]**, see also **[Pos04]**, which shows existence of a self-adjoint operator with the resolvent of the appropriate form, after that one verifies that this operator coincides with the above  $H_{\alpha,\Gamma}$ .

**REMARK 2.5.** Since the mentioned resolvent formula analogous to that of Proposition 2.3 will be used in the following we will describe it at least briefly. It contains again traces of the free resolvent which is now given by

$$(2.13) \quad G_k(x-y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

however, the use of Posilicano's abstract result requires to interpret the embedding operators involved not as maps between  $L^2$  spaces but rather the last factor  $R_\Gamma^k$  as  $W^{2,2}(\mathbb{R}^3) \rightarrow L^2(\mathbb{R})$  and its counterpart  $\check{R}_\Gamma^k$  as the Banach space dual to  $\overline{R_\Gamma^k}$ . Then

$$(2.14) \quad R^k = R_0^k - \check{R}_\Gamma^k (Q^k - \alpha)^{-1} R_\Gamma^k,$$

where the modified position of  $\alpha$  in this formula corresponds to the usual convention about the coupling parameter for two-dimensional point interactions **[AGHH04]** reflected in (2.12); roughly speaking, it is an inverse of the one appearing in (2.6). The operator  $Q^k$  is the counterpart to  $R_{m,m}^k$  of Proposition 2.3 but we use on purpose a different symbol to stress that we cannot write it simply as an integral operator and a renormalization is needed, cf. **[EK02]** for more details.

### 3. Geometrically induced properties

We said in the opening that even if we think about  $\Gamma$  in the usual quantum graph model as embedded in  $\mathbb{R}^d$ , the shapes of the edges do not influence the spectrum. Leaky graphs are different as one can illustrate in various ways.

**3.1. Bound states due to non-straightness.** Consider again a leaky graph in  $\mathbb{R}^2$ . If  $\Gamma = \Gamma_0$  is a straight line corresponding to  $\gamma_0(s) = as + b$  for some  $a, b \in \mathbb{R}^2$  with  $|a| = 1$ , we can separate variables and show that

$$(3.1) \quad \sigma(H_{\alpha,\Gamma_0}) = \left[ -\frac{1}{4}\alpha^2, \infty \right)$$

is purely absolutely continuous. We are going to show that a bend or deformation produces, within a wide class of curves  $\Gamma$ , a non-void discrete spectrum. To be specific we assume that the generating function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  is continuous and

piecewise  $C^1$  (or, in terms of the assumption (g1)–(g3), a graph which may have vertices but no branchings) satisfying the following conditions<sup>4</sup>:

- (a1) **a lower distance bound:** there is  $c \in (0, 1)$  such that  $|\gamma(s) - \gamma(s')| \geq c|s - s'|$ . In particular,  $\Gamma$  has no cusps and self-intersections, and its possible asymptotes are not parallel to each other.
- (a2) **asymptotic straightness:** there are positive  $d, \mu > \frac{1}{2}$ , and  $\omega \in (0, 1)$  such that the inequality

$$1 - \frac{|\gamma(s) - \gamma(s')|}{|s - s'|} \leq d [1 + |s + s'|^{2\mu}]^{-1/2}$$

holds true in the sector  $S_\omega := \{(s, s') : \omega < \frac{s}{s'} < \omega^{-1}\}$ .

- (a3) **non-triviality:** we excluded the case  $\Gamma = \Gamma_0$ . Recall that

$$|\gamma(s) - \gamma(s')| \leq |s - s'|$$

holds for any  $s, s' \in \mathbb{R}$ , hence we request in other words that the last inequality is sharp at least for some  $s, s' \in \mathbb{R}$ .

Then we have the following result:

**THEOREM 3.1.** *Let  $\alpha > 0$  and suppose that  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  satisfies the above assumptions. Then the essential spectrum is the same as for the straight line,  $\sigma_{\text{ess}}(H_{\alpha, \Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ , but  $H_{\alpha, \Gamma}$  has at least one isolated eigenvalue below  $-\frac{1}{4}\alpha^2$ .*

**SKETCH OF THE PROOF.** Observe first that in view of (a2) it is not difficult to construct a Weyl sequence to  $H_{\alpha, \Gamma}$  showing that any non-negative number belongs to  $\sigma_{\text{ess}}$ . To deal with the negative part, we use the generalized BS principle of Proposition 2.3; the idea is to treat the difference between the operator  $\mathcal{R}_{\alpha, \Gamma}^\kappa := \alpha R_{m, m}^{i\kappa}$  on  $L^2(\mathbb{R})$  and its counterpart corresponding to  $\Gamma_0$  as a perturbation. The integral kernel of the operator is

$$\mathcal{R}_{\alpha, \Gamma}^\kappa(s, s') = \frac{\alpha}{2\pi} K_0(\kappa|\gamma(s) - \gamma(s')|),$$

where  $K_0$  is the Macdonald function; for  $\Gamma = \Gamma_0$  one has to replace  $|\gamma(s) - \gamma(s')|$  by  $|s - s'|$ . In the last named case the operator is of convolution type and using Fourier transformation it is easy to check that its spectrum is absolutely continuous covering the interval  $[0, \alpha/2\kappa]$  in correspondence with (3.1).

The key observation is that the kernel of  $\mathcal{D}_\kappa := \mathcal{R}_{\alpha, \Gamma}^\kappa - \mathcal{R}_{\alpha, \Gamma_0}^\kappa$  satisfies

$$(3.2) \quad \mathcal{D}_\kappa(s, s') := \frac{\alpha}{2\pi} \left( K_0(\kappa|\gamma(s) - \gamma(s')|) - K_0(\kappa|s - s'|) \right) \geq 0$$

in view of (a1) and the monotonicity of  $K_0$ , the inequality being strict for at least some values of the variables  $s, s'$ . We shall then argue in three steps:

- (1) a variational argument in combination with (3.2) shows that the spectrum is “pushed up” by the perturbation,  $\sup \sigma(\mathcal{R}_{\alpha, \Gamma}^\kappa) > \frac{\alpha}{2\kappa}$  if  $\Gamma$  is not straight.
- (2) in view of (a2)  $\mathcal{D}_\kappa$  is Hilbert-Schmidt for  $\mu > \frac{1}{2}$ , and therefore compact.
- (3) the map  $\kappa \mapsto \mathcal{R}_{\alpha, \Gamma}^\kappa$  is operator-norm continuous and  $\mathcal{R}_{\alpha, \Gamma}^\kappa \rightarrow 0$  as  $\kappa \rightarrow \infty$ .

<sup>4</sup>If  $\gamma \in C^2$  we have a sufficient condition for (a2) in terms of the signed curvature introduced in Sec. 2.2: it is valid with  $\mu > \frac{1}{2}$  if  $k(s) = \mathcal{O}(|s|^{-\beta})$  with  $\beta > \frac{5}{4}$  as  $|s| \rightarrow \infty$ , cf. [E101].

The compactness of  $\mathcal{D}_\kappa$  implies, in particular, in combination with Proposition 2.3 the claim about the negative part of the essential spectrum.

The discrete spectrum part can be also derived from the generalized BS principle. It follows from the above claims that there are spectral points of  $\mathcal{R}_{\alpha,\Gamma}^\kappa$  above  $\alpha/2\kappa$  and they cannot be anything but eigenvalues of a finite multiplicity. Moreover, every such eigenvalue depends continuously on  $\kappa$  and tends to zero as  $\kappa \rightarrow \infty$ , hence it crosses one at a value  $\kappa_0 > \frac{1}{2}\alpha$  giving rise to the sought eigenvalue of the operator  $H_{\alpha,\Gamma}$ ; for details of the argument see [EI01].  $\square$

It may seem that the result covers only a rather particular class of graphs. Using the minimax principle, however, we arrive at the following easy consequence:

**COROLLARY 3.2.** *Suppose that  $\Gamma$  has a subgraph in the form of an infinite curve satisfying the assumptions of the theorem, and  $\sigma_{\text{ess}}(H_{\alpha,\Gamma}) = [-\frac{1}{4}\alpha^2, \infty)$ , then the discrete spectrum of  $H_{\alpha,\Gamma}$  is non-empty.*

It is important that the assumption about preservation of the essential spectrum can be often verified easily, for instance, in the situation when  $\Gamma$  has outside a compact a finite number of straight edges separated by non-trivial wedges.

**3.2. An example: leaky star graphs.** To illustrate the last made claim let us investigate in more detail a particular class of such graphs, namely the situation when  $\Gamma$  is of a *star shape*. Given an integer  $N \geq 2$ , consider an  $(N-1)$ -tuple  $\beta = \{\beta_1, \dots, \beta_{N-1}\}$  of positive numbers such that

$$\beta_N := 2\pi - \sum_{j=1}^{N-1} \beta_j > 0.$$

Denote  $\vartheta_j := \sum_{i=1}^j \beta_i$  and  $\vartheta_0 := 0$ . Let  $L_j$  be the radial half-line with the endpoint at the origin,  $L_j := \{x \in \mathbb{R}^2 : \arg x = \vartheta_j\}$ , naturally parametrized by its arc length  $s = |x|$ . These half-lines will be the edges of  $\Gamma \equiv \Gamma_\beta := \bigcup_{j=0}^{N-1} L_j$  to which we associate<sup>5</sup> the Hamiltonian  $H_N(\beta) := H_{\alpha,\Gamma_\beta}$ . Trivial examples are

- (1)  $H_2(\pi)$  corresponding to a straight line which has obviously a purely a.c. spectrum covering the interval  $[-\frac{1}{4}\alpha^2, \infty)$ .
- (2)  $H_4(\beta_s)$  with  $\beta_s = \{\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\}$  corresponding to cross-shaped  $\Gamma$  allows again a separation of variables. The a.c. part of its spectrum is the same as above, and in addition, there is a single isolated eigenvalue  $-\frac{1}{2}\alpha^2$  corresponding to the eigenfunction  $(2\alpha)^{-1}e^{-\alpha(|x|+|y|)/2}$ .

Star-shaped graphs have the property indicated above:

**PROPOSITION 3.3.**  $\sigma_{\text{ess}}(H_N(\beta)) = [-\frac{1}{4}\alpha^2, \infty)$  holds for any  $N$  and  $\beta$ .

**SKETCH OF THE PROOF.** The inequality  $\inf \sigma_{\text{ess}}(H_N(\beta)) \geq -\frac{1}{4}\alpha^2$  is obtained by Neumann bracketing dissecting the plane outside a compact into semi-infinite strips with  $L_j$  in the middle and “empty” wedges. The fact that  $[-\frac{1}{4}\alpha^2, \infty)$  belongs to the spectrum is checked by means of a family of Weyl sequences, cf. [EN03].  $\square$

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<sup>5</sup>Properties of  $H_N(\beta)$  depend on the order of the angles, however, operators related by a cyclic permutation are unitarily equivalent by an appropriate rotation of the plane.

By Corollary 3.2  $\sigma_{\text{disc}}(H_N(\beta))$  is nonempty unless  $N = 2$  and  $\beta = \pi$ . Using direct methods one can establish various other properties of the discrete spectrum<sup>6</sup>.

**PROPOSITION 3.4.** *Fix  $N$  and a positive integer  $n$ . If at least one of the angles  $\beta_j$  is small enough,  $\#\sigma_{\text{disc}}(H_N(\beta)) \geq n$ . In particular, the number of bound states can exceed any fixed integer for  $N$  large enough.*

**SKETCH OF THE PROOF.** By the minimax principle it is sufficient to check the claim for  $H_2(\beta)$ . We choose the coordinate system in such a way that the two “arms” correspond to  $\arg \theta = \pm\beta/2$  and employ trial functions of the form  $\Phi(x, y) = f(x)g(y)$  supported in the strip  $L \leq x \leq 2L$ , with  $f \in C^2$  satisfying  $f(L) = f(2L) = 0$ , and

$$g(y) = \begin{cases} 1 & \dots & |y| \leq 2d \\ e^{-\alpha(|y|-2d)} & \dots & |y| \geq 2d \end{cases}$$

with  $d := L \tan(\beta/2)$ . Evaluating the quadratic form of  $H_2(\beta)$  and using minimax principle again we get the result, see [EN03] for details.  $\square$

**REMARK 3.5.** The above variational estimate also shows that the bound state number for a sharply broken line is roughly proportional to the inverse angle,

$$n \gtrsim \frac{3^{3/2}}{8\pi\sqrt{5}} \beta^{-1}$$

as  $\beta \rightarrow 0$ . This can be regarded as an expected result, since the number is given by the length of the effective potential well which exists in the region where the two lines are so close that they roughly double the depth of the transverse well.

Let us see how the BS equation looks like explicitly for star graphs. Define

$$(3.3) \quad d_{ij}(s, s') \equiv d_{ij}^\beta(s, s') = \sqrt{s^2 + s'^2 - 2ss' \cos |\vartheta_j - \vartheta_i|}$$

with  $\vartheta_j - \vartheta_i = \sum_{l=i+1}^j \beta_l$ , in particular,  $d_{ii}(s, s') = |s - s'|$ . By  $\mathcal{R}_{ij}^\kappa(\beta) = \mathcal{R}_{ji}^\kappa(\beta)$  we denote the operator  $L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$  with the integral kernel

$$\mathcal{R}_{ij}^\kappa(s, s'; \beta) := \frac{\alpha}{2\kappa} K_0(\kappa d_{ij}(s, s')) ;$$

then the (discrete part of the) spectral problem for the operator  $H_N(\beta)$  is by Proposition 2.3 equivalent to the matrix integral-operator equation

$$(3.4) \quad \sum_{j=1}^N (\mathcal{R}_{ij}^\kappa(\beta) - \delta_{ij}I) \phi_j = 0, \quad i = 1, \dots, N,$$

on  $\bigoplus_{j=1}^N L^2(\mathbb{R}^+)$ . Notice that the “entries” of the above kernel have a monotonicity property,  $\mathcal{R}_{ij}^\kappa(\beta) > \mathcal{R}_{ij}^\kappa(\beta')$  if  $|\vartheta_j - \vartheta_i| < |\vartheta'_j - \vartheta'_i|$ . This fact has the following easy consequence [EN03]:

**PROPOSITION 3.6.** *Each isolated eigenvalue  $\lambda_n(\beta)$  of  $H_2(\beta)$  is an increasing function of the angle  $\beta$  between the two half-lines in  $(0, \pi)$ .*

<sup>6</sup>Variational methods can be also used to establish the existence of discrete spectrum in some range of parameters, see the paper [BEW07] in this volume.

**3.3. Higher dimensions.** If we restrict ourselves to the physically interesting case of three-dimensional configuration space, there are two possible ways how to extend the above results. One refers to the situation when the interaction is supported by a surface. Here unfortunately only a particular result is known to the date and we will mention it at the appropriate place, in Sec. 4.2 below.

Let us thus consider the second possibility when  $\Gamma$  is an infinite piecewise  $C^1$  curve in  $\mathbb{R}^3$ . The argument is similar to that of the previous section but it needs care due to the more singular character of the interaction. If  $\Gamma = \Gamma_0$  the spectrum of  $H_{\alpha,\Gamma}$  is found by separation of variables; the known properties of the two-dimensional point interaction [AGHH04] imply that

$$\sigma(H_{\alpha,\Gamma}) = \sigma_{\text{ac}}(H_{\alpha,\Gamma}) = [\zeta_\alpha, \infty),$$

where  $\zeta_\alpha = -4e^{2(-2\pi\alpha + \psi(1))}$  with  $-\psi(1) \approx 0.577$  being Euler's number. To rephrase it in terms of the BS operator  $\mathcal{Q}_0^\kappa := Q_0^{i\kappa}$  notice that the latter equals  $\mathcal{T}_\kappa + \ln 2 + \psi(1)$  where  $\mathcal{T}_\kappa$  in the momentum representation acts as multiplication<sup>7</sup> by  $\ln(p^2 + \kappa^2)^{1/2}$ . Consequently, the spectrum of  $\mathcal{Q}_0^\kappa$  is purely absolutely continuous and equal to  $(-\infty, s_\kappa]$  where  $s_\kappa := \frac{1}{2\pi}(\psi(1) - \ln(\kappa/2))$ .

To proceed we need assumptions about the curve  $\Gamma$ . We retain (a1) and (a3) from the previous section, while (a2) will be replaced by

**(a2')** *ASLS*: there are positive  $d$ ,  $\mu > \frac{1}{2}$ , and  $\omega \in (0, 1)$  such that

$$1 - \frac{|\gamma(s) - \gamma(s')|}{|s - s'|} \leq d \frac{|s - s'|}{(1 + |s - s'|)(1 + (s^2 + s'^2)^\mu)^{1/2}}$$

holds true in the sector  $S_\omega$ , the same as before.

In addition to the asymptotic straightness<sup>8</sup> we require newly some local smoothness of the curve. Now we can follow proof of Theorem 3.1 step by step; after checking that the essential spectrum is preserved, and denoting  $\mathcal{Q}^\kappa := Q^{i\kappa}$ , we prove that

(1)  $\sup \sigma(\mathcal{Q}^\kappa) > s_\kappa$  by a variational argument using the sign definiteness,

$$\mathcal{D}_\kappa(s, s') = G_{i\kappa}(\gamma(s) - \gamma(s')) - G_{i\kappa}(s - s') \geq 0,$$

with a sharp inequality at least for some values of the variables  $s, s'$ .

(2) in view of (a2')  $\mathcal{D}_\kappa$  is Hilbert-Schmidt for  $\mu > \frac{1}{2}$ , hence compact, and the corresponding norm  $\|\mathcal{D}_\kappa\|_{\text{HS}}$  is uniformly bounded w.r.t.  $\kappa \geq |\zeta_\alpha|^{1/2}$ .

(3) the function  $\kappa \rightarrow \mathcal{Q}^\kappa$  is operator-norm continuous in  $(|\zeta_\alpha|^{1/2}, \infty)$  and  $\mathcal{Q}^\kappa \rightarrow -\infty$  holds as  $\kappa \rightarrow \infty$ .

Working out details of this scheme [EK02] we arrive at the following conclusion:

**THEOREM 3.7.** *Fix  $\alpha > 0$  and suppose that the generating function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$  of  $\Gamma$  satisfies the stated assumptions. Then  $\sigma_{\text{ess}}(H_{\alpha,\Gamma}) = [\zeta_\alpha, \infty)$ , and the operator  $H_{\alpha,\Gamma}$  has at least one isolated eigenvalue in the interval  $(-\infty, \zeta_\alpha)$ .*

Let us remark that the strengthened hypothesis in (a2') is not needed to prove the existence of the geometrically induced spectrum but rather its character; due to the more strongly singular character of the interaction in the codimension two case the local smoothness is required to guarantee its discreteness.

<sup>7</sup>Comparing to  $(2\pi)^{1/2}(p^2 + \kappa^2)^{-1/2}$  in the codimension one case. This is why the kernel of  $\mathcal{Q}_0^\kappa$  makes sense as a distribution only and a renormalization mentioned in Remark 2.5 is needed.

<sup>8</sup>For  $C^2$  smooth curves we have a sufficient condition for the appropriate large-distance behavior analogous to that mentioned in the footnote to assumption (a2).

**3.4. Geometric perturbations.** Let us turn to another way in which the leaky character of our graphs is manifested. Consider a graph  $\Gamma \subset \mathbb{R}^\nu$  with two edges the endpoints of which are close to each other; we can think of this situation as of a single edge having a hiatus. In the standard quantum graph setting it is only the topology which matters, either the two edges are connected or not. Here, in contrast, the distance of the two endpoints plays a role.

Let us first analyze the general codimension one situation, the  $\nu$ -dimensional Schrödinger operators with a  $\delta$ -interaction supported by a  $(\nu - 1)$ -dimensional smooth surface having a “puncture”. We are going to show, formally speaking, that up to an error term the eigenvalue shift resulting from removing an  $\epsilon$ -neighborhood of a surface point is the same as that of adding a repulsive  $\delta$  interaction at this point, with the coupling constant proportional to the puncture “area”.

Let  $\Gamma$  be a compact  $C^r$ -smooth surface in  $\mathbb{R}^\nu$  with  $r \geq \frac{1}{2}\nu$ ; without loss of generality we may suppose that it contains the origin. Let further  $\{\mathcal{P}_\epsilon\}_{\epsilon \geq 0}$  be a family of subsets of  $\Gamma$  which obeys the following requirements:

**(p1) measurability:**  $\mathcal{P}_\epsilon$  is measurable with respect to the  $(\nu - 1)$ -dimensional Lebesgue measure on  $\Gamma$  for any  $\epsilon$  small enough.

**(p2) shrinking:**  $\sup_{x \in \mathcal{P}_\epsilon} |x| = \mathcal{O}(\epsilon)$  as  $\epsilon \rightarrow 0$ .

Consider the operators  $H_{\alpha, \Gamma}$  and  $H_{\alpha, \Gamma_\epsilon}$  corresponding to  $\Gamma_\epsilon := \Gamma \setminus \mathcal{P}_\epsilon$  defined as in Sec. 2.1. Since  $\Gamma_\epsilon$  is bounded, we have

$$\sigma_{\text{ess}}(H_{\alpha, \Gamma_\epsilon}) = [0, \infty) \quad \text{and} \quad \#\sigma_{\text{disc}}(H_{\alpha, \Gamma}) < \infty.$$

By the minimax principle there is a unique  $\alpha^* \geq 0$  such that  $\sigma_{\text{disc}}(H_{\alpha, \Gamma})$  is non-empty if  $\alpha > \alpha^*$  while the reverse is true for  $\alpha \leq \alpha^*$ ; it is not difficult to check that  $\alpha^* = 0$  when  $\nu = 2$  and  $\alpha^* > 0$  for  $\nu \geq 3$ , see [BEKŠ94, Thm 4.2].

Let  $N$  be the number of negative eigenvalues of  $H_{\alpha, \Gamma}$ . Using the convergence of the corresponding quadratic forms (2.3) on  $W^{1,2}(\mathbb{R}^\nu)$  as  $\epsilon \rightarrow 0$  we find by [Ka76, Thm VIII.3.15] that for all  $\epsilon$  small enough  $H_{\alpha, \Gamma_\epsilon}$  has the same number  $N$  of negative eigenvalues, which we denote as  $\lambda_1(\epsilon) < \lambda_2(\epsilon) \leq \dots \leq \lambda_N(\epsilon)$ , and moreover

$$\lambda_j(\epsilon) \rightarrow \lambda_j(0) \quad \text{as} \quad \epsilon \rightarrow 0 \quad \text{for} \quad 1 \leq j \leq N.$$

Let  $\{\varphi_j(x)\}_{j=1}^N$  be an orthonormal system of eigenfunctions of  $H_{\alpha, \Gamma}$  corresponding to these eigenvalues; without loss of generality we may suppose that  $\varphi_1(x) > 0$  in  $\mathbb{R}^\nu$ . Using the Sobolev trace theorem one can check that each function  $\varphi_j$  is continuous on a  $\Gamma$ -neighborhood of the origin. Given  $\mu \in \sigma_{\text{disc}}(H_{\alpha, \Gamma})$  we denote as  $m(\mu)$  and  $n(\mu)$  the smallest and largest index value  $j$ , respectively, for which  $\mu = \lambda_j(0)$ , and we introduce the positive matrix

$$(3.5) \quad C(\mu) := \left( \overline{\varphi_i(0)} \varphi_j(0) \right)_{m(\mu) \leq i, j \leq n(\mu)}.$$

denoting by  $s_{m(\mu)} \leq s_{m(\mu)+1} \leq \dots \leq s_{n(\mu)}$  its eigenvalues. In particular, if  $\mu = \lambda_j(0)$  is a simple eigenvalue of  $H_{\alpha, \Gamma}$ , we have  $m(\mu) = n(\mu) = j$  and  $s_j = |\varphi_j(0)|^2$ . With these prerequisites, we can make the following claim:

**THEOREM 3.8.** *Assume (p1), (p2), and suppose that  $\alpha > \alpha^*$ . For a given  $\mu \in \sigma_{\text{disc}}(H_{\alpha, \Gamma})$  we have the asymptotic formula*

$$\lambda_j(\epsilon) = \mu + \alpha m_\Gamma(\mathcal{P}_\epsilon) s_j + o(\epsilon^{\nu-1}), \quad m(\mu) \leq j \leq n(\mu), \quad \text{as} \quad \epsilon \rightarrow 0,$$

where  $m_\Gamma(\cdot)$  stands for the  $(\nu - 1)$ -dimensional Lebesgue measure on  $\Gamma$ .

To derive the expansion, one has to note that due to the singular character of the perturbation a direct use of the asymptotic perturbation theory of quadratic forms [Ka76, Sec. VIII.4] is not possible. Indeed, we have

$$\mathcal{E}_{-\alpha m_{\Gamma_\epsilon}}[\psi] = \mathcal{E}_{-\alpha m_\Gamma}[\psi] + \alpha m_\Gamma(\mathcal{P}_\epsilon)|\psi(0)|^2 + \mathcal{O}(\epsilon^\nu) \quad \text{as } \epsilon \rightarrow 0$$

for  $\psi \in C_0^\infty(\mathbb{R}^\nu)$  and the quadratic form  $C_0^\infty(\mathbb{R}^\nu) \ni \psi \mapsto |\psi(0)|^2 \in \mathbb{R}$  does not extend to a bounded form on  $W^{1,2}(\mathbb{R}^\nu)$ , because the set of  $\psi \in C_0^\infty(\mathbb{R}^\nu)$  vanishing at the origin is dense in  $W^{1,2}(\mathbb{R}^\nu)$ . A way to eliminate this difficulty is to employ the compactness of the map  $W^{1,2}(\mathbb{R}^\nu) \ni f \mapsto f|_\Gamma \in L^2(\Gamma)$ ; we refer to [EY03] for a detailed description of such a proof<sup>9</sup>.

Let us observe further that while the compactness of  $\Gamma$  was used in formulation of the theorem it played essentially no role in the proof. This allows us to treat other situations, for instance, eigenvalues corresponding to non-straight curves as described in Sec. 3.1. Suppose we replace such a  $\Gamma$  by a family of curves with a hiatus,  $\Gamma_\epsilon$  given by the same function  $\gamma$  where, however, the argument runs over  $\mathbb{R} \setminus (-\epsilon, \epsilon)$ . Using the above introduced notation, we then have the following result:

**COROLLARY 3.9.** *Suppose that  $\Gamma$  satisfies the assumptions of Theorem 3.1, then the eigenvalues of  $H_{\alpha,\Gamma}$  and  $H_{\alpha,\Gamma_\epsilon}$  obey the asymptotic formula*

$$\lambda_j(\epsilon) = \lambda_j(0) + 2\alpha\epsilon s_j + o(\epsilon), \quad m(\mu) \leq j \leq n(\mu), \quad \text{as } \epsilon \rightarrow 0$$

**REMARK 3.10.** As before things look differently in the case of codimension two. If we have, for instance, a simple eigenvalue of  $H_{\alpha,\Gamma}$  with the eigenfunction  $\varphi$ , where  $\Gamma$  is a curve in  $\mathbb{R}^3$  and perturb the latter by making a  $2\epsilon$ -hiatus in it, the leading term in the perturbation expansion is again proportional to  $|\varphi(0)|^2$ , however, this time it comes multiplied not by  $\epsilon$  but rather  $\epsilon \ln \epsilon$  – cf. [EK07].

**3.5. An isoperimetric problem.** The above results do not exhaust ways in which the edge shapes influence spectral properties of leaky-graph Hamiltonians. Let us mention one more; for simplicity we restrict ourselves again to the planar case,  $\Gamma \subset \mathbb{R}^2$ . If  $\Gamma$  is of a finite length, we have  $\sigma_{\text{ess}}(H_{\alpha,\Gamma}) = [0, \infty)$  while the discrete spectrum is nonempty and finite, so that

$$(3.6) \quad \lambda_1 \equiv \lambda_1(\alpha, \Gamma) := \inf \sigma(H_{\alpha,\Gamma}) < 0.$$

Suppose now that  $\Gamma$  is a loop of a fixed length and ask which shape of it makes the above principal eigenvalue maximal.

Let us make the assumptions more precise. We suppose that  $\gamma : [0, L] \rightarrow \mathbb{R}^2$  is a closed  $C^1$ , piecewise  $C^2$  smooth curve,  $\gamma(0) = \gamma(L)$ ; we allow self-intersections provided the curve meets itself at a non-zero angle. Furthermore, we introduce the equivalence relation: the loops  $\Gamma$  and  $\Gamma'$  belong to the same class if one can be obtained from the other by a Euclidean transformation of the plane. Spectral properties of the corresponding operators  $H_{\alpha,\Gamma}$  and  $H_{\alpha,\Gamma'}$  are obviously the same, hence we will speak about a curve  $\Gamma$  having in mind the corresponding equivalence class. The stated assumptions are satisfied, in particular, by the circle, say  $\mathcal{C} := \{(\frac{L}{2\pi} \cos s, \frac{L}{2\pi} \sin s) : s \in [0, L]\}$ , and its equivalence class.

**THEOREM 3.11.** *Within the above described class of loops, the principal eigenvalue  $\lambda_1(\alpha, \Gamma)$  is for any fixed  $\alpha > 0$  and  $L > 0$  sharply maximized by the circle.*

<sup>9</sup>There are other ways to derive such asymptotic expansions, for instance, the technique of matching of asymptotic expansions [II92]. The advantage of the sketched approach is that it requires no self-similarity properties for the family of shrinking sets  $\mathcal{P}_\epsilon$ .

We will need the following geometric result about means of chords:

PROPOSITION 3.12. *Let  $\Gamma$  have the properties described above, then*

$$\int_0^L |\gamma(s+u) - \gamma(s)|^p ds \leq \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L} \quad \text{for } p \in (0, 2].$$

The right-hand side of the inequality is obviously the value of the integral for the circle. Notice that the same is true for loops in  $\mathbb{R}^{\nu}$  and a similar *reverse* inequality holds for negative powers  $p \in [-2, 0)$ . To prove this it is only necessary to establish the result for  $p = 2$  which was done in various ways in [Lu66, ACF03, EHL06], see also [Ex05b] for a local maximum proof.

SKETCHED PROOF OF THEOREM 3.11. We shall rely again on Proposition 2.3 which relates our eigenvalue problem to the integral equation  $\mathcal{R}_{\alpha, \Gamma}^{\kappa} \phi = \phi$  on  $L^2([0, L])$  with  $\mathcal{R}_{\alpha, \Gamma}^{\kappa}$  defined similarly as in the proof of Theorem 3.1; we note that the operator-valued function  $\kappa \mapsto \mathcal{R}_{\alpha, \Gamma}^{\kappa}$  is strictly decreasing in  $(0, \infty)$  and  $\|\mathcal{R}_{\alpha, \Gamma}^{\kappa}\| \rightarrow 0$  as  $\kappa \rightarrow \infty$ . By a positivity improving argument the maximum eigenvalue of  $\mathcal{R}_{\alpha, \Gamma}^{\kappa}$  is simple, and the same is true by Proposition 2.3 for the ground state of  $H_{\alpha, \Gamma}$ . If  $\Gamma$  is a circle the latter exhibits rotational symmetry, and using Proposition 2.3 again we see that the respective eigenfunction of  $\mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1}$  corresponding to the unit eigenvalue is constant,  $\tilde{\phi}_1(s) = L^{-1/2}$ . Then we have

$$\max \sigma(\mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1}) = (\tilde{\phi}_1, \mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha, \mathcal{C}}^{\tilde{\kappa}_1}(s, s') ds ds',$$

while for a general  $\Gamma$  a simple variational estimate gives

$$\max \sigma(\mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}) \geq (\tilde{\phi}_1, \mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1} \tilde{\phi}_1) = \frac{1}{L} \int_0^L \int_0^L \mathcal{R}_{\alpha, \Gamma}^{\tilde{\kappa}_1}(s, s') ds ds';$$

hence to check that the circle is a maximizer it sufficient to show that

$$\int_0^L \int_0^L K_0(\kappa |\Gamma(s) - \Gamma(s')|) ds ds' \geq \int_0^L \int_0^L K_0(\kappa |\mathcal{C}(s) - \mathcal{C}(s')|) ds ds'$$

holds for all  $\kappa > 0$  and  $\Gamma$  of the considered class. By a simple change of variables we find that this is equivalent to positivity of the functional

$$F_{\kappa}(\Gamma) := \int_0^{L/2} du \int_0^L ds \left[ K_0(\kappa |\Gamma(s+u) - \Gamma(s)|) - K_0(\kappa |\mathcal{C}(s+u) - \mathcal{C}(s)|) \right],$$

where the second term is equal to  $K_0\left(\frac{\kappa L}{\pi} \sin \frac{\pi u}{L}\right)$ . Now we employ the (strict) convexity of  $K_0$  which yields by means of the Jensen inequality the estimate

$$\frac{1}{L} F_{\kappa}(\Gamma) \geq \int_0^{L/2} \left[ K_0 \left( \frac{\kappa}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)| ds \right) - K_0 \left( \frac{\kappa L}{\pi} \sin \frac{\pi u}{L} \right) \right] du,$$

where the inequality is sharp unless  $\int_0^L |\Gamma(s+u) - \Gamma(s)| ds$  is independent of  $s$ . Finally, we note that  $K_0$  is decreasing in  $(0, \infty)$ , hence the result follows from the geometric inequality of Proposition 3.12 with  $p = 1$ , see [Ex05b] for details.  $\square$

**3.6. Scattering.** The investigation of leaky graphs is not exhausted, of course, by analysis of their discrete spectrum. Another important problem concerns scattering on graphs having semi-infinite edges. Our knowledge about this subject is far from satisfactory at present and we will concentrate here on a particular situation. First of all, we consider again planar graphs,  $\Gamma \subset \mathbb{R}^2$ , only. Secondly, we restrict ourself to graphs which can be regarded as a local modification of a straight line. And finally, we will analyze the situation which is from the point of view of our physical motivation the most interesting, namely the negative part of the spectrum where the scattering states are “guided” along the graph edges.

The unperturbed graph is thus the straight line  $\Sigma = \{(x_1, 0) : x_1 \in \mathbb{R}\}$  for which  $H_{\alpha, \Sigma}$  allows a separation of variables; the spectrum is purely a.c. and, in particular, the generalized eigenfunctions corresponding to  $\lambda \in (-\frac{1}{4}\alpha^2, 0)$  are

$$(3.7) \quad \omega_\lambda(x_1, x_2) = e^{i(\lambda + \alpha^2/4)^{1/2}x_1} e^{-\alpha|x_2|/2}$$

and its complex conjugate  $\bar{\omega}_\lambda$ . If we perturb the line  $\Sigma$  locally we will get a nontrivial scattering but the essentially one-dimensional character of the motion will remain asymptotically preserved. Let us specify the perturbation:

(s1) **locality:** there is a compact  $M \subset \mathbb{R}^2$  such that  $\Gamma \setminus M = \Sigma \setminus M$ .

(s2) **finiteness:**  $\Gamma \setminus \Sigma$  is a finite graph with the properties (g1) and (g2).

We will treat the operator  $H_{\alpha, \Gamma}$  defined by the prescription given in Sec. 2.1 as a singular perturbation of  $H_{\alpha, \Sigma}$  supported by the set

$$(3.8) \quad \Lambda = \Lambda_0 \cup \Lambda_1 \quad \text{with} \quad \Lambda_0 := \Sigma \setminus \Gamma, \quad \Lambda_1 := \Gamma \setminus \Sigma = \bigcup_{i=1}^N \Gamma_i;$$

the coupling constant of the perturbation will take the positive value  $\alpha$  on the “erased” part  $\Lambda_0$  and negative one,  $-\alpha$ , on added edges  $\Lambda_1$ .

For our present purpose we need a more suitable resolvent expression than that given by Proposition 2.3; instead of (2.10) we will use the resolvent of  $H_{\alpha, \Sigma}$  as the comparison operator. The latter can be expressed, of course, again by Proposition 2.3: we have  $R_\Sigma^k = R_0^k + \alpha R_\mu^k (I - \alpha R_{\mu, \mu}^k)^{-1} (R_\mu^k)^*$ , where for simplicity we write  $R_{dx, \mu}^k \equiv R_\mu^k$  and  $\mu := m_\Sigma$  is the measure associated with the line by (2.5). A direct calculation then yields the expression

$$(3.9) \quad R_\Sigma^k(x-y) = G_k(x-y) + \frac{\alpha}{4\pi^3} \int_{\mathbb{R}^3} \frac{e^{ipx - ip'y}}{(p^2 - k^2)(p'^2 - k^2)} \frac{\tau_k(p_1)}{2\tau_k(p_1) - \alpha} dp dp'_2$$

with  $\tau_k(p_1) := (p_1^2 - k^2)^{1/2}$  for any  $k$  with  $\text{Im } k > 0$  such that  $k^2 \in \mathbb{C} \setminus [-\frac{1}{4}\alpha^2, \infty)$ . To get the above indicated expression for the resolvent of  $H_{\alpha, \Gamma}$  we decompose the Dirac measure associated with the perturbation by (2.5) as

$$(3.10) \quad \nu \equiv \nu_\Lambda = \nu_0 + \sum_{i=1}^N \nu_i,$$

where  $\nu_0$  corresponds to the “erased” edge  $\Lambda_0$  and  $\nu_i$  to  $\Gamma_i$ . The associated integral operator will act in the Hilbert space  $\mathfrak{h} := L^2(\nu)$  which inherits from (3.10) the decomposition  $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$  with  $\mathfrak{h}_0 := L^2(\nu_0)$  and  $\mathfrak{h}_1 := \bigoplus_{i=1}^N L^2(\nu_i)$ . We will again need the trace maps, this time of the operator associated with (3.9), given by

$$(3.11) \quad R_{\Sigma, \nu}^k : \mathfrak{h} \rightarrow L^2, \quad R_{\Sigma, \nu}^k f = R_\Sigma^k * f \nu \quad \text{for} \quad f \in \mathfrak{h}$$

together with the adjoint  $(R_{\Sigma, \nu}^k)^* : L^2 \rightarrow \mathfrak{h}$  and  $R_{\Sigma, \nu \nu}^k$  which is the operator-valued matrix in  $\mathfrak{h}$  with the “block elements”  $R_{\Sigma, ij}^k : L^2(\nu_j) \rightarrow L^2(\nu_i)$  defined as the appropriated embeddings of (3.9). They have properties analogous to those of Proposition 2.3 which can be checked in a similar way, cf. [EK05].

PROPOSITION 3.13. *The operator  $R_{\Sigma, \nu}^{i\kappa}$  is bounded for any  $\kappa \in (\frac{1}{2}\alpha, \infty)$ . Moreover, to any  $\sigma > 0$  there is a  $\kappa_\sigma > 0$  such that  $\|R_{\Sigma, \nu}^{i\kappa}\| < \sigma$  holds for  $\kappa > \kappa_\sigma$ .*

To express the resolvent we introduce an operator-valued matrix in  $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ ,

$$(3.12) \quad \Theta_\alpha^k := -(\alpha^{-1}\mathbb{I} + R_{\Sigma, \nu \nu}^k) \quad \text{with} \quad \mathbb{I} := \begin{pmatrix} I_0 & 0 \\ 0 & -I_1 \end{pmatrix},$$

where  $I_i$  are the unit operators in  $\mathfrak{h}_i$ . By Proposition 3.13 the operator  $\Theta_\alpha^{i\kappa}$  is boundedly invertible for  $\kappa$  large enough and we have the following theorem [EK05]:

THEOREM 3.14. *Suppose that  $(\Theta_\alpha^k)^{-1} \in \mathcal{B}(\mathfrak{h})$  hold for  $k \in \mathbb{C}^+$  and the operator*

$$R_\Gamma^k = R_\Sigma^k + R_{\Sigma, \nu}^k (\Theta_\alpha^k)^{-1} (R_{\Sigma, \nu}^k)^*$$

*is defined everywhere in  $L^2(\mathbb{R}^2)$ . Then  $k^2$  belongs to  $\rho(H_{\alpha, \Gamma})$  and the resolvent  $(H_{\alpha, \Gamma} - k^2)^{-1}$  coincides with  $R_\Gamma^k$ .*

A simple estimate shows that the operator  $R_{\Sigma, \nu}^k$  is Hilbert–Schmidt under our assumptions and since the other two factors are bounded by Proposition 3.13 we get as a consequence stability of the essential spectrum.

COROLLARY 3.15.  $\sigma_{\text{ess}}(H_{\alpha, \Gamma}) = \sigma_{\text{ess}}(H_{\alpha, \Sigma}) = [-\frac{1}{4}\alpha^2, \infty)$ .

Let us turn now to the proper topic of this section which is the scattering theory for the pair  $(H_{\alpha, \Gamma}, H_{\alpha, \Sigma})$ . To establish existence of the wave operators we need to check that the resolvent difference  $B^k := R_{\Sigma, \nu}^k (\Theta_\alpha^k)^{-1} (R_{\Sigma, \nu}^k)^*$  is of the trace class.

PROPOSITION 3.16.  *$B^{i\kappa}$  is a trace class operator for all  $\kappa$  sufficiently large.*

SKETCH OF THE PROOF. The idea is inspired by [BT92]. We estimate the (sign-indefinite) operators  $(\Theta_\alpha^{i\kappa})^{-1}$  and  $B^{i\kappa}$  from above and below by a positive and negative operator, respectively, which are obtained by taking both signs in the matrix multiplying  $\alpha^{-1}$  in (3.12) the same. Then we can integrate the kernel diagonal of the estimates to  $B^{i\kappa}$  with a suitable cut-off, using the lemma following Thm XI.31 in [RSi79], and to show subsequently that the trace-class property is preserved when the cut-off is removed, cf. [EK05] for details.  $\square$

The existence of wave operators which follows from Proposition 3.16 does not tell us much, and we have to find also the on-shell S-matrix relating the incoming and outgoing asymptotic solutions. In particular, for scattering in the negative part of the spectrum with a fixed  $\lambda \in (-\frac{1}{4}\alpha^2, 0)$  corresponding to the effective momentum  $k_\alpha(\lambda) := (\lambda + \alpha^2/4)^{1/2}$ , the latter are combinations of  $\omega_\lambda$  and  $\bar{\omega}_\lambda$  given by (3.7). These generalized eigenfunctions and their analogues  $\omega_z$  for complex values of the energy parameter are  $L^2$  only locally, of course, but we can use the standard trick and approximate them by regularized functions, for instance

$$\omega_z^\delta(x) = e^{-\delta x_1^2} \omega_z(x) \quad \text{with} \quad z \in \rho(H_{\alpha, \Sigma}),$$

which naturally belong to the domain  $D(H_{\alpha, \Sigma})$ . Now we are looking for a function  $\psi_z^\delta$  such that  $(-\Delta_\Gamma - z)\psi_z^\delta = (-\Delta_\Sigma - z)\omega_z^\delta$ . Computing the right-hand side and

taking the limit  $\lim_{\epsilon \rightarrow 0} \psi_{\lambda+i\epsilon}^\delta =: \psi_\lambda^\delta$  in the topology of  $L^2$  we find that  $\psi_\lambda^\delta$  still belongs to  $D(H_{\alpha,\Gamma})$ , and moreover

$$\psi_\lambda^\delta = \omega_\lambda^\delta + R_{\Sigma,\nu}^{k_\alpha(\lambda)} (\Theta^{k_\alpha(\lambda)})^{-1} I_\Lambda \omega_\lambda^\delta,$$

where  $I_\Gamma$  is the standard embedding from  $W^{1,2}$  to  $\mathfrak{h} = L^2(\nu_\Lambda)$  and  $R_{\Sigma,\nu}^{k_\alpha(\lambda)}$  is the integral operator acting on the Hilbert space  $\mathfrak{h}$ , analogous to (3.11), with the kernel

$$(3.13) \quad R_\Sigma^{k_\alpha(\lambda)}(x-y) := \lim_{\epsilon \rightarrow 0} R_\Sigma^{k_\alpha(\lambda+i\epsilon)}(x-y);$$

similarly  $\Theta^{k_\alpha(\lambda)} := -\alpha^{-1}\mathbb{I} - R_{\Sigma,\nu\nu}^{k_\alpha(\lambda)}$  are the operators on  $\mathfrak{h}$  with  $R_{\Sigma,\nu\nu}^{k_\alpha(\lambda)}$  being the embeddings defined by means of (3.13). When we remove the regularization, the pointwise limit  $\psi_\lambda := \lim_{\delta \rightarrow 0} \psi_\lambda^\delta$  ceases to be square integrable, however, it still belongs locally to  $L^2$  and yields the generalized eigenfunction of  $H_{\alpha,\Gamma}$ , namely

$$(3.14) \quad \psi_\lambda = \omega_\lambda + R_{\Sigma,\nu}^{k_\alpha(\lambda)} (\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda,$$

where  $J_\Lambda \omega_\lambda$  is the embedding of  $\omega_\lambda$  to  $L^2(\nu_\Lambda)$ . The on-shell S-matrix can be then found by inspecting the asymptotic behavior of the function  $\psi_\lambda$  as  $|x_1| \rightarrow \infty$ . Using the explicit form of the kernel (3.13) derived in [EK02] one arrives by a direct computation at the following result:

**THEOREM 3.17.** *For a fixed  $\lambda \in (-\frac{1}{4}\alpha^2, 0)$  the generalized eigenfunctions of  $H_{\alpha,\Gamma}$  behave under the assumptions (s1), (s2) asymptotically as*

$$\psi_\lambda(x) \approx \begin{cases} \mathcal{T}(\lambda) e^{ik_\alpha(\lambda)x_1} e^{-\alpha|x_2|/2} & \text{for } x_1 \rightarrow +\infty \\ e^{ik_\alpha(\lambda)x_1} e^{-\alpha|x_2|/2} + \mathcal{R}(\lambda) e^{-ik_\alpha(\lambda)x_1} e^{-\alpha|x_2|/2} & \text{for } x_1 \rightarrow -\infty \end{cases}$$

where  $k_\alpha(\lambda) := (\lambda + \alpha^2/4)^{1/2}$  is the effective momentum along  $\Sigma$  and  $\mathcal{T}(\lambda), \mathcal{R}(\lambda)$  are the transmission and reflection amplitudes, respectively, given by

$$\mathcal{R}(\lambda) = 1 - \mathcal{T}(\lambda) = \frac{i\alpha}{8k_\alpha(\lambda)} \left( (\Theta^{k_\alpha(\lambda)})^{-1} J_\Lambda \omega_\lambda, J_\Lambda \bar{\omega}_\lambda \right)_\mathfrak{h}.$$

#### 4. Strong-coupling asymptotics

The coupling constant in  $H_{\alpha,\Gamma}$  determines how is the particle attracted to the graph, and this in turn implies, in particular, which is the ‘‘spread’’ of possible eigenfunctions in the direction transverse to the edges. It is thus natural to ask what happens in the case of a strong coupling when the wave functions are transversally sharply localized. We are going now to show that if the interaction support is a sufficiently smooth manifold various asymptotic formulæ can be derived.

**4.1. Interactions supported by curves.** In distinction to the previous considerations it is rather the dimension that codimension of  $\Gamma$  which will be important. As usual we begin with the case of planar curves, at first finite ones.

**THEOREM 4.1.** *Suppose that  $\gamma : [0, L] \rightarrow \mathbb{R}^2$  is a  $C^4$  smooth function,  $|\dot{\gamma}| = 1$ , which defines a curve  $\Gamma$ ; then the relation*

$$\sharp \sigma_{\text{disc}}(H_{\alpha,\Gamma}) = \frac{\alpha L}{2\pi} + \mathcal{O}(\ln \alpha)$$

holds as  $\alpha \rightarrow \infty$ . In addition, if  $\Gamma$  is a closed curve without self-intersections, then the  $j$ -th eigenvalue of the operator  $H_{\alpha,\Gamma}$  behaves asymptotically as

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha),$$

where  $\mu_j$  is the  $j$ -th eigenvalue of the operator  $S_\Gamma := -\frac{d}{ds^2} - \frac{1}{4}k(s)^2$  on  $L^2(0, L)$  with periodic b.c., counted with multiplicity, and  $k(s)$  is the signed curvature of  $\Gamma$ .

SKETCH OF THE PROOF. Suppose first that  $\Gamma$  is closed, without self-intersections, and consider its strip neighborhood analogous to (2.7), in other words, the set  $\Sigma_a$  onto which the function  $\Phi_a : [0, L] \times (-a, a) \rightarrow \mathbb{R}^2$  defined by

$$(4.1) \quad (s, u) \mapsto (\gamma_1(s) - u\gamma_2'(s), \gamma_2(s) + u\gamma_1'(s))$$

maps, diffeomorphically for all  $a > 0$  small enough. The main idea is to apply to  $H_{\alpha,\Gamma}$  the Dirichlet-Neumann bracketing at the boundary of  $\Sigma_a$ ,

$$(-\Delta_{\Lambda_a}^N) \oplus L_{a,\alpha}^- \leq H_{\alpha,\Gamma} \leq (-\Delta_{\Lambda_a}^D) \oplus L_{a,\alpha}^+$$

where  $\Lambda_a = \Lambda_a^{\text{in}} \cup \Lambda_a^{\text{out}}$  is the exterior domain, and  $L_{a,\alpha}^\pm$  are self-adjoint operators associated with the forms

$$q_{a,\alpha}^\pm[f] = \|\nabla f\|_{L^2(\Sigma_a)}^2 - \alpha \int_\Gamma |f(x)|^2 dS$$

where  $f \in W_0^{1,2}(\Sigma_a)$  and  $W^{1,2}(\Sigma_a)$  for  $\pm$ , respectively. The exterior  $\mathbb{R}^2 \setminus \bar{\Sigma}_a$  does not contribute to the negative part of the spectrum, so we may consider  $L_{a,\alpha}^\pm$  only.

We use the curvilinear coordinates  $(s, u)$ , the same as in (2.7), passing from  $L_{a,\alpha}^\pm$  to unitarily equivalent operators given by quadratic forms

$$\begin{aligned} b_{a,\alpha}^+[f] &= \int_0^L \int_{-a}^a (1 + uk(s))^{-2} \left| \frac{\partial f}{\partial s} \right|^2 (s, u) du ds + \int_0^L \int_{-a}^a \left| \frac{\partial f}{\partial u} \right|^2 (s, u) du ds \\ &\quad + \int_0^L \int_{-a}^a V(s, u) |f(s, u)|^2 ds du - \alpha \int_0^L |f(s, 0)|^2 ds \end{aligned}$$

with  $f \in W^{1,2}((0, L) \times (-a, a))$  satisfying periodic boundary conditions in the variable  $s$  and Dirichlet b.c. at  $u = \pm a$ , and

$$b_{a,\alpha}^-[f] = b_{a,\alpha}^+[f] - \sum_{j=0}^1 \frac{1}{2} (-1)^j \int_0^L \frac{k(s)}{1 + (-1)^j ak(s)} |f(s, (-1)^j a)|^2 ds,$$

where  $V$  is the usual curvature induced potential [EŠ89]

$$(4.2) \quad V(s, u) = -\frac{k(s)^2}{4(1+uk(s))^2} + \frac{uk''(s)}{2(1+uk(s))^3} - \frac{5u^2k'(s)^2}{4(1+uk(s))^4}.$$

We may employ rougher bounds squeezing  $H_{\alpha,\Gamma}$  between  $\tilde{H}_{a,\alpha}^\pm = U_a^\pm \otimes 1 + 1 \otimes T_{a,\alpha}^\pm$  with decoupled variables. Here  $U_a^\pm$  are self-adjoint operators on  $L^2(0, L)$  given by

$$U_a^\pm = -(1 \mp a\|k\|_\infty)^{-2} \frac{d^2}{ds^2} + V_\pm(s)$$

with periodic b.c., where  $V_-(s) \leq V(s, u) \leq V_+(s)$  with an  $\mathcal{O}(a)$  error, and the transverse operators are associated with the forms

$$t_{a,\alpha}^+[f] = \int_{-a}^a |f'(u)|^2 du - \alpha |f(0)|^2$$

and

$$t_{a,\alpha}^- [f] = t_{a,\alpha}^- [f] - \|k\|_\infty (|f(a)|^2 + |f(-a)|^2),$$

where  $f \in W_0^{1,2}(-a, a)$  and  $W^{1,2}(-a, a)$  for the  $\pm$  sign, respectively. Their negative spectrum can be localized with an exponential precision: there is a  $c > 0$  such that  $T_{\alpha,a}^\pm$  has for  $\alpha$  large enough a single negative eigenvalue  $\kappa_{\alpha,a}^\pm$  satisfying

$$(4.3) \quad -\frac{\alpha^2}{4} \left(1 + c e^{-\alpha a/2}\right) < \kappa_{\alpha,a}^- < -\frac{\alpha^2}{4} < \kappa_{\alpha,a}^+ < -\frac{\alpha^2}{4} \left(1 - 8e^{-\alpha a/2}\right)$$

To finish the proof one has to check that the eigenvalues of  $U_a^\pm$  differ by  $\mathcal{O}(a)$  from those of the comparison operator, then we choose  $a = 6\alpha^{-1} \ln \alpha$  as the neighborhood width and putting the estimates together we get the eigenvalue asymptotic formula; for details see [EY02a]. If  $\Gamma$  is not closed, the same can be done with the comparison operators  $S_\Gamma^{\text{D,N}}$  having the appropriate b.c., Dirichlet or Neumann, at the endpoints of  $\Gamma$ ; this gives the estimate on  $\sharp \sigma_{\text{disc}}(H_{\alpha,\Gamma})$ .  $\square$

The case of a finite curve in  $\mathbb{R}^3$  is similar but we have to be more cautious about the regularity of the curve. It will be again a graph of a  $C^4$  smooth function,  $\gamma : [0, L] \rightarrow \mathbb{R}^3$  with  $|\dot{\gamma}(s)| = 1$ . To construct the three-dimensional counterpart of the ‘‘straightening’’ transformation used in the above proof, we suppose for simplicity that  $\Gamma$  possesses a global Frenet frame<sup>10</sup> and consider the map  $\phi_a : [0, L] \times \mathcal{B}_a \rightarrow \mathbb{R}^3$

$$\phi_a(s, r, \theta) = \gamma(s) - r [n(s) \cos(\theta - \beta(s)) + b(s) \sin(\theta - \beta(s))],$$

where  $\mathcal{B}_a$  is the circle of radius  $a$  centered at the origin and the function  $\beta$  has to be specified; for small enough  $a$  it is a diffeomorphic map on a tubular neighborhood  $\Sigma_a$  of  $\Gamma$  which does not intersect itself. The geometry of  $\Sigma_a$  is naturally described in terms of the metric tensor  $g_{ij}$  expressed by means of the curvature  $k$  and torsion  $\tau$  of  $\Gamma$ . In particular, in the neighborhood with a circular cross section we can always choose the so-called Tang coordinate system,  $\dot{\beta} = \tau$ , in which the tensor  $g_{ij}$  is diagonal, i.e. the longitudinal and transverse variable decouple [DE95].

To state the result we have to note that in the codimension two case the strong coupling means large *negative* values of the parameter  $\alpha$ .

**THEOREM 4.2.** *For curves  $\Gamma$  without self-intersections described above we have*

$$\sharp \sigma_{\text{disc}}(H_{\alpha,\Gamma}) = \frac{L}{\pi} (-\zeta_a)^{1/2} (1 + \mathcal{O}(e^{\pi\alpha})).$$

as  $\alpha \rightarrow -\infty$ , where we put again  $\zeta_\alpha := -4e^{2(-2\pi\alpha + \psi(1))}$ . If, in addition,  $\Gamma$  is a closed curve, the  $j$ -th eigenvalue of the operator  $H_{\alpha,\Gamma}$  behaves asymptotically as

$$\lambda_j(\alpha) = \zeta_\alpha + \mu_j + \mathcal{O}(e^{\pi\alpha})$$

where  $\mu_j$  is the  $j$ -th eigenvalue of the same operator  $S_\Gamma$  as in Theorem 4.1.

**SKETCH OF THE PROOF.** The argument follows the same scheme. We use Dirichlet–Neumann bracketing at the boundary of  $\Sigma_a$  and estimate the internal part using the Tang coordinate system. The effective potential replacing (4.2) is known from [DE95]; it is important that the torsion does not contribute to its leading order as  $a \rightarrow 0$  which is the same as in the two-dimensional case. Also (4.3) has to be replaced by the appropriate two-dimensional estimate which is again exponentially precise, see [EK04a] for details.  $\square$

<sup>10</sup>This assumption can be weakened, see the footnote in Sec. 2.4.

The technique used in these proofs can be applied to many other cases. If  $\Gamma$  is an infinite curve the threshold of the essential spectrum is moved and the estimates on  $\#\sigma_{\text{disc}}(H_{\alpha,\Gamma})$  are no longer relevant. On the other hand, the eigenvalue asymptotic formulae remain valid under mild additional assumptions [EY01, EK04a]:

**THEOREM 4.3.** *Suppose that  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^\nu$ ,  $\nu = 2, 3$ , satisfies hypotheses of Theorems 3.1 and 3.7, respectively. In addition, assume that  $\dot{k}(s)$  and  $\ddot{k}(s)^{1/2}$  are  $\mathcal{O}(s^{-1-\epsilon})$  as  $|s| \rightarrow \infty$ , and  $\tau, \dot{\tau} \in L^\infty(\mathbb{R})$  for  $\nu = 3$ . Then the asymptotic expansions from Theorems 4.1 and 4.2 hold for all the eigenvalues  $\lambda_j(\alpha)$  of  $H_{\alpha,\Gamma}$ , when  $S_\Gamma := -\frac{d}{ds^2} - \frac{1}{4}k(s)^2$  is now the operator on  $L^2(\mathbb{R})$  with the domain  $W^{2,2}(\mathbb{R})$ .*

**REMARK 4.4.** In this case we need not care about the multiplicity because the spectrum of  $S_\Gamma$  in  $L^2(\mathbb{R})$  is simple. This may not be true, of course, in the more general case to which the results extend easily, namely for  $\Gamma$ 's consisting of disconnected  $C^4$  smooth edges, i.e. curves which do not touch or cross each other. On the other hand, situation becomes considerably more complicated in presence of angles or branchings; we will comment on it in Sec. 7.12 below.

**4.2. Surfaces in  $\mathbb{R}^3$ .** The method works also for interactions supported by surfaces but the geometric part is naturally different. Let us consider first the case of a  $C^4$  smooth compact and closed Riemann surface  $\Gamma \subset \mathbb{R}^3$  of a finite genus  $g$ . In the usual way [Kli78], the geometry of  $\Gamma$  is encoded in the metric tensor  $g_{\mu\nu}$  and Weingarten tensor  $h_\mu{}^\nu$ . The eigenvalues  $k_\pm$  of the latter are the principal curvatures which determine the Gauss curvature  $K$  and mean curvature  $M$  by

$$K = \det(h_\mu{}^\nu) = k_+ k_-, \quad M = \frac{1}{2} \text{Tr}(h_\mu{}^\nu) = \frac{1}{2}(k_+ + k_-).$$

The operator  $H_{\alpha,\Gamma}$  is defined as in Sec. 2.1. For a compact  $\Gamma$  the essential spectrum is  $[0, \infty)$  and we ask about the asymptotic behavior of the negative eigenvalues as  $\alpha \rightarrow \infty$ . It will be expressed again in terms of a comparison operator: the  $S_\Gamma$  of Theorem 4.1 has to be now replaced by

$$(4.4) \quad S_\Gamma := -\Delta_\Gamma + K - M^2$$

on  $L^2(\Gamma, d\Gamma)$ , where  $\Delta_\Gamma = -g^{-1/2} \partial_\mu g^{1/2} g^{\mu\nu} \partial_\nu$  is the Laplace-Beltrami operator on  $\Gamma$ . The  $j$ -th eigenvalue  $\mu_j$  of  $S_\Gamma$  is bounded from above by that of  $\Delta_\Gamma$  because

$$K - M^2 = -\frac{1}{4}(k_+ - k_-)^2 \leq 0$$

in analogy with the curve case; in distinction to the latter the two coincide when  $\Gamma$  is a sphere. With these prerequisites we can make the following claim:

**THEOREM 4.5.** *Under the stated assumptions,  $\#\sigma_{\text{disc}}(H_{\alpha,\Gamma}) \geq j$  for any fixed integer  $j$  if  $\alpha$  is large enough. The  $j$ -th eigenvalue  $\lambda_j(\alpha)$  of  $H_{\alpha,\Gamma}$  has the expansion*

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

as  $\alpha \rightarrow \infty$ , where  $\mu_j$  is the  $j$ -th eigenvalue of  $S_\Gamma$ . Moreover, the counting function  $\alpha \mapsto \#\sigma_{\text{disc}}(H_{\alpha,\Gamma})$  behaves asymptotically as

$$\#\sigma_{\text{d}}(H_{\alpha,\Gamma}) = \frac{|\Gamma|}{16\pi} \alpha^2 + \mathcal{O}(\alpha),$$

where  $|\Gamma|$  is the Riemann area of the surface  $\Gamma$ .

SKETCH OF THE PROOF. To employ the bracketing we need to construct a family of layer neighborhoods of  $\Gamma$ . Let  $\{n(x) : x \in \Gamma\}$  be a field of unit vectors normal to the manifold; such a field exists globally because  $\Gamma$  is orientable. We define a map  $\mathcal{L}_a : \Gamma \times (-a, a) \rightarrow \mathbb{R}^3$  by  $\mathcal{L}_a(x, u) = x + un(x)$ . Due to the assumed smoothness of  $\Gamma$  it is a diffeomorphism for all  $a$  small enough, mapping onto the sought layer neighborhood  $\Omega_a = \{x \in \mathbb{R}^3 : \text{dist}(x, \Gamma) < a\}$ .

By bracketing we get a two-sided estimate for the negative spectrum of  $H_{\alpha, \Gamma}$  by means of the layer part operators. The latter can be analyzed by means of the curvilinear coordinates following [DEK01], see [Ex03] for details. One arrives at estimates through operators with decoupled variables,  $S_a^\pm \otimes I + I \otimes T_{\alpha, a}^\pm$  with

$$S_a^\pm := -C_\pm(a)\Delta_\Gamma + C_\pm^{-2}(a)(K - M^2) \pm va$$

and the transverse part which is the same as in the proof of Theorem 4.1. Here  $C_\pm(a) := (1 \pm a\varrho^{-1})^2$  with  $\varrho := \max(\{\|k_+\|_\infty, \|k_-\|_\infty\})^{-1}$  and  $v$  is a suitable constant. The rest of the argument is again analogous to Theorem 4.1; to get the counting function one has to employ the appropriate Weyl formula [Ch99].  $\square$

REMARK 4.6. The connectedness assumption is made for simplicity; the claim remains valid if  $\Gamma$  is a finite disjoint union of  $C^4$  smooth compact Riemann surfaces of finite genera. Moreover, the asymptotic formula for  $\sharp\sigma_d(H_{\alpha, \Gamma})$  is preserved if  $\Gamma$  has a nonempty and smooth boundary, see [Ex03] for a more detailed discussion.

Under additional assumptions the technique can be applied also to interactions supported by infinite surfaces. One possible set of hypotheses looks as follows:

- (as1) **injectivity:** the map  $\mathcal{L}_a : \Gamma \times (-a, a) \rightarrow \Omega_a \subset \mathbb{R}^3$  defined above is injective for all  $a$  small enough.
- (as2) **uniform ellipticity:**  $c_-\delta_{\mu\nu} \leq g_{\mu\nu} \leq c_+\delta_{\mu\nu}$  for some  $c_\pm > 0$ .
- (as3) **asymptotic planarity:**  $K, M \rightarrow 0$  as the geodesic radius  $r \rightarrow \infty$ .

One can also replace the last requirement by a stronger assumption which implies, however, at the same time the validity of (as1):

- (as3') **asymptotic direction:** the normal vector  $n \rightarrow n_0$  as the geodesic radius  $r \rightarrow \infty$ , where  $n_0$  is a fixed vector.

Then one can prove in a similar way as above the following result [EK03]:

THEOREM 4.7. (i) Assume (as1) and (as3), or alternatively (as3'), then we have  $\inf \sigma_{\text{ess}}(H_{\alpha, \Gamma}) = \epsilon(\alpha)$ , where  $\epsilon(\alpha) + \frac{1}{4}\alpha^2 = \mathcal{O}(\alpha^2 e^{-\alpha a/2})$  as  $\alpha \rightarrow \infty$ .

(ii) In addition, assume (as2). Unless  $\Gamma$  is a plane, there is at least one isolated eigenvalue of  $H_{\alpha, \Gamma}$  below the threshold of the essential spectrum for all  $\alpha$  large enough, and moreover, the eigenvalues  $\lambda_j(\alpha)$  of  $H_{\alpha, \Gamma}$  have the following asymptotic expansion,

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

as  $\alpha \rightarrow \infty$ , where  $\mu_j$  the  $j$ -th eigenvalue of the corresponding operator (4.4) counted with multiplicity.

In addition to the eigenvalue expansion we have established also the existence of curvature-induced bound states for non-planar  $\Gamma$  with suitable spatial asymptotic properties in the situation when the particle is attracted to  $\Gamma$  sufficiently strongly.

**4.3. Periodic curves.** Other large class of leaky graph Hamiltonians for which we can investigate the strong-coupling asymptotics in the described way concerns periodic manifolds. Let us start with a planar curve  $\Gamma$  being the graph of a  $C^4$  smooth function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ . In view of the smoothness, the signed curvature  $k$  is a  $C^2$  function; we assume

**(as1) curvature periodicity:** there is  $L > 0$  such that  $k(s + L) = k(s)$ .

**(as2) curve periodicity:**  $\int_0^L k(s) ds = 0$ . We may suppose that the normal at  $s = 0$  is  $(1, 0)$ , then  $\Gamma(\cdot + L) - \Gamma(\cdot) = (l_1, l_2)$  where period-shift components are  $l_j := \int_0^L \sin\left(\frac{\pi}{2}(2-j) - \int_0^t k(u) du\right) dt$ . Again without loss of generality, we may suppose that  $l_1 > 0$ .

**(as3) period cell match:** the map (4.1) is injective for all  $a$  small enough and  $\Phi_a((0, L) \times (-a, a)) \subset \Lambda := (0, l_1) \times \mathbb{R}$ .

As usual in a periodic situation we have to perform the Floquet-Bloch decomposition. The operator  $H_{\alpha, \Gamma}(\theta)$  on  $L^2(\Lambda)$  is for a  $\theta \in [-\pi, \pi)$  defined through the quadratic form as in Sec. 2.1; its domain consists of functions  $u \in W^{1,2}(\Lambda)$  satisfying the boundary conditions  $u(l_1, l_2 + \cdot) = e^{i\theta} u(0, \cdot)$ . In a standard way [EY01] one proves existence of a unitary  $\mathcal{U} : L^2(\mathbb{R}^2) \rightarrow \int_{[-\pi, \pi)}^{\oplus} L^2(\Lambda) d\theta$  such that

$$\mathcal{U} H_{\alpha, \Gamma} \mathcal{U}^{-1} = \int_{[-\pi, \pi)}^{\oplus} H_{\alpha, \Gamma}(\theta) d\theta \quad \text{and} \quad \sigma(H_{\alpha, \Gamma}) = \bigcup_{[-\pi, \pi)} \sigma(H_{\alpha, \Gamma}(\theta));$$

since  $\gamma((0, L))$  is compact we have  $\sigma_{\text{ess}}(H_{\alpha, \Gamma}(\theta)) = [0, \infty)$ . Next we need a comparison operator on the curve analogous to  $S_{\Gamma}$  of Theorem 4.1. For a fixed  $\theta \in [-\pi, \pi)$  it is given by the same symbol,

$$S_{\Gamma}(\theta) = -\frac{d^2}{ds^2} - \frac{1}{4}k(s)^2 \quad \text{on} \quad L^2((0, L))$$

with the domain  $\{u \in W^{2,2}((0, L)) : u(L) = e^{i\theta} u(0), u'(L) = e^{i\theta} u'(0)\}$ . We denote by  $\mu_j(\theta)$  the  $j$ -th eigenvalue of  $S_{\Gamma}(\theta)$  counted with multiplicity. Modifying the method of Sec. 4.1 to the present situation we get following result [EY01]:

**THEOREM 4.8.** *To any  $j \in \mathbb{N}$  there is an  $\alpha_j > 0$  such that  $\#\sigma_{\text{disc}}(H_{\alpha, \Gamma}(\theta)) \geq j$  holds for  $\alpha \geq \alpha_j$  and any  $\theta \in [-\pi, \pi)$ . The  $j$ -th eigenvalue of  $H_{\alpha, \Gamma}(\theta)$  counted with multiplicity has the asymptotic expansion*

$$\lambda_j(\alpha, \theta) = -\frac{1}{4}\alpha^2 + \mu_j(\theta) + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

as  $\alpha \rightarrow \infty$ , where the error term is uniform with respect to  $\theta \in [-\pi, \pi)$ .

Combining this result with Borg's theorem on the inverse problem for Hill's equation, we can make a claim about gaps of  $\sigma(H_{\alpha, \Gamma})$ :

**COROLLARY 4.9.** *Assume that  $\Gamma$  is not a straight line,  $k \neq 0$ , then the spectrum of  $H_{\alpha, \Gamma}$  contains open gaps for all  $\alpha$  large enough.*

In an exactly similar manner one can treat periodic curves in  $\mathbb{R}^3$  applying the technique to the fiber operator in the Floquet-Bloch decomposition of  $H_{\alpha, \Gamma}$  [EK04a]; more care is needed only when defining  $H_{\alpha, \Gamma}(\theta)$  since quadratic forms cannot be now used.

**THEOREM 4.10.** *Let  $\Gamma$  be a periodic curve, without self-intersections and with the global Frenet frame, given by a  $C^4$  smooth function  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ . Suppose, in addition, that the period cells  $\Gamma_p$  of  $\Gamma$  and  $\Lambda$  referring to the corresponding operator  $H_{\alpha,\Gamma}$  match in the sense that  $\Gamma_p = \Gamma \cap \Lambda$ . Then  $\sigma_{\text{disc}}(H_{\alpha,\Gamma}(\theta))$  has the properties analogous to those of the previous theorem, in particular, the  $j$ -th eigenvalue of  $H_{\alpha,\Gamma}(\theta)$  has the asymptotic expansion of the form*

$$\lambda_j(\alpha, \theta) = \zeta_\alpha + \mu_j(\theta) + \mathcal{O}(e^{\pi\alpha}) \quad \text{as } \alpha \rightarrow -\infty,$$

where  $\mu_j(\theta)$  is the  $j$ -th eigenvalue of  $S_\Gamma(\theta)$  and the error is uniform w.r.t.  $\theta$ .

**REMARKS 4.11.** (i) Corollary 4.9 has, of course, its three-dimensional analogue. The number of gaps which can be open depends on the shape of  $\Gamma$ . Notice that the operator  $\int_{[-\pi,\pi]}^\oplus S_\Gamma(\theta) d\theta$  has generically infinitely many open gaps in which case the corresponding  $H_{\alpha,\Gamma}$  can have any finite number for  $\alpha$  large enough.

(ii) The assumption about a match between the periodic decompositions of the curve  $\Gamma$  and the corresponding  $H_{\alpha,\Gamma}$  may seem restrictive. One should realize, however, the period cell need not be rectangular as one usually supposes when periodic Schrödinger operators are considered. What we actually need is a complete “tiling” of  $\mathbb{R}^\nu$  by domains with piecewise smooth boundaries. In the case  $\nu = 3$  such “bricks” need not even be simply connected: remember what your grandmother was doing with her crotchet to get an example of a curve which cannot be topologically deformed into a line, or if you want, unwind by pulling the ends.

(iii) The case we have discussed above, namely a single infinite curve in  $\mathbb{R}^\nu$  periodic in a given direction, is the simplest possibility. In a similar way one can treat infinite families of curves periodic in  $r \leq \nu$  directions, the only restriction is that their components have to satisfy individually the listed assumptions and the distances between them must have a uniform positive lower bound. The case  $r = \nu$  differs from  $r < \nu$  because then the periodic cell is precompact, and therefore the spectrum of each  $H_{\alpha,\Gamma}(\theta)$  is purely discrete.

(iv) A particular situation occurs when a periodic  $\Gamma$  consists of disjoint compact components. The asymptotic expansions are valid again but now the eigenvalues of the fiber comparison operator are independent of the parameter  $\theta$  so we have

$$\lambda_j(\alpha, \theta) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

and the respective expansion in the three-dimensional case (when the topology may be again nontrivial – imagine a chain of interlocked rings). Naturally, the chances to have open gaps in this situation are generally better than in the connected case.

(v) The last comment concerns interpretation of these results. Recall that the deviation of  $\sigma(H_{\alpha,\Gamma})$ , in the negative part at least, from the one described by the comparison operator is due to quantum tunneling. Hence it must be sensitive to the appropriate parameter, i.e. the Planck’s constant if we reintroduce it into the picture. However, the operator  $-h^2\Delta - v\delta(x - \Gamma)$  is the  $h^2$  multiple of (1.1) if we denote  $\alpha := vh^{-2}$ ; in this sense therefore the obtained asymptotic formulæ represent a semiclassical approximation.

Let us mention one more consequence of these considerations [**BDE03**].

**THEOREM 4.12.** *Suppose that the curve  $\Gamma \subset \mathbb{R}^\nu$  satisfies the assumptions of Theorems 4.8 and 4.10 for  $\nu = 2, 3$ , respectively. If  $\nu = 2$  to any  $\lambda > 0$  there is an  $\alpha_\lambda > 0$  such the spectrum of the operator  $H_{\alpha,\Gamma}$  is absolutely continuous in*

$(-\infty, -\frac{1}{4}\alpha^2 + \lambda]$  as long as  $\alpha > \alpha_\lambda$ . The same is true for  $\nu = 3$  with  $-\frac{1}{4}\alpha^2$  replaced by  $\zeta_\alpha$  provided  $-\alpha > \alpha_\lambda$ .

**SKETCH OF THE PROOF.** It is easy to check that  $\{H_{\alpha,\Gamma}(\theta) : \theta \in [-\pi, \pi]\}$  is a type A analytic family. The spectral interval in question contains a finite number of eigenvalue branches, each is a real analytic function which can be written through one of the above asymptotic expansions. The functions  $\mu_j(\cdot)$  are nonconstant, hence the same is true for  $\lambda_j(\alpha, \cdot)$  provided  $(-1)^\nu \alpha$  is large enough.  $\square$

To appreciate this result notice recall that if the period cell of the operator  $H_{\alpha,\Gamma}$  is precompact – cf. Remark 4.11(iv) above – there is a way to establish the (global) absolute continuity of such operators [BSŠ00, SŠ01] while the situation with  $r < \nu$  is more difficult; we will mention related results in Sec. 6.1 below.

**4.4. Periodic surfaces.** The treatment of operators  $H_{\alpha,\Gamma}$  corresponding to periodic surfaces is similar and we describe it only briefly referring to [Ex03] for details. We consider discrete translations of  $\mathbb{R}^3$  generated by an  $r$ -tuple  $\{l_i\}$ , where  $r = 1, 2, 3$ . We decompose  $\Gamma$ , supposed to be a  $C^4$  smooth Riemann surface, not necessarily connected, and  $\mathbb{R}^3$  into period cells  $\Gamma_p$  and  $\Lambda$  assuming again that they match mutually. The Floquet-Bloch decomposition proceeds as above: we define the fiber operators  $H_{\alpha,\Gamma}(\theta)$  on  $L^2(\Lambda)$  through quadratic forms defined on functions satisfying the appropriate boundary condition; after that we prove existence of a unitary  $\mathcal{U} : L^2(\mathbb{R}^3) \rightarrow \int_{[-\pi, \pi]^r}^\oplus L^2(\Lambda) d\theta$  such that

$$\mathcal{U}H_{\alpha,\Gamma}\mathcal{U}^{-1} = \int_{[-\pi, \pi]^r}^\oplus H_{\alpha,\Gamma}(\theta) d\theta \quad \text{and} \quad \sigma(H_{\alpha,\Gamma}) = \bigcup_{[-\pi, \pi]^r} \sigma(H_{\alpha,\Gamma}(\theta)).$$

The spectrum of  $H_{\alpha,\Gamma}(\theta)$  is purely discrete if  $r = 3$  while  $\sigma_{\text{ess}}(H_{\alpha,\Gamma}(\theta)) = [0, \infty)$  if  $r = 1, 2$ ; the eigenvalues are continuous functions of the quasi-momenta  $\theta_\mu$ .

As before we need a comparison operator. Its fibers act on  $L^2(\Gamma_p, d\Gamma)$  being defined, for instance, by the following prescription

$$S_\Gamma(\theta) := g^{-1/2}(-i\partial_\mu + \theta_\mu)g^{1/2}g^{\mu\nu}(-i\partial_\nu + \theta_\nu) + K - M^2$$

with the domain consisting of  $\phi \in W^{1,2}(\Gamma_p)$  such that  $\Delta_\Gamma \phi \in L^2(\Gamma_p, d\Gamma)$  satisfying periodic b.c. Since  $\Gamma_p$  is precompact and the curvatures involved are bounded, the spectrum of  $S_\Gamma(\theta)$  is purely discrete for each  $\theta \in [-\pi, \pi]^r$ ; we denote the  $j$ -th eigenvalue, counting multiplicity, as  $\mu_j(\theta)$ . In the same way as above we get

**THEOREM 4.13.** *Under the stated assumptions the following claims are valid:*

(a) Fix  $\lambda$  as an arbitrary number if  $r = 3$  and a non-positive one for  $r = 1, 2$ . To any  $j \in \mathbb{N}$  there is  $\alpha_j > 0$  such that  $H_{\alpha,\Gamma}(\theta)$  has at least  $j$  eigenvalues below  $\lambda$  for any  $\alpha > \alpha_j$  and  $\theta \in [-\pi, \pi]^r$ . The  $j$ -th eigenvalue  $\lambda_j(\alpha, \theta)$  has then the expansion

$$(4.5) \quad \lambda_j(\alpha, \theta) = -\frac{1}{4}\alpha^2 + \mu_j(\theta) + \mathcal{O}(\alpha^{-1} \ln \alpha)$$

as  $\alpha \rightarrow \infty$ , where the error term is uniform with respect to  $\theta$ .

(b) If the set  $\sigma(S) := \bigcup_{\theta \in [-\pi, \pi]^r} \sigma(S_\Gamma(\theta))$  has a gap separating a pair of bands, then the same is true for  $\sigma(H_{\alpha,\Gamma})$  provided  $\alpha$  is large enough.

**4.5. Magnetic loops.** Up to now we considered systems without external fields which leaves out many situation of experimental interest. For instance, one of the often studied features of mesoscopic systems are persistent currents in rings threaded by a magnetic flux. For a charged particle (an electron) confined to a loop  $\Gamma$  the effect is manifested by the dependence of the corresponding eigenvalues  $\lambda_n$  on the flux  $\phi$  through the loop, conventionally measured in the units of flux quanta,  $2\pi\hbar c|e|^{-1}$ . The derivative  $\partial\lambda_n/\partial\phi$  equals  $-\frac{1}{c}I_n$ , where  $I_n$  is the persistent current in the  $n$ -th state. For the ideal loop, e.g., the eigenvalues in absence of other than magnetic potential are proportional to  $(n + \phi)^2$  so the currents depend linearly on the applied field. The question is what can we say when the confinement is of the type discussed here realized through an attractive interaction on the loop.

We add a homogeneous magnetic field with vector potential  $A = \frac{1}{2}B(-x_2, x_1)$  to our considerations and investigate the Hamiltonian formally given by

$$(4.6) \quad H_{\alpha,\Gamma}(B) := (-i\nabla - A)^2 - \alpha\delta(x - \Gamma)$$

in  $L^2(\mathbb{R}^2)$ . To define it properly we use quadratic form analogous to (2.3),

$$\mathcal{E}_{-\alpha m, B}[\psi] = \left\| \left( -i\partial_x + \frac{1}{2}By \right) \psi \right\|^2 + \left\| \left( -i\partial_y - \frac{1}{2}Bx \right) \psi \right\|^2 - \alpha \int_{\mathbb{R}^2} |(I_m\psi)(x)|^2 dx$$

with the domain  $W^{1,2}(\mathbb{R}^2)$ . It is straightforward to check that the form is closed and below bounded; we identify the self-adjoint operator associated to it with  $H_{\alpha,\Gamma}(B)$ .

We use the same technique based on bracketing and estimating the operator in the strip in suitable coordinates. We need again a comparison operator, this time

$$S_\Gamma(B) = -\frac{d}{ds^2} - \frac{1}{4}k(s)^2$$

on  $L^2(0, L)$  with  $\psi(L-) = e^{iB|\Omega|}\psi(0+)$ ,  $\psi'(L-) = e^{iB|\Omega|}\psi'(0+)$ , where  $\Omega$  is the area encircled by  $\Gamma$ . Using it we can state the following result which establishes, in particular, the existence of persistent currents on a leaky loop for  $\alpha$  large enough.

**THEOREM 4.14.** *Let  $\Gamma$  be a  $C^4$ -smooth curve without self-intersections. For a fixed  $j \in \mathbb{N}$  and a compact interval  $I$  we have  $\sharp\sigma(H_{\alpha,\Gamma}(B)) \geq j$  for  $B \in I$  if  $\alpha$  is large enough, and the  $j$ -th eigenvalue behaves as*

$$\lambda_j(\alpha, B) = -\frac{1}{4}\alpha^2 + \mu_j(B) + \mathcal{O}(\alpha^{-1} \ln \alpha),$$

where  $\mu_j(B)$  is the  $j$ -th eigenvalue of  $S_\Gamma(B)$  and the error term is uniform in  $B$ . In particular, for a fixed  $j$  and  $\alpha$  large enough the function  $\lambda_j(\alpha, \cdot)$  cannot be constant.

**SKETCH OF THE PROOF.** The argument is closely similar to the analysis of fiber operators in Theorem 4.8, the magnetic flux replacing Floquet parameter, with small technical differences for which we refer to [EY02b].  $\square$

## 5. A discrete analogue

It is often useful to investigate in parallel with leaky graphs analogous discrete structures in which the attractive interaction is supported by suitably arranged families of point interactions. Let us briefly recall the basic notions, for more information and a rich bibliography we refer to [AGHH04]. Consider a set<sup>11</sup> of points  $Y = \{y_n\}_{n \in I} \subset \mathbb{R}^\nu$ ,  $\nu = 2, 3$ ; if  $I$  is infinite we suppose that  $Y$  can accumulate

<sup>11</sup>For simplicity we use the symbol  $Y$  both for the map  $I \rightarrow \mathbb{R}^\nu$  and its range.

only at infinity. The operators of interest are point-interaction Hamiltonians  $H_{\alpha,Y}$ , typically with the same interaction “strength” at each point, which are defined by means of the boundary conditions

$$(5.1) \quad L_1(\psi, y_j) - \alpha L_0(\psi, y_j) = 0, \quad j \in I,$$

expressed in terms of the generalized boundary values

$$L_0(\psi, y) := \lim_{|x-y| \rightarrow 0} \frac{\psi(x)}{\phi_d(x-y)}, \quad L_1(\psi, y) := \lim_{|x-y| \rightarrow 0} [\psi(x) - L_0(\psi, y) \phi_d(x-y)],$$

where  $\phi_d$  are the appropriate fundamental solutions, namely  $\phi_2(x) = -\frac{1}{2\pi} \ln|x|$  and  $\phi_3(x) = (4\pi|x|)^{-1}$  related to the free Green’s functions (2.10) and (2.13), respectively. The resolvent of  $H_{\alpha,Y}$  is given by Krein’s formula,

$$(5.2) \quad (-H_{\alpha,Y} - k^2)^{-1} = G_k + \sum_{j,j' \in I} [\Gamma_{\alpha,Y}(k)]_{j,j'}^{-1} \left( \overline{G_k(\cdot - y_{j'})}, \cdot \right) G_k(\cdot - y_j)$$

for  $k^2 \in \rho(H_{\alpha,Y})$  with  $\text{Im } k > 0$ , where  $\Gamma_{\alpha,Y}(k)$  is a closed operator (which is bounded in our case) on  $\ell^2(I)$  the matrix representation of which is

$$\Gamma_{\alpha,Y}(k) := \left[ (\alpha - \xi_d^k) \delta_{jj'} - G_k(y_j - y_{j'} (1 - \delta_{jj'})) \right]_{j,j' \in I},$$

where  $\xi_d^k$  is the regularized Greens’s function,  $\xi_2^k = -\frac{1}{2\pi} (\ln \frac{k}{2i} - \psi(1))$  and  $\xi_3^k = \frac{ik}{4\pi}$ . If  $\alpha$  is independent of  $j$ , the map  $k \mapsto \Gamma_{\alpha,Y}(k)$  is analytic in the open upper halfplane. Moreover,  $\Gamma_{\alpha,Y}(k)$  is boundedly invertible for  $\text{Im } k > 0$  large enough, while for  $k \in \mathbb{C}^+$  not too far from the real axis it may have a nontrivial null-space. By (5.2) the latter determines the spectrum of the original operator  $H_{\alpha,Y}$  on the negative halfline in view the following result analogous to Proposition 2.3.

- PROPOSITION 5.1. (i) A point  $-\kappa^2 < 0$  belongs to  $\rho(H_{\alpha,Y})$  iff  $\ker \Gamma_{\alpha,Y} = \{0\}$ .  
(ii) If the operator-valued function  $\kappa \mapsto \Gamma_{\alpha,Y}(i\kappa)^{-1}$  has bounded values in an open interval  $J \subset \mathbb{R}_+$  with the exception of a point  $\kappa_0 \in J$ , where  $\dim \ker \Gamma_{\alpha,Y}(i\kappa) = n$ , then  $-\kappa_0^2$  is an isolated eigenvalue of  $H_{\alpha,Y}$  of multiplicity  $n$ .  
(iii) an eigenfunction of  $H_{\alpha,Y}$  associated with such an eigenvalue  $-\kappa_0^2$  is equal to  $\psi = \sum_{j \in I} d_j G_{i\kappa_0}(\cdot - y_j)$ , where  $d = \{d_j\}$  solves the equation  $\Gamma_{\alpha,Y}(i\kappa_0)d = 0$ .

Let us now review discrete analogues of some results derived above.

**5.1. Curved polymers.** The operator  $H_{\alpha,Y}$  referring to a straight equidistant array  $Y$  is called a polymer model in [AGHH04]. Let us look what happens if we abandon the straightness. We adopt the following hypotheses:

- (ad) *analogue of (a1)–(a3)* : Let  $Y = \{y_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}^\nu$ ,  $\nu = 2, 3$ , be such that  $|y_j - y_{j+1}| = \ell$  for some  $\ell > 0$ . It implies  $|y_j - y_{j'}| \leq \ell |j - j'|$ ; we suppose that the inequality is sharp for some  $j, j' \in \mathbb{Z}$ . Next we assume that there is a  $c_1 \in (0, 1)$  such that  $|y_j - y_{j'}| \geq c_1 \ell |j - j'|$ , and moreover, that there are  $c_2 > 0$ ,  $\mu > \frac{1}{2}$ , and  $\omega \in (0, 1)$  such that the inequality

$$1 - \frac{|y_j - y_{j'}|}{|j - j'|} \leq c_2 [1 + |j + j'|^{2\mu}]^{-1/2}$$

holds if  $(j, j')$  belongs to the sector  $S_\omega$  of assumption (a2).

Recall first known facts [AGHH04] about a straight polymer,  $|y_j - y_{j'}| = \ell |j - j'|$  for all  $j, j' \in \mathbb{Z}$ . Its spectrum is purely absolutely continuous and consists of two bands which may overlap if  $\alpha$  is not large enough negative. Its threshold  $E_\nu^{\alpha, \ell}$  is always negative; in the three-dimensional case it is known explicitly,

$$E_3^{\alpha, \ell} = \frac{1}{\ell^2} \left[ \ln \left( 1 + \frac{1}{2} e^{-4\pi\alpha\ell} + e^{-2\pi\alpha\ell} \sqrt{1 + \frac{1}{4} e^{-4\pi\alpha\ell}} \right) \right]^2,$$

while for  $\nu = 2$  we have  $E_2^{\alpha, \ell} = -\kappa_{\alpha, \ell}^2$ , where  $\kappa_{\alpha, \ell}$  solves the equation

$$\alpha + \frac{1}{2\pi} (\gamma - \ln 2) = g_{i\kappa}(0)$$

with  $g_k(\theta) := \frac{1}{2\pi} \lim_{N \rightarrow \infty} \left\{ \sum_{n=-N}^N \frac{1}{2} \left[ \left( n + \frac{\theta\ell}{2\pi} \right)^2 - \left( \frac{k\ell}{2\pi} \right)^2 \right]^{-1/2} - \ln N \right\}$ .

**THEOREM 5.2.** *Let  $Y$  satisfy the assumptions (ad), then  $\sigma_{\text{ess}}(H_{\alpha, Y})$  is the same as for the corresponding straight polymer, and the operator  $H_{\alpha, Y}$  has at least one isolated eigenvalue below  $E_\nu^{\alpha, \ell}$  for any  $\alpha \in \mathbb{R}$ .*

**SKETCH OF THE PROOF.** The argument is a direct counterpart of that used in the proof of Theorem 3.1. One checks that the perturbation is sign-definite, pushing the spectrum of  $\Gamma_{\alpha, Y}$  down, and compact; the result then follows by continuity and Proposition 5.1, cf. [Ex01] for more details.  $\square$

**5.2. Isoperimetric problem.** Let us mention also a discrete analogue of the problem discussed in Sec. 3.5; we will think now of the point interactions as of beads on a loop-shaped string. To be precise, suppose the curve  $\Gamma$  is the range of a function  $\gamma : [0, L] \rightarrow \mathbb{R}^\nu$  which is continuous, piecewise  $C^1$  and such that  $\gamma(0) = \gamma(L)$ , and furthermore,  $|\dot{\gamma}(s)| = 1$  holds for any  $s \in [0, L]$  for which  $\dot{\gamma}(s)$  exists. We consider the set  $Y = \{y_j\}$  given by

$$(5.3) \quad y_j := \gamma \left( \frac{jL}{N} \right), \quad j = 0, 1, \dots, N-1,$$

with the indices regarded as integers,  $y_j = y_{j \pmod N}$ . A distinguished element of the described class is a regular polygon  $\tilde{\mathcal{P}}_N$  for which the points  $y_j$  lie in a plane  $\subset \mathbb{R}^\nu$  (this is trivial if  $\nu = 2$ ) at a circle of radius  $\frac{L}{N} (2 \sin \frac{\pi}{N})^{-1}$ .

We will suppose that the Hamiltonian  $H_{\alpha, Y}$  has the property analogous to (3.6), namely that it possesses a negative principal eigenvalue  $\lambda_1(\alpha, Y)$ ; this is automatically satisfied if  $\nu = 2$  while for  $\nu = 3$  it is true if  $-\alpha$  is large enough [AGHH04]. A counterpart to Theorem 3.11 now reads

**THEOREM 5.3.** *Under the stated assumptions the eigenvalue  $\lambda_1(\alpha, Y_\Gamma)$  is for a fixed  $\alpha$  and  $L > 0$  globally sharply maximized by a regular polygon,  $\Gamma = \tilde{\mathcal{P}}_N$ .*

We need a geometric result analogous to Proposition 3.12 which was proved in different ways in [Lu66, Ex06], see also [Ex05a] for a local proof.

**PROPOSITION 5.4.** *Let the set  $Y$  be such that  $|y_{j+1} - y_j| \leq \frac{L}{N}$ , then for any  $k$  and  $p \in (0, 2]$  the following inequality is valid*

$$\sum_{j=1}^N |y_{j+k} - y_j|^p \leq \frac{N^{1-p} L^p \sin^p \frac{\pi m}{N}}{\sin^p \frac{\pi}{N}}$$

Notice that the right-hand side is now the value of the sum for the regular polygon, and that a similar reverse inequality holds for negative powers  $p \in [-2, 0)$ .

**SKETCHED PROOF OF THEOREM 5.3.** The argument is analogous to that in the proof of Theorem 3.11. All the elements are in place: the ground state is non-degenerate and for the regular polygon it has a symmetry, this time with respect to a discrete group of rotations which implies that the corresponding eigenfunction of  $\Gamma_{\alpha, Y}(i\kappa)$  is  $N^{-1/2}(1, \dots, 1)$ . Using it to make a variational estimate and employing the strict convexity and monotony of the resolvent kernel we find that the chord-sum inequality of Proposition 5.4 has to be valid with  $p = 1$ ; a detailed account of the proof can be found in [Ex05a].  $\square$

**5.3. Approximation by point interactions.** Discrete “leaky graphs” described here are useful not only as mathematical objects analogous to the main topic of this review. As we are going to mention now, suitable families of them can be used to approximate the “true” leaky graphs. Importance of such an approximation stems from the fact that apart of particular cases where symmetry allows for separation of variables we have no efficient method to find spectral properties of the operator  $H_{\alpha, \Gamma}$ . It is true that Proposition 2.3 makes it possible to rephrase the original PDE problem as solution of an integral equation of Birman-Schwinger type but this is in general a task which not easy either; the purpose of the approximation is to convert it into an essentially algebraic problem.

To get an idea how to proceed in constructing the approximation one can compare the spectra of  $H_{\alpha, \Gamma}$  corresponding to a straight line  $\Gamma$  to that of a straight polymer mentioned in Sec. 5.1 above. We let the spacing between point interactions go to zero. If the two spectra should coincide in the limit the coupling parameter must be inversely proportional to the spacing. It looks queer at a glance but one has to keep in mind that the coupling described by the boundary conditions (5.1) becomes *weaker* as  $\alpha$  increases. We have the following result:

**THEOREM 5.5.** *Let  $\Gamma \subset \mathbb{R}^2$  be a finite graph obeying the assumptions (g1), (g2) and  $\alpha > 0$ . Choose  $k$  with  $\text{Im } k > 0$  such that the equation  $\sigma - \alpha R_{m, m}^k \sigma = \alpha R_{m, dx}^k \psi$  has for any  $\psi \in L^2(\mathbb{R}^2)$  a unique solution  $\sigma$  which has a bounded and continuous representative on  $\Gamma$ . Suppose next that there is a family  $\{Y_n\}_{n=1}^{\infty}$  of non-empty finite subsets of  $\Gamma$  such that  $|Y_n| := \sharp Y_n \rightarrow \infty$  and the following relations hold*

$$\frac{1}{|Y_n|} \sum_{y \in Y_n} f(y) \rightarrow \int_{\mathbb{R}^2} f(x) dm = \int_{\Gamma} f(s) ds$$

for any bounded continuous function  $f : \Gamma \rightarrow \mathbb{C}$ , and furthermore

$$\sup_{n \in \mathbb{N}} \frac{1}{|Y_n|} \sup_{x \in Y_n} \sum_{y \in Y_n \setminus \{x\}} G_k(x - y) < \frac{1}{\alpha |\Gamma|},$$

$$\sup_{x \in Y_n} \left| \frac{1}{|Y_n|} \sum_{y \in Y_n \setminus \{x\}} \sigma(y) G_k(x - y) - (R_{dx, m}^k \sigma)(x) \right| \rightarrow 0$$

as  $n \rightarrow \infty$ , where  $|\Gamma|$  is the sum of all the edge lengths in  $\Gamma$ . Then the family of the operators  $H_{\alpha_n, Y_n}$  with  $\alpha_n := |Y_n|(\alpha |\Gamma|)^{-1}$  approximates the leaky-graph Hamiltonian,  $H_{\alpha_n, Y_n} \rightarrow H_{\alpha, \Gamma}$  in the strong resolvent sense as  $n \rightarrow \infty$ .

SKETCH OF THE PROOF. The argument is straightforward even if executing it needs some effort. We have on one hand the resolvent of  $H_{\alpha,\Gamma}$  given by Proposition 2.3, on the other hand the resolvent of the approximating point-interaction Hamiltonians given by (5.2); one has to show that their difference applied to any  $\psi \in L^2(\mathbb{R}^2)$  tends to zero as  $n \rightarrow \infty$ , see [EN03] for details.  $\square$

REMARKS 5.6. (i) A similar results holds for approximations of leaky surfaces in  $\mathbb{R}^3$  by families of three-dimensional point interaction, cf. the paper [BFT98] where such an approximation was studied for the first time. Both the two-dimensional and three-dimensional results are valid for more general sets  $\Gamma$  and couplings which are non-constant functions over such a  $\Gamma$ .

(ii) Validity of the theorem extends to two-dimensional systems exposed to a magnetic field perpendicular to the plane. The field need not be homogeneous but it has to be sufficiently smooth [Ož06].

(iii) As a mean for numerical calculations the above theorem is not optimal. The convergence of eigenvalues is slow, roughly as  $\mathcal{O}(n^{-1/2})$ , and only in the strong resolvent sense. These flaws can be overcome by considering the family  $\epsilon^2\Delta^2 + H_{\alpha_n, Y_n}$  which converges to  $\epsilon^2\Delta^2 + H_{\alpha,\Gamma}$  in the norm-resolvent sense, and the limit approximates  $H_{\alpha,\Gamma}$  in the same sense as  $\epsilon \rightarrow 0$  [BO07]. In addition, the numerical approximations obtained by this method converge faster.

**5.4. Edge currents in the absence of edges.** Magnetic system exhibit interesting transport properties manifested, in particular, through edge currents discovered in [Ha82, MDS84] and studied in numerous subsequent papers. The effect is very robust and can be observed also in situations when the “edge” is thin indeed consisting of just an array of point interactions. Consider the situation when the interaction sites are arranged equidistantly along a line, for which we can without loss of generality take the  $x$ -axis in  $\mathbb{R}^2 \ni x = (x_1, x_2)$ , and a charged particle interacting with them is exposed to a homogeneous magnetic field  $B$  perpendicular to the plane. In such a situation it is natural to use the Landau gauge in which the Hamiltonian can be formally written as

$$(-i\partial_{x_1} + Bx_2)^2 - \partial_{x_2}^2 + \sum_j \tilde{\alpha}_j \delta(x_1 - x_1^{(0)} - j\ell),$$

where  $\ell$  is the interaction sites spacing; we write  $\tilde{\alpha}_j$  to stress that this formal constant is not identical with the “true” coupling parameter which enters the boundary conditions analogous to (5.1) in a proper definition of the operator.

The first thing to observe is that the infinitely degenerate eigenvalues which the Hamiltonian has in the absence of the point interactions are preserved in the spectrum, because one can construct eigenfunctions vanishing at  $(x_1^{(0)} + j\ell, 0)$ ,  $j \in \mathbb{Z}$ . Since the system is  $\ell$  periodic, one can perform Floquet decomposition and analyze the fiber operators on the strip, which plays here the role of period cell, with a single point interaction by Krein’s formula analogous to (5.2), arriving thus at the following conclusion [EJK99]:

**THEOREM 5.7.** *For any  $\alpha \in \mathbb{R}$  the spectrum of the indicated operator consists of the Landau levels  $B(2n+1)$ ,  $n = 0, 1, 2, \dots$ , and absolutely continuous spectral bands; between each adjacent Landau levels there is one such band and the lowest one lies below  $B$ , the unperturbed spectral threshold.*

Moreover, one can compute the probability current associated with the generalized eigenfunction for a fixed value of the quasi-momentum to see that it is a nontrivial vector field describing transport along the array [EJK99]. If the point interaction are arranged along a non-straight line, an explicit solution is no longer possible but the effect persists – see, e.g., [ChE03] for a regular polygon arrangement. This is interesting in connection with the result mentioned in Remark 5.6(ii): arranging point interactions densely around the loop we can approximate the operators discussed in Sec. 4.5, the “edge” currents are then nothing but the persistent currents considered there in the particular case of a strong coupling.

## 6. Other results

**6.1. Periodically modulated wires.** In Remark 2.1(i) we have mentioned that the definition extends to the situation with a non-constant coupling referring to the formal expression (1.1). Similarly we proceed in the case of codimension two; one has to replace  $\alpha$  in the boundary condition of (2.12) by  $\alpha(s)$ . We will denote such operators again  $H_{\alpha,\Gamma}$  where we have now  $\alpha : \mathbb{R} \rightarrow (0, \infty)$  for  $\nu = 2$  and  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  for  $\nu = 3$ . The following result is of a particular interest<sup>12</sup>:

**THEOREM 6.1.** *Suppose that  $\alpha \in L^\infty(\mathbb{R})$  is a periodic function, then the spectrum of  $H_{\alpha,\Gamma}$  is purely absolutely continuous; its negative part is non-empty and consists of at most finite number of bands.*

**SKETCH OF THE PROOF.** The argument is based of investigation of the scattering for the pair  $(H_{\alpha,\Gamma}, -\Delta)$ . Using the Floquet decomposition one establishes the existence of wave operators for the fibers and the limiting absorption principle; the method employs complexification of the quasi-momentum à la Thomas. To fill the details in the case  $\nu = 2$  one has to proceed as in [Fr03, Fr06, FS04] – note that the analogous result for a regular potential “ditch” was derived in [FK104] – the full proof for  $\nu = 3$  can be found in [EF07c].  $\square$

Let us mention also that in the case  $\nu = 3$  one has an analogous result in the situation when a periodic interaction is supported by an infinite family by parallel lines arranged equidistantly in a plane  $\subset \mathbb{R}^3$ , cf. [EF07c].

**6.2. A line-and-points model.** The number of explicitly solvable models in this area is not large, in particular, if we exclude those which can be treated by separation of variables. For instance, Theorems 3.1 and 3.17 tell us about the discrete spectrum and scattering due to a local perturbation of a straight leaky wire but it is difficult to find the eigenvalues or the on-shell S-matrix for a particular shape of the deformation. One can achieve more in a caricature model in which a straight line is perturbed just by a finite family of point interactions, so the Hamiltonian in  $L^2(\mathbb{R}^2)$  can be formally written as

$$-\Delta - \alpha\delta(x - \Sigma) + \sum_{i=1}^n \tilde{\beta}_i \delta(x - y^{(i)}),$$

where  $\Sigma := \{(x_1, 0); x_1 \in \mathbb{R}\}$ ,  $\alpha > 0$ , and  $Y := \{y^{(i)}\}$  is the perturbation support; we use again tilde to stress that the  $\tilde{\beta}_i$  are not the “true” coupling parameters appearing in the boundary conditions analogous to (5.1).

<sup>12</sup>Recall that for a non-straight periodic  $\Gamma$  and a constant  $\alpha$  we proved the absolute continuity only at the bottom of the spectrum provided the interaction is strong enough, cf. Theorem 4.12.

To analyze such an operator, denoted  $H_{\alpha,\beta}$ , properly defined, we need its resolvent. It can be expressed in a way similar to Proposition 2.3 with the auxiliary Hilbert space being now  $L^2(\mathbb{R}) \oplus \mathbb{C}^n$ , or alternatively in analogy with Theorem 3.14, i.e. by Krein's formula as a rank  $n$  perturbation to the resolvent of  $H_{\alpha,\Sigma}$ ; we refrain from stating the explicit formulæ which can be found with proofs in [EK04b]. Due to the finite rank of the perturbation the analysis reduces to an essentially algebraic problem. This allows us to prove various results, in particular

**THEOREM 6.2.** *Let  $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n$  and  $\alpha > 0$ . The operator  $H_{\alpha,\beta}$  has a non-empty discrete spectrum,  $1 \leq \sharp\sigma(H_{\alpha,\beta}) \leq n$ ; the number of eigenvalues is exactly  $n$  if all the numbers  $-\beta_i$  are large enough. In particular, if  $n = 1$  and  $y = (0, a)$ , there is a single eigenvalue; it tends to  $-\frac{1}{4}\alpha^2$  as  $|a| \rightarrow \infty$  when  $\zeta_\beta := -4e^{2(-2\pi\beta + \psi(1))} \in (-\frac{1}{4}\alpha^2, 0)$  and to  $\zeta_\beta$  in the opposite case.*

Recall the usual definition of a resonance as a pole in the analytical continuation of the resolvent across (a part of) the continuous spectrum to the lower half-plane.

**THEOREM 6.3.** *Let again  $n = 1$  and  $y = (0, a)$ , and suppose that  $\zeta_\beta > -\frac{1}{4}\alpha^2$ . Then for all  $|a|$  large enough the Hamiltonian has a unique resonance,  $z(a) = \mu(a) - i\nu(a)$  with  $\nu(a) > 0$ , which in the limit  $|a| \rightarrow \infty$  behaves as*

$$\mu(a) = \zeta_\beta + \mathcal{O}(e^{-a\zeta_\beta}), \quad \nu(a) = \mathcal{O}(e^{-a\zeta_\beta}).$$

We will not give details of the proofs referring to [EK04b] for a full exposition. In this paper also other spectral and scattering properties of this system are discussed as well as an extension to the three-dimensional situation with the interaction supported by a plane and a finite number of points. Furthermore, resonances in the case  $n > 1$  are discussed in [EIK07] where one analyzes also the decay of an eigenstate of  $H_{0,\beta}$  due to the presence of the leaky line – see also Sec. 7.18.

**6.3. Numerical results.** The emphasis in this review is on analytic approaches to the problem, hence numerical methods will be mentioned only briefly. As we have mentioned the cases in which the spectral problem can be solved by separation of variables are rare and in a sense trivial. Sometimes, however, one can divide  $\mathbb{R}^d$  into regions exterior to  $\Gamma$  in which such a separation is possible and to find the solutions by the method known in physics as “mode matching”. Consider, e.g., a circular  $\Gamma$  with an angle-dependent coupling; one can use inside and outside the circle the Ansätze  $\sum_{m \in \mathbb{Z}} c_m^{(\pm)} f_m^{(\pm)}(r) e^{im\varphi}$  where  $f_m^{(\pm)}$  are suitable Bessel functions and to find the coefficients  $c_m^{(\pm)}$  using the boundary conditions which couple the two parts of the plane – see [ET04] for this and similar examples.

The most versatile method, however, is based on the approximation of leaky-graph Hamiltonians by point interactions as we have discussed in Sec. 5.3; examples of spectra obtained in this way can be found in [EN01, EN03, Ož06]. A case of particular interest concerns resonances for  $H_{\alpha,\Gamma}$  when  $\Gamma$  is a straight line with a buckling having a narrow bottleneck. Of course, the approximation method applies to finite  $\Gamma$ 's only but one can use the approach popular among the physicists – rigorously justified so far only for one-dimensional potential scattering [HM00] – in which one cuts the system to a finite length and observes the dependence of eigenvalues on the cut-off size; resonances are manifested by a pattern in which intervals of almost constancy are interlaced with steep jumps. In our case resonances appear when the bottleneck half-width becomes comparable with the characteristic transverse size of the generalized eigenfunction (3.7), i.e.  $(2\alpha)^{-1}$ , cf. [EN03].

## 7. Open problems

While we have reviewed numerous results in the previous sections, the topic we are discussing here is new and many questions remain still open. In this closing section we are going to list some problems the reader may wish to learn about, or better, to attack. As usual in mathematics the number of generalizations is unlimited and we restrict only to those which we regard as both meaningful and reasonably close to the above exposition. Even in such a class one can find very different problems. Some of those listed below are rather technical and require mostly perspiration to achieve the result; others are more challenging when we do not know what the result might be, which method to apply, or possibly neither of these. Presenting the problems, however, we will not rank them according to these criteria but rather list them in the order they appear in the text.

**7.1. Approximation by regular potential “ditches”.** This is an important link between our singular model and description of realistic systems of “wires” of a small but finite width. In Theorem 2.2 we considered the simplest case of a single  $C^2$  curve  $\Gamma \subset \mathbb{R}^2$ . If  $\Gamma$  has angles, or even it is a nontrivial graph with branchings, one expects the analogous result to be valid, however, the method used in the proof is no longer applicable and it seems likely that the edge and vertex parts of the approximating potentials have to be treated separately. Similar results can be expected if the “edges” of  $\Gamma$  are surfaces in  $\mathbb{R}^3$ .

A related question concerns existence of curvature-induced bound states in such “fat” leaky graphs, as well as counterparts of other results discussed in Sec. 3. Using the above mentioned convergence result, combined possibly with minimax estimates, one can establish the analogue of Theorem 3.1 provided the potential “ditches” in question are sufficiently deep and narrow; it is less clear whether the claim will remain true generally outside the asymptotic regime.

On the other hand, a potential approximation to  $H_{\alpha,\Gamma}$  is more delicate in case of codimension two. A hint can be obtained from approximations of a two-dimensional point interaction [AGHH04] where one starts from a potential well having a zero-energy resonance and scales it in a particular nonlinear way. One can conjecture that this would yield an approximation for a single smooth  $\Gamma$  if we take such a potential family in the normal plane to  $\Gamma$ , while the problem may be more complicated in presence of angles and branchings.

**7.2. More singular leaky graphs.** The previous problem brings to mind a related question. Transverse to the graph edges we choose the  $\delta$  interaction as the mean to describe the way in which the particle is attracted to  $\Gamma$ . This is not the only possibility, though, even if we restrict ourselves to interactions supported by a single point. We know that such singular interactions form a four-parameter family containing some prominent cases such as the  $\delta'$  interaction [AGHH04]. Let us recall that the latter are not mere mathematical artefacts as it can be seen, e.g., from the fact that they are approximated by triples of properly scaled  $\delta$  interactions [CS98] and even by regular potentials [AN00, ENZ01].

It is appropriate at this place to mention that we have not discussed here counterparts of leaky-graph Hamiltonians with a *repulsive* interaction, formally  $-\Delta + \alpha\delta(x - \Gamma)$  with  $\alpha > 0$ , which can be introduced in the case  $\text{codim } \Gamma = 1$  as in Sec. 2.1; the reason is that the model has different spectral and scattering properties as well as physical interpretation. A combination of the attractive and

repulsive interactions on a “triplicated” graph where each edge of  $\Gamma$  is replaced by three close edges approximating in the cut the  $\delta'$  interaction according to [CS98] makes perfect sense, however, and justifies interest in such models.

For  $\delta'$  and more general leaky graphs we do not have at our disposal a “natural” quadratic form definition analogous to (2.3) or a Birman–Schwinger–type expression for the resolvent, hence it is a priori not clear which ones of the spectral and scattering properties discussed here can be extended to this case.

**7.3. More on curvature induced spectra.** The existence of a discrete spectrum for an infinite curve  $\Gamma$  which is not straight but it is asymptotically straight in a suitable sense as in Theorem 3.1 is an interesting result, however, a full understanding requires more. We already know that  $\#\sigma_{\text{disc}}(H_{\alpha,\Gamma})$  can be made arbitrarily large finite for a curve as simple as  $H_2(\beta)$  of Sec. 3.2. It is intuitively clear that a rich discrete spectrum is to be expected when the edges of  $\Gamma$  come close to each other at numerous places or over long stretches; it would be desirable to have a more quantitative expression of this intuitive statement.

It is useful to stress at this place that the leaky curves have a lot in common with *quantum waveguides* [EŠ89, DE95]. In both cases we study solutions to the Schrödinger equation localized in a tubelike region, here in a “soft” way through an attraction to a curve in contrast to the “hard” way with Dirichlet boundary conditions. This analogy makes it easier to understand effects like existence of localized solutions due to curvature because they are similar in both cases<sup>13</sup>. This analogy suggests various questions, for instance, about small bending asymptotic behavior. Take a curve which is straight outside a compact and differs only slightly from the straight line; an archetype of such a behavior is a broken line corresponding to  $H_2(\pi - \theta)$ . Using the method from the proof of Theorem 3.1 one can check that for  $\theta$  small enough such a  $H_{\alpha,\Gamma}$  has a single eigenvalue  $\lambda(\theta)$ . We conjecture that

$$(7.1) \quad \lambda(\theta) = -\frac{1}{4}\alpha^2 - c\theta^4 + \mathcal{O}(\theta^5)$$

in analogy with the slightly curved waveguide [ABGM91, DE95]. The positive constant  $c$  here depends on  $\alpha$ , in particular, for  $H_2(\pi - \theta)$  we have  $c = c'\alpha^2$ . The methods discussed in this paper, however, do not give a way to prove the conjectured asymptotic behavior and another approach is needed.

The assumption (a2) in Sec. 3.1 was used to ensure that the curvature induced spectrum is discrete. The proof of Theorem 3.1 shows, however, that any departure from a straight line pushes the spectrum down<sup>14</sup>, the question is which character does it have below  $-\frac{1}{4}\alpha^2$ . The cases to be analyzed include, e.g., curves with a slow curvature decay or a line with sparse deformations. Analogies with one-dimensional Schrödinger operators suggest that one might get different spectral types, however, they have to be used with caution since the other dimension may play a role<sup>15</sup>.

<sup>13</sup>This is not to say that all properties are the same. As example is provided by nodal lines of the discrete–spectrum eigenfunctions: in thin waveguides they can be described by means of the one-dimensional comparison operator [FKr07] while for leaky curves they extend over the plane and it is more complicated to analyze their behavior.

<sup>14</sup>Or alternatively, for a pure mathematician, to the left — the same for the previous text.

<sup>15</sup>Interesting effects can be seen also in the positive part of the spectrum. Recall the situation where  $\Gamma$  is an infinite family of concentric, equidistantly spaced circles, then the  $\sigma_{\text{ess}}(H_{\alpha,\Gamma})$  consists of interlaced intervals of absolutely continuous and dense pure point spectrum [EF07b].

The above question belongs to those which are open also in the quantum waveguide setting mentioned above. Another problem of this sort concerns the spectral multiplicity, specifically, one would like to know whether – or possibly under which conditions – is the discrete spectrum of Theorem 3.1 simple.

**7.4. More on star graphs.** The example discussed in Sec. 3.2 is simple, nevertheless, some open questions remain. Notice that the important parameters are the angles only, because a star-shaped  $\Gamma$  is self-similar and a change of  $\alpha$  is equivalent to a modification of the length scale. One can ask, for instance, about the configuration which maximized the ground state; a natural conjecture is that this happens in the case of the maximum symmetry,  $\beta_i = 2\pi/N$ ,  $i = 1, \dots, N$ . Less obvious is the answer to the question about existence of closed nodal lines which are expected if  $N$  is large enough. A numerical example [EN03] based on Theorem 5.5 suggests this happens, e.g., with the fourth eigenfunction of  $H_{10}(\pi/5)$ . One can ask what is the minimum  $N$  for which a leaky-star Hamiltonian has an eigenfunction with a closed nodal line, about stability of nodal patterns w.r.t. the angles, etc.

**7.5. Bound states for curved surfaces.** The analogy with quantum waveguides extends to the case of a surface in  $\mathbb{R}^3$ : by Theorem 4.7 we know that non-planar strongly attractive surfaces satisfying (as1)–(as3), or (as2) and (as3') give rise to a non-empty discrete spectrum. Recall that for Dirichlet quantum layers we also do not have an universal result; they bind if their width is small enough, if the total Gauss curvature is non-positive, or under various symmetry assumptions [DEK01, CEK04]. The thinness assumption is the analogue of a strong coupling,  $\alpha \rightarrow \infty$ , discussed here. One is interested whether and under which conditions does the curvature imply existence of bound states for a fixed coupling parameter  $\alpha$ .

Guided again by the quantum layer analogy [EKr01] one can ask about the weak-coupling asymptotics of bound states corresponding to mildly curved surfaces. In distinction to the conjecture (7.1) one guesses that in the two-dimensional case the binding will be exponentially weak with respect to a suitable deformation parameter. The method to prove this result is not clear, though.

**7.6. Bound states of nontrivial graphs of codimension two.** We have seen repeatedly that leaky graphs of codimension two require more subtle analysis. This applies already to the definition of  $H_{\alpha,\Gamma}$  which we have briefly described in Sec. 2.4 for the case when  $\Gamma$  is a curve. There is little doubt that the boundary conditions (2.12) can be used for a non-trivial graphs  $\Gamma$  as well, however, the proof using the resolvent formula (2.14) has to be worked out properly.

Another question concerns the possible analogue of Corollary 3.2. Due to the absence of a “natural” quadratic form associated with  $H_{\alpha,\Gamma}$  it is not obvious that bound states are preserved when edges are added to a non-straight curve creating a non-trivial graph, although one expects that this will be the case.

**7.7. Geometric perturbations for general graphs.** The results discussed in Sec. 3.4 are formulated for a single smooth curve or manifold. With an additional effort one can extend validity of the asymptotic formulæ to a general graph  $\Gamma$  provided the point at which the hiatus is centered is in the interior of an edge. While a similar claim may be still true for edges “narrowly disconnected” in a vertex, a more subtle analysis is needed to see whether such a claim is valid.

**7.8. More on isoperimetric problems.** Theorem 3.11 raises naturally various questions about possible extensions. A modification for a loop in  $\mathbb{R}^3$  is relatively straightforward and one can ask also whether there are local maxima for loops of a fixed knot topology. On the other hand, the corresponding problem for a closed surface in  $\mathbb{R}^3$  seems to be more complicated and direct generalizations do not work. Returning to the two-dimensional situation, and referring to Sec. 7.4 above, one can ask more generally about configurations which maximize the ground state for  $\Gamma$  with a fixed topology and edge lengths, etc.

**7.9. More on absolute continuity.** The case of  $H_{\alpha,\Gamma}$  with a periodic curve  $\Gamma$  remains to be one of the important challenges in this area. One certainly expects that the spectrum of such an operator will be purely absolutely continuous, however, proof of this assertion is missing. So far we have only the result in the related but different case expressed by Theorem 6.1, and a partial answer to the question given by Theorem 4.12; let us note that the absolute continuity is expected to be valid even for  $\Gamma$  consisting of infinitely many disconnected finite components for which the strong-coupling argument does not work.

**7.10. More on scattering.** The setting we choose in Sec. 3.6 has the advantage of simplicity offered by the straight-line comparison operator  $H_{\alpha,\Sigma}$ . Scattering for a general leaky graph with straight leads outside a compact can be treated similarly, but needs a more elaborate formulation: one has to compare motion on each external lead with that of  $H_{\alpha,\Sigma}$  using a suitable identification operator. The situation becomes even more complicated when the leads are not straight but only asymptotically straight, in particular, if the “curvature decay” is slow so the curvature-induced interaction is of a long range.

A different class of problems concerns scattering at positive energies. Here one has to distinguish two cases. If  $\Gamma$  is compact we deal with standard scattering on a potential, albeit a singular one; this situation was discussed in [BT92]. On the other hand, if there are infinite edges the situation is more complicated due to presence of guided states; note that they can exist also at positive energies, just take (3.7) with  $\lambda > 0$ . This case remains so far largely untreated.

**7.11. Strong coupling behavior of scattering.** Speaking about scattering one may ask whether there are asymptotic formulæ analogous to those we derived for the discrete spectrum in Sec. 4. We again suppose that  $\Gamma$  is a connected infinite  $C^4$  smooth curve straight outside a compact. For large  $\alpha$  the wave function is localized in a small neighborhood of  $\Gamma$  and one can expect that the reflection and transmission amplitudes which determine the scattering in the negative part of the spectrum will be determined in the leading order by the local geometry of the curve.

Let  $\mathcal{S}_{\Gamma,\alpha}(\lambda)$  be the on-shell scattering matrix at energy  $\lambda$  for the operator  $H_{\alpha,\Gamma}$ , compared to the free dynamics in the leads, and denote by  $\mathcal{S}_{S_\Gamma}(\lambda)$  the corresponding quantity for the one-dimensional comparison operator  $S_\Gamma$  of Theorem 4.3. One conjectures that for a fixed  $k \neq 0$  and  $\alpha \rightarrow \infty$  we have the relation

$$\mathcal{S}_{\Gamma,\alpha}\left(k^2 - \frac{1}{4}\alpha^2\right) \rightarrow \mathcal{S}_{S_\Gamma}(k^2);$$

to prove it and to find the convergence rate one has to show that the corresponding generalized eigenfunction of  $H_{\alpha,\Gamma}$  and that of  $S_\Gamma$  multiplied by the transverse eigenfunction analogous to that in (3.7) converge to each other, and how fast.

**7.12. More on strong coupling asymptotics.** This brings us to strong-coupling asymptotic results of Sec. 4. One question left open there concerns manifolds with boundaries; for simplicity let us consider a finite curve  $\Gamma$  which is not closed. As we have pointed out in the proof of Theorem 4.1 the direct use of the bracketing technique gives the first assertion of the theorem but they are not precise enough to yield an asymptotic formula. We conjecture that it is again of the form

$$\lambda_j(\alpha) = -\frac{1}{4}\alpha^2 + \mu_j + \mathcal{O}(\alpha^{-1} \ln \alpha),$$

where  $\mu_j$  is now the  $j$ -th eigenvalue of the operator given by the same expression, but with *Dirichlet* boundary conditions. A way to prove this result would be to consider a bracketing on prolonged tubular neighborhoods extending to the distance of order  $a$  over the endpoints of  $\Gamma$ ; a trouble to overcome is that one cannot then separate variables in the leading order.

All the considerations of Sec. 4 required a sufficient smoothness of  $\Gamma$ . If this is not true the results are no longer valid. As an example consider again the operator  $H_2(\beta)$  of Sec. 3.2; using a scaling transformation we find that  $\mu_j$  in the above formula has to be replaced by  $-(\lambda_j + \frac{1}{4})\alpha^2$  where  $\lambda_j$  is the  $j$ -th eigenvalue of  $H_2(\beta)$  corresponding to  $\alpha = 1$ . We conjecture that if an otherwise smooth  $\Gamma$  has one angle equal to  $\pi - \beta$ , the  $j$ -th eigenvalue asymptotics of the corresponding  $H_{\alpha,\Gamma}$  is again  $\lambda_j\alpha^2 + o(\alpha^2)$ ; if there are more angles the situation is similar but the numbering of the eigenvalues changes in the appropriate way.

**7.13. Strong-coupling graph limit.** So far we have considered strong coupling for a single curve. If  $\Gamma$  is a nontrivial graph with branchings, one expects a behavior similar to that of a curve with angles: each vertex will contribute to the asymptotics below  $-\frac{1}{2}\alpha^2$  by  $\lambda_j\alpha^2 + o(\alpha^2)$  where  $\lambda_j$ 's are now the eigenvalues of the corresponding leaky star-graph Hamiltonian  $H_n(\beta)$  with  $n$  being the vertex degree.

This leads us to the question, whether in the limit  $\alpha \rightarrow \infty$  one can get a meaningful expression of the standard quantum graph type, i.e. an operator on  $L^2(\Gamma)$ . The question makes sense, of course, only if we perform a suitable energy renormalization. The most natural way to do that is to stay in the vicinity of the “transverse threshold”, i.e. to subtract the diverging factor  $-\frac{1}{4}\alpha^2$ . If the above conjecture is valid the limit will be generically trivial, i.e. a graph with Dirichlet-decoupled edges. There could be nontrivial limits, however, in cases when the involved family of leaky-graph Hamiltonians has a threshold resonance (or a singularity which stays in the vicinity of such a resonance in the limiting process). The problem is analogous to the squeezing of Dirichlet fat graphs mentioned in the introduction, cf. [Po05, CE07, Gr07], and it is likely to be no less difficult. One can also conjecture that if the reference point is chosen instead as  $\lambda\alpha^2$  with  $\lambda > -\frac{1}{4}\alpha^2$  one gets generically a nontrivial limit with the vertex conditions determined by the scattering on the appropriate leaky star graph in analogy with [MV07].

**7.14. More on resonances.** The presence or absence of resonances is one of the most important features in scattering. In this survey we touched that subject but not very deeply: we were able to establish the existence of resonances in the caricature model of Sec. 6.2 and we quoted numerical results which suggest that resonances are present in some other situations. The first problem here is whether one is able to establish the existence of resonances for a wider class of operators  $H_{\alpha,\Gamma}$  with an infinite  $\Gamma$  having some number of (asymptotically) straight “leads”.

The question has to be put more precisely. The resonances in Theorem 6.3 are understood in the common sense as poles of the analytically continued resolvent. For the model in question one can check easily [EK04b] that they are at the same time singularities of the scattering matrix. A similar equivalence between the resolvent and scattering resonances is expected to be valid for other leaky-graph Hamiltonians as well, and its verification should accompany the existence proof.

One more problem concerns the method we mentioned in Sec. 6.3, called usually the  $L^2$ -*approach* by the physicists, which allows to identify resonances by spectral methods. As we have noted the method is rigorously justified for the one-dimensional potential scattering only [HM00] and it is naturally desirable to find an appropriate formulation and proof of it in the present situation.

**7.15. More on magnetic leaky graphs.** This is another subject not much explored so far, and also one where the analogy between leaky graphs and quantum waveguides sometimes fails, e.g., a waveguide in a homogeneous field has dominantly a continuous spectrum due to edge states “skipping” along the boundary while the spectrum of its leaky-graph counterpart is a point one being dominated by the Landau levels. A numerical example worked out in [ET04] using mode matching indicates that the asymptotic behavior described in Theorem 4.14 can be destroyed if we keep  $\alpha$  fixed, albeit large, and make the magnetic field strong.

In connection to that one can ask, e.g., what will happen with the curvature-induced bound states of Theorem 3.1 if the system is exposed to a magnetic field à la (4.6), in particular, whether they will survive an arbitrarily strong field. In a similar vein, it is possible to ask what will happen with the curious magnetic transport described in Theorem 5.7 if the point interaction array in question is not straight but “bent” and only asymptotically straight.

A much weaker perturbation is represented by local magnetic fields. Here again one can derive an inspiration from the waveguide theory: it is known that spectrum of a waveguide with such a field is, contrary to the non-magnetic case, stable with respect to small perturbation as a consequence of a Hardy-type inequality [EKo05]. One can ask whether a similar result is true for leaky curves slightly different from a straight line if a local magnetic field, regular or singular, is present.

**7.16. Perturbations of periodic graphs.** In addition to the absolute continuity problem mentioned above periodic graphs pose other questions. The spectrum is expected to have a gap structure, and in some cases one is able to establish existence of open gaps as, e.g., in Corollary 4.9. In such a case one can ask about the effect of local perturbation to such a periodic  $\Gamma$ , in particular, under which conditions they give rise to eigenvalues in gaps, what is the number of the latter and their dependence on the perturbation parameters, etc.

**7.17. Random leaky graphs.** Quantum graphs with various sorts of randomness have been recently an object of intense interest – see, e.g., [ASW06, EHS07, KP07, GV07] and references therein. In contrast, not much was done about the analogous problem for leaky graphs which are certainly of interest because they mix elements of a one-dimensional and multidimensional behavior. One naturally expects a localization if either the edge shapes or the coupling constants on them become random, and there is a numerical evidence supporting these expectations [ET04]. The truly important question, however, concerns the existence and properties of a mobility edge in such systems.

The effects of randomness can be also studied through a point interaction counterpart to the operators  $H_{\alpha,\Gamma}$ . If we take a point-interaction polygon with a magnetic field mentioned in Sec. 5.4 and randomize the coupling constant, one can observe numerically how the transport is destroyed. One can ask therefore what can be proven about the spectrum of the model from Theorem 5.7 if the coupling constants are made random, whether there is a localization and whether a part of the absolutely continuous spectrum will survive – recall that for a similar model with an array replaced by a rectangular point-interaction lattice a localization was proved in the low-lying spectral bands [DMP99].

**7.18. Time evolution.** So far we spoke about the stationary aspect of the problem, even when we discussed problems such as scattering. We do not know much about the way the wave functions evolve in leaky-graph systems. One can ask, for instance, about smoothing properties analogous to those of the usual Schrödinger operators. In some cases this is true — take the example of  $H_{\alpha,\Gamma}$  corresponding to a straight line in the plane — but a caution is needed since it is known that the time evolution in systems with singular interactions can have sometimes rather counterintuitive properties [EF07a].

A more specific question is associated with resonances in quantum graphs. The usual duality between resonances and unstable states motivates us to ask what happens if we fix a resonant state, typically an eigenstate of an unperturbed operator associated with  $H_{\alpha,\Gamma}$  embedded into the continuum, at an initial instant, and ask about the way in which it decays. For the caricature model of Sec. 6.2 it can be done [EIK07]; one would welcome to learn more about the decay processes in physically more interesting cases.

This problem list is in no case exhaustive. If the reader made it to this point — my greetings to such a persistent colleague — I am sure he or she managed to formulate many additional questions on the way, and is ready to address them. It remains for me only to express a good luck wish in such an effort.

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