ABSOLUTELY CONTINUOUS SPECTRUM OF ONE RANDOM ELLIPTIC OPERATOR

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1. Formulation of the main result

In dimension $d \geq 5$, we consider the differential operator

$$H_0 = -\Delta + \tau\zeta(x)|x|^{-\varepsilon}(-\Delta_\theta), \qquad \varepsilon > 0,\ \tau > 0, \tag{1.1}$$

where Δ_θ is the Laplace-Beltrami operator on the unit sphere $\mathbb{S} = \{x \in \mathbb{R}^d : \ |x| = 1\}$ and ζ is the characteristic function of the complement to the unit ball $\{x \in \mathbb{R}^d : \ |x| \leq 1\}$. The standard argument with separation of variables allows one to define this operator as the orthogonal sum of one-dimensional Schrödinger operators, which implies that H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$.

The spectrum of this operator has an absolutely continuous component, which coincides with the positive half-line $[0,\infty)$ as a set. We perturb now the operator H_0 by a real valued potential $V = V_\omega$ that depends on the random parameter $\omega = \{\omega_n\}_{n\in\mathbb{Z}^d}$:

$$V_\omega = \sum_{n\in\mathbb{Z}^d} \omega_n \chi(x-n).$$

The function χ in this formula is the characteristic function of the cube $[0,1)^d$, and ω_n are independent random variables taking their values in the interval $[-1,1]$. We will assume that $\mathbb{E}[\omega_n] = 0$ for all n. This condition guarantees the presence of oscillations of V. Set

$$H_\pm = H_0 \pm V_\omega$$

Theorem 1.1. *Let $0 < \varepsilon < 2/(d+1)$. Then, for any $\tau > 0$, the operator $H_\pm$ has an absolutely continuous spectrum, whose essential support covers the positive half-line $[0,\infty)$. In other words, the spectral projection $E_{H_\pm}(\delta)$ corresponding to a set $\delta \subset \mathbb{R}_+$ of positive Lebesgue measure, is different from zero.*

Proof. We shall consider only the case when $\tau = 1$, because the only property of τ that matters is that it is positive. The proof of the theorem is based on two sufficiently deep observations.

1. The entropy of the spectral measure of the operator H_+ (as well as H_-) can be estimated by the negative eigenvalues of the operators H_+ and

H_-. Let us clarify this statement. Let μ be the spectral measure of H_+, constructed for the element f, which means that

$$((H_+ - z)^{-1} f, f) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{t-z}$$

Let $\lambda_j(\mp V)$ be the negative eigenvalues of the operator $H_\pm$. Then one can find such an element f of the space $L^2(\mathbb{R}^d)$, that the measure constructed for this element will satisfy the condition

$$\int_a^b \log \mu'(\lambda)\, d\lambda \geq -C(1 + \sum_j |\lambda_j(V)|^{1/2} + \sum_j |\lambda_j(-V)|^{1/2}), \tag{1.2}$$

where $0 < a < b < \infty$ and the constant C depends only on a and b. The proof of this statement can be found in [5]. Due to Jensen's inequality, the integral $\int_a^b \log \mu'(\lambda)\, d\lambda$ can diverge only to $-\infty$. But, if it converges, then $\mu'(\lambda) > 0$ almost everywhere on $[a, b]$, which leads to a certain conclusion about the absolutely continuous spectrum of H_+.

Let us draw attention of the reader to the main difficulty of application of (1.2): it is derived only for compactly supported perturbations and one has to make sure that it survives in the limit, when V is approximated by compactly supported functions.

2. Within the conditions of the theorem,

$$\mathbb{E}\Big[\sum_j |\lambda_j(\pm V)|^{1/2}\Big] < \infty, \tag{1.3}$$

which implies

$$\sum_j |\lambda_j(\pm V)|^{1/2} < \infty, \qquad \text{almost surely}.$$

Actually, it is much better to take the expectation in both sides of (1.2) and then talk about approximations of V by compactly supported functions, instead of doing it directly.

Let us introduce the notation $\bar{V}$ for the mean value of V_ω over the sphere of radius $|x|$

$$\bar{V}(x) = \frac{1}{|\mathbb{S}|} \int_{\mathbb{S}} V_\omega(|x|\theta)\, d\theta.$$

In order to establish (1.3) we will show that $V_\omega = \bar{V} + \operatorname{div} Q$ where Q is a vector potential having no radial component, i.e. $< x, Q(x) >= 0,\ \forall x$. Besides this, we will show that Q can be chosen in such a way that

$$\mathbb{E}\Big[\int_{|x|=R} |Q|^p dS\Big] \leq C\mathbb{E}\Big[\int_{|x|=R} |V_\omega|^p dS\Big], \qquad R > 2, \tag{1.4}$$

where C depends only on the dimension d and the parameter $p \geq 2$.

Our arguments will be based on the fact that the operator

$$H_0 \pm 2V \mp 2\bar{V} + 4|x|^{-2+\varepsilon}Q^2 \geq 0 \tag{1.5}$$

is positive, and therefore it does not have negative eigenvalues. The reason why the relation (1.5) holds is that the operator in its left hand side is representable in the form

$$-\Delta + \Big(|x|^{-\varepsilon/2}\nabla_\theta \mp 2|x|^{-1+\varepsilon/2}Q\Big)^* \Big(|x|^{-\varepsilon/2}\nabla_\theta \mp 2|x|^{-1+\varepsilon/2}Q\Big)$$

We will keep the relation (1.5) in mind and leave it for the moment. In order to apply (1.5) we have to understand how the eigenvalue sums $\sum_j |\lambda_j(\pm V)|^{1/2}$ behave. We will show that

$$\sum_j |\lambda_j(V_1 + V_2)|^\gamma \leq \sum_j |\lambda_j(\varepsilon^{-1}V_1)|^\gamma + \sum_j |\lambda_j((1-\varepsilon)^{-1}V_2)|^\gamma \tag{1.6}$$

In order to do that we need to recall the Birman-Schwinger principle that reduces the study of eigenvalues $\lambda_j(V)$ to the study of the spectrum of a certain compact operator. For any self adjoint operator T and $s > 0$ we define

$$n_+(s,T) = \mathrm{rank} E_T(s, +\infty),$$

where $E_T(\cdot)$ denotes the spectral measure of T. Recall the following relation (see [4])

$$n_+(s+t, T+S) \leq n_+(s,T) + n_+(t,S); \tag{1.7}$$

The next statement is known as the Birman-Schwinger principle.

Lemma 1.1. *Let V be a real valued function defined on the space $\mathbb{R}^d$. Let $N(\lambda, V)$ be the number of eigenvalues of $H_0 - V$ below $\lambda < 0$. Then*

$$N(\lambda, V) = n_+(1, (H_0 - \lambda)^{-1/2} V (H_0 - \lambda)^{-1/2}).$$

Combining this lemma with (1.7) we obtain

Corollary 1.1. *For any $\varepsilon \in (0,1)$*

$$N(\lambda, V_1 + V_2) \leq N(\lambda, \varepsilon^{-1}V_1) + N(\lambda, (1-\varepsilon)^{-1}V_2). \tag{1.8}$$

Now, since $\sum_j |\lambda_j(V)|^\gamma = \int_0^\infty \gamma s^{\gamma-1} N(-s, V)\, ds$, the inequality (1.6) holds for the Lieb-Thirring sums.

Due to the representation $V = (V - \bar{V} - 2|x|^{-2+\varepsilon}Q^2) + \bar{V} + 2|x|^{-2+\varepsilon}Q^2$, we obtain from (1.6), that

$$\sum |\lambda_j(V)|^\gamma \leq \sum |\lambda_j(2V - 2\bar{V} - 4|x|^{-2+\varepsilon}Q^2)|^\gamma + \sum |\lambda_j(2\bar{V} + 4|x|^{-2+\varepsilon}Q^2)|^\gamma. \tag{1.9}$$

Since the operator (1.5) is positive, the first sum in the right hand side of (1.9) equals zero. Thus

$$\sum|\lambda_j(V_\omega)|^\gamma\le\sum|\lambda_j(2\bar V+4|x|^{-2+\varepsilon}Q^2)|^\gamma. \tag{1.10}$$

Now formula (1.10) and the classical Lieb-Thirring estimate (see [6] and [7]) lead to the following important intermediate result:

$$\sum_j|\lambda_j(V_\omega)|^\gamma\le C\Bigl(\int||x|^{-1+\varepsilon/2}Q|^{d+2\gamma}dx+\int|\bar V|^{d/2+\gamma}dx\Bigr). \tag{1.11}$$

Indeed,

Theorem 1.2. [Lieb-Thirring] *If $d\ge3$. Then the negative eigenvalues ν_j of $-\Delta-V$ satisfy the estimate*

$$\sum_j|\nu_j|^\gamma\le C\int|V(x)|^{\gamma+d/2}\,dx,\qquad\gamma\ge0.$$

What is left to prove at this moment? If we take the expectation in both sides of (1.11), then we shall reduce the problem to the proof of the two relations:

$$\mathbb{E}\Bigl[\int||x|^{-1+\varepsilon/2}Q|^{d+1}dx\Bigr]<\infty \tag{1.12}$$

and

$$\mathbb{E}\Bigl[\int|\bar V|^{(d+1)/2}dx\Bigr]<\infty. \tag{1.13}$$

The relation (1.12) for $\varepsilon<2/(d+1)$ immediately follows from (1.4). We shall establish (1.13) in the next section.

2. Proof of (1.4) and (1.13)

Let us now prove the necessary estimates (1.4) and (1.13). We shall begin with the following statement.

Lemma 2.1. *The relation* (1.4) *holds for even integers $p=2q$ with $q\ge1$.*

Proof. The mapping $V\mapsto Q$ is given by the formula

$$Q=|x|\nabla_\theta\Delta_\theta^{-1}(V-\bar V) \tag{2.1}$$

The kernel $k(x,y)$ of this mapping has a singularity of order $|x-y|^{-(d-2)}$ on the diagonal, but this singularity is integrable. Thus,

$$k(x,y)=\frac{k_0(\theta_1,\theta_2)}{|x-y|^{d-2}},\qquad\theta_1=\frac{x}{|x|},\theta_2=\frac{y}{|y|},$$

where $k_0 \in L^\infty$,

$$Q(x) = \int_{\{y\colon\ |y|=|x|\}} k(x,y)V_\omega(y)dS(y).$$

All the statements about the kernel k will be proved in the appendix section (see Proposition 4.1).

We represent Q in the form of a sum $Q = Q_1 + Q_2$,where

$$Q_1 = \int_{\{y\colon\ |y|=|x|\}} \frac{k_0}{|x-y|^{d-2}}\chi_0(x-y)V(y)dS(y)$$

and χ_0 is the characteristic function of the unit ball $\{x:\ |x|<1\}$. We will establish the estimates

$$\int_{|x|=R} \mathbb{E}\Big[|Q_j|^{2q}\Big]dS(x) \le CR^{d-1}, \qquad j=1,2, \tag{2.2}$$

separately. The estimate (2.2) for Q_1 is obvious. Indeed,

$$|Q_1(x)| \le C\int_{\{y\colon\ |y|=|x|\}} \frac{\chi_0(x-y)}{|x-y|^{d-2}}dS(y) \le C_1$$

Let us prove estimate (2.2) for Q_2. Fix $x \in \mathbb{R}^d$ and denote $\Delta_n = ([0,1)^d + n) \cap \{y \in \mathbb{R}^d:\ |y| = |x|\}$ Since $\mathbb{E}[\omega_n] = 0$, we obtain that

$$\mathbb{E}[Q_2^{2q}(x)] \le C_2 \sum_{m_1+\dots+m_k=2q} \prod_j \frac{2q!}{m_1!\dots m_k!} \sum_n \Big(\int_{\Delta_n} \frac{dS(y)}{(1+|x-y|)^{d-2}}\Big)^{m_j}$$

where all numbers m_j are even. Applying the Hölder inequality for L^p-functions, we get

$$\sum_n \Big(\int_{\Delta_n} \frac{dS(y)}{(1+|x-y|)^{d-2}}\Big)^{m_j} \le C_3 \sum_n \Big(\int_{\Delta_n} \frac{dS(y)}{(1+|x-y|)^{2(d-2)}}\Big)^{m_j/2} \le$$

$$\le C_4 \sum_n \int_{\Delta_n} \frac{dS(y)}{(1+|x-y|)^{2(d-2)}} = C_4 \int_{\{y\colon\ |y|=|x|\}} \frac{dS(y)}{(1+|x-y|)^{2(d-2)}}$$

simply because all $m_j \ge 2$ and Δ_n are uniformly bounded.

Consequently,

$$\mathbb{E}[Q_2^{2q}(x)] \le C_5$$

Integrating this inequality with respect to x we obtain (2.2) for $j = 2$. Thus the statement of the lemma follows from the triangle inequality in the Banach space L^{2q}

$$\Big(\int \mathbb{E}[Q^{2q}(x)]\, dS\Big)^{1/2q} \le \Big(\int \mathbb{E}[Q_1^{2q}(x)]\, dS\Big)^{1/2q} + \Big(\int \mathbb{E}[Q_2^{2q}(x)]\, dS\Big)^{1/2q}.$$

(Recall that $\mathbb{E}[f] = \int_\Omega f(\omega)\, d\omega$.) $\square$

Estimate (1.4) is proved only for even integer p. It follows for arbitrary $p \geq 2$ by interpolation arguments. Indeed, consider the analytic function

$$f(z) = \mathbb{E}\Big[\int_{\{x:\ |x|=R\}} |Q|^z dS\Big].$$

Since this function is bounded by the constant $C_p R^{d-1}$ on each vertical line Re $z = 2p$, with $p \in \mathbb{N}$, it is bounded by $C_{\mathrm{Re}(z)} R^{d-1}$ for Re $z \geq 2$ according to the following statement:

Theorem 2.1. *Let $f(z)$ be a bounded analytic function defined on an open domain containing the region $a \leq$ Re $z \leq b$. Suppose that it is bounded on the two vertical lines by the constants C_a and C_b:*

$$|f(a+it)| \leq C_a, \qquad |f(b+it)| \leq C_b, \qquad \forall t \in \mathbb{R}.$$

Then

$$|f(z)| \leq C_a^\theta C_b^{1-\theta}, \qquad \textit{for} \quad \mathrm{Re}\, z = a\theta + b(1-\theta), \quad \theta \in (0,1).$$

Now we obtain a certain information about the mean values of V over the sphere of radius R.

Lemma 2.2. *Under conditions of Theorem 1.1*

$$\mathbb{E}\Big[\int |\bar V|^{(d+1)/2} dx\Big] < \infty. \tag{2.3}$$

Proof. First we shall prove that

$$\mathbb{E}[|\bar V|^{2q}(x)] \leq C(1+|x|)^{-(d-1)q}. \tag{2.4}$$

for any positive integer q. Then we will interpolate two such inequalities to obtain (2.4) for $q = (d+1)/4$. The mapping $V \mapsto \bar V$ is given by the formula

$$\bar V(x) = \frac{c_0}{|x|^{d-1}} \int_{\{y:\ |y|=|x|\}} V_\omega(y) dS(y).$$

Fix $x \in \mathbb{R}^d$ and denote $\Delta_n = [0,1)^d + n \cap \{y \in \mathbb{R}^d :\ |y| = |x|\}$ Since $\mathbb{E}[\omega_n] = 0$, we obtain that

$$\mathbb{E}[|\bar V|^{2q}(x)] \leq C_5 \sum_{m_1+\dots+m_k=2q} \prod_j \frac{2q!}{m_1!\dots m_k!} \sum_n \Big(\int_{\Delta_n} \frac{dS(y)}{(1+|x|)^{d-1}}\Big)^{m_j}$$

where all numbers m_j are even. It is clear that

$$\sum_n \Big(\int_{\Delta_n} \frac{dS(y)}{(1+|x|)^{d-1}}\Big)^{m_j} \leq$$

$$\leq C_6 \sum_n \frac{1}{(1+|x|)^{m_j(d-1)}} = C_7 \frac{1}{(1+|x|)^{(d-1)(m_j-1)}}$$

simply because Δ_n are uniformly bounded.

Consequently, (2.4) holds for any positive integer q and therefore for any $q \geq 1$ (according to Theorem 2.1). Integrating this inequality with respect to x with $q = (d+1)/4$, we obtain (2.3). $\square$

3. Remarks and references

There are some impressive results devoted to the operator

$$-\Delta + (1+|x|)^{-s}V_\omega, \qquad s > 1/2$$

in the mathematical literature. One can study either the discrete or the continuous model of this operator. For the discrete model , the presence of the absolutely continuous spectrum in dimension $d = 2$ was proved by Bourgain [2], [3]. In dimension $d = 3$, the corresponding result was obtained by Denissov [1].

In spite the fact that the main result of this paper pertains to the theory of random operators, we would like now to formulate a result of the deterministic type. This result will be related to the operator

$$-\Delta - \varepsilon\Gamma + V \tag{3.1}$$

where

$$\Gamma u(x) = \frac{1}{|\mathbb{S}|}\int_{\mathbb{S}} u(|x|\theta)\, d\theta$$

The potential V in this model is not random: on the contrary, it is a fixed potential.

Theorem 3.1. *Let* $d \geq 2$ *and* $\varepsilon > 0$. *Assume that* $V \in L^\infty$*,*

$$\int_{\mathbb{S}} V(r\theta)\, d\theta = 0, \qquad \forall r > 1, \tag{3.2}$$

and

$$\int \frac{V^2(x)}{(1+|x|)^{d-1}} dx < \infty. \tag{3.3}$$

Then the absolutely continuous spectrum of the operator (3.1) *is essentially supported by* $[-\varepsilon, \infty)$.

Proof. The proof of this theorem relies on the fact that the negative eigenvalues $\beta_j(\mp V)$ of the operator $-\Delta - \varepsilon\Gamma + \varepsilon I \pm V$ satisfy the condition

$$\sum_j |\beta_j(\mp V)|^{1/2} < \infty. \tag{3.4}$$

The proof of (3.4) is based on the circumstance that the behaviour of the eigenvalues near zero depends only on the structure of the edge of the spectrum of the unperturbed operator. But in the suggested model, this edge has the same structure as the one of the one-dimensional Schrödinger operator. Let us introduce the notation P_1 for the spectral projection of the operator $A = -\Delta - \varepsilon\Gamma + \varepsilon I$ corresponding to the interval $[0, \varepsilon)$. Set also $P_2 = I - P_1$. Then

$$V = 2\text{Re } P_1 V P_2 + P_2 V P_2,$$

due to the condition (3.2) on the mean values of V. It was noticed before that

$$\sum_j |\beta_j(V)|^{1/2} = \int_0^\infty n_+(1, (A+\lambda)^{-1/2} V (A+\lambda)^{-1/2}) \frac{d\lambda}{2\sqrt{\lambda}}.$$

Besides the distribution function n_+ we shall need distribution functions of singular values of non-selfadjoint operators

$$n(s, T) = n_+(s^2, T^*T), \qquad s > 0$$

(here T is a compact operator). Two of the important properties of this function are

$$n(s, TS) \leq n(\frac{s}{||S||}, T),$$

and

$$n(s_1 + s_2, T_1 + T_2) \leq n(s_1, T_1) + n(s_2, T_2).$$

Using these properties, we obtain

$$\sum_j |\beta_j(V)|^{1/2} \leq \int_0^\infty n(c_1, (A+\lambda)^{-1/2} P_1 V) \frac{d\lambda}{\sqrt{\lambda}} +$$
$$\int_0^\infty n_+(c_2, (A+\lambda)^{-1/2} P_2 V P_2 (A+\lambda)^{-1/2}) \frac{d\lambda}{2\sqrt{\lambda}}$$

Let us remark that the second term equals zero if the norm $||V||_{L_\infty}$ is sufficiently small. In the general case, this term can be well estimated by the integral (3.3). It remains to consider the first term

$$\int_0^\infty n(c_1, (A+\lambda)^{-1/2} P_1 V) \frac{d\lambda}{\sqrt{\lambda}} =$$
$$\int_0^\infty n_+(c_1^2, (A+\lambda)^{-1/2} P_1 V^2 P_1 (A+\lambda)^{-1/2}) \frac{d\lambda}{\sqrt{\lambda}} \leq$$
$$\int_0^\infty n_+(c_1^2, (A+\lambda)^{-1/2} \Gamma V^2 \Gamma (A+\lambda)^{-1/2}) \frac{d\lambda}{\sqrt{\lambda}} = 2 \sum_j |\Lambda_j|^{1/2},$$

where Λ_j are the eigenvalues of the operator $-\Delta\Gamma - c_1^{-2}\Gamma V^2 \Gamma$, which, as a matter of fact, is a one dimensional Schrödinger operator with the potential $\alpha_d/r^2 - \overline{V^2}$. Therefore, according to the Lieb-Thirring bound for a one dimensional operator (see [6], [7]), the sum $\sum_j |\Lambda_j|^{1/2}$ can be estimated by the integral (3.3). $\square$

At the end of this section the author would like to mention one idea, which might be useful for the reader in the study of Anderson's model. Let us look at the result from a different point of view. Consider an operator which is in a certain sense close to the operator

$$(3.5) \qquad -\Delta + V_\omega, \qquad \text{where} \quad V_\omega = \sum_n \omega_n \chi(x-n).$$

First, we introduce the class $\mathfrak{S}$ of perturbations B for which the wave operators exist:

$$B \in \mathfrak{S} \quad \text{if} \quad \text{and} \quad \text{only} \quad \text{if} \quad \exists\, \text{s}-\lim_{t\to\pm\infty} \exp(-it(-\Delta+B))\exp(it(-\Delta))$$

Note, that this class is very rich (meaning "large") and it can include even differential operators whose coefficients do not decay at infinity. For example, the operator

$$-\zeta(x)|x|^{-s}\Delta_\theta, \qquad s>1,$$

belongs to the class $\mathfrak{S}$ (here ζ is the characteristic function of the exterior of the unit ball), but the coefficients of this operator behave at infinity as $|x|^{2-s}$.

Theorem 3.2. *Let $\varepsilon > 0$ and let $d \geq 3$. Assume that ω_n are bounded independent random variables with the property $\mathbb{E}[\omega_n] = 0$, for all n. Then for almost every ω, there exists a perturbation $B \in \mathfrak{S}$ such that the operator $-\Delta + B + (1+|x|)^{-\varepsilon} V_\omega$ has a.c. spectrum allover the positive half line $[0,\infty)$.*

Proof . Indeed, let Q and $\bar V$ be the same as in the proof of Theorem 1.1. In particular, it means that $< x, Q(x) >= 0, \quad \forall x,$

$$\text{div} Q = V_\omega, \qquad \text{and} \qquad \bar V = |\mathbb{S}|^{-1} \int_{\mathbb{S}} V_\omega(|x|\theta)\, d\theta.$$

Define

$$B = -\zeta(x)|x|^{-(1+\varepsilon)}\Delta_\theta - \zeta(x)(1+|x|)^{-\varepsilon}\bar V + (1+|x|)^{-1-\varepsilon} Q^2.$$

It is easy to check that $B \in \mathfrak{S}$. On the other hand the operator $-\Delta + B + \zeta(x)(1+|x|)^{-\varepsilon} V_\omega$ is positive. So, there is no necessity to estimate eigenvalues of this operator and the trace formula obtained in [5] gives the relation

$$\mathbb{E}\Big(\int_a^b \log(\mu'(\lambda))\, d\lambda\Big) \geq -C\Big(1 + \int (1+|x|)^{-1-\varepsilon}\mathbb{E}\big[Q^2\big]\, |x|^{1-d} dx\Big)$$

for the spectral measure μ of the operator $-\Delta + B + \zeta(x)(1+|x|)^{-\varepsilon} V_\omega$.

4. Appendix

Finally consider a technical and less interesting question about the kernel of the operator $\nabla_\theta \Delta_\theta^{-1}$ on the unit sphere. Let Γ be the orthogonal projection in $L^2(\mathbb{S})$ onto the subspace of constant functions.

Proposition 4.1. *The kernel of the operator $\nabla_\theta \Delta_\theta^{-1}(I-\Gamma)$ acting in $L^2(\mathbb{S})$ is a function of the form*

$$k(x,y) = \frac{k_0(x,y)}{|x-y|^{d-2}}, \qquad x,y \in \mathbb{S},$$

where k_0 is bounded.

Proof. Let us fix the point y. Clearly, the kernel $k(x,y)$ possesses the symmetry with respect to the axis connecting y and $-y$. Let s be the distance between x and y along the geodesic curve. Then $k(x,y) = \rho(s)\mathbf{e}(s)$, where $\rho(s)$ is a certain scalar function and $\mathbf{e}(x)$ is the unit vector, tangent to the mentioned geodesic curve at the point x. That means $s = 2\arcsin(|x-y|/2)$. Since div$k = \delta(x-y) - 1/|\mathbb{S}|$, we obtain that ρ is a solution of an equation of the form

$$\rho' + q(s)\rho = -1/|\mathbb{S}|, \qquad s \in (0,\pi)$$

where the function $q(s) = \operatorname{div}(\mathbf{e})$ has two singularities: at the point $s=0$ corresponding to y and at the point $s=\pi$ corresponding to $-y$. Moreover the character of the singularities at both points y and $-y$ is the same, the only difference is the sign of the leading term:

$$q(s) \sim \frac{d-2}{s}, \qquad s \to 0,$$
$$q(s) \sim \frac{d-2}{s-\pi}, \qquad s \to \pi.$$

Indeed, if $x \in \mathbb{S}$ is close to y, then $\mathbf{e}$ is close to the vector $\frac{x-y}{|x-y|}$. Therefore div$(\mathbf{e}) \sim \frac{d-2}{|x-y|}$, The function $\rho(s)$ must be smooth at the point $s=\pi$, therefore

$$\rho = \frac{1}{|\mathbb{S}| f(s)} \int_s^\pi f(s)\,ds, \quad \text{where} \quad f(s) = \exp(\int q(s)\,ds)$$

Let us clarify the situation with the point $s=0$. Since $\operatorname{div}\rho\mathbf{e} = -1/|\mathbb{S}|$ everywhere except the point y, we conclude automatically that the function div $\rho\mathbf{e}$ has to have a singularity at y. The only possible singularity is the one of the type $\rho \sim \frac{c}{f(s)}$ as $s \to 0$, with some constant c. In other words,

$$k(x,y) \sim \frac{c}{|x-y|^{d-2}}$$

as $x \to y$. $\square$

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