# Eulerian limit for 2D Navier-Stokes equation and damped/driven KdV equation as its model

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*Dedicated to Vladimir Igorevich Arnold on his 70-th birthday*

**Abstract**

We discuss the inviscid limits for the randomly forced 2D Navier-Stokes equation and the damped/driven KdV equation. The former describes the space-periodic 2D turbulence in terms of a special class of solutions for the free Euler equation, and we view the latter as its model. We review and revise recent results on the inviscid limit for the perturbed KdV and use them to suggest a setup which could be used to make a next step in the study of the inviscid limit of 2D NSE. The proposed approach is based on an ergodic hypothesis for the flow of the 2D Euler equation on iso-integral surfaces. It invokes a Whitham equation for the 2D Navier-Stokes equation, written in terms of the ergodic measures.

*Keywords*: Navier-Stokes equations, 2D turbulence, Eulerian limit, KdV, stationary measure, disintegration, Whitham equation.

# 0 Introduction

We consider the 2D Navier-Stokes equation (NSE) under the periodic boundary conditions. The equation is perturbed by a Gaussian random force which is smooth in the space variable, while as a function of time it is a white noise. We are interested in the inviscid limit for the NSE, i.e. in the behaviour of its solutions when the viscosity goes to zero. It is not hard to see that in order to have a limit of order one the force should be proportional to the square root of the viscosity (see [Kuk06], Section 10.3). Accordingly, we consider the following equation:

$$\dot{u} - \nu\Delta u + (u \cdot \nabla)u + \nabla p = \sqrt{\nu}\,\eta(t,x), \quad 0 < \nu \le 1, \tag{0.1}$$

$$\operatorname{div} u = 0, \quad u = u(t,x) \in \mathbb{R}^2, \quad p = p(t,x), \quad x \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2).$$

It is assumed that $\int u\,dx \equiv \int \eta\,dx \equiv 0$, that the force $\eta$ is divergence-free and is non-degenerate being interpreted as a random process in a function space (see Section 1). It is known that equation (0.1) defines a Markov process in the function space

$$\mathcal{H} = \{u(x) \in L^2(\mathbb{T}^2;\mathbb{R}^2) \mid \operatorname{div} u = 0, \ \int_{\mathbb{T}^2} u\,dx = 0\}\,,$$

and that this process has a unique stationary measure $\mu_\nu$. This is a probability Borel measure in $\mathcal{H}$ which attracts distributions of all solutions for (0.1). Let $u_\nu(t,x)$ be a corresponding stationary solution, i.e.

$$\mathcal{D}u_\nu(t) \equiv \mu_\nu.$$

It is proved in [Kuk04] (also see [Kuk06]) that when $\nu \to 0$ along a subsequence, the random field $u_\nu(t,x)$ converges in distribution to a non-trivial limit $U(t,x)$ (a priori depending on the subsequence), which is a random field, stationary in $t$, $H^2$-smooth in $x$, and such that each its realisation $U(t,x)$ satisfies the free Euler equation (i.e. eq. $(0.1)_{\nu=0}$). Accordingly, the marginal distribution $\mu_0 = \mathcal{D}U(0)$ is an invariant measure for the Euler equation. The process $U$ is called an *Eulerian limit.* See below Theorem 1.1.

The estimates for the stationary solutions $u_\nu$ (see (1.2) in Section 1) imply that $\mathbf{E}\int |u_\nu(t,x)|^2 dx \sim 1$ for all $\nu$. That is, the characteristic size of the solution $u_\nu$ remains $\sim 1$ when $\nu \to 0$. Since the characteristic space-scale also is $\sim 1$, then the Reynolds number of $u_\nu$ grows as $\nu^{-1}$ when $\nu$ decays to zero. Hence, the Eulerian limits $U(t,x)$ describe the transition to turbulence for stationary in time space-periodic 2D flows. [1] So the study of the Euler limits is equivalent (at least, is closely related) to the study of the stationary in time, periodic in space 2D turbulence.

The Euler equation under the periodic boundary conditions has the infinite-dimensional integral of motion $I = I(u)$, where $u(\cdot) \mapsto I(u(\cdot))$ is a map,

[1] This kind of 2D turbulence usually is modeled by the 2D NSE, stirred by a stationary random force.

valued in certain metric space $B$. In Theorem 1.2 we show that the measure $\mu_0 = \mathcal{D}(U(0))$ may be disintegrated as[2]

$$\mu_0 = \int_B \gamma_{\mathbf{b}} \, d\lambda(\mathbf{b}) \,. \tag{0.2}$$

Here $\lambda = I \circ \mu_0$ is the image of the measure $\mu_0$ under the map $I$ and $\gamma_{\mathbf{b}}$, $\mathbf{b} \in B$, is a measure on the iso-integral set $\{I(u) = \mathbf{b}\}$, invariant for the Euler flow.

The Eulerian Limit Theorem 1.1 supports the popular claim that the 2D Euler equation describes the 2D turbulence, while the measure $\mu_0$ and its disintegration (0.2) specify this claim. Accordingly, properties of solutions for the Euler equation are relevant for the 2D turbulence if they correspond to a set of vector-fields $u$ in $\mathcal{H}$ of positive $\mu_0$-measure. For example, it is known that some steady solutions of the Euler equation are its Lyapunov-stable equilibria (see [Arn89], Addendum 2, and [AK01]). They are relevant for the 2D turbulence if the corresponding set of values of the vector-integral $I$ has positive $\lambda$-measure.

The task to study the limiting measure $\mu_0$ and the measures $\gamma_b$ and $\lambda$ is (very) difficult. To develop corresponding intuition, in Section 2 we discuss as a model for (0.1) the damped-driven KdV equation under the periodic boundary conditions:

$$\begin{gathered} \dot{u} - \nu u_{xx} + u_{xxx} - 6uu_x = \sqrt{\nu}\eta(t,x) \,, \\ x \in \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z} \,, \quad \int u \, dx \equiv \int \eta \, dx \equiv 0 \,. \end{gathered} \tag{0.3}$$

This equation is obtained by replacing in the NSE (0.1) the Euler equation by the KdV equation $\dot{u} + u_{xxx} - 6uu_x = 0$ . I.e., by replacing one Hamiltonian PDE with infinitely many integrals of motion by another. The inviscid limit for eq. (0.3) is studied in [KP06]. It is shown there that the limiting solutions are stationary processes $U(t,x)$, formed by smooth solutions of KdV. In this case the disintegration (0.2) simplifies significantly: now the measures $\gamma_{\mathbf{b}}$ are the Haar measures in infinite-dimensional tori, and $\lambda$ is a measure on the octant $\mathbb{R}^\infty_+$, which is a stationary measure for an SDE, obtained as the Whitham averaging of the equation (0.3). See Theorem 2.1 in Section 2.

In Section 3 we use the results for the inviscid limit in eq. (0.3) as a pilot-model to study the disintegration (0.2) and suggest three assertions, describing the objects, involved there.

[2]In Theorem 1.2 the result is stated in an equivalent form, where the Euler equation is written in terms of vorticity

**Notations.** $\mathcal{P}(M)$ denotes the set of probability Borel measures on a metric space $M$; $\mathcal{D}(\xi)$ denotes the distribution of a random variable $\xi$; $f \circ \mu$ stands for the image of a measure $\mu$ under a map $f$, and the symbol $\rightharpoonup$ indicates $*$-weak convergence of Borel measures.

# 1 The Eulerian limit and its vorticity

We first specify the force $\eta$ in the 2D NSE (0.1). Let $(e_s, s \in \mathbb{Z}^2 \setminus \{0\})$ be the standard trigonometric basis of $H$:

$$e_s(x) = \frac{\sin(s \cdot x)}{\sqrt{2}\pi|s|} \begin{bmatrix} -s_2 \\ s_1 \end{bmatrix} \quad \text{or} \quad e_s(x) = \frac{\cos(s \cdot x)}{\sqrt{2}\pi|s|} \begin{bmatrix} -s_2 \\ s_1 \end{bmatrix},$$

depending whether $s_1 + s_2\delta_{s_1,0} > 0$ or $s_1 + s_2\delta_{s_1,0} < 0$. The force $\eta$ is

$$\eta = \frac{d}{dt}\zeta(t,x), \quad \zeta = \sum_{s \in \mathbb{Z}^2 \setminus \{0\}} b_s \beta_s(t) e_s(x)\,,$$

where $\{b_s\}$ is a set of real constants, satisfying

$$b_s = b_{-s} \neq 0 \quad \forall\, s, \qquad \sum |s|^2 b_s^2 < \infty\,,$$

and $\{\beta_s(t)\}$ are standard independent Wiener processes. By $\mathcal{H}^l, l \geq 0$, we will denote the Sobolev space $\mathcal{H} \cap H^l(\mathbb{T}^2; \mathbb{R}^2)$, given the norm

$$\|u\|_l = \Big( \int \big((-\Delta)^{l/2} u(x)\big)^2\, dx \Big)^{1/2}. \tag{1.1}$$

Let $\mu_\nu$ be the stationary measure for (0.1) and $u_\nu(t,x)$ be a corresponding stationary in time solution. It is known to be stationary (=homogeneous) in $x$, and a straightforward application of Ito's formula to $\|u_\nu(t)\|_0^2$ and $\|u_\nu(t)\|_1^2$ implies that

$$\mathbf{E}\, \|u_\nu(t)\|_1^2 \equiv \frac{1}{2}\, B_0\,, \quad \mathbf{E}\|u_\nu(t)\|_2^2 \equiv \frac{1}{2}\, B_1\,, \tag{1.2}$$

where for $l \in \mathbb{R}$ we denote $B_l = \sum |s|^{2l} b_s^2$ (note that $B_0, B_1 < \infty$ by assumption); e.g. see in [Kuk06].

The theorem below describes what happens to the stationary solutions $u_\nu(t,x)$ as $\nu \to 0$. For its proof see [Kuk04, Kuk06].

**Theorem 1.1.** *Any sequence $\tilde{\nu}_j \to 0$ contains a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}U(\cdot) \quad in \quad \mathcal{P}\big(C(0,\infty;\mathcal{H}^1)\big)\,.$$

*The limiting process $U(t) \in \mathcal{H}^1$, $U(t) = U(t,x)$, is stationary in $t$ and in $x$. Moreover,*

*1)a) every its trajectory $U(t,x)$ is such that*

$$U(\cdot) \in L_{2\,loc}(0,\infty;\mathcal{H}^2), \quad \dot{U}(\cdot) \in L_{1\,loc}(0\,\infty;\mathcal{H}^1)\,,$$

*b) it satisfies the free Euler equation*

$$\dot{u}(t,x) + (u\cdot\nabla)u + \nabla p = 0\,, \quad \operatorname{div} u = 0\,, \tag{1.3}$$

*and c) $\|U(t)\|_0$ and $\|U(t)\|_1$ are time-independent quantities. If $g$ is a bounded continuous function, then*

$$\int_{\mathbb{T}^2} g(\operatorname{rot} U(t,x))\,dx \tag{1.4}$$

*also is a time-independent quantity.*

*2) For each $t \geq 0$ we have $\mathbf{E}\|U(t)\|_1^2 = \frac{1}{2}B_0$, $\mathbf{E}\|U(t)\|_2^2 \leq \frac{1}{2}B_1$ and $\mathbf{E}\exp\big(\sigma\|U(t)\|_1^2\big) \leq C$ for some $\sigma > 0, C \geq 1$.*

Due to 1b), the measure $\mu_0 = \mathcal{D}U(0)$ is invariant for the Euler equation. By 2) it is supported by the space $\mathcal{H}^2$ and is not the $\delta$-measure at the origin.

Below we denote by $H^l = H^l(\mathbb{T}^d)$ the Sobolev space of functions with zero mean-value on the torus $\mathbb{T}^d, d = 1$ or 2. The norm in this space is denoted $\|\cdot\|_l$, i.e. as the norm in $\mathcal{H}^l$, and is defined as in (1.1).

Let us write the Euler equation (1.3) in terms of the vorticity $\xi(t,x) = \operatorname{rot} u(t,x) = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$:

$$\dot{\xi} + (u\cdot\nabla)\xi = 0\,, \quad u = \nabla^\perp(-\Delta)^{-1}\xi\,. \tag{1.5}$$

Here $\nabla^\perp = (\partial/\partial x_2, -\partial/\partial x_1)^t$ and $\Delta$ is the Laplacian, operating on functions on $\mathbb{T}^2$ with zero mean value. By Theorem 1.1, $V(t,x) = \operatorname{rot} U$ satisfies (1.5) for every value of the random parameter. Now we show that $V(t)$ belongs to a certain functions space $K$ where (1.5), supplemented by an initial condition $\xi(0) = \xi_0 \in K$, has a unique solution, continuously depending on $\xi_0$. To define this space we first set

$$\mathcal{K} = \{u \in L_{2\,loc}(\mathbb{R};\mathcal{H}^2) \mid \dot{u} \in L_{1\,loc}(\mathbb{R};\mathcal{H}^1)\}\,,$$

given the topology of uniform convergence on bounded time-intervals. This is a Polish space (i.e., a complete separable metric space). Next we define $\tilde{\mathcal{K}}$ as the set of solutions for (1.3), belonging to $\mathcal{K}$. This is a closed subset of $\mathcal{K}$, so also a Polish space. The group of the flow-maps of the Euler equation acts on $\tilde{\mathcal{K}}$ by time-shifts which are its continuous homeomorphisms. Now consider the continuous map

$$\pi : \tilde{\mathcal{K}} \to H^0(\mathbb{T}^2)\,, \quad u(t,x) \mapsto \operatorname{rot} u(0,x)\,.$$

Due to an uniqueness theorem of the Yudovich type (see Lemma 3.5 in [Kuk04]), $\pi$ is an embedding. We set

$$K = \pi(\tilde{\mathcal{K}})$$

and provide $K$ with the distance, induced from $\tilde{\mathcal{K}}$. It makes $K$ a Polish space (we do not know if $K$ is a linear space or not, i.e. whether it is invariant with respect to the usual linear operations). It follows from the results of [BKM84] that $H^2(\mathbb{T}^2) \subset K$. So

$$H^2(\mathbb{T}^2) \subset K \subset H^0(\mathbb{T}^2)\,, \tag{1.6}$$

where the embeddings are continuous.

Due to what was said above, the Euler equation defines a group of continuous homeomorphisms

$$S_t : K \to K\,, \qquad t \in \mathbb{R}\,. \tag{1.7}$$

Theorem 1.1 shows that $U(\cdot) \in \tilde{\mathcal{K}}$ for each value of the random parameter. Therefore the measure

$$\theta = \mathcal{D}(V(0)) = \operatorname{rot} \circ \mu_0$$

is supported by $K$ (i.e., $\theta(K) = 1$). Since $S_t \circ \theta = \mathcal{D}(V(t))$ and $V(t)$ is a stationary process, then $\theta$ is an invariant measure for the dynamical system (1.7). By the estimates in item 2) of Theorem 1.1, it is supported by the space $K \cap H^1(\mathbb{T}^2)$ and $\int \exp\big(\sigma \|v\|_0^2\, \theta(dv) < \infty$.

Our next goal is to express the assertion 1c) of Theorem 1.1 in terms of the measure $\theta$. Let us denote by $\mathcal{P}(\mathbb{R})$ the set of probability Borel measures on $\mathbb{R}$, furnished with the Lipschitz-dual distance

$$\operatorname{dist}(m', m'') = \sup_{f \in \mathcal{L}} |\langle m', f\rangle - \langle m'', f\rangle|\,,$$

where $\mathcal{L}$ is the set of all Lipschitz functions $f$ on $\mathbb{R}$ such that $\mathrm{Lip}\,(f) \le 1$ and $|f| \le 1$. This is a Polish space, and convergence in the introduced distance is equivalent to the $*$-weak convergence of measures, see [Dud02]. Due to (1.6) the map

$$M : K \to \mathcal{P}(\mathbb{R})\,, \qquad \xi \mapsto \xi \circ \big((2\pi)^{-2}dx\big)\,,$$

is continuous. Indeed, for any $f \in \mathcal{L}$ we have

$$\begin{aligned} |\langle M\xi_1, f\rangle - \langle M\xi_2, f\rangle| &= \Big| \int (f(\xi_1(x)) - f(\xi_2(x))\,(2\pi)^{-2}dx \Big| \\ &\le \int |\xi_1 - \xi_2|\,(2\pi)^{-2}dx = o(1) \quad \text{as} \;\; \mathrm{dist}(\xi_1, \xi_2) \to 0\,, \end{aligned}$$

where we use (1.6) to get the last equality. Therefore, the map

$$\Psi : K \to \mathcal{P}(\mathbb{R}) \times \mathbb{R}_+ =: B\,, \quad \xi(\cdot) \mapsto (M(\xi), \|\xi\|_{-1})\,,$$

also is continuous.

Repeating (say) the arguments in [Kuk04, Kuk06] which prove assertion 1c) of Theorem (1.1), we get that each trajectory $u(t,x)$ of (1.3), belonging to the space $\mathcal{K}$, satisfies $\Psi(u(t)) = \text{const}$. Recalling the definition of the flow-maps $S_t$, we get that they commute with $\Psi$. That is,

$$S_t : K_{\mathbf{b}} \to K_{\mathbf{b}} \qquad \forall\, t \in \mathbb{R}\,, \tag{1.8}$$

for every $\mathbf{b} \in B$. Here

$$K_{\mathbf{b}} = \Psi^{-1}(\mathbf{b}) \subset K\,, \quad \mathbf{b} \in B\,.$$

Clearly $K_{\mathbf{b}}$ is a closed subset of $K$, and it is not hard to see that it is not compact. We provide $K_{\mathbf{b}}$ with the induced topology.

Let us denote $\lambda = \Psi \circ \theta$. This is a measure on Borel subsets of the Polish space $B$.

**Theorem 1.2.** *There exists a family $\{\theta_{\mathbf{b}}, \mathbf{b} \in B\}$ of measures on Borel subsets of $K$ such that*

*1) $\theta_{\mathbf{b}}(K_{\mathbf{b}}) = 1$ for each $\mathbf{b}$, for any Borel set $A \subset K$ the function $\mathbf{b} \mapsto \theta_{\mathbf{b}}(A)$ is Borel-measurable on $B$, and*

$$\theta(A \cap \Psi^{-1}(D)) = \int_D \theta_{\mathbf{b}}(A)\, d\lambda(\mathbf{b})\,, \tag{1.9}$$

*for any Borel set $D \subset B$.*

*2) For $\lambda$-a.a. $\mathbf{b} \in B$ the measure $\theta_{\mathbf{b}}$ (interpreted as a measure on Borel subsets of $K_{\mathbf{b}}$) is invariant for the dynamical system* (1.8).

*Proof.* 1) This group of assertions makes the statement of the Disintegration Theorem, applied to a measurable map

$$\Psi : (K, \mathcal{A}, \theta) \to (B, \mathcal{B}, \lambda)\,,$$

where $\mathcal{A}$ and $\mathcal{B}$ are the Borel sigma-algebras for Polish spaces $K$ and $B$. For the case when $K$ and $B$ are locally compact sets, the theorem is proved in [Bou59]. The case of arbitrary Polish spaces reduces to that one since the measure spaces $(K, \mathcal{A})$ and $(B, \mathcal{B})$ both are isomorphic to the unit segment [0,1] with the Borel sigma-algebra, by means of some measurable isomorphism, see [Dud02] (also see [Par77], section 46).

2) Since the measure $\theta$ is invariant for the dynamical system (1.7) and since $\Psi$ commutes with the flow-maps $S_t$, then for any Borel sets $A \subset K$ and $D \subset B$ we have $\int_D \theta_{\mathbf{b}}(A)\, d\lambda(\mathbf{b}) = \int_D \theta_{\mathbf{b}}(S_t A)\, d\lambda(\mathbf{b})$. That is,

$$\int_D \theta_{\mathbf{b}}\, d\lambda(\mathbf{b}) = \int_D (S_t \circ \theta_{\mathbf{b}})\, d\lambda(\mathbf{b})\,,$$

where the integrands are vector-functions, valued in the linear space of signed Borel measures on $K$. Hence, $\theta_{\mathbf{b}} = S_t \circ \theta_{\mathbf{b}}$ for $\lambda$-a.a. $\mathbf{b}$, as stated. □

# 2 The KdV-model

In this section we discuss the inviscid limit in the equation (0.3) as a model for the Eulerian limit. The results, described below, are obtained in [KP06].

The force $\eta$ in (0.3) is assumed to have the same form as the force in Section 1:

$$\eta = \frac{\partial}{\partial t} \sum_{s \in \mathbb{Z} \setminus \{0\}} b_s \beta_s(t) e_s(x)\,.$$

Here

$$e_s(x) = \begin{cases} \cos sx, & s > 0\,, \\ \sin sx, & s < 0\,, \end{cases}$$

$\{\beta_s(t)\}$ are standard independent Wiener processes and the coefficients $b_s$ satisfy

$$b_s = b_{-s} \neq 0 \;\; \forall\, s\,; \qquad b_s \leq C_m |s|^{-m} \quad \forall\, m, s$$

with suitable constants $C_m$.

For the same reasons as for the NSE, eq. (0.3) has a unique stationary measure. Let $u_\nu(t,x)$, $t\geq 0$, be a corresponding stationary solution. It is also stationary in $x$, and its Sobolev norms satisfy the following estimates uniformly in $\nu>0$:

$$\mathbf{E}\, e^{\sigma\|u_\nu(t)\|_0^2}\leq C_\sigma<\infty, \quad \mathbf{E}\,\|u_\nu\|_m^k\leq C_{m,k}<\infty \tag{2.1}$$

(for a suitable $\sigma>0$ and for all $m,k$); see in [KP06]. These estimates allow to prove for eq. (0.3) an analogy of Theorem 1.1. That is, to establish that any sequence $\tilde\nu_j\to 0$ contains a subsequence $\nu_j\to 0$ such that

$$\mathcal{D}u_{\nu_j}(\cdot)\rightharpoonup \mathcal{D}U(\cdot) \quad \text{in} \quad \mathcal{P}(C(0,\infty;H^m))\,,$$

for each $m$. The limiting process $U(t)=U(t,x)$ is stationary in $t$ and in $x$ and satisfies estimates (2.1). Its marginal distribution

$$\theta=\mathcal{D}U(0)$$

is an invariant measure for KdV, and every its realisation $U(t,x)$ is a smooth solution for KdV. Solutions of that equation satisfy infinitely many integrals of motion. The realisation of these integrals, studied in [MT76, KP03] (also see in [KP06]), is the most convenient for our purposes. These integrals are non-negative analytic functions $I_1,I_2,\dots$ on the space $H^0$, and in big difference with the integrals for the Euler equation, now the structure of the iso-integral sets

$$T_I=\{u\mid I_j(u)=I_j\geq 0\quad\forall\, j\}$$

is well understood. Namely, each $T_I$ is an analytic torus in $H^0$ of dimension $|\mathcal{J}(I)|\leq\infty$, where $\mathcal{J}(I)=\{j\mid I_j>0\}$. Moreover, the torus $T_I$ carries a cyclic coordinate $\varphi=\{\varphi_j, j\in\mathcal{J}(I)\}$ such that in the coordinates $(I,\varphi)$ the KdV-dynamics takes the integrable form

$$\dot I=0\,,\quad \dot\varphi=W(I)\,,$$

where the frequency vector $W(I)$ is analytic and non-degenerate in $I$. The Haar measure $d\varphi$ on any torus $T_I$ is invariant for this dynamics. Since the map $I\mapsto W(I)$ is analytic non-degenerate, then for a 'typical' $I$ this is the unique invariant Borel measure on $T_I$. It turns out that the limiting measure $\theta$ is supported by the set $\{u\in H^\infty=\cap H^m\mid I_j(u)>0\quad\forall\, j\}$

which is measurably isomorphic to a Borel subset of $\mathbb{R}_+^\infty \times \mathbb{T}^\infty$. Under this isomorphism $\theta$ can be written as

$$\theta = \lambda \times d\varphi\,, \tag{2.2}$$

where $\lambda = \mathcal{D}\big(I(U(0)\big)$ is a Borel measure on $\mathbb{R}_+^\infty$. This is a KdV-analogy of the disintegration (1.9). It simplifies compare the NSE-case since, firstly, the iso-integral sets $T_I$ are isomorphic to tori, while the topology of the sets $K_{\mathbf{b}}$ is unknown, and, secondly, any set $T_I$ carries the simple KdV-invariant measure $d\varphi$ which typically is the unique invariant measure, while the structure of the invariant measures $\theta_{\mathbf{b}}$ in (1.9) is unknown.

To describe the measure $\lambda$ in (2.2) we apply Ito's formula to eq. (0.3) and the map $u \mapsto I(u)$, and pass in the obtained equation to the fast time $\tau = \nu^{-1}t$. We get a system

$$dI_k(\tau) = F_k(u)\,d\tau + \sum_j \sigma_{kj}(u)\,d\beta_j(\tau)\,, \quad k = 1, 2, \dots \tag{2.3}$$

(see [KP06], Section 3). This is an infinite-dimensional SDE with the drift $F_k$ and the diffusion matrix $a_{kl} = \sum_j \sigma_{kj}\sigma_{lj}$. Clearly $I_\nu(t) = I(u_\nu(t))$ is a stationary solution for this system. Let us define the averaged drift and the averaged diffusion matrix as follows:

$$\langle F_k\rangle(I) = \int_{T_I} F_k(u(\varphi))\,d\varphi\,, \quad \langle a_{kl}\rangle(I) = \int_{T_I} a_{kl}(u(\varphi))\,d\varphi \tag{2.4}$$

($u(\varphi)$ is a point in $T_I$ with the coordinate $\varphi$).

**Theorem 2.1.** *Every sequence $\tilde\nu_j \to 0$ contains a subsequence $\nu_j \to 0$ such that the process $I(u_{\nu_j}(\tau))$ (where $\tau = \nu^{-1}t$) $*$-weakly converges in distribution in the space $\mathcal{P}\big(C(0,\infty; H^0)\big)$ to a stationary process $I(\tau)$, which is a solution for the martingale problem with the drift and diffusion as in (2.4). The law $\mathcal{D}(I(0))$ equals $\lambda = \mathcal{D}I(U(0))$. Its finite-dimensional projections* [3] *are absolutely continuous with respect to the Lebesgue measure, and all moments of all norms $|I|_m = \sum |I_j| j^m$ with respect to the measure $\lambda$ are finite.*

Thus, the averaged martingale problem, which can be formally written as

$$dI(\tau) = \langle F\rangle(I)\,d\tau + \big(\langle a\rangle\langle a\rangle^t\big)^{1/2}(I)\,d\beta(\tau) \tag{2.5}$$

[3]i.e., images of $\lambda$ under the maps $I \mapsto (I_1, \dots, I_M) \subset \mathbb{R}_+^M$, $M \in \mathbb{N}$.

describes the limiting behaviour of the stationary measures $\mathcal{D}u_{\nu_j}(t)$. Indeed, the limiting (as $\nu_j \to 0$) measure may be written in the form (2.2), where $\lambda$ is a stationary measure for (2.5). It is plausible that (2.5) has a unique stationary measure $\lambda$. If so, then $\mathcal{D}(u_\nu(t)) \rightharpoonup \lambda \times d\,\varphi$ as $\nu \to 0$.

# 3 Ergodic hypothesis for the Eulerian limit

We believe that the damped/driven KdV equation (0.3) is a right model for the Eulerian limit (that is, a right model for the space-periodic stationary 2D turbulence). Guided by this belief, below we make statements **1-3** which specify the results in Section 1 concerning the Eulerian limit. Crucial among them is the first one, which is an ergodic hypothesis for the Euler equation on a typical iso-integral set $K_{\mathbf{b}}$. We cannot prove the three statements, but, firstly, we believe that the validity of their analogies for the KdV-model justifies the assumptions up to some extend and, secondly, we can prove some fragments of the picture, given by the assumptions (the corresponding results will be published elsewhere).

**1.** Every non-empty set $K_{\mathbf{b}}$ carries a measure $m_{\mathbf{b}}$, invariant for the Euler flow (1.8), such that for a.a. $u \in \mathcal{H}$ with respect to any stationary measure $\mu_\nu$ we have

$$\lim_{T\to\infty} T^{-1} \int_0^T f(S_t u)\, dt = \langle f, m_{\mathbf{b}} \rangle\,, \quad \mathbf{b} = \Psi(u)\,. \tag{3.1}$$

For $\lambda$-a.a. $\mathbf{b}$ the measure $m_{\mathbf{b}}$ coincides with $\theta_{\mathbf{b}}$ in (1.9).

Let $u(t,x)$ be a solution of (0.1). Then the vector $\mathbf{b}(t) = \Psi(u(t)) \in B$ satisfies a SDE. To describe it we introduce the space $\mathcal{C} = C_b(\mathbb{R}) \times \mathbb{R}$ and denote by $\langle \cdot, \cdot \rangle$ its natural pairing with $\mathcal{P}(\mathbb{R}) \times \mathbb{R} \supset B$. For any $\mathbf{f} \in \mathcal{C}$ we set

$$\Psi_{\mathbf{f}}(u) = \langle \Psi(u), \mathbf{f} \rangle\,.$$

Applying Ito's formula to $u_{\mathbf{f}} = \Psi_{\mathbf{f}}(u)$, where $u(t)$ satisfies (0.1), and passing to the fast time $\tau = \nu t$, we get

$$d\,u_{\mathbf{f}}(\tau) = F_{\mathbf{f}}(u(\tau))\, d\tau + \sum_s \sigma_{\mathbf{f}s}(u(\tau))\, d\beta_s(\tau)\,. \tag{3.2}$$

Here $F_{\mathbf{f}}$ and $\{\sigma_{\mathbf{f}s}, s \in \mathbb{Z}^2 \setminus 0\}$ are smooth functions on $\mathcal{H}$ (depending on the coefficients $b_s$) and $\{\beta_s(\tau)\}$ are new standard Wiener processes.

**2.** Let $u_\nu(t)$ be a stationary solution of (0.1). Then along a subsequence $\nu_j \to 0$ the process $\Psi(u_\nu(\tau))$ converges in distribution to a limiting process $\mathbf{b}(\tau)$. This is a stationary Ito process in $B$ such that for any $\mathbf{f} \in \mathcal{C}$ the process $b_{\mathbf{f}}(\tau) = \langle \mathbf{b}(\tau), \mathbf{f} \rangle$ has the drift

$$\langle F \rangle_{\mathbf{f}}(\mathbf{b}) = \langle F_{\mathbf{f}}(u), m_{\mathbf{b}} \rangle = \int_{K_{\mathbf{b}}} F_{\mathbf{f}}(u)\, m_{\mathbf{b}}(du)\,, \tag{3.3}$$

and for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{C}$ processes $b_{\mathbf{f}_1}$ and $b_{\mathbf{f}_2}$ have the covariation

$$\langle a \rangle_{\mathbf{f}_1\, \mathbf{f}_2}(\mathbf{b}) = \Big\langle \sum_s \sigma_{\mathbf{f}_1 s}(u) \sigma_{\mathbf{f}_2 s}(u), m_{\mathbf{b}} \Big\rangle\,. \tag{3.4}$$

I.e., the processes $\mathbf{b}_{\mathbf{f}_1}(\tau) - \int_0^\tau \langle F \rangle_{\mathbf{f}_1}(\mathbf{b}(s))\, ds$ and $\mathbf{b}_{\mathbf{f}_2}(\tau) - \int_0^\tau \langle F \rangle_{\mathbf{f}_2}(\mathbf{b}(s))\, ds$ are martingales and their bracket equals $\langle a \rangle_{\mathbf{f}_1 \mathbf{f}_2}(\mathbf{b}(\tau))$.

That is to say, the limiting process $\mathbf{b}(\tau)$ is a martingale solution for a stochastic differential equation in the space $B$ with the drift $\langle F \rangle(\mathbf{b})$ and the covariance $\langle a \rangle(\mathbf{b})$. This is the *Whitham equation for the 2D NSE* (0.1).

In difference with the KdV case, space $B$ does not have a natural basis, and we cannot *naturally* write this equation as a system of infinitely many SDE. Still we can find a countable system of vectors $f_j \in \mathcal{C}$ such that their linear combinations are dense in $\mathcal{C}$, and use these vectors in (3.3), (3.4). In this way we write the Whitham equation above as an over-determined system of SDE.

We can write down the Whitham equation, using in (3.3), (3.4) the measures $\theta_{\mathbf{b}}$ instead of the measures $m_{\mathbf{b}}$, i.e. without invoking the Ergodic Hypothesis **1**. But if the hypothesis fails, then the measures $\theta_{\mathbf{b}}$ may depend on the sequence $\nu_j \to 0$ in Theorem 1.1. In this case the Whitham equation seems to be a useless object.

**3.** Distribution of the limiting process $\mathbf{b}(\tau)$ is independent from the sequence $\{\nu_j \to 0\}$. Accordingly, the measure $\theta = \mathcal{D}\mathbf{b}(0)$ also is independent from the sequence, and

$$\Psi \circ \mu_\nu \rightharpoonup \theta\,, \quad \mu_\nu \rightharpoonup \mu_0 = \int m_{\mathbf{b}}\, \theta(d\mathbf{b})$$

as $\nu \to 0$.

The crucial assumption for the scenario above is the Ergodic Hypothesis **1**. We assume there that (3.1) holds for $\mu_\nu$-a.a. initial data (rather than for all of them) since analogy with the damped/driven KdV suggests that on some atypical sets $K_{\mathbf{b}}$ the dynamics $\{S_t\}$ may be non-ergodic (i.e., 'resonant').

The task of proving hypothesis **1** and finding out properties of the measures $m_{\mathbf{b}}$ seems to be very difficult. In particular, it is not clear for us which role in this problem plays the fact that the group of area preserving diffeomorphisms of $\mathbb{T}^2$ transitively acts on the set of equidistributed functions $K_m = \bigcup_{b\in\mathbb{R}} K_{(m,b)}$, $m \in \mathcal{P}(\mathbb{R})$.

Despite the measures $m_{\mathbf{b}}$ are unknown, the averaged drift and covariance (3.3) and (3.4) which characterise the limiting equation can be calculated by replacing the ensemble-average by the time-average (see (3.1)). So validity of the suggested scenario **1-3** may be verified numerically by comparing $\Psi(u_\nu(\tau))$ with solutions for the limiting equation.

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