AGMON-TYPE ESTIMATES FOR A CLASS OF DIFFERENCE OPERATORS

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Abstract. We analyze a general class of difference operators $H_\varepsilon = T_\varepsilon + V_\varepsilon$ on $\ell^2((\varepsilon\mathbb{Z})^d)$, where V_ε is a one-well potential and ε is a small parameter. We construct a Finslerian distance d induced by H_ε and show that short integral curves are geodesics. Then we show that Dirichlet eigenfunctions decay exponentially with a rate controlled by the Finsler distance to the well. This is analog to semiclassical Agmon estimates for Schrödinger operators.

1. Introduction

The central topic of this paper is the investigation of a rather general class of families of difference operators H_ε on the Hilbert space $\ell^2((\varepsilon\mathbb{Z})^d)$, as the small parameter $\varepsilon > 0$ tends to zero.

The operator H_ε is given by

$$H_\varepsilon = (T_\varepsilon + V_\varepsilon), \quad T_\varepsilon = \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} a_\gamma\tau_\gamma, \quad (a_\gamma\tau_\gamma u)(x,\varepsilon) = a_\gamma(x,\varepsilon)u(x+\gamma) \quad \text{for} \quad x,\gamma\in(\varepsilon\mathbb{Z})^d \tag{1.1}$$

where V_ε is a multiplication operator, which in leading order is given by $V_0 \in \mathscr{C}^\infty(\mathbb{R}^d)$.

We remark that the limit $\varepsilon \to 0$ is analog to the semiclassical limit $\hbar \to 0$ for the Schrödinger operator $-\hbar^2\Delta + V$. This paper is the first in a series of papers; the aim is to develop an analytic approach to the semiclassical eigenvalue problem and tunneling for H_ε which is comparable in detail and precision to the well known analysis for the Schrödinger operator (see Simon [19], [20] and Helffer-Sjöstrand [14]). Our motivation comes from stochastic problems (see Bovier-Eckhoff-Gayrard-Klein [8], [9]). A large class of discrete Markov chains analyzed in [9] with probabilistic techniques falls into the framework of difference operators treated in this article.

We recall that sharp semiclassical Agmon estimates describing the exponential decay of eigenfunctions of appropriate Dirichlet realizations of the Schrödinger operator are crucial to analyze tunneling for the Schrödinger operator. Agmon realized in [3] that for a large class of second order differential operators the exponential rate at which eigenfunctions decay is given by the geodesic distance in the Agmon metric. This is the Riemannian metric from Jacobi's theorem in classical mechanics: For a Hamilton function whose kinetic energy is a positive definite quadratic form in the momenta, the projection to configuration space of an integral curve of the Hamiltonian vector field is a geodesic in the Agmon (Jacobi) metric.

This paper contains analog results for the class of operators H_ε, including a generalization of Jacobi's theorem. It is essential that we consider these operators as semiclassical quantization of suitable Hamilton functions and investigate the relation of these Hamilton functions to Finsler geometry.

If $\mathbb{T}^d := \mathbb{R}^d/(2\pi)\mathbb{Z}^d$ denotes the d-dimensional torus and $a \in \mathscr{C}^\infty\left(\mathbb{R}^d\times\mathbb{T}^d\times(0,1]\right)$, a pseudo-differential operator $\mathrm{Op}_\varepsilon^{\mathbb{T}^d}(a) : \mathcal{K}\left((\varepsilon\mathbb{Z})^d\right) \longrightarrow \mathcal{K}'\left((\varepsilon\mathbb{Z})^d\right)$ is defined by

$$\mathrm{Op}_\varepsilon^{\mathbb{T}^d}(a)\, v(x) := (2\pi)^{-d} \sum_{y\in(\varepsilon\mathbb{Z})^d} \int_{[-\pi,\pi]^d} e^{\frac{i}{\varepsilon}(y-x)\xi} a(x,\xi;\varepsilon)v(y)\, d\xi\,, \tag{1.2}$$

where

$$\mathcal{K}\left((\varepsilon\mathbb{Z})^d\right) := \{u : (\varepsilon\mathbb{Z})^d \to \mathbb{C} \mid u \text{ has compact support}\} \tag{1.3}$$

and $\mathcal{K}'\left((\varepsilon\mathbb{Z})^d\right) := \{f : (\varepsilon\mathbb{Z})^d \to \mathbb{C}\ \}$ is dual to $\mathcal{K}\left((\varepsilon\mathbb{Z})^d\right)$ by use of the scalar product $\langle u\,,\, v\rangle_{\ell^2} := \sum_x \bar{u}(x)v(x)$.

We assume that $a_\gamma(x,\varepsilon) = a_\gamma^{(0)}(x) + \varepsilon a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x,\varepsilon)$, where $R_\gamma^{(2)}(x,\varepsilon) = O(\varepsilon^2)$ uniformly with respect to x and γ. Then, with some furhter assumptions on a_γ, the difference operator T_ε

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(the kinetic energy), leads to a symbol $t \in \mathscr{C}^\infty\left(\mathbb{R}^d \times \mathbb{T}^d \times [0,1)\right)$. t can be considered as a function on $\mathbb{R}^{2d} \times [0,1)$, which is 2π-periodic with respect to ξ. Denoting this function by t also we have

$$t(x,\xi,\varepsilon) = \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} a_\gamma(x,\varepsilon) \exp\left(-\frac{i}{\varepsilon}\gamma\cdot\xi\right) . \tag{1.4}$$

Furthermore by the expansion of $a_\gamma(x,\varepsilon)$ with respect to ε we can write

$$\begin{aligned} t(x,\xi,\varepsilon) &= t_0(x,\xi) + \varepsilon\, t_1(x,\xi) + t_2(x,\xi,\varepsilon)\,, \qquad \text{with} \\ t_j(x,\xi) &:= \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} \varepsilon^j a_\gamma^{(j)}(x) e^{-\frac{i}{\varepsilon}\gamma\xi}\,, \qquad j=0,1 \\ t_2(x,\xi,\varepsilon) &:= \sum_{\gamma\in(\varepsilon\mathbb{Z})^d} R_\gamma^{(2)}(x,\varepsilon) e^{-\frac{i}{\varepsilon}\gamma\xi}\,. \end{aligned} \tag{1.5}$$

Thus, in leading order the symbol of H_ε is $h_0 = t_0 + V_0$. In its original form, neither Jacobi's theorem applies to $h_0(x,\xi)$ nor Agmon estimates to H_ε. Our analysis is motivated by the remark in Agmon's book [3] to develop part of the theory of the Agmon metric in the more general context of Finsler geometry. It turns out that the Hamilton function $\tilde{h}_0(x,\xi) := -h_0(x,i\xi)$ (the transformation from h_0 to $\tilde{h}_0$ is analog to the procedure in the case of the Schrödinger operator) in a natural way induces a Finsler metric and an associated Finsler distance d on $\mathbb{R}^d$. This allows to formulate and prove a generalization of Jacobi's theorem (which might be some kind of lesser known folk wisdom in mathematical physics, which, however, we were unable to find in the literature) and prove an analog of the semiclassical Agmon estimates for H_ε. We remark that Finsler distances have been used for higher order elliptic differential operators in the analysis of decay of resolvent kernels and/or heat kernels, see Tintarev [21] and Barbatis [7], [6].[1] However, these papers do not develop a generalization of Jacobi's theorem, which turns out to be crucial in our semiclassical analysis.

We will now state our assumptions on H_ε and formulate our results more precisely.

HYPOTHESIS 1.1 (a) *The coefficients $a_\gamma(x,\varepsilon)$ in* (1.1) *are functions*

$$a : (\varepsilon\mathbb{Z})^d \times \mathbb{R}^d \times [0,1) \to \mathbb{R}\,, \qquad (\gamma, x, \varepsilon) \mapsto a_\gamma(x,\varepsilon)\,, \tag{1.6}$$

satisfying the following conditions:

(i) *they have an expansion*

$$a_\gamma(x,\varepsilon) = a_\gamma^{(0)}(x) + \varepsilon\, a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x,\varepsilon)\,, \tag{1.7}$$

where $a_\gamma^{(i)} \in \mathscr{C}^\infty(\mathbb{R}^d)$ and $|a_\gamma^{(j)}(x) - a_\gamma^{(j)}(x+h)| = O(|h|)$ for $j=0,1$ uniformly with respect to $\gamma \in (\varepsilon\mathbb{Z})^d$ and $x \in \mathbb{R}^d$. Furthermore $R_\gamma^{(2)} \in \mathscr{C}^\infty(\mathbb{R}^d \times [0,1))$ for all $\gamma \in (\varepsilon\mathbb{Z})^d$.

(ii) *there exists a function*

$$\tilde{a} : \mathbb{Z}^d \times \mathbb{R}^d \ni (\eta, x) \mapsto \tilde{a}_\eta(x) \in \mathbb{R}\,, \quad \textit{such that} \quad a_\gamma^{(0)}(x) = \tilde{a}_{\frac{\gamma}{\varepsilon}}(x) \tag{1.8}$$

(iii) $\sum_\gamma a_\gamma^{(0)} = 0$ *and* $a_\gamma^{(0)} \leq 0$ *for* $\gamma \neq 0$

(iv) $a_\gamma(x,\varepsilon) = a_{-\gamma}(x+\gamma,\varepsilon)$ *for* $x \in \mathbb{R}^d, \gamma \in (\varepsilon\mathbb{Z})^d$

(v) *for any $c > 0$ there exists $C > 0$ such that for $j = 0,1$ uniformly with respect to $x \in (\varepsilon\mathbb{Z})^d$ and ε*

$$\| e^{\frac{c|.|}{\varepsilon}} a_{.}^{(j)}(x) \|_{\ell^2_\gamma((\varepsilon\mathbb{Z})^d)} \leq C \qquad \textit{and} \qquad \| e^{\frac{c|.|}{\varepsilon}} R_{.}^{(2)}(x) \|_{\ell^2_\gamma((\varepsilon\mathbb{Z})^d)} \leq C\varepsilon^2 \tag{1.9}$$

(vi) $\operatorname{span}\{\gamma \in (\varepsilon\mathbb{Z})^d \,|\, a_\gamma^{(0)}(x) < 0\} = \mathbb{R}^d$ *for all* $x \in \mathbb{R}^d$.

(b) *The potential energy V_ε is the restriction to $(\varepsilon\mathbb{Z})^d$ of a function $\widehat{V}_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}^d)$, which has an expansion*

$$\widehat{V}_\varepsilon(x) = \sum_{l=0}^N \varepsilon^l V_l(x) + R_{N+1}(x;\varepsilon)\,,$$

[1] M.K. thanks S. Agmon for the reference to [21], where prior to the publication of Agmons book a Finslerian approach was used to obtain estimates on the kernel of the resolvent and the decay of the heat kernel for higher order elliptic operators, following ideas of Agmon

where $V_\ell \in \mathscr{C}^\infty(\mathbb{R}^d)$, $R_{N+1} \in \mathscr{C}^\infty(\mathbb{R}^d \times (0,\varepsilon_0])$ *for some* $\varepsilon_0 > 0$ *and for any compact set* $K \subset \mathbb{R}^d$ *there exists a constant* C_K *such that* $\sup_{x\in K} |R_{N+1}(x;\varepsilon)| \leq C_K \varepsilon^{N+1}$. *There exist constants* $R, C > 0$ *such that* $V_\varepsilon(x) > C$ *for all* $|x| \geq R$ *and* $\varepsilon \in (0,\varepsilon_0]$. *In addition* $V_0(x)$ *has exactly one non-degenerate minimum at* $x_0 = 0$ *with the value* $V_0(0) = 0$.

We denote by $H_\varepsilon = T_\varepsilon + V_\varepsilon$ *the self adjoint realization of the operator defined in* (1.1) *on the maximal domain of* V_ε, *which we denote by* $\mathscr{D}(H_\varepsilon) \subset \ell^2\left((\varepsilon\mathbb{Z})^d\right)$. *The associated symbol is denoted by* $h_\varepsilon(x,\xi;\varepsilon)$.

The following Lemma couples the assumptions on the coefficients a_γ given in Hypothesis 1.1 with properties of the symbol t and the kinetic energy T_ε.

LEMMA 1.2 *Assume Hypothesis 1.1 and let* t *and* $t_j, j = 0,1,2$ *be defined in* (1.4) *and* (1.5) *respectively. Then*

(a) $t \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{T}^d \times [0,1))$ *and the estimate* $\sup_{x,\xi} |\partial_x^\alpha \partial_\xi^\beta t(x,\xi,\varepsilon)| \leq C_{\alpha,\beta}$ *holds for all* $\alpha,\beta \in \mathbb{N}^d$ *uniformly with respect to* ε. *Furthermore* t_0 *and* t_1 *are bounded and* $\sup_{x,\xi} |t_2(x,\xi,\varepsilon)| = O(\varepsilon^2)$.

(b) *the* 2π*-periodic function* $\mathbb{R}^d \ni \xi \mapsto t_0(x,\xi)$ *is even and has an analytic continuation to* $\mathbb{C}^d$.

(c) *at* $\xi = 0$, *for fixed* $x \in \mathbb{R}^d$ *the function* t_0 *defined in* (1.5) *has an expansion*

$$t_0(x,\xi) = \langle \xi\,,\, B(x)\xi\rangle + O\left(|\xi|^4\right) \qquad as\ |\xi| \to 0\,, \tag{1.10}$$

where $B : \mathbb{R}^d \to \mathcal{M}(d\times d,\mathbb{R})$ *is positive definite and symmetric.*

(d) *The operator* T_ε *defined in* (1.1) *is symmetric, bounded (uniformly in* ε*) and* $\langle u\,,\, T_\varepsilon u\rangle_{\ell^2} \geq -C\varepsilon\|u\|^2$ *for some* $C > 0$. *Furthermore* $T_\varepsilon = \mathrm{Op}_\varepsilon^{\mathbb{T}^d}(t)$ *(see* (1.2)*).*

We write

$$\tilde{h}_0(x,\xi) = \tilde{t}_0(x,\xi) - V_0(x) \,:\, \mathbb{R}^{2d} \to \mathbb{R}\,, \tag{1.11}$$

where

$$\tilde{t}_0(x,\xi) := -t_0(x,i\xi) = -\sum_\gamma a_\gamma^{(0)}(x)\cosh\left(\tfrac{1}{\varepsilon}\gamma\cdot\xi\right)\ . \tag{1.12}$$

We shall now describe, how Hamilton functions such as $\tilde{h}_0$ for fixed energy E_0 introduce a Finsler geometry in configuration space.

HYPOTHESIS 1.3 *Let* M *be a* d*-dimensional smooth manifold. Let* $h \in \mathscr{C}^\infty(T^*M,\mathbb{R})$ *be hyperregular and even and strictly convex in each fibre. Furthermore let* $h(.,0)$ *be bounded from above. For* $E_0 \in \mathbb{R}$ *set* $\widetilde{M} := M \setminus \{h(x,0) \geq E_0\}$. *Denoting the fibre derivative of* h *by* $\mathcal{D}_F h$, *we associate to* h *the energy function* $E_h(x,v) := h \circ (\mathcal{D}_F h)^{-1}(x,v)$ *on* TM.

The notion of fibre derivative and hyperregular are standard (see Abraham-Marsden [2]). For convenience of the reader, they are repeated in Definition 2.6 .

Now Theorem 2.10 states that assuming Hypothesis 1.3

$$\ell_{h,E_0}(x,v) := (\mathcal{D}_F h)^{-1}(x,\tilde{v})\cdot v\,, \tag{1.13}$$

where $\tilde{v}$ is chosen such that $E_h(x,\tilde{v}) = E_0$, is a Finsler function on $\widetilde{M}$. The most important property of ℓ_{h,E_0} is the homogeneity

$$\ell_{h,E_0}(x,\lambda v) = |\lambda|\ell_{h,E_0}(x,v)\,, \qquad \lambda \in \mathbb{R}\,,$$

which is analog to the homogeneity of $|v| = \sqrt{g(v,v)}$ in the case of a Riemannian metric. This is essential to define a curve length associated to ℓ_{h,E_0} as described in Definition 2.2 by

$$s_{\ell_{h,E_0}}(\gamma) := \int_a^b \ell_{h,E_0}(\gamma(t),\dot{\gamma}(t))\,dt\,.$$

A Finsler geodesic is then a curve γ on M, for which $s_{\ell_{h,E_0}}$ is extremal (see Def. 2.3).

The following Theorem establishes the connection between geodesics with respect to the Finsler function ℓ_{h,E_0} for a given hyperregular Hamilton function h and the integral curves of the associated Hamiltonian vector field X_h. It amplifies the Maupertuis principle in classical mechanics.

THEOREM 1.4 *Let* h, E_0 *and* $\widetilde{M}$ *satisfy Hypothesis 1.3. Let* $\ell_h := \ell_{h,E_0}$ *be as defined in* (1.13) *(see Theorem 2.10 for details).*

(a) *Let* $\gamma_0 : [a,b] \to \widetilde{M}$ *be a base integral curve of the Hamiltonian vector field* X_h *with energy* E_0 *(i.e.* $E_h(\gamma_0(t), \dot\gamma_0(t)) = E_0$ *for all* $t \in [a,b]$*). Then* γ_0 *is a geodesic on* $\widetilde{M}$ *with respect to* ℓ_h.

(b) *Conversely, if* γ_0 *is a geodesic on* $\widetilde{M}$ *with respect to* ℓ_h *with energy* $E_h(\gamma_0, \dot\gamma_0) = E_0$*, then* γ_0 *is a base integral curve of* X_h.

The Hamilton function $\tilde h_0$ introduced in (1.11) actually satisfies Hypothesis 1.3 with respect to the energy $E_0 = 0$ (see Corollary 2.14) and thus induces a Finsler function $\ell := \ell_{\tilde h_0, 0}$ and a Finsler distance defined by

$$d_\ell(x_0, x_1) = \inf_{\gamma \in \Gamma_{0,1}(x_0,x_1)} \int_0^1 \ell(\gamma(t), \dot\gamma(t))\, dt\,, \tag{1.14}$$

where $\Gamma_{0,1}(x_0, x_1)$ denotes the set of regular curves γ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$.

Theorem 1.5 *There exists a neighborhood* Ω *of* 0 *such that* $d^0(x) := d_\ell(0,x)$*, with* d_ℓ *defined in* (1.14)*, fulfills the generalized eikonal equation*

$$\tilde h_0(x, \nabla d^0(x)) = 0\,, \qquad x \in \Omega\,.$$

Furthermore

$$d^0(x) - \sum_{1\leq k \leq N} \varphi_k(x) = O(|x|^{N+1}) \quad \text{as} \quad x \to 0\,, \tag{1.15}$$

where each φ_k *is an homogeneous polynomial of degree* $k+2$.

In addition d_ℓ *is locally Lipschitz continuous, i.e.*

$$|d_\ell(x,y)| \leq C\,|x-y|\,, \qquad x, y \in \mathbb{R}^d\,, \tag{1.16}$$

where C *is locally uniform in* x *and* y.

The eikonal inequality

$$\tilde h_0(x, \nabla d^0(x)) \leq 0 \tag{1.17}$$

holds almost everywhere in $\mathbb{R}^d$.

To analyze eigenfunctions concentrated at the potential minimum $x_0 = 0$, we introduce a Dirichlet operator H_ε^Σ as follows.

Definition 1.6 *For* $\Sigma \subset \mathbb{R}^d$ *we set* $\Sigma_\varepsilon := \Sigma \cap (\varepsilon\mathbb{Z})^d$*. Any function* $u \in \ell^2(\Sigma_\varepsilon)$ *can by zero extension, i.e. via* $u(x) = 0$ *for* $x \notin \Sigma_\varepsilon$*, be embedded in* $\ell^2((\varepsilon\mathbb{Z})^d)$*. If we denote this embedding by* i_{Σ_ε}*, we can define the space* $\ell^2_{\Sigma_\varepsilon} := i_{\Sigma_\varepsilon}\left(\ell^2(\Sigma_\varepsilon)\right) \subset \ell^2((\varepsilon\mathbb{Z})^d)$ *and the Dirichlet operator*

$$H_\varepsilon^\Sigma := \mathbf{1}_{\Sigma_\varepsilon} H_\varepsilon|_{\ell^2_{\Sigma_\varepsilon}} \;:\; \ell^2_{\Sigma_\varepsilon} \to \ell^2_{\Sigma_\varepsilon} \tag{1.18}$$

with domain $\mathscr{D}(H_\varepsilon^\Sigma) = \{u \in \ell^2_{\Sigma_\varepsilon} \,|\, V_\varepsilon u \in \ell^2_{\Sigma_\varepsilon}\}$.

We now formulate our estimates of weighted ℓ^2-norms of eigenfunctions of the Dirichlet operator H_ε^Σ. We will show that they decay exponentially at a rate controlled by the Finsler distance $d^0(x)$. Theorem 1.5 is crucial to prove these estimates.

Theorem 1.7 *Let* $\Sigma \subset \mathbb{R}^d$ *be a bounded open region including the point* 0 *such that* $d^0 \in \mathscr{C}^2(\overline{\Sigma})$*, where* $d^0(x) := d_\ell(0,x)$ *is defined by* (1.14).

Let $E \in [0, \varepsilon R_0]$ *for* R_0 *fixed, assume Hypothesis 1.1 and let* H_ε^Σ *denote the Dirichlet operator introduced in (1.18).*

Then there exist constants $\varepsilon_0, B, C > 0$*, such that for all* $\varepsilon \in (0, \varepsilon_0]$ *and real* $u \in \ell^2_{\Sigma_\varepsilon}$

$$\left\| \left(1 + \tfrac{d^0}{\varepsilon}\right)^{-B} e^{\frac{d^0}{\varepsilon}} u \right\|_{\ell^2} \leq C \left[\varepsilon^{-1} \left\| \left(1 + \tfrac{d^0}{\varepsilon}\right)^{-B} e^{\frac{d^0}{\varepsilon}} \left(H_\varepsilon^\Sigma - E\right) u \right\|_{\ell^2} + \|u\|_{\ell^2} \right]\,. \tag{1.19}$$

In particular, let $u \in \ell^2_{\Sigma_\varepsilon}$ *be a normalized eigenfunction of* H_ε^Σ *with respect to the eigenvalue* $E \in [0, \varepsilon R_0]$*. Then there exist constants* $B, C > 0$*, such that for all* $\varepsilon \in (0, \varepsilon_0]$

$$\left\| \left(1 + \frac{d^0}{\varepsilon}\right)^{-B} e^{\frac{d^0}{\varepsilon}} u \right\|_{\ell^2} \leq C\,. \tag{1.20}$$

The plan of the paper is as follows.

Section 2 is devoted to the construction and properties of a Finsler function associated to a hyperregular Hamilton function. In particular, in Subsection 2.1 we introduce the general notion of a Finsler manifold, the associated curve length and Finsler geodesics. In Subsection 2.2 we construct the absolute homogeneous Finsler function ℓ_{h,E_0} with respect to an hyperregular Hamilton function h and a fixed energy E_0. In particular, we prove Theorem 2.10. The proof of Theorem 1.4 is given in Subsection 2.3. In Subsection 2.4 we prove Lemma 1.2 and we show that we can apply the results derived up to this point to the Hamilton function $\tilde{h}_0$ defined in (1.11). Subsection 2.5 contains the proof of Theorem 1.5.

In Section 3 we show the exponential decay of the eigenfunctions of the low lying spectrum of H_ε with a rate controlled by the Finsler distance constructed in Section 2. In particular, in Subsection 3.1 we show three basic lemmata and in Subsection 3.2 we prove Theorem 1.7.

2. Finsler Distance associated to H_ε

2.1. **Definition and Properties of Finsler Manifold and Finsler Metric.** We introduce the general notion of a Finsler manifold and Finsler distance (for detailed description of Finsler manifolds we refer e.g. to Bao-Chern-Shen [5], Abate-Patrizio [1]).

For a manifold M, $\pi : TM \to M$ denotes the tangent bundle with fibre $T_xM = \pi^{-1}(x)$. We denote an element of TM by (x, v) where $x \in M$ and $v \in T_xM$. Analogously, (x, ξ) with $\xi \in T^*_xM$ denotes a point in the cotangent bundle $\pi^* : T^*M \to M$.

The canonical pairing between an element $v \in T_xM$ and $\xi \in T^*_xM$ is written as $v \cdot \xi$.

Definition 2.1 *Let M be a d-dimensional $\mathscr{C}^\infty$-manifold and $TM \setminus \{0\} := \{(x, v) \in TM \,|\, v \neq 0\}$ the slit tangent bundle.*

(a) *A (Lagrange)-function $F : TM \to [0, \infty)$ is called a Finsler function on M, if:*
  1) *F is of class $\mathscr{C}^\infty(TM \setminus \{0\})$.*
  2) *$F(x, \lambda v) = \lambda F(x, v)$ for $\lambda > 0$, i.e., F is positive homogeneous of order 1 in each fibre.*
  3) *$F(x, v) > 0$ for $v \neq 0$.*

(b) *A Finsler function F is said to be absolute homogeneous, if*
  4) *$F(x, \lambda v) = |\lambda| F(x, v)$ for all $\lambda \in \mathbb{R}$,*

(c) *A manifold together with a Finsler function, (M, F), is called a Finsler manifold.*

It is shown in Bao-Chern-Shen [5] that for any Finsler function F the triangle inequality holds:

$$F(x, v + \tilde{v}) \leq F(x, v) + F(x, \tilde{v}), \quad (x, v), (x, \tilde{v}) \in TM\,. \tag{2.1}$$

A Finsler function induces a curve length on M as follows. A curve $\gamma : [a, b] \to M$ on M is called regular, if it is $\mathscr{C}^2$ and $\dot{\gamma}(t) \neq 0$ for all $t \in [a, b]$. We introduce the Banach-manifold

$$\Gamma_{a,b}(x_1, x_2) := \{\gamma \in \mathscr{C}^2([a, b], M) \,|\, \gamma \text{ is regular and } \gamma(a) = x_1,\, \gamma(b) = x_2\,\}\,. \tag{2.2}$$

Definition 2.2 *For any Finsler function F on M, the curve length $s_F : \Gamma_{a,b}(x_1, x_2) \to \mathbb{R}$ associated to F is defined as*

$$s_F(\gamma) := \int_a^b F(\gamma(t), \dot{\gamma}(t))\, dt\,.$$

For any $\delta > 0$, a regular variation of $\gamma \in \Gamma_{a,b}(x_1, x_2)$ is a $\mathscr{C}^2$-map $\gamma_\delta : [a, b] \times (-\delta, \delta) \to M$, such that $\gamma_\delta(t, 0) = \gamma(t)$ for all $t \in [a, b]$ and $\gamma_\delta(., u)$ is regular for each $u \in (-\delta, \delta)$.

Each $\mathscr{C}^2$-map $\gamma_\delta : (-\delta, \delta) \to \Gamma_{a,b}(x_1, x_2)$ with $\gamma_\delta(0) = \gamma$ can be considered as a regular variation of γ with fixed endpoints (i.e. with $\gamma_\delta(a, u) = x_1$ and $\gamma_\delta(b, u) = x_2$ for all $u \in (-\delta, \delta)$). Therefore the tangent space of $\Gamma_{a,b}(x_1, x_2)$ at a point η is given by

$$T_\eta \Gamma_{a,b}(x_1, x_2) = \{\partial_u \eta_\delta|_{u=0} \,|\, \eta_\delta \text{ is a regular variation of } \eta \text{ with fixed endpoints}\}\,, \tag{2.3}$$

where $\partial_u \eta_\delta|_{u=0}$ is considered as a vector field along η, i.e., as a function $\partial_u \eta_\delta|_{u=0} : [a, b] \to TM$ such that $\partial_u \eta_\delta|_{u=0}(t) \in T_{\eta(t)}M$. Since the variation η_δ has fixed endpoints, it follows that $\partial_u \eta_\delta|_{u=0}(a) = \partial_u \eta_\delta|_{u=0}(b) = 0$.

Definition 2.3 *$\gamma \in \Gamma_{a,b}(x_1, x_2)$ is called a geodesic with respect to the Finsler function F (or a Finsler geodesic), if $ds_F|_\gamma = 0$.*

Definition 2.4 *Let (M, F) denote a Finsler manifold.*

(a) *The Finsler distance* $d_F(x_1, x_2) : M \times M \to [0, \infty]$ *between the points* x_1 *and* x_2 *is defined by*

$$d_F(x_1, x_2) := \inf_{\gamma \in \Gamma_{0,1}(x_1, x_2)} s_F(\gamma) \, .$$

If $\Gamma_{0,1}(x_1, x_2)$ *is empty, the distance is defined to be infinity.*

(b) *A geodesic* γ *between two points* x_1 *and* x_2 *is called minimal, if* $s_F(\gamma) = d(x_1, x_2)$*.*

It follows easily from the definitions of a Finsler function F and the associated Finsler distance d_F that $d_F(x_1, x_2) \geq 0$, where equality holds if and only if $x_1 = x_2$. Furthermore by (2.1) the triangle inequality $d_F(x_1, x_3) \leq d_F(x_1, x_2) + d_F(x_2, x_3)$ holds. If in addition the Finsler function F is absolute homogeneous, then $d_F(x_1, x_2) = d_F(x_2, x_1)$. Thus for an absolute homogeneous Finsler function, (M, d_F) is a metric space.

Definition 2.5 *We denote by* $SM := TM/\sim_S$ *the sphere bundle, where*

$$(x, v) \sim_S (y, w) \, , \quad \textit{if} \quad x = y \quad \textit{and} \quad v = \lambda w \quad \textit{for any} \quad \lambda > 0 \, .$$

Let $\pi_s : TM \to SM$ *denote the projection* $\pi_s(x, v) = [x, v]$*.*

2.2. **The Finsler Function of a hyperregular Hamilton function.** To define a Finsler distance for which an analog of Jacobi's Theorem holds, we briefly introduce the notion of fibre derivatives, hyperconvexity and hyperregularity of h. It is shown in Proposition 2.8 that hyperconvexity of h is a sufficient condition for hyperregularity.

Definition 2.6 (a) *Let* M *be a manifold and* $f \in \mathscr{C}^\infty(T^*M, \mathbb{R})$*. Then for* $f_x := f|_{T^*_x M}$ *the map* $\mathcal{D}_F f : T^*M \to TM$ *defined by* $\mathcal{D}_F f(x, \xi) := Df_x(\xi)$ *is called the fibre derivative of* f*.*

(b) *Analogously, the fibre derivative of a function* $g \in \mathscr{C}^\infty(TM, \mathbb{R})$ *is defined as* $\mathcal{D}_F g : TM \to T^*M$, $\mathcal{D}_F g(x, v) := Dg_x(v)$*.*

(c) *A function* $f : SM \to TM$ *is called strictly fibre preserving, if* $f([x, u]) \in [x, u]$*, where* $[x, u]$ *denotes the equivalence class with respect to* $\sim_S$*.*

(d) *A smooth function* $h : T^*M \to \mathbb{R}$ *(or* $L : TM \to \mathbb{R}$*) is said to be hyperregular, if its fibre derivative* $\mathcal{D}_F h : T^*M \to TM$ *(or* $\mathcal{D}_F L : TM \to T^*M$*) is a diffeomorphism. For a hyperregular function* $h \in \mathscr{C}^\infty(T^*M)$*, we sometimes use the notation*

$$\xi_h(x, v) := \left(\mathcal{D}_F h\right)^{-1}(x, v) \quad \textit{and} \quad v_h(x, \xi) := \mathcal{D}_F h(x, \xi) \, . \tag{2.4}$$

Definition 2.7 *For a normed vector space* V *we call a function* $L \in \mathscr{C}^2(V, \mathbb{R})$ *hyperconvex, if there exists a constant* $\alpha > 0$ *such that*

$$D^2 L|_{v_0}(v, v) \geq \alpha \|v\|^2 \quad \textit{for all} \quad v_0, v \in V \, .$$

We recall that a strictly convex function $L \in \mathscr{C}^2(V, \mathbb{R})$ has the properties

$$L(v_1) - L(v_2) \geq DL(v_2)(v_1 - v_2) \tag{2.5}$$

$$D^2 L|_{v_0}[v, v] > 0 \tag{2.6}$$

$$\left(DL(v_1) - DL(v_2)\right)(v_1 - v_2) > 0 \tag{2.7}$$

Proposition 2.8 *If a real valued function* $h \in \mathscr{C}^\infty(T^*M)$ *is hyperconvex in each fibre* $T^*_x M$*, it is hyperregular.*

Proof. By definition, $\mathcal{D}_F h$ is fibre preserving, thus in the coordinates on TM and T^*M induced from local coordinates on M at x_0, its derivative $D\mathcal{D}_F h|_{(x_0, \xi_0)}$ is given by the $2d \times 2d$-matrix

$$\begin{pmatrix} \mathbf{1} & 0 \\ * & M \end{pmatrix} , \tag{2.8}$$

where M is the matrix representation of $D^2_\xi h|_{(x_0, \xi_0)} = D\left(\mathcal{D}_F h|_{T^*_x M}\right)$.

Since h was assumed to be hyperconvex in each fibre, M is positive definite and thus it follows from (2.8) that $\mathcal{D}_F h$ is a local diffeomorphism.

We claim that $Dh_x : T^*_x M \to T_x M$ is bijective for all $x \in M$. Since $\mathcal{D}_F h$ is fibre preserving, this shows that $\mathcal{D}_F h$ is a global diffeomorphism and finishes the proof. Thus we fix any $x \in M$ and analyze Dh_x.

Since h_x is strictly convex for each $x \in M$, by (2.7)

$$(\xi - \eta)\left(Dh_x(\xi) - Dh_x(\eta)\right) > 0 \, , \qquad \xi, \eta \in T^*_x M, \, \eta \neq \xi$$

holds and thus $\mathcal{D}_F h(x,.) = Dh_x$ is injective.

To show the surjectivity, we claim that, for any $v_0 \in T_xM$, the initial value problem

$$v_0 = \frac{d}{dt} Dh_x(\xi(t)) = D^2 h_x(\xi(t)) \cdot \dot{\xi}(t) \,, \qquad \xi(0) = 0 \,. \tag{2.9}$$

has a solution $\xi(t)$ for all $t \in [0,1]$. Then

$$v_0 = \int_0^1 D^2 h_x(\xi(t)) \cdot \dot{\xi}(t)\, dt = Dh_x(\xi(1)) - Dh_x(\xi(0)) = Dh_x(\xi(1))$$

and thus Dh_x is surjective.

Since h is hyperconvex, the inverse $\left(D^2 h_x|_{\xi(t)}\right)^{-1}$ exists, thus (2.9) can be rewritten as

$$\dot{\xi}(t) = \left(D^2 h_x|_{\xi(t)}\right)^{-1} \cdot v_0 \,, \qquad \xi(0) = 0 \,. \tag{2.10}$$

Thus 2.10 is of the form $\dot{\xi} = F(\xi)$, where F is locally Lipschitz. Therefore for any $v_0 \in T_xM$, (2.10) has a solution, which either exists for all $t \geq 0$ or becomes infinite for a finite value of t.

In order to exclude that the curve ξ reaches infinity for some $t < 1$, we need the hyperconvexity of h. We choose a norm $\|.\|_{T_x^*M}$ on T_x^*M and denote by $\|.\|_{T_xM}$ the norm on T_xM, which is induced by duality. Since for fixed $\eta \in T_x^*M = T_\xi(T_x^*M)$ the second derivative $D^2 h_x|_\xi(\eta)$ can be regarded as an element of T_xM, it follows by the hyperconvexity of h that there exists a constant $\alpha > 0$ such that for all $\xi \in T_x^*M$

$$\|D^2 h_x|_\xi(\eta)\|_{T_xM} = \sup_{\mu \in T_x^*M} \frac{|D^2 h_x|_\xi(\eta,\mu)|}{\|\mu\|_{T_x^*M}} \geq \frac{|D^2 h_x|_\xi(\eta,\eta)|}{\|\eta\|_{T_x^*M}} \geq \alpha \|\eta\|_{T_x^*M} \,, \quad \eta \in T_\xi(T_x^*M) \tag{2.11}$$

and therefore

$$\|v\|_{T_xM} = \|D^2 h_x|_\xi \left(D^2 h_x|_\xi\right)^{-1}(v)\|_{T_xM} \geq \alpha \| \left(D^2 h_x|_\xi\right)^{-1}(v)\|_{T_x^*M} \,, \qquad v \in T_xM \,. \tag{2.12}$$

(2.10) together with (2.12) yields

$$\|\dot{\xi}(t)\|_{T_x^*M} = \| \left(D^2 h_x|_{\xi(t)}\right)^{-1}(v_0)\|_{T_x^*M} \leq \frac{1}{\alpha}\|v_0\| \,. \tag{2.13}$$

Therefore the curve $\xi(t)$ exists for all $t \in [0,1]$ and $\mathcal{D}_F h(\xi(1)) = v_0$.

□

For any hyperregular Hamilton function $h \in \mathscr{C}^\infty(T^*M)$, we define the energy function E_h on TM by

$$E_h(x,v) := h \circ (\mathcal{D}_F h)^{-1}(x,v) (= h(x, \xi_h(x,v))) \tag{2.14}$$

and the action

$$A_h : TM \to \mathbb{R} \,, \quad A_h(x,v) := (\mathcal{D}_F h)^{-1}(x,v) \cdot v (= \xi_h(x,v) \cdot v) \,. \tag{2.15}$$

Then the Lagrange function

$$L_h : TM \to \mathbb{R} \quad \text{defined by} \quad L_h(x,v) = A_h(x,v) - E_h(x,v) \tag{2.16}$$

(the Legendre transform of h) is hyperregular on TM and

$$\mathcal{D}_F L_h(x,v) = (\mathcal{D}_F h)^{-1}(x,v) \tag{2.17}$$

(by Theorem 3.6.9 in [2], the hyperregular Lagrange functions on TM and the hyperregular Hamilton functions on T^*M are in bijection). In particular, by (2.15) and (2.17),

$$A_h(x,v) = \mathcal{D}_F L_h(x,v) \cdot v \,. \tag{2.18}$$

DEFINITION 2.9 *For a smooth manifold M, a $\mathscr{C}^\infty$-function $h : T^*M \to \mathbb{R}$ and $E_0 \in \mathbb{R}$, we define the singular set $S_h(E_0)$ by*

$$S_h(E_0) := \{ x \in M \mid h(x,0) \geq E_0 \} \,.$$

Since $h(.,0)$ is continuous, $S_h(E_0)$ is closed. Thus $\widetilde{M} := M \setminus S_h(E_0)$ is again a smooth manifold.

For h hyperregular, we shall now introduce an associated Finsler function on $\widetilde{M}$.

THEOREM 2.10 *Let M and $h \in \mathscr{C}^\infty(T^*M)$ satisfy Hypothesis 1.3 and let E_0, $S_h(E_0)$ and $\widetilde{M}$ be as described in Definition 2.9.*

i) *Then there exists a strictly fibre preserving* $\mathscr{C}^\infty$*-function* $\tau_{E_0} : S\widetilde{M} \to T\widetilde{M}$*, which is uniquely determined by the condition*

$$h \circ (\mathcal{D}_F h)^{-1} \circ \tau_{E_0} = E_0 \ . \tag{2.19}$$

ii) *Let* $\tilde{\tau}_{E_0} := \tau_{E_0} \circ \pi_S : T\widetilde{M} \to T\widetilde{M}$ *and let* $\ell_{h,E_0} : T\widetilde{M} \to \mathbb{R}$ *be defined by*

$$\ell_{h,E_0}(x, v) := (\mathcal{D}_F h)^{-1} \circ \tilde{\tau}_{E_0}(x, v) \cdot v \ .$$

Then ℓ_{h,E_0} *is an absolute homogeneous Finsler function on* $\widetilde{M}$.

iii) *For any regular curve* $\gamma : [a, b] \to \widetilde{M}$*, there exists a unique* $\mathscr{C}^1$*-function* $\lambda : [a, b] \to \mathbb{R}_+$ *such that*

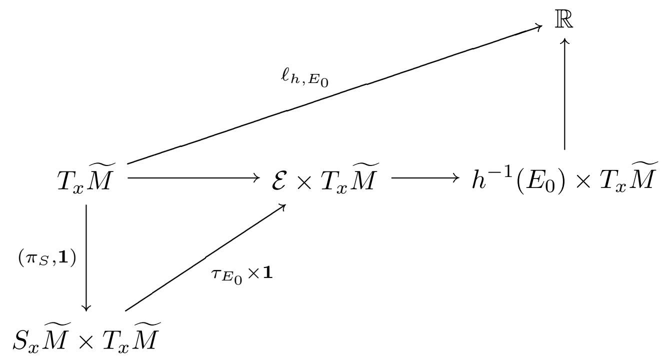
$$\tilde{\tau}_{E_0}(\gamma(t), \dot{\gamma}(t)) = (\gamma(t), \lambda(t)\dot{\gamma}(t)) \ . \tag{2.20}$$

Remark 2.11 (a) *Since* ℓ_{h,E_0} *is defined on* $\widetilde{M} = M \setminus S(E_0)$*, we call* (M, ℓ_{h,E_0}) *a Finsler manifold with singularities.*

(b) *If we continuously extend* ℓ_{h,E_0} *from* $\widetilde{M}$ *to* M *by setting* $\ell_{h,E_0}(x, v) = 0$ *for* $x \in S(E_0)$*, the associated distance* d_ℓ *is well defined on all of* M*. Nevertheless contrary to the case of a Finsler manifold without singularities (as described for example in Bao-Chern-Shen* [5]*), the geodesic curves with respect to* ℓ_{h,E_0} *may have kinks at the "singular points", which are the connected components of* $S_h(E_0)$.

(c) *Geometrically, the function* $\tilde{\tau}_{E_0}$ *projects an element* (x, v) *of the tangent bundle* $T\widetilde{M}$ *to an element* $(x, \lambda v)$ *in the* $(2d - 1)$*-dimensional submanifold* $\mathcal{E} = E_h^{-1}(E_0)$.

(d) *Schematically the functions occurring in Theorem 2.10 are illustrated in the following diagram.*



$$v \xrightarrow{(\tilde{\tau}_{E_0}, \mathbf{1})} (\tilde{v}, v) \xrightarrow{(\mathcal{D}_F h)^{-1} \times \mathbf{1}} (\xi_h(\tilde{v}), v)$$

(e) *With the notation* (2.4)*,* $\ell_{h,E_0}(x, v)$ *can be written as*

$$\ell_{h,E_0}(x, v) = \xi_h(x, \tilde{v}) \cdot v \quad \textit{where} \quad (x, \tilde{v}) = \tilde{\tau}_{E_0}(x, v) \in \mathcal{E} \ . \tag{2.21}$$

To prove Theorem 2.10, we need the following lemma.

Lemma 2.12 *In the setting of Theorem 2.10 fix* $x \in \widetilde{M}$ *and* $u \in T_x\widetilde{M} \setminus \{0\}$*. Then for* $E_h : TM \to \mathbb{R}$ *defined by (2.14), the function*

$$E_u : [0, \infty) \to \mathbb{R} \ , \qquad E_u(\lambda) := E_h(x, \lambda u)$$

is strictly increasing with $\frac{d}{d\lambda} E_u > 0$ *for* $\lambda > 0$*. Furthermore* $E_u(0) \leq E_0$ *and* $\lim_{\lambda \to \infty} E_u(\lambda) = \infty$.

Proof of Lemma 2.12. Since h_x is even, $Dh_x(0) = 0$, thus $v_h(x, 0) = 0$, $\xi_h(x, 0) = 0$ and

$$E_u(0) = E_h(x, 0) = h(x, 0) < E_0 \ . \tag{2.22}$$

To show that E_u is strictly increasing, we will analyze the derivative of E_u for $\lambda > 0$.
By definition $\mathcal{D}_F h(x, \xi) = Dh_x(\xi)$, thus

$$\frac{dE_u}{d\lambda}|_\lambda = Dh_x|_{(Dh_x)^{-1}(\lambda u)} \cdot D\,(Dh_x)^{-1}|_{\lambda u}(u) \ . \tag{2.23}$$

We notice that

$$Dh_x|_{(Dh_x)^{-1}(\lambda u)} = \lambda u \tag{2.24}$$

and for L_h defined in (2.16) it follows from (2.17) that

$$D\left(Dh_x\right)^{-1}|_{\lambda u}(u) = D\left(DL_{h,x}\right)|_{\lambda u}(u)\,. \tag{2.25}$$

Inserting (2.25) and (2.24) in (2.23) yields

$$\frac{dE_u}{d\lambda}|_\lambda = \lambda u\cdot D\left(DL_{h,x}\right)|_{\lambda u}(u) = \lambda\, D^2 L_{h,x}|_{\lambda u}(u,u)\,, \tag{2.26}$$

where we identify linear maps from T_xM to T^*_xM with bilinear forms on T_xM. If h is strictly convex in each fibre, the same is true for L_h. Therefore by (2.26) the first derivative of E_u is strictly positive for $\lambda\in(0,\infty)$ and thus E_u is strictly increasing.

The fact that $\lim_{\lambda\to\infty}E_u(\lambda)=\infty$ can be seen as follows. From the strict convexity of h and since $h(x,\xi)\geq h(x,0)$ for all $\xi\in T^*M$, it follows that $\lim_{|\xi|\to\infty}h(x,\xi)=\infty$. Since h is hyperregular, $\mathcal{D}_F h(x,.) = Dh_x : T_xM\to T^*_xM$ is a global diffeomorphism. Thus for any norm $\|.\|_{T_xM}$ on T_xM and the induced norm $\|.\|_{T^*_xM}$ on T^*_xM, we have $\|Dh_x(v_n)\|_{T^*_xM}\to\infty$ for any sequence (v_n) in T_xM satisfying $\|v_n\|_{T_xM}\to\infty$ (any global diffeomorphism is proper). Thus

$$\lim_{\lambda\to\infty}E_u(\lambda) = \lim_{\lambda\to\infty}h(x,\xi_h(x,\lambda u)) = \lim_{\|\xi\|\to\infty}h(x,\xi) = \infty\,.$$

□

Proof of Theorem 2.10. i) From Lemma 2.12 it follows that for fixed $x\in\widetilde{M}$, each ray $[x,u]\in S\widetilde{M}$ intersects the hypersurface $\mathcal{E}_x := E_h^{-1}(E_0)\cap T_x\widetilde{M}$ in exactly one point v, i.e. for each ray $[x,u]$ there is exactly one point $(x,v)\in T_x\widetilde{M}$ such that $E_h(x,v)=E_0$. Thus $[x,u]\mapsto(x,v)$ defines a map τ_{E_0}, which is uniquely determined by (2.19). Clearly τ_{E_0} is strictly fibre preserving.

To analyze the regularity of τ_{E_0}, we will use the Implicit Function Theorem. We remark that there exists an isomorphism $\psi : T\widetilde{M}\setminus\{0\}\to S\widetilde{M}\times\mathbb{R}_+$, such that $\psi^{-1}(\{[x,u]\}\times\mathbb{R}_+)=[x,u]\subset T\widetilde{M}\setminus\{0\}$. Setting $\widehat{E}_h := E_h\circ\psi^{-1}$ and $\widehat{\tau}_{E_0} := \psi\circ\tau_{E_0}$, we get $\widehat{\tau}_{E_0}(s) = (s,\lambda(s)), s\in S\widetilde{M}$, for some $\lambda : S\widetilde{M}\to\mathbb{R}_+$ and therefore $\widehat{E}_h(s,\lambda(s)) = E_0$. By Lemma 2.12, $\frac{d\widehat{E}_h}{d\lambda}(s_0,\lambda_0)>0$ for all $(s_0,\lambda_0)\in S\widetilde{M}\times\mathbb{R}_+$. Smoothness of λ and thus of τ_{E_0} now follows from the Implicit Function Theorem.

iii) By i), for $t\in[a,b]$ fixed, there is a unique $\lambda(t)\in\mathbb{R}_+$ with (2.20). Since τ_{E_0} is smooth (as composition of smooth maps), $t\mapsto(\gamma(t),\lambda(t)\dot\gamma(t))$ is a $\mathscr{C}^1$-curve. Thus $\lambda\in\mathscr{C}^1([a,b],\mathbb{R})$.

ii) To show that $\ell_{h,E_0} : T\widetilde{M}\to\mathbb{R}$ is a Finsler function on $\widetilde{M}$, we check the defining properties.

1) The regularity $\ell_{h,E_0}\in\mathscr{C}^\infty(T\widetilde{M}\setminus\{0\})$ follows from the fact that h is hyperregular and the function $\tilde\tau_{E_0}$ is $\mathscr{C}^\infty$.
2) To show $\ell_{h,E_0}(x,\lambda v) = \lambda\ell_{h,E_0}(x,v)$, $(\lambda>0)$, we notice that by construction $\tilde\tau_{E_0}(x,\lambda v) = \tilde\tau_{E_0}(x,v)$ for any $\lambda>0$. Thus $(\mathcal{D}_F h)^{-1}\circ\tilde\tau_{E_0}$ is homogeneous of order zero in each fibre. Since $\xi\cdot v$ is bilinear, it follows that

$$\ell_{h,E_0}(x,\lambda v) = (\mathcal{D}_F h)^{-1}\circ\tilde\tau_{E_0}(x,\lambda v)\,\cdot\,\lambda v = \lambda\ell_{h,E_0}(x,v)\,.$$

3) To show $\ell_{h,E_0}(x,v)>0$, $(v\neq 0)$, we define

$$a_h = A_h\circ\mathcal{D}_F h\,:\,T^*M\to\mathbb{R}\,,\qquad a_h(x,\xi) = \xi\cdot\mathcal{D}_F h(x,\xi)\,.$$

Since h was assumed to be strictly convex in each fibre, one obtains from (2.7)

$$(\xi-\eta)\cdot(\mathcal{D}_F h(x,\xi)-\mathcal{D}_F h(x,\eta))>0\,,\quad \xi,\eta\in T^*_xM,\ \eta\neq\xi\,.$$

Therefore choosing $\xi=-\eta$ and using that h is even in each fibre (thus $\mathcal{D}_F h$ is odd) yields

$$2\xi\cdot(\mathcal{D}_F h(x,\xi)-\mathcal{D}_F h(x,-\xi)) = 4\xi\cdot\mathcal{D}_F h(x,\xi) = 4a_h(x,\xi)>0\,,\quad\text{for}\quad \xi\neq 0\,. \tag{2.27}$$

Since h is even and strictly convex, it takes its absolute minimum at $\xi=0$ and thus $\mathcal{D}_F h(x,0)=0$. Since furthermore $\mathcal{D}_F h$ is a global diffeomorphism, we get $(\mathcal{D}_F h)^{-1}(x,v)\neq 0$ for $v\neq 0$. By (2.27)

$$A_h(x,v) = a_h(x,(\mathcal{D}_F h)^{-1}(x,v))>0\,,\qquad v\neq 0\,. \tag{2.28}$$

Setting $\tilde{\tau}_{E_0}(x,v) =: (x,\tilde{v})$, it follows from the fact that τ_{E_0} is strictly fibre preserving that there exists a $\lambda > 0$ such that $v = \lambda\tilde{v}$. Thus by (2.28) for $v \neq 0$

$$\ell_{h,E_0}(x,v) = (\mathcal{D}_F h)^{-1}(x,\tilde{v}) \cdot \lambda\tilde{v} = \lambda A_h(x,\tilde{v}) > 0\,.$$

Obviously $\ell_{h,E_0}(x,0) = 0$.

4) It remains to show that ℓ_{h,E_0} is absolute homogeneous of order one. Since h is even in each fibre, $\mathcal{D}_F h$ and $(\mathcal{D}_F h)^{-1}$ are odd. Thus for $(x,v) \in \mathcal{E}_x$

$$h \circ (\mathcal{D}_F h)^{-1}(x,-v) = h \circ (\mathcal{D}_F h)^{-1}(x,v) = E_0 \tag{2.29}$$

and (2.29) yields

$$(x,v) \in \mathcal{E} \qquad \Longrightarrow \qquad (x,-v) \in \mathcal{E}\,. \tag{2.30}$$

Since τ_{E_0} is strictly fibre preserving, we have for $(x,v) \in T_x\widetilde{M}$ and some $\lambda > 0$, using (2.30)

$$\tilde{\tau}_{E_0}(-v) = \lambda(-v) = -\tilde{\tau}_{E_0}(v)\,. \tag{2.31}$$

By the fact that $(\mathcal{D}_F h)^{-1}$ is odd and (2.31) we can conclude that

$$\ell_{h,E_0}(-v) = (\mathcal{D}_F h)^{-1} \circ \tilde{\tau}_{E_0}(-v) \cdot (-v) = (\mathcal{D}_F h)^{-1} \circ \tilde{\tau}_{E_0}(v) \cdot v = \ell_{h,E_0}(v)\,. \tag{2.32}$$

By (2.32), ℓ_{h,E_0} is even in each fibre and thus for any $\lambda \in \mathbb{R}$

$$\ell_{h,E_0}(x,\lambda v) = \ell_{h,E_0}(x,|\lambda|v) = |\lambda|\ell_{h,E_0}(x,v)\,.$$

□

2.3. **Proof of Theorem 1.4.** Step 4 of our proof will use the Maupertuis principle (at least implicitly). It is adapted from Abraham-Marsden [2].

Step 1:
We will show that

$$\begin{aligned}\Gamma(x_1,x_2,[a,b],E_0) := \{(\gamma,\alpha) \mid \alpha : [a,b] \to \mathbb{R} \quad \text{is} \quad \mathscr{C}^2, \quad \tfrac{d\alpha}{dt} > 0\,, \quad \alpha(a) = 0\,,\\ \gamma \in \Gamma_{0,\alpha(b)}(x_1,x_2) \quad \text{such that} \quad E_h(\gamma(\alpha(t)),\dot{\gamma}(\alpha(t))) = E_0 \quad \text{for all} \quad t \in [a,b]\,\}\,,\end{aligned} \tag{2.33}$$

where $\Gamma_{a,b}(x_1,x_2)$ was introduced in (2.2), is a Banach-manifold.
$\Gamma(x_1,x_2,[a,b],E_0)$ is the set of all pairs (γ,α), where γ is a regular curve on $\widetilde{M}$ joining the points x_1 and x_2 and α is a change of parameter, ensuring that the curve $(\gamma\circ\alpha,\dot{\gamma}\circ\alpha) \in TM$ (which is not equal to the lifted curve $(\gamma\circ\alpha,\frac{d}{dt}(\gamma\circ\alpha))$) lies on the energy shell $\mathcal{E} = E_h^{-1}(E_0)$.

Set $A := \{\alpha : [a,b] \to \mathbb{R} \mid \frac{d\alpha}{dt} > 0 \quad \text{and} \quad \alpha(a) = 0\}$ and denote by $\Gamma_{0,\infty}$ the space of all regular curves $\gamma : [0,\infty) \to \widetilde{M}$. Then $\Gamma_{0,\infty} \times A$ is a Banach manifold. We consider the $\mathscr{C}^1$-mapping

$$g : \Gamma_{0,\infty} \times A \to \widetilde{M} \times \widetilde{M}\,, \quad (\gamma,\alpha) \mapsto (\gamma\circ\alpha(a),\gamma\circ\alpha(b))\,.$$

Then $(x_1,x_2) \in \widetilde{M} \times \widetilde{M}$ is a regular value of g and

$$\Gamma([a,b],x_1,x_2) := g^{-1}(x_1,x_2) = \{(\gamma,\alpha) \in \Gamma_{0,\infty} \times A \mid \gamma(\alpha(a)) = x_1, \gamma(\alpha(b)) = x_2\}$$

is a submanifold of $\Gamma_{0,\infty} \times A$. This follows by the fact that the Inverse Function Theorem holds in Banach manifolds (see Hamilton [13]). We introduce

$$\tilde{E}_h : \mathscr{C}^1([a,b],T\widetilde{M}) \to \mathscr{C}^1([a,b],\mathbb{R})\,, \quad \tilde{E}_h(\eta)(t) := E_h(\eta(t))$$

and for

$$\Gamma_{a,b}^{T\widetilde{M}}(x_1,x_2) := \{(\gamma,k\,\dot{\gamma}) \mid \gamma \in \Gamma_{a,b}(x_1,x_2),\ k \in \mathscr{C}^1([a,b],\mathbb{R}_+)\}$$

we set

$$\Phi : \Gamma([a,b],x_1,x_2) \to \Gamma_{a,b}^{T\widetilde{M}}(x_1,x_2)\,, \quad (\gamma,\alpha) \mapsto (\gamma\circ\alpha,\dot{\gamma}\circ\alpha)\,.$$

Then Φ is a diffeomorphism. In fact by a straightforward calculation it is bijective, with inverse $\Phi^{-1}(\eta,k\dot{\eta}) = (\eta\circ\alpha^{-1},\alpha)$, where $\alpha(t) = \int_a^t (k(s))^{-1}\,ds$.

Identifying E_0 with the constant function $E_0(t) = E_0$, we obtain $\Gamma(x_1,x_2,[a,b],E_0) = f^{-1}(E_0)$ for $f := \tilde{E}_h \circ \Phi$. To show that E_0 is a regular value of f it is sufficient to show that it is a regular value of $\tilde{E}_h$, i.e., that for each $v \in \mathscr{C}^1([a,b],\mathbb{R})$ (considered as a vector field along $E_0 \in \mathscr{C}^1([a,b],\mathbb{R})$), there is a vector field X along $\eta \in \tilde{E}_h^{-1}(E_0)$ with $d\tilde{E}_h|_\eta X = v$. Note that $\eta(t) = (x(t),v(t))$ with $v(t) \neq 0$.

Since $DE_h(x,v) \neq 0$ for $v \neq 0$, there is a covering of $[a,b]$ by open intervals $I_j, j \in J$ and vector fields X_j along $\eta|_{I_j}$ with $DE_h|_{\eta(t)} X_j(t) = v(t)$ for all $t \in I_j$. Choosing a partition of unity (χ_j) subordinate to (I_j), we set $X(t) = \sum_{j\in J} \chi_j(t) X_j(t)$. Then $d\tilde{E}_h|_\eta X = v$. Thus E_0 is a regular value of f and $\Gamma([a,b], x_1, x_2, E_0)$ is a Banach-manifold.

Step 2:
We construct a diffeomorphism $b_{E_0} : \Gamma_{a,b}(x_1,x_2) \to \Gamma(x_1,x_2,[a,b],E_0)$.

By Theorem 2.10, there exists for any $\eta \in \Gamma_{a,b}(x_1,x_2)$ a unique $\mathscr{C}^1$-function $\lambda : [a,b] \to \mathbb{R}_+$, such that

$$E_h(\eta(t), \lambda(t)\dot\eta(t)) = E_0\ .$$

Set

$$\alpha(t) := \int_a^t \frac{1}{\lambda(s)}\, ds \quad \text{and} \quad \gamma = \eta \circ \alpha^{-1} : [0, \alpha(b)] \to \widetilde{M}\ , \tag{2.34}$$

then $\alpha : [a,b] \to \mathbb{R}$ with $\dot\alpha > 0$. From $\dot\eta(t) = \dot\gamma(\alpha(t)) \cdot \dot\alpha(t)$ it follows that

$$E_h(\gamma(\alpha(t)), \dot\gamma(\alpha(t))) = E_h(\eta(t), \lambda(t)\dot\eta(t)) = E_0\ , \tag{2.35}$$

i.e. $(\gamma,\alpha) \in \Gamma(x_1,x_2,[a,b],E_0)$. We can conclude that there is a bijection between Banach manifolds given by

$$b_{E_0} : \Gamma_{a,b}(x_1,x_2) \to \Gamma(x_1,x_2,[a,b],E_0)$$

$$b_{E_0}(\eta) = (\eta\circ\alpha^{-1}, \alpha) \quad \text{with} \quad \alpha(t) := \int_a^t (\lambda(s))^{-1}\, ds \quad \text{for} \quad \tilde\tau_{E_0}(\eta,\dot\eta) = (\eta, \lambda\dot\eta)\ . \tag{2.36}$$

On the other hand, if we start with $(\gamma,\alpha) \in \Gamma(x_1,x_2,[a,b],E_0)$, then $E_h(\gamma(s),\dot\gamma(s)) = E_0$ with $s = \alpha(t)$. Setting $\eta := \gamma\circ\alpha : [a,b] \to \widetilde{M}$ it follows from $\dot\gamma(s) = \dot\eta(t)\,(\dot\alpha(t))^{-1}$ that

$$E_h(\eta(t), (\dot\alpha(t))^{-1}\,\dot\eta(t)) = E_0 \qquad \text{and thus} \qquad \lambda(t) = (\dot\alpha(t))^{-1}\ . \tag{2.37}$$

Thus the inverse function $b_{E_0}^{-1}$ is given by $b_{E_0}^{-1}(\gamma,\alpha) = \gamma\circ\alpha$.

Step 3:
We show that the critical points of the length functional s_{ℓ_h} defined in Definition 2.2 (i.e., the geodesics of ℓ_h) are in bijection with the critical points of the action integral

$$I : \Gamma(x_1,x_2,[a,b],E_0) \to \mathbb{R}\,, \quad I(\gamma,\alpha) := \int_{\alpha(a)}^{\alpha(b)} A_h(\gamma(s),\dot\gamma(s))\, ds\ , \tag{2.38}$$

where A_h denotes the action with respect to h defined in (2.15). Setting $s = \alpha(t)$ and using (2.4) gives

$$\int_{\alpha(a)}^{\alpha(b)} A_h(\gamma(s),\dot\gamma(s))\, ds = \int_a^b \xi_h(\gamma(\alpha(t)), \dot\gamma(\alpha(t))) \cdot \dot\gamma(\alpha(t))\dot\alpha(t)\, dt\ . \tag{2.39}$$

Setting $\eta(t) = \gamma(\alpha(t))$ and using (2.37) and the definition of ℓ_h and s_{ℓ_h}, we obtain from (2.38) and (2.39)

$$\begin{aligned} I(\gamma,\alpha) &= \int_a^b \xi_h(\eta(t), \dot\eta(t)(\dot\alpha(t))^{-1}) \cdot \dot\eta(t)\, dt = \int_a^b \xi_h \circ \tilde\tau_{E_0}(\eta(t),\dot\eta(t)) \cdot \dot\eta(t)\, dt \\ &= \int_a^b \ell_{h,E_0}(\eta(t),\dot\eta(t))\, dt = s_{\ell_h}(\eta)\ . \end{aligned}$$

Since $b_{E_0}(\gamma\circ\alpha) = (\gamma,\alpha)$, it follows that

$$s_{\ell_h} = I \circ b_{E_0} \quad \text{and thus} \quad ds_{\ell_h}|_\eta = dI|_{b_{E_0}(\eta)} \circ db_{E_0}|_\eta\ . \tag{2.40}$$

Since b_{E_0} is a diffeomorphism, we get

$$ds_{\ell_h}|_\eta = 0 \qquad \Longleftrightarrow \qquad dI|_{b_{E_0}(\eta)} = 0 \qquad \eta \in \Gamma_{a,b}(x_1,x_2)\ . \tag{2.41}$$

Step 4:
We show (a).

We set $\gamma_0(a) = x_1$, $\gamma_0(b) = x_2$. If γ_0 is a base integral curve of the Hamiltonian vector field X_h with $E_h(\gamma_0(t), \dot\gamma_0(t)) = E_0$ for all $t \in [a,b]$, then $b_{E_0}(\gamma_0) = (\gamma_0, \mathbf{1})$, where $\mathbf{1} : [a,b] \to [a,b]$ is defined by $\mathbf{1}(t) = t$.

Thus by (2.41) it remains to show that $dI|_{(\gamma_0,\mathbf{1})} = 0$ for any base integral curve $\gamma_0 \in \Gamma_{a,b}(x_1,x_2)$ of the Hamiltonian vector field X_h with energy E_0. The tangent space of $\Gamma(x_1,x_2,[a,b],E_0)$ at a point (γ,α) can be described by use of variations as

$$T_{(\gamma,\alpha)}\Gamma(x_1,x_2,[a,b],E_0) = \{\partial_u(\gamma,\alpha)_\delta|_{u=0} \,|\, (\gamma,\alpha)_\delta : (-\delta,\delta) \to \Gamma(x_1,x_2,[a,b],E_0) \quad \text{is } \mathscr{C}^2 \quad \text{with} \quad (\gamma,\alpha)_\delta(0) = (\gamma,\alpha)\} \, . \tag{2.42}$$

We start analyzing $dI|_{(\gamma,\alpha)}$. We use the notation $(\gamma(\,.\,),\alpha(\,.\,))_\delta(u) =: (\gamma_\delta(\,.\,,u), \alpha_\delta(\,.\,,u))$. From (2.42) and (2.33) it follows that $\alpha_\delta(a,u) = 0$. Furthermore $\gamma_\delta(0,u) = x_1$ and $\gamma_\delta(\alpha_\delta(b,u),u) = x_2$ for all $u \in (-\delta,\delta)$. This leads to

$$\frac{d}{du}\gamma_\delta(0,u) = 0 = \frac{d}{du}\gamma_\delta(\alpha_\delta(b,u),u) \, . \tag{2.43}$$

Since $A_h = L_h + E_h$ by the definition (2.16) of the Lagrange function L_h, it follows from the definition (2.33) of $\Gamma(x_1,x_2,[a,b],E_0)$ that

$$A_h(\gamma_\delta(\alpha_\delta(t,u),u), \dot\gamma_\delta(\alpha_\delta(t,u),u)) = L_h(\gamma_\delta(\alpha_\delta(t,u),u), \dot\gamma_\delta(\alpha_\delta(t,u),u)) + E_0 \, , \tag{2.44}$$

thus the definition (2.38) of I and (2.44) yield

$$dI|_{(\gamma,\alpha)}\left(\partial_u(\gamma,\alpha)_\delta|_{u=0}\right) = \partial_u I\left((\gamma_\delta,\alpha_\delta)\right)|_{u=0} = \frac{d}{du}\int_{\alpha_\delta(a,u)}^{\alpha_\delta(b,u)} \left(L_h(\gamma_\delta(s,u),\dot\gamma_\delta(s,u)) + E_0\right) ds\bigg|_{u=0} . \tag{2.45}$$

We get using $\gamma_\delta(t,0) = \gamma(t)$ and $\alpha_\delta(t,0) = \alpha(t)$

$$\begin{aligned}\frac{d}{du}\int_{\alpha_\delta(a,u)}^{\alpha_\delta(b,u)} &\left(L_h(\gamma_\delta(s,u),\dot\gamma_\delta(s,u)) + E_0\right) ds\bigg|_{u=0} \\ &= \left[\left(L_h(\gamma(\alpha(t)),\dot\gamma(\alpha(t))) + E_0\right)\cdot \partial_u\alpha_\delta|_{u=0}(t)\right]_a^b \\ &\qquad + \int_0^{\alpha(b)} \frac{d}{du}L_h(\gamma_\delta(s,u),\dot\gamma_\delta(s,u))\bigg|_{u=0} ds \, . \end{aligned} \tag{2.46}$$

For the integrand on the right hand side of (2.46) we get

$$\begin{aligned}\frac{d}{du}L_h(\gamma_\delta(s,u),\dot\gamma_\delta(s,u))\bigg|_{u=0} &= D_\gamma L_h(\gamma(s),\dot\gamma(s))\cdot\partial_u\gamma_\delta|_{u=0}(s) \\ &\qquad + D_{\dot\gamma}L_h(\gamma(s),\dot\gamma(s))\partial_u\dot\gamma_\delta|_{u=0}(s) \, . \end{aligned} \tag{2.47}$$

Since $\partial_u\dot\gamma_\delta|_{u=0}(s) = \partial_s\partial_u\gamma_\delta|_{u=0}(s)$, integration by parts for the second summand on the right hand side of (2.47) gives

$$\begin{aligned}\int_{\alpha(a)}^{\alpha(b)} \frac{d}{du}L_h(\gamma_\delta(s,u),\dot\gamma_\delta(s,u))\bigg|_{u=0} ds &= \left[D_{\dot\gamma}L_h(\gamma(s),\dot\gamma(s))\cdot\partial_u\gamma_\delta(s,u)|_{u=0}\right]_0^{\alpha(b)} \\ &- \int_0^{\alpha(b)}\left(D_\gamma L_h(\gamma(s),\dot\gamma(s)) + \frac{d}{ds}D_{\dot\gamma}L_h(\gamma(s),\dot\gamma(s))\right)\cdot\partial_u\gamma_\delta|_{u=0}(s)\, ds \, . \end{aligned} \tag{2.48}$$

It follows from (2.43) that

$$\partial_u\gamma_\delta(\alpha(t),u)|_{u=0} = -\dot\gamma(\alpha(t))\partial_u\alpha_\delta|_{u=0}(t)\, , \quad t = a,b \, . \tag{2.49}$$

Since by (2.18) we have $A_h(\gamma,\dot\gamma) = D_{\dot\gamma}L_h(\gamma,\dot\gamma)\cdot\dot\gamma$, we get by (2.44)

$$-\left[D_{\dot\gamma}L_h(\gamma(\alpha(t)),\dot\gamma(\alpha(t)))\cdot\dot\gamma(\alpha(t))\partial_u\alpha_\delta|_{u=0}(t)\right]_a^b = -\left[\left(L_h(\gamma(\alpha(t)),\dot\gamma(\alpha(t))) + E_0\right)\cdot\partial_u\alpha_\delta|_{u=0}(t)\right]_a^b \, . \tag{2.50}$$

Using (2.49) and (2.50), the boundary terms on the right hand side of (2.46) and (2.48) cancel. Combining (2.45), (2.46) and (2.48) yields

$$dI|_{(\gamma,\alpha)}\left(\partial_u(\gamma,\alpha)_\delta|_{u=0}\right) = \int_0^{\alpha(b)} \left(D_\gamma L_h(\gamma(s),\dot\gamma(s)) - \frac{d}{ds} D_{\dot\gamma} L_h(\gamma(s),\dot\gamma(s))\right)\cdot \partial_u \gamma_\delta|_{u=0}(s)\, ds\,. \quad (2.51)$$

For $(\gamma,\alpha) = (\gamma_0, \mathbf{1})$, the integrand is zero, since the integral curve $(\gamma_0,\dot\gamma_0)$ of X_h solves Lagranges equation and thus

$$dI|_{(\gamma_0,\mathbf{1})} = 0\,.$$

Step 5:
We show (b).

If γ_0 is a Finslerian geodesic with energy E_0, then $b_{E_0}(\gamma_0) = (\gamma_0,\mathbf{1})$ and by (2.41) the integral (2.51) is zero for each tangent vector $\partial_u\gamma_{0,\delta}|_{u=0}$. By standard arguments it follows that $(\gamma_0,\dot\gamma_0)$ solves Lagranges equation. Thus γ_0 is a base integral curve of X_h. □

2.4. **Application to H_ε.** We start with

Proof of Lemma 1.2. (a):
These estimates and the regularity follow at once by Hypothesis 1.1, (a), (i) and (iii).
(b):
By standard Fourier theory, t_0 is even with respect to ξ, i.e. $t_0(x,\xi) = t_0(x,-\xi)$ if and only if for all $\gamma\in(\varepsilon\mathbb{Z})^d$

$$a_\gamma^{(0)}(x) = a_{-\gamma}^{(0)}(x)\,. \quad (2.52)$$

To show (2.52) we use that by Hypothesis 1.1,(a),(iv) for all $\gamma\in(\varepsilon\mathbb{Z})^d$, $x\in\mathbb{R}^d$ and $\varepsilon\in[0,1)$

$$a_\gamma^{(0)}(x) + \varepsilon\, a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x,\varepsilon) = a_{-\gamma}^{(0)}(x+\gamma) + \varepsilon\, a_{-\gamma}^{(1)}(x+\gamma) + R_{-\gamma}^{(2)}(x+\gamma,\varepsilon) \quad (2.53)$$

By Hypothesis 1.1,(a),(i) and (v) we have for any $\delta > 0$

$$a_{-\gamma}^{(0)}(x+\gamma) = a_{-\gamma}^{(0)}(x) + \begin{cases} O(\varepsilon^{1-\delta}) & \text{if} \quad |\gamma| \leq C\varepsilon^{1-\delta} \\ O(e^{-C\varepsilon^{-\delta}}) & \text{if} \quad |\gamma| > C\varepsilon^{1-\delta}\end{cases}\,. \quad (2.54)$$

Inserting (2.54) into (2.53) gives $|a_\gamma^{(0)}(x) - a_{-\gamma}^{(0)}(x)| = O(\varepsilon^{1-\delta})$ and since the left hand side is independent of ε (2.52) follows.

The analytic continuation of t_0 follows at once from Hypothesis 1.1,(a),(v), since $a_\gamma^{(0)}(x)$ are the Fourier-coefficients of $t_0(x,\xi)$.
(c):
By (a), $t_0(x,\xi) = \sum_{\eta\in\mathbb{Z}^d}\tilde a_\eta(x)\cos(\eta\cdot\xi)$, thus its Taylor-expansion at $\xi = 0$ yields by Hyp.1.1(a)(ii)

$$\sum_{\eta\in\mathbb{Z}^d}\tilde a_\eta(x)\left(1 - \frac12(\eta\cdot\xi)^2 + O\left(|\xi|^4\right)\right) = \langle\xi\,,\,B(x)\xi\rangle + O\left(|\xi|^4\right)\,, \quad (2.55)$$

where the symmetric $d\times d$-matrix B is given by

$$-\frac12\sum_\eta \tilde a_\eta(x)\eta_\nu\eta_\mu = B_{\nu\mu}(x) \qquad \text{for} \quad \mu,\nu\in\{1,\dots,d\}\,, x\in\mathbb{R}^d\,. \quad (2.56)$$

Since $\langle v\,,\,B(x)v\rangle = -\sum_\gamma a_\gamma^{(0)}(x)(v\cdot\gamma)^2$, by Hypothesis 1.1,(a),(iii) and (vi) the matrix B is positive definite.
(d):
First we mention that by a short calculation $\mathrm{Op}_\varepsilon^{\mathbb{T}^d}(e^{-\frac{i}{\varepsilon}\gamma\cdot\xi}) = \tau_\gamma$. This implies $T_\varepsilon = \sum_\gamma a_\gamma\tau_\gamma = \mathrm{Op}_\varepsilon^{\mathbb{T}^d}(t)$ as operator on $u\in\mathcal{K}((\varepsilon\mathbb{Z})^d)$ for t given in (1.4).

Boundedness:
For $u \in \ell^2\left((\varepsilon\mathbb{Z})^d\right)$, by the Cauchy-Schwarz inequality the l^2-norm of $T_\varepsilon u$ can be estimated as

$$\begin{aligned}
\|T_\varepsilon u\|^2_{\ell^2} &\leq \sum_{x\in(\varepsilon\mathbb{Z})^d}\left(\sum_{\gamma\in(\varepsilon\mathbb{Z})^d}|a_\gamma(x,\varepsilon)u(x+\gamma)|\right)^2 \\
&\leq \sum_{x\in(\varepsilon\mathbb{Z})^d}\left(\sum_{\gamma\in(\varepsilon\mathbb{Z})^d}|a_\gamma(x,\varepsilon)|^2\left(\frac{|\gamma|}{\varepsilon}\right)^{d+1}\right)^{\frac12}\left(\sum_{\gamma\in(\varepsilon\mathbb{Z})^d}\left(\frac{|\gamma|}{\varepsilon}\right)^{-(d+1)}|u(x+\gamma)|^2\right)^{\frac12} . \quad (2.57)
\end{aligned}$$

By (1.9), the first factor on the right hand side of (2.57) is bounded uniformly in x. Thus

$$\|T_\varepsilon u\|^2_{\ell^2} \leq C\sum_{\eta\in\mathbb{Z}^d}|\eta|^{-(d+1)}\sum_{x\in(\varepsilon\mathbb{Z})^d}|u(x+\varepsilon\eta)|^2 \leq C\|u\|^2_{\ell^2} .$$

Thus T_ε is a bounded operator on $\ell^2((\varepsilon\mathbb{Z})^d)$.
Symmetry:
Since T_ε is bounded, it is symmetric if and only if for any $x, \gamma \in (\varepsilon\mathbb{Z})^d$

$$\langle T_\varepsilon\delta_x\,,\,\delta_{x+\gamma}\rangle_{\ell^2} = \langle \delta_x\,,\,T_\varepsilon\delta_{x+\gamma}\rangle_{\ell^2}\ , \qquad (2.58)$$

where $\delta_x(y) := \delta_{xy}$. Since the left hand side of (2.58) is equal to $a_{-\gamma}(x+\gamma,\varepsilon)$ and the right hand side equals $a_\gamma(x,\varepsilon)$, the statement follows by Hyp.1.1(a)(iv).
Boundedness from below:
For $u \in \mathcal{K}((\varepsilon\mathbb{Z})^d)$, we write

$$\begin{aligned}
\langle u\,,\,T_\varepsilon u\rangle_{\ell^2} &= A[u] + B[u] \qquad \text{where} \qquad (2.59)\\
A[u] &:= \sum_{x\in(\varepsilon\mathbb{Z})^d}\left\{a_0^{(0)}(x)|u(x)|^2 + \sum_{\gamma\neq0}a_\gamma^{(0)}(x)u(x+\gamma)\bar{u}(x)\right\}\\
B[u] &:= \sum_{x,\gamma\in(\varepsilon\mathbb{Z})^d}\left(\varepsilon\, a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x,\varepsilon)\right)\bar{u}(x)u(x+\gamma)\,.
\end{aligned}$$

Then by the exponential decay of $a^{(1)}$ and $R^{(2)}$ with respect to γ (Hyp.1.1(a)(v))

$$|B[u]| \leq \sum_{x,\gamma\in(\varepsilon\mathbb{Z})^d}\left|\varepsilon\, a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x,\varepsilon)\right|\left(|\bar{u}(x)|^2 + |u(x+\gamma)|^2\right) \leq \varepsilon C\|u\|^2\,. \qquad (2.60)$$

By Hypothesis 1.1(iii) we have

$$\begin{aligned}
A[u] &= \sum_x\sum_{\gamma\neq0}a_\gamma^{(0)}(x)\left(u(x+\gamma)\bar{u}(x) - |u(x)|^2\right) \qquad (2.61)\\
&= \frac12\left\{\sum_{\substack{x\\ \gamma\neq0}}a_\gamma^{(0)}(x)\left(u(x+\gamma)\bar{u}(x) - |u(x)|^2\right) + \sum_{\substack{\tilde{x}\\ \tilde{\gamma}\neq0}}a_{-\tilde{\gamma}}^{(0)}(\tilde{x}+\tilde{\gamma})\left(u(\tilde{x})\bar{u}(\tilde{x}+\tilde{\gamma}) - |u(\tilde{x}+\tilde{\gamma})|^2\right)\right\}\\
&= -\frac12\sum_{\substack{x\\ \gamma\neq0}}a_\gamma^{(0)}(x)\,|u(x) - u(x+\gamma)|^2 \geq 0\ ,
\end{aligned}$$

where for the second step we used the symmetry of T_ε and the substitution $\tilde{x} = x+\gamma$ and $\tilde{\gamma} = -\gamma$ and the last estimate follows from Hyp.1.1(a)(iii). Inserting (2.60) and (2.61) in (2.59) gives the stated result. □

Definition 2.1 and Theorem 2.10 allow to define a metric adapted to the Hamilton operator H_ε as follows.

Proposition 2.13 *The Hamilton function* $\tilde{h}_0 : \mathbb{R}^{2d} \cong T^*\mathbb{R}^d \to \mathbb{R}$ *defined in* (1.11) *is hyperconvex in each fibre.*

Proof. We have to show that there exists a constant $\alpha > 0$ such that

$$\left\langle v\,,\, D^2_\xi \tilde{h}_0(x,\xi) v \right\rangle \geq \alpha \|v\|^2 \quad \text{for all} \quad x, \xi, v \in \mathbb{R}^d\,. \tag{2.62}$$

For simplicity of notation, we will skip the x-dependence of $\tilde{h}_0$ and $a^{(0)}_\gamma$. We have for $\tilde{a}$ defined in (1.8)

$$\left\langle v\,,\, D^2_\xi \tilde{h}_0(\xi) v \right\rangle = - \sum_{\eta\in\mathbb{Z}^d} \tilde{a}_\eta (\gamma \cdot v)^2 \cosh(\gamma\cdot\xi)\,, \qquad \xi, v \in \mathbb{R}^d\,. \tag{2.63}$$

By Hypothesis 1.1,(vi), we can choose a basis $\{\eta^1, \ldots, \eta^d\}$, $\eta^j \in \mathbb{Z}^d$ of $\mathbb{R}^d$ with $\tilde{a}_{\eta^i} < 0$. Since by Hyp.1.1(a)(iv) each summand in (2.63) has positive sign

$$\left\langle v\,,\, D^2_\xi \tilde{h}_0(\xi) v \right\rangle \geq - \sum_{k=1}^d \tilde{a}_{\eta^k} (\eta^k \cdot v)^2 \cosh(\eta^k\cdot\xi)\,, \qquad \xi, v \in \mathbb{R}^d\,. \tag{2.64}$$

We have $C = \min_k \left(-\tilde{a}_{\eta^k}\right) > 0$, thus (2.64) yields

$$\left\langle v\,,\, D^2_\xi \tilde{h}_0(\xi) v \right\rangle \geq C \sum_{k=1}^d (\eta^k \cdot v)^2 = \langle v\,,\, Mv\rangle \geq 0\,, \quad \text{for} \quad M = \left(C \sum_k \eta^k_i \eta^k_j \right)\,.$$

The sum can take the value 0 only if $v = 0$ since $\{\eta^k\}$ is a basis of $\mathbb{R}^d$. Thus $\tilde{h}_0$ is hyperconvex (the lowest eigenvalue of M gives the lower bound for its second derivative). □

Proposition 2.13 leads by Proposition 2.8 and Lemma 1.2 to the following corollary.

Corollary 2.14 *The Hamilton function* $\tilde{h}_0 : \mathbb{R}^{2d} \to \mathbb{R}$ *defined in* (1.11) *is hyperregular and even and strictly convex in each fibre.*

In the setting of Theorem 2.10, we choose $M = \mathbb{R}^d$, $E_0 = 0$ and $h = \tilde{h}_0 = \tilde{t}_0 - V_0$. Recall that by Hypothesis 1.1, the set $S(0)$ of singular points with respect to the energy $E_0 = 0$ is given by $S(0) = \{0\}$.

Definition 2.15 *For the hyperregular Hamilton function* $\tilde{h}_0$ *given in* (1.11)*, we define*

$$\ell(x,v) := \begin{cases} \ell_{\tilde{h}_0,0}(x,v)\,, & x \in \widetilde{M} := \mathbb{R}^d \setminus \{0\},\, v \in T_x\widetilde{M} \\ 0 & x = 0\,. \end{cases} \tag{2.65}$$

The associated Finsler metric $d_\ell : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ *is given by*

$$d_\ell(x_0, x_1) = \inf_{\gamma_{0,1}\in\Gamma(x_0,x_1)} \int_0^1 \ell(\gamma(t), \dot{\gamma}(t))\, dt\,. \tag{2.66}$$

We notice that it follows from the Definition of $\tilde{\tau}_0$ that $\lim_{x\to 0} \tilde{\tau}_0(x,v) = (0,0)$. Thus $\ell : \mathbb{R}^{2d} \to \mathbb{R}$ defined in (2.65) is continuous.

2.5. **Proof of Theorem 1.5.**

In order to prove Theorem 1.5, we notice that if d_ℓ is locally Lipschitz continuous, it is differentiable almost everywhere in both arguments (Rademacher Theorem).

Step 1: We prove (1.16).
By the triangle inequality and the definition of $d_\ell(x,y)$, we have for any $v \in \mathbb{R}^d$ with $|v| = 1$ and $\delta > 0$

$$d^0(x + \delta v) - d^0(x) \leq d_\ell(x, x+\delta v) \leq \int_0^1 \ell(\gamma_0(t), \dot{\gamma}_0(t))\, dt\,, \tag{2.67}$$

where $\gamma_0(t) = x + t\delta v$. For this special curve we get by the homogeneity of ℓ

$$\int_0^1 \ell(\gamma_0(t), \dot{\gamma}_0(t))\, dt \leq \sup_{t\in[0,1]} \ell(x + t\delta v, \delta v) = \delta \sup_{t\in[0,1]} \ell(x + t\delta v, v)\,, \tag{2.68}$$

where by a slight abuse of notation v is considered as an element of $T_{x+t\delta v}\mathbb{R}^d$. Thus (2.67) together with (2.68) proves (1.16).

Step 2: We prove (1.17).
By (2.67) and (2.68) we have for any $v \in \mathbb{R}^d$ with $|v| = 1$ almost everywhere in $x \in \mathbb{R}^d$

$$\nabla d^0(x) \cdot v = \partial_v d^0(x) = \lim_{\delta \to 0} \frac{d^0(x + \delta v) - d^0(x)}{\delta} \leq \lim_{\delta \to 0} \sup_{t \in [0,1]} \ell(x + t\delta v, v) = \ell(x, v) \, . \tag{2.69}$$

Note that $\nabla d^0(x)$ can be considered as an element of $T_x^*\mathbb{R}^d$. Since both sides in (2.69) are positive homogeneous of order one with respect to v, we can extend the inequality to all $v \in \mathbb{R}^d$. Using (2.21), the Finsler function ℓ can be written as $\ell(x, v) = \xi_{\tilde{h}_0}(x, \tilde{v}) \cdot v$, where v is considered as an element of T_xM and will be written as (x, v). It follows from (2.69) that

$$\big(\xi_{\tilde{h}_0}(x, \tilde{v}) - \nabla d^0(x)\big) \cdot v \geq 0 \, , \qquad (x, v) \in TM \;\; a.e.\, \text{on}\, M \, . \tag{2.70}$$

Since $\tilde{h}_0(x, \xi)$ is differentiable, real valued and convex in each fibre, by (2.5) the inequality

$$\tilde{h}_0(x, \xi) \geq \tilde{h}_0(x, \eta) + D_\eta \tilde{h}_0(x, \eta) \cdot (\xi - \eta)$$

holds for all $x, \xi, \eta \in \mathbb{R}^d$. Thus by setting $\xi = \xi_{\tilde{h}_0}(x, \tilde{v})$ and $\eta = \nabla d^0(x)$, we get for all $(x, v) \in TM$ the estimate

$$\tilde{h}_0(x, \xi_{\tilde{h}_0}(x, \tilde{v})) \geq \tilde{h}_0(x, \nabla d^0(x)) + D_\xi \tilde{h}_0(x, \nabla d^0(x)) \cdot (\xi_{\tilde{h}_0}(x, \tilde{v}) - \nabla d^0(x)) \, , \tag{2.71}$$

where $(x, \tilde{v}) \in \mathcal{E}$ is associated to (x, v). The left hand side of (2.71) is by definition of $\tilde{v}$ equal to zero. Choosing $(x, v) = D_\xi \tilde{h}_0(x, \nabla d^0(x))$ in equation (2.71) yields

$$0 \geq \tilde{h}_0(x, \nabla d^0(x)) + v \cdot (\xi_{\tilde{h}_0}(x, \tilde{v}) - \nabla d^0(x)) \, .$$

Using (2.70), this proves (1.17).

Step 3: We prove (1.15) (the eikonal equality):
We consider the generalized eikonal equation

$$\tilde{h}_0(x, \nabla\varphi(x)) = \tilde{t}_0(x, \nabla\varphi(x)) - V_0(x) = 0 \, . \tag{2.72}$$

Choose coordinates, such that $t_0(x_0, \xi) = |\xi|^2 + O\left(|\xi|^3\right)$ and $V_0(x) = \sum_{\nu=1}^d \lambda_\nu^2 x_\nu^2 + O\left(|x|^3\right)$. It is proven in [18] that there exists a unique positive $\mathscr{C}^\infty$-function φ defined in a neighborhood Ω of 0, solving (2.72), such that φ has an expansion as asymptotic series

$$\varphi(x) \sim \sum_{\nu=1}^d \frac{\lambda_\nu}{2} x_\nu^2 + \sum_{k \geq 1} \varphi_k(x) \, , \qquad x \in \Omega \, , \tag{2.73}$$

where each φ_k is an homogeneous polynomial of order $k + 2$.

In particular, denote by F_t the flow of the Hamiltonian vector field $X_{\tilde{h}_0}$. Then the Local Stable Manifold Theorem ([2]) tells us that there is an open neighborhood $\mathscr{N}$ of $(0, 0)$, such that the two submanifolds

$$\Lambda_\pm\left(X_{\tilde{h}_0}, (0,0)\right) := \left\{ (x, \xi) \in T^*\mathbb{R}^d \,\middle|\, F_t(x, \xi) \to (0, 0) \quad \text{for} \quad t \to \mp\infty \right\} \tag{2.74}$$

exist and are unique in $\mathscr{N}$. They are called stable (Λ_-) and unstable (Λ_+) manifold of $X_{\tilde{h}_0}$ of the critical point $(0, 0)$. Moreover they are of dimension d and contained in $\tilde{h}_0^{-1}(0)$. It is shown in [18] that $\Lambda_\pm$ are Lagrangian manifolds in $T^*\mathbb{R}^d$ and that the outgoing manifold can be parametrized as $\Lambda_+ = \{(x, \nabla\varphi(x)) \,|\, x \in \Omega\}$. Thus for a given $x \in \Omega$ there exists an integral curve $\widehat{\gamma}_0 := (\gamma_0, \nabla\varphi(\gamma_0)) \subset \Lambda_+$ of the Hamiltonian vector field $X_{\tilde{h}_0}$, parametrized by $[-\infty, 0]$, such that $\widehat{\gamma}_0(0) = (x, \nabla\varphi(x))$ and $\lim_{t \to -\infty} \widehat{\gamma}_0(t) = (0, 0)$. Since $\widehat{\gamma}_0$ is an integral curve of $X_{\tilde{h}_0}$, it follows from Hamilton's equations that

$$(\gamma_0, \dot{\gamma}_0) = \mathcal{D}_F \tilde{h}_0 \left(\gamma_0, \nabla\varphi(\gamma_0)\right)$$

and therefore

$$\left(\mathcal{D}_F \tilde{h}_0\right)^{-1} (\gamma_0, \dot{\gamma}_0) = (\gamma_0, \nabla\varphi(\gamma_0)) \, .$$

Thus

$$\frac{d}{dt} \varphi \circ \gamma_0 = \nabla\varphi|_{\gamma_0} \cdot \dot{\gamma}_0 = \left(\mathcal{D}_F \tilde{h}_0\right)^{-1} (\gamma_0, \dot{\gamma}_0) \cdot \dot{\gamma}_0 \, . \tag{2.75}$$

Since $\widehat{\gamma}_0$ is an integral curve, $(\gamma_0(t), \dot\gamma_0(t)) \in \mathcal{E}$ for all t. Therefore $\tilde\tau_0(\gamma_0, \dot\gamma_0) = (\gamma_0, \dot\gamma_0)$ and it follows at once from (2.75) and the definition of ℓ that

$$\frac{d}{dt}\varphi \circ \gamma_0 = \ell(\gamma_0, \dot\gamma_0) \tag{2.76}$$

The point $x = 0$ is a singular point of the Finsler manifold $(\mathbb{R}^d, \ell)$, thus the base integral curve $\gamma_0 : [-\infty, 0] \to \Omega \ni 0$ of $X_{\tilde h_0}$ is not a regular curve on a Finsler manifold in the sense of Definition 2.2. To avoid this difficulty, we restrict the curve γ_0 to $[-T, 0]$ and set $y_T := \gamma_0(-T)$. Then by (2.76)

$$\varphi(x) - \varphi(y_T) = \int_{-T}^{0} \ell(\gamma_0(t), \dot\gamma_0(t))\, dt \tag{2.77}$$

By Proposition 1.4 the base integral curve γ_0 of $X_{\tilde h_0}$ is a geodesic with respect to the associated Finsler function ℓ. It is a basic theorem in Finsler Geometry (see Abate-Patrizio [1], Theorem 1.6.6), that geodesics, which are short enough, actually minimize the curve length among all $\mathscr{C}^\infty$-curves (or $\mathscr{C}^2$-curves) with the same endpoints. Thus the length of any short geodesic joining x and y is for $|x-y|$ sufficiently small equal to the Finsler distance $d_\ell(x, y)$ and

$$\varphi(x) - \varphi(y_T) = d_\ell(y_T, x)\ . \tag{2.78}$$

Since $y_T \to 0, T \to \infty$ and d_ℓ and φ are continuous in $x = 0$, we get

$$\varphi(x) = d^0(x) \quad x \in \Omega$$

for $|x|$ sufficiently small. □

3. Weighted estimates for Dirichlet eigenfunctions

3.1. **Preliminary Results.**

Lemma 3.1 *Assume Hypothesis 1.1 and, for $\Sigma \subset \mathbb{R}^d$, let H_ε^Σ denote the Dirichlet operator introduced in Definition 1.6. Let $\varphi : \Sigma \to \mathbb{R}$ be Lipschitz and constant outside some bounded set. Then for any real valued $v \in \mathscr{D}(H_\varepsilon^\Sigma)$*

$$\begin{aligned}\left\langle \left(e^{\frac{\varphi}{\varepsilon}} H_\varepsilon^\Sigma e^{-\frac{\varphi}{\varepsilon}}\right) v\, ,\, v\right\rangle_{\ell^2} \quad &= \quad \left\langle \left(V_\varepsilon + V_{\varepsilon,\Sigma}^\varphi\right) v\, ,\, v\right\rangle_{\ell^2} \\ &\quad -\frac{1}{2}\sum_{x\in\Sigma}\sum_{\gamma\in\Sigma'_\varepsilon(x)} a_\gamma(x,\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}\varphi(x) - \varphi(x+\gamma)\right)(v(x) - v(x+\gamma))^2\, ,\end{aligned}$$

where $\Sigma'_\varepsilon(x) := \{\gamma \in (\varepsilon\mathbb{Z})^d \,|\, x+\gamma \in \Sigma\}$ and

$$V_{\varepsilon,\Sigma}^\varphi(x) := \sum_{\gamma\in\Sigma'_\varepsilon(x)} a_\gamma(x,\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}(\varphi(x) - \varphi(x+\gamma))\right)\ , \tag{3.1}$$

where the sum on the right hand side converges.

Proof. By use of the symmetry of T_ε (Lemma 1.2) and since v and φ are assumed to be real valued and $e^{\pm\frac{\varphi}{\varepsilon}} v \in \mathscr{D}(H_\varepsilon^\Sigma)$, we have

$$\begin{aligned}\left\langle \left(e^{\frac{\varphi}{\varepsilon}} \mathbf{1}_{\Sigma_\varepsilon} T_\varepsilon \mathbf{1}_{\Sigma_\varepsilon} e^{-\frac{\varphi}{\varepsilon}}\right) v\, ,\, v\right\rangle_{\ell^2} &= \frac{1}{2}\left[\left\langle \mathbf{1}_{\Sigma_\varepsilon} T_\varepsilon \mathbf{1}_{\Sigma_\varepsilon} e^{-\frac{\varphi}{\varepsilon}} v\, ,\, e^{\frac{\varphi}{\varepsilon}} v\right\rangle_{\ell^2} + \left\langle e^{-\frac{\varphi}{\varepsilon}} v\, ,\, \mathbf{1}_{\Sigma_\varepsilon} T_\varepsilon \mathbf{1}_{\Sigma_\varepsilon} e^{\frac{\varphi}{\varepsilon}} v\right\rangle_{\ell^2}\right] \\ &= \frac{1}{2}\sum_{x, x+\gamma\in\Sigma} a_\gamma(x,\varepsilon)\left(e^{\frac{1}{\varepsilon}(\varphi(x)-\varphi(x+\gamma))} + e^{-\frac{1}{\varepsilon}(\varphi(x)-\varphi(x+\gamma))}\right) v(x+\gamma)v(x)\end{aligned}$$

Writing $v(x+\gamma)v(x) = v^2(x) - \frac{1}{2}\left(2v^2(x) - 2v(x+\gamma)v(x)\right)$ yields by the definition of $V_{\varepsilon,\Sigma}^\varphi$

$$\begin{aligned}&\left\langle \left(e^{\frac{\varphi}{\varepsilon}} T_\varepsilon e^{-\frac{\varphi}{\varepsilon}}\right) v\, ,\, v\right\rangle_{\ell^2} \\ &= \left\langle V_{\varepsilon,\Sigma}^\varphi v\, ,\, v\right\rangle_{\ell^2} - \frac{1}{2}\sum_{x, x+\gamma\in\Sigma} a_\gamma(x,\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}(\varphi(x) - \varphi(x+\gamma))\right)(2v^2(x) - 2v(x)v(x+\gamma))\ .\end{aligned} \tag{3.2}$$

By Hypothesis 1.1 we have $a_\gamma(x,\varepsilon) = a_{-\gamma}(x+\gamma,\varepsilon)$. Thus by use of the substitutions $x' = x+\gamma$ and $\gamma' = -\gamma$ together with the fact that cosh is even

$$\sum_{x,x+\gamma\in\Sigma} a_\gamma(x,\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}(\varphi(x)-\varphi(x+\gamma))\right)v^2(x)$$
$$= \sum_{x',x'+\gamma'\in\Sigma} a_{-\gamma'}(x'+\gamma',\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}(\varphi(x'+\gamma')-\varphi(x'))\right)v^2(x'+\gamma')$$
$$= \sum_{x',x'+\gamma'\in\Sigma} a_{\gamma'}(x',\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}(\varphi(x')-\varphi(x'+\gamma'))\right)v^2(x'+\gamma') \quad (3.3)$$

Inserting (3.3) into (3.2) gives

$$\left\langle \left(e^{\frac{\varphi}{\varepsilon}}\mathbf{1}_{\Sigma_\varepsilon}T_\varepsilon\mathbf{1}_{\Sigma_\varepsilon}e^{-\frac{\varphi}{\varepsilon}}\right)v\,,\,v\right\rangle_{\ell^2} =$$
$$\left\langle V^\varphi_{\varepsilon,\Sigma}v\,,\,v\right\rangle_{\ell^2} - \frac{1}{2}\sum_{x,x+\gamma\in\Sigma} a_\gamma(x,\varepsilon)\cosh\left(\tfrac{1}{\varepsilon}(\varphi(x)-\varphi(x+\gamma))\right)\left(v^2(x)-2v(x)v(x+\gamma)+v^2(x+\gamma)\right)\,.$$

Since V_ε commutes with $e^{-\frac{\varphi}{\varepsilon}}$, the stated equality follows. The convergence of the series in (3.1) follows from the decay of $a_\gamma(x,\varepsilon)$ with respect to γ (Hyp.1.1(v)) together with the assumptions on φ and the mean value theorem. □

Lemma 3.1 leads to the following norm estimate, which will be used later on to prove Theorem 1.7.

Lemma 3.2 *Assume Hypothesis 1.1 and, for $\Sigma\subset\mathbb{R}^d$, let H^Σ_ε denote the Dirichlet operator introduced in Definition 1.6. For $E\geq 0$ fixed, let $F_\pm:\Sigma\to[0,\infty)$ be a pair of functions such that $F(x) := F_+(x)+F_-(x) > 0$ and*

$$F^2_+(x) - F^2_-(x) = \widehat{V}_\varepsilon(x) + V^\varphi_{\varepsilon,\Sigma}(x) - E\,, \qquad x\in\Sigma\,, \quad (3.4)$$

where $V^\varphi_{\varepsilon,\Sigma}(x)$ is given in (3.1). Then for $v\in\mathscr{D}(H^\Sigma_\varepsilon)$ real-valued with $Fv\in\ell^2_{\Sigma_\varepsilon}$ and $\varphi:\Sigma\to\mathbb{R}$ Lipschitz and constant outside some bounded set, we have for some $C>0$

$$\|Fv\|^2_{\ell^2} \leq 4\left\|\tfrac{1}{F}\left(e^{\frac{\varphi}{\varepsilon}}(H^\Sigma_\varepsilon - E)e^{-\frac{\varphi}{\varepsilon}}\right)v\right\|^2_{\ell^2} + 8\|F_-v\|^2_{\ell^2} + C\varepsilon\|v\|^2\,. \quad (3.5)$$

Proof. First observe that

$$\|Fv\|^2_{\ell^2} \leq 2\left(\|F_+v\|^2_{\ell^2} + \|F_-v\|^2_{\ell^2}\right) = 2\left(\|F_+v\|^2_{\ell^2} - \|F_-v\|^2_{\ell^2}\right) + 4\|F_-v\|^2_{\ell^2}\,. \quad (3.6)$$

By (3.4) one has

$$\|F_+v\|^2_{\ell^2} - \|F_-v\|^2_{\ell^2} = \left\langle(\widehat{V}_\varepsilon + V^\varphi_{\varepsilon,\Sigma} - E)v\,,\,v\right\rangle_{\ell^2}\,. \quad (3.7)$$

Hyp.1.1(a)(iii) and (v) yields by straightforward calculation

$$-\frac{1}{2}\sum_{x,x+\gamma\in\Sigma} a_\gamma(x,\varepsilon)\cosh\left(\frac{1}{\varepsilon}(\varphi(x)-\varphi(x+\gamma))\right)(v(x)-v(x+\gamma))^2 \geq -C\varepsilon\|v\|^2\,, \quad (3.8)$$

since $|\varphi(x+\gamma)-\varphi(x)| \leq |\gamma|\sup_{y\in K}|D\varphi(y)|$ for some compact set $K\subset\Sigma$. Thus it follows from Lemma 3.1 and (3.8) that

$$\left\langle(\widehat{V}_\varepsilon + V^\varphi_{\varepsilon,\Sigma} - E)v\,,\,v\right\rangle_{\ell^2} - C\varepsilon\|v\|^2 \leq \left\langle\left(e^{\frac{\varphi}{\varepsilon}}(H^\Sigma_\varepsilon - E)e^{-\frac{\varphi}{\varepsilon}}\right)v\,,\,v\right\rangle_{\ell^2}\,. \quad (3.9)$$

(3.7) and (3.9) yield by use of the Cauchy-Schwarz inequality

$$\begin{aligned} 2\left(\|F_+v\|^2_{\ell^2} - \|F_-v\|^2_{\ell^2}\right) - C\varepsilon\|v\|^2 &\leq 2\left\langle\left(e^{\frac{\varphi}{\varepsilon}}(H^\Sigma_\varepsilon - E)e^{-\frac{\varphi}{\varepsilon}}\right)v\,,\,v\right\rangle_{\ell^2} \quad (3.10)\\ &\leq 2\sqrt{2}\left\|\tfrac{1}{F}\left(e^{\frac{\varphi}{\varepsilon}}(H^\Sigma_\varepsilon - E)e^{-\frac{\varphi}{\varepsilon}}\right)v\right\|_{\ell^2}\frac{1}{\sqrt{2}}\|Fv\|_{\ell^2}\\ &\leq 2\left\|\tfrac{1}{F}\left(e^{\frac{\varphi}{\varepsilon}}(H^\Sigma_\varepsilon - E)e^{-\frac{\varphi}{\varepsilon}}\right)v\right\|^2_{\ell^2} + \frac{1}{2}\|Fv\|^2_{\ell^2}\,. \end{aligned}$$

Inserting (3.10) into (3.6) we get

$$\|Fv\|^2_{\ell^2} \leq 2\left\|\tfrac{1}{F}\left(e^{\frac{\varphi}{\varepsilon}}(H^\Sigma_\varepsilon - E)e^{-\frac{\varphi}{\varepsilon}}\right)v\right\|^2_{\ell^2} + \frac{1}{2}\|Fv\|^2_{\ell^2} + 4\|F_-v\|^2_{\ell^2} + C\varepsilon\|v\|^2\,.$$

This proves (3.5). □

LEMMA 3.3 *Let $\Sigma \subset \mathbb{R}^d$ be an open bounded region including the point 0 and such that $d^0 \in \mathscr{C}^2(\overline{\Sigma})$, where $d^0(x) := d_\ell(0,x)$ is defined in (2.66). Let $\chi \in \mathscr{C}^\infty(\mathbb{R}_+, [0,1])$, such that $\chi(r) = 0$ for $r \leq \frac{1}{2}$ and $\chi(r) = 1$ for $r \geq 1$. In addition we assume that $0 \leq \chi'(r) \leq \frac{2}{\log 2}$. For $B > 0$ we define $g : \Sigma \to [0,1]$ by*

$$g(x) := \chi\left(\frac{d^0(x)}{B\varepsilon}\right) , \qquad x \in \Sigma \tag{3.11}$$

and set

$$\Phi(x) := d^0(x) - \frac{B\varepsilon}{2}\log\left(\frac{B}{2}\right) - g(x)\frac{B\varepsilon}{2}\log\left(\frac{2d^0(x)}{B\varepsilon}\right) , \qquad x \in \Sigma . \tag{3.12}$$

Then there exists a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$

$$|\partial_\nu \partial_\mu \Phi(x)| \leq C . \tag{3.13}$$

Furthermore for any $B > 0$ there is $C' > 0$ such that

$$e^{\frac{d^0(x)}{\varepsilon}} \frac{1}{C'} \left(1 + \frac{d^0(x)}{\varepsilon}\right)^{-\frac{B}{2}} \leq e^{\frac{\Phi(x)}{\varepsilon}} \leq e^{\frac{d^0(x)}{\varepsilon}} C' \left(1 + \frac{d^0(x)}{\varepsilon}\right)^{-\frac{B}{2}} . \tag{3.14}$$

Proof. We write for simplicity $d(x) := d^0(x)$. First we notice that there exists a $C > 0$ such that for $\alpha \in \mathbb{N}^d, |\alpha| \leq 2$

$$|\partial^\alpha g(x)| \leq C\varepsilon^{-\frac{\alpha}{2}} , \qquad x \in \Sigma . \tag{3.15}$$

Here one uses that by (1.15) $d(x) = O(|x|^2)$ and $\nabla d(x) = O(|x|)$ as $|x| \to 0$, thus $x = O(\sqrt{\varepsilon})$ and $\nabla d(x) = O(\sqrt{\varepsilon})$ on $\operatorname{supp} \nabla g \subset \{x \in \Sigma \,|\, \frac{B\varepsilon}{2} \leq d(x) \leq B\varepsilon\}$. By the definition (3.12) of Φ we have

$$\partial_\nu \partial_\mu \Phi(x) = \partial_\nu \partial_\mu d(x) - \partial_\nu \partial_\mu \left(g(x)\frac{B\varepsilon}{2}\log\left(\frac{2d(x)}{B\varepsilon}\right)\right) = A_1 + A_2 + A_3 \tag{3.16}$$

with

$$\begin{aligned}
A_1 &:= \partial_\nu \partial_\mu d(x) \\
A_2 &:= -\left\{(\partial_\nu \partial_\mu g)(x)\frac{B\varepsilon}{2}\log\left(\frac{2d(x)}{B\varepsilon}\right) + (\partial_\nu g)(x)\frac{B\varepsilon}{2d(x)}(\partial_\mu d)(x) + (\partial_\mu g)(x)\frac{B\varepsilon}{2d(x)}(\partial_\nu d)(x)\right\} \\
A_3 &:= g(x)\frac{B\varepsilon}{2d(x)}\left(\frac{(\partial_\nu d)(x)(\partial_\mu d)(x)}{d(x)} + (\partial_\nu \partial_\mu d)(x)\right) .
\end{aligned}$$

Since Σ is bounded, all derivatives of d are at least bounded by a constant independent of ε, thus A_1 is bounded.

Each summand in A_2 includes a derivative of g and is therefore supported in the region $\frac{B\varepsilon}{2} < d(x) < B\varepsilon$. Thus $1 < \frac{2d(x)}{B\varepsilon} < 2$ and from (1.15), it follows as above that $\partial_\nu d(x) = O(\sqrt{\varepsilon})$. By (3.15) A_2 is bounded.

To estimate A_3, we introduce a constant $\delta > 0$ such that $\{x \in \Sigma \,|\, d(x) < \delta\} \subset \Omega$ and $\delta \geq \varepsilon_0 B$ and analyze the regions $d(x) < \delta$ and $d(x) \geq \delta$ separately.

Case 1: $d(x) < \delta$:
By Theorem 1.5, we have $\partial_\nu d(x) = O(|x|)$ and $\partial_\nu \partial_\mu d(x) = O(1)$. Thus there exists a constant $M > 0$ such that

$$\sum_{\nu,\mu} \left|\frac{(\partial_\nu d)(x)(\partial_\mu d)(x)}{d(x)}\right| + |(\partial_\nu \partial_\mu d)(x)| < M \qquad \text{for} \quad \delta \quad \text{small enough.}$$

Since in addition for $d(x) > \frac{B\varepsilon}{2}$ (on the support of g), the term $\frac{B\varepsilon}{2d}$ is bounded by 1, A_3 is bounded by a constant independent of ε.

Case 2: $d(x) \geq \delta$:
We use that the derivatives of d are bounded on Σ and that $\frac{1}{d(x)} \leq \frac{1}{\delta}$.

Combining Case 1 and 2 we get the boundedness of A_3 and thus (3.13).
To see (3.14), we first note that by definition

$$e^{\frac{\Phi(x)}{\varepsilon}} = e^{\frac{d(x)}{\varepsilon}} \left(\frac{B}{2}\right)^{-\frac{B}{2}(g(x)-1)} \left(\frac{d(x)}{\varepsilon}\right)^{-\frac{B}{2}g(x)} . \tag{3.17}$$

We notice that for any $y \geq 0$ and for any $B > 0$ there exists $\tilde{C} > 0$ such that

$$\frac{1}{\tilde{C}} \leq \frac{y^{\chi(\frac{y}{B})}}{1+y} \leq \tilde{C}\,.$$

Setting $y = \frac{d(x)}{\varepsilon}$, this leads to (3.14). □

3.2. **Proof of Theorem 1.7.** We partly follow the ideas in the proof of Proposition 5.5 in Helffer-Sjöstrand [14].

Let

$$t_0^\Sigma(x,\xi) := \sum_{\gamma\in\Sigma'_\varepsilon(x)} a_\gamma^{(0)}(x)\cos\left(\tfrac{1}{\varepsilon}\gamma\cdot\xi\right)\ , \qquad (x,\xi)\in\Sigma\times\mathbb{T}^d\,, \tag{3.18}$$

where $\Sigma'_\varepsilon(x) := \{\gamma\in(\varepsilon\mathbb{Z})^d \,|\, x+\gamma\in\Sigma\}$. We notice that

$$t_0(x,i\xi) \leq t_0^\Sigma(x,i\xi)\ , \tag{3.19}$$

since $a_\gamma^{(0)} \leq 0$ for $\gamma\neq 0$. In the following we write for simplicity $d(x) := d^0(x)$. By Theorem 1.5, for any $B>0$ we may choose $\varepsilon_B > 0$, such that for all $\varepsilon < \varepsilon_B$

$$V_0(x) + t_0(x, i\nabla d(x)) = 0\,, \qquad x\in\Sigma\cap d^{-1}([0,B\varepsilon))\,, \tag{3.20}$$

By (3.11) and (3.12)

$$\nabla\Phi(x) = \nabla d(x)\left\{1 - \frac{B\varepsilon}{2d(x)}\chi\left(\frac{d(x)}{B\varepsilon}\right) - \frac{1}{2}\chi'\left(\frac{d(x)}{B\varepsilon}\right)\log\left(\frac{2d(x)}{B\varepsilon}\right)\right\}\,. \tag{3.21}$$

Step 1: We shall show that there is $C_0>0$ independent of B such that

$$V_0(x) + t_0^\Sigma(x,i\nabla\Phi) \geq \begin{cases} 0\,, & x\in\Sigma\cap d^{-1}([0,B\varepsilon]) \\ \frac{B}{C_0}\varepsilon\,, & x\in\Sigma\cap d^{-1}([B\varepsilon,\infty)) \end{cases} \tag{3.22}$$

Case 1: $d(x)\leq\frac{B\varepsilon}{2}$

Since $\chi(x)=\chi'(x)=0$ and the eikonal equation (2.72) holds, we get

$$V_0(x) + t_0(x,i\nabla\Phi(x)) = V_0(x) + t_0(x,i\nabla d(x)) = 0\,, \qquad x\in\Sigma\cap d^{-1}([0,\tfrac{B\varepsilon}{2}])\,. \tag{3.23}$$

which by (3.19) leads at once to the first estimate in (3.22) in Case 1.

Case 2: $d(x)\geq B\varepsilon$

Since $\chi'(x)=0$ in this region, we have by (3.21)

$$\nabla\Phi(x) = \nabla d(x)\left(1-\frac{B\varepsilon}{2d(x)}\right)\,. \tag{3.24}$$

By Lemma 2.13, $t_0(x,i\xi) = -\tilde{t}_0(x,\xi)$ is concave with respect to ξ, therefore

$$t_0(x,\lambda i\xi + (1-\lambda)i\eta) \geq \lambda t_0(x,i\xi) + (1-\lambda)t_0(x,i\eta) \quad \text{for} \quad 0\leq\lambda\leq 1,\ \xi,\eta\in\mathbb{R}^d\,. \tag{3.25}$$

We have $0 \leq (1-\frac{B\varepsilon}{2d(x)}) \leq 1$. Thus chosing $\lambda = (1-\frac{B\varepsilon}{2d(x)})$ and $\eta=0$ in (3.25) and using $t_0(x,0)=0$ for all $x\in(\varepsilon\mathbb{Z})^d$, by (3.24) we get the estimate

$$\begin{aligned} V_0(x) + t_0(x,i\nabla\Phi(x)) &\geq V_0(x) + \left(1-\frac{B\varepsilon}{2d(x)}\right)t_0(x,i\nabla d(x)) \\ &\geq V_0(x)\left(1-\left(1-\frac{B\varepsilon}{2d(x)}\right)\right) \\ &= V_0(x)\frac{B\varepsilon}{2d(x)}\,, \end{aligned} \tag{3.26}$$

where for the second estimate we used that by Theorem 1.5 the eikonal inequality $t_0(x,i\nabla d(x)) \geq -V_0(x)$ holds. It follows from Theorem 1.5 and Hypothesis (1.1)(b) respectively that $d(x) = O(|x|^2)$ and $V_0(x) = O(|x|^2)$ for $|x|\to 0$. Since the region Σ was assumed to be bounded, it thus follows that there exists a constant $C_0>0$ such that

$$C_0^{-1} \leq \frac{V_0(x)}{2d(x)} \leq C_0\,, \qquad x\in\Sigma\cap d^{-1}([B\varepsilon,\infty))\,. \tag{3.27}$$

Combining (3.19), (3.26) and (3.27), we finally get the second estimate in (3.22).

Case 3: $\frac{B\varepsilon}{2} < d(x) < B\varepsilon$
We define

$$f_1(x) := \frac{B\varepsilon}{2d(x)}\chi\left(\frac{d(x)}{B\varepsilon}\right) \quad \text{and} \quad f_2(x) := \frac{1}{2}\chi'\left(\frac{d(x)}{B\varepsilon}\right)\log\left(\frac{2d(x)}{B\varepsilon}\right) ,$$

such that by (3.21)

$$\nabla\Phi(x) = \nabla d(x)(1 - f_1(x) - f_2(x)) . \tag{3.28}$$

Since $1 < \frac{2d(x)}{B\varepsilon} < 2$, f_1 and f_2 are non-negative and therefore $1 - f_1(x) - f_2(x) \leq 1$. In addition it follows that $0 \leq f_1(x) \leq 1$ and by the assumption $\chi'(r) \leq \frac{2}{\log 2}$ we get $0 \leq f_2(x) \leq 1$. Therefore $0 \leq f_1(x) + f_2(x) \leq 2$ and thus the estimate

$$|1 - f_1(x) - f_2(x)| \leq 1 \tag{3.29}$$

holds. Setting $\lambda(x) := 1 - f_1(x) - f_2(x)$ it follows from (3.28) and (3.29) that

$$\nabla\Phi(x) = \lambda(x)\nabla d(x) \qquad \text{with} \quad |\lambda(x)| \leq 1 \quad x \in \mathbb{R}^d . \tag{3.30}$$

Thus again from (3.25) (with $\eta = 0$ and $\xi = \nabla d(x)$) together with (3.30), (3.19) and the fact that t_0 is even with respect to $i\xi$ it follows that

$$V_0(x) + t_0^\Sigma(x, i\nabla\Phi(x)) \geq V_0(x) + |\lambda(x)|t_0(x, i\nabla d(x)) \geq V_0(1 - |\lambda(x)|) , \tag{3.31}$$

where for the second step we used (3.20). Since $|\lambda(x)| \leq 1$ and $V_0 \geq 0$, (3.31) gives the first estimate in (3.22) in Case 3.

Step 2: We shall show

$$\widehat{V}_\varepsilon(x) + V^\Phi(x) \geq \begin{cases} -C_5\,\varepsilon & \text{for} \quad x \in \Sigma \cap d^{-1}([0, B\varepsilon]) \\ \left(\frac{B}{C_0} - C_5\right)\varepsilon & \text{for} \quad x \in \Sigma \cap d^{-1}([B\varepsilon, \infty)) . \end{cases} \tag{3.32}$$

for some $C_5 > 0$ independent of B, where $V^\Phi := V^\Phi_{\varepsilon,\Sigma}$ is defined in (3.1).

We write

$$\widehat{V}_\varepsilon(x) + V^\Phi(x) = \left(\widehat{V}_\varepsilon(x) - V_0(x)\right) + \left(V^\Phi(x) - t_0^\Sigma(x, i\nabla\Phi(x))\right) + \left(V_0(x) + t_0^\Sigma(x, i\nabla\Phi(x))\right) \tag{3.33}$$

and give estimates for the differences in the first two brackets on the right hand side.

By Hypothesis 1.1 and since Σ is bounded, there exists a constant $C_1 > 0$ such that

$$\widehat{V}_\varepsilon(x) - V_0(x) \geq -C_1\varepsilon , \qquad x \in \Sigma . \tag{3.34}$$

We shall show that

$$\left|V^\Phi(x) - t_0^\Sigma(x, i\nabla\Phi(x))\right| \leq \varepsilon C_4 . \tag{3.35}$$

Then inserting (3.35), (3.34) and (3.22) in (3.33) proves (3.32).

Setting (see (3.1))

$$V_0^\Phi(x) := \sum_{\gamma\in\Sigma'(x)} a_\gamma^{(0)}(x)\cosh\left(\tfrac{1}{\varepsilon}(\Phi(x) - \Phi(x+\gamma))\right) ,$$

we write

$$V^\Phi(x) - t_0^\Sigma(x, i\nabla\Phi(x)) = \left(V^\Phi(x) - V_0^\Phi(x)\right) + \left(V_0^\Phi(x) - t_0^\Sigma(x, i\nabla\Phi(x))\right) =: D_1(x) + D_2(x)$$

and analyze the two Summands on the right hand side separately. Since Φ is Lipschitz and constant outside of some bounded set, it follows from Hypothesis 1.1(a) (as in the proof of (3.8)) that for some $\tilde{C} > 0$

$$|D_1(x)| = \left|\sum_{\gamma\in\Sigma'(x)} \left(\varepsilon\, a_\gamma^{(1)}(x) + R_\gamma^{(2)}(x, \varepsilon)\right)\cosh\left(\tfrac{1}{\varepsilon}(\Phi(x) - \Phi(x+\gamma))\right)\right| \leq \tilde{C}\varepsilon . \tag{3.36}$$

uniformly with respect to x.

We have for $x \in \Sigma$

$$|D_2(x)| \leq \sum_{\gamma\in\Sigma'_\varepsilon(x)} |a_\gamma^{(0)}(x)|\left|\cosh\left(\frac{1}{\varepsilon}(\Phi(x) - \Phi(x+\gamma))\right) - \cosh\left(\frac{1}{\varepsilon}\gamma\nabla\Phi(x)\right)\right| . \tag{3.37}$$

By the mean value theorem for $\cosh z$ with $z_0 = \frac{1}{\varepsilon}\gamma\nabla\Phi(x)$ and $z_1 = \frac{1}{\varepsilon}(\Phi(x) - \Phi(x+\gamma))$, we get from $|\sinh x| \leq e^{|x|}$

$$\left|\cosh\left(\frac{1}{\varepsilon}(\Phi(x) - \Phi(x+\gamma))\right) - \cosh\left(\frac{1}{\varepsilon}\gamma\nabla\Phi(x)\right)\right| \leq \sup_{t\in[0,1]} e^{\left|\frac{1}{\varepsilon}\{(\Phi(x)-\Phi(x+\gamma))t+\gamma\nabla\Phi(x)(1-t)\}\right|}\left|\frac{1}{\varepsilon}\{(\Phi(x) - \Phi(x+\gamma)) + \gamma\nabla\Phi(x)\}\right| . \tag{3.38}$$

By (1.16) and the definition (3.12) of Φ there exist constants $c_1, c_2 > 0$ such that

$$|\Phi(x) - \Phi(x+\gamma)| \leq c_1|\gamma| \quad \text{and} \quad |\gamma\nabla\Phi(x)| \leq c_2|\gamma| , \qquad x \in \Sigma,\, \gamma \in \Sigma'_\varepsilon(x) . \tag{3.39}$$

(3.39) gives a constant $D > 0$, such that

$$e^{\left|\frac{1}{\varepsilon}\{(\Phi(x)-\Phi(x+\gamma))t+\gamma\nabla\Phi(x)(1-t)\}\right|} \leq e^{\frac{D}{\varepsilon}|\gamma|} . \tag{3.40}$$

By second order Taylor-expansion

$$\frac{1}{\varepsilon}\left|(\Phi(x) - \Phi(x+\gamma)) + \gamma\nabla\Phi(x)\right| \leq \sup_{t\in[0,1]}\frac{1}{\varepsilon}\sum_{\nu,\mu=1}^{d}\left|\gamma_\nu\gamma_\mu\partial_\nu\partial_\mu\Phi(x+t\gamma)\right| . \tag{3.41}$$

Inserting (3.13) into (3.41) shows that there exists a constant $C_3 > 0$ independent of the choice of B such that for all $\varepsilon \in (0, \varepsilon_0]$

$$\frac{1}{\varepsilon}\left|(\Phi(x) - \Phi(x+\gamma)) + \gamma\nabla\Phi(x)\right| \leq \frac{C_3}{\varepsilon}|\gamma|^2 . \tag{3.42}$$

By (1.9), inserting (3.40) and (3.42) in (3.37) we get for any $A > 0$ with $\eta = \frac{\gamma}{\varepsilon}$

$$\left|V_0^\Phi(x) - t_0^\Sigma(x, -i\nabla\Phi)\right| \leq \sum_{\frac{\eta}{\varepsilon}\in\Sigma'_\varepsilon(x)} e^{-A|\eta|}e^{D|\eta|}C_3\varepsilon|\eta|^2 \leq \varepsilon\sum_{\eta\in\mathbb{Z}^d} e^{-|\eta|D'}C_3|\eta|^2 \leq \varepsilon C_4 ,$$

where $A - D = D' > 0$. This together with (3.36) gives (3.35).

Step 3: We prove (1.19) by use of Lemma 3.2.

Choosing $B \geq C_0(1 + R_0 + C_5)$, we have

$$\left(\frac{B}{C_0} - C_5\right)\varepsilon - E \geq \varepsilon , \qquad E \in [0, \varepsilon R_0] . \tag{3.43}$$

Let

$$\Omega_- := \{x \in \Sigma \,|\, \widehat{V}_\varepsilon(x) + V^\Phi(x) - E < 0\} \qquad \text{and} \qquad \Omega_+ := \Sigma \setminus \Omega_- , \tag{3.44}$$

then from (3.43) it follows that $\Omega_- \subset \{d(x) < \varepsilon B\}$ and by (3.32)

$$|\widehat{V}_\varepsilon(x) + V^\Phi(x)| \leq \varepsilon \max\{C_5, R_0\} \qquad \text{for all} \quad x \in \Omega_- . \tag{3.45}$$

We define the functions $F_\pm : \Sigma \to [0, \infty)$ by

$$F_+(x) := \sqrt{\varepsilon\mathbf{1}_{\{d(x)<B\varepsilon\}}(x) + (\widehat{V}_\varepsilon(x) + V^\Phi(x) - E)\mathbf{1}_{\Omega_+}(x)} \tag{3.46}$$

and

$$F_-(x) := \sqrt{\varepsilon\mathbf{1}_{\{d(x)<B\varepsilon\}}(x) + (E - \widehat{V}_\varepsilon(x) - V^\Phi(x))\mathbf{1}_{\Omega_-}(x)} . \tag{3.47}$$

Then $F_\pm$ are well defined and furthermore there exists a constant $C > 0$ such that

$$F := F_+ + F_- \geq C\sqrt{\varepsilon} > 0 , \qquad F_- = O(\sqrt{\varepsilon}) \qquad \text{and} \qquad F_+^2 - F_-^2 = \widehat{V}_\varepsilon + V^\Phi - E . \tag{3.48}$$

Lemma 3.2 yields with the choice $v = e^{\frac{\Phi}{\varepsilon}}u$

$$\left\|Fe^{\frac{\Phi}{\varepsilon}}u\right\|_{\ell^2}^2 \leq 4\left\|\tfrac{1}{F}e^{\frac{\Phi}{\varepsilon}}\left(H_\varepsilon^\Sigma - E\right)u\right\|_{\ell^2}^2 + 8\left\|F_-e^{\frac{\Phi}{\varepsilon}}u\right\|_{\ell^2}^2 + C\varepsilon\|u\|^2 . \tag{3.49}$$

By (3.14) and (3.48)

$$\left\|Fe^{\frac{\Phi}{\varepsilon}}u\right\|_{\ell^2}^2 \geq C\varepsilon\left\|\left(1 + \tfrac{d}{\varepsilon}\right)^{-\frac{B}{2}}e^{\frac{d}{\varepsilon}}u\right\|_{\ell^2}^2 \tag{3.50}$$

and

$$\left\|\tfrac{1}{F}e^{\frac{\Phi}{\varepsilon}}\left(H_\varepsilon^\Sigma - E\right)u\right\|_{\ell^2}^2 \leq C\varepsilon^{-1}\left\|\left(1 + \tfrac{d}{\varepsilon}\right)^{-\frac{B}{2}}e^{\frac{d}{\varepsilon}}\left(H_\varepsilon^\Sigma - E\right)u\right\|_{\ell^2}^2 . \tag{3.51}$$

Since $\Omega_- \subset \{d(x) < B\varepsilon\}$ it follows from the definition of F_- that $\frac{d(x)}{\varepsilon} \leq C$ on the support of F_-. Therefore by (3.14) and (3.48) there exists a constant $C > 0$ such that

$$\left\| F_- e^{\frac{\Phi}{\varepsilon}} u \right\|_{\ell^2}^2 \leq C\varepsilon \left\| u \right\|_{\ell^2}^2 . \tag{3.52}$$

Inserting (3.50), (3.51) and (3.52) in equation (3.49) yields with $\tilde{B} := \frac{B}{2}$

$$\tilde{C}\varepsilon \left\| \left(1 + \tfrac{d}{\varepsilon}\right)^{-\tilde{B}} e^{\frac{d}{\varepsilon}} u \right\|_{\ell^2}^2 \leq \varepsilon^{-1} \left\| \left(1 + \tfrac{d}{\varepsilon}\right)^{-\tilde{B}} e^{\frac{d}{\varepsilon}} \left(H_\varepsilon^\Sigma - E\right) u \right\|_{\ell^2}^2 + \varepsilon \left\| u \right\|_{\ell^2}^2 .$$

This proves (1.19).

Step 4: We prove (1.20).

If u is an eigenfunction of H_ε^Σ with eigenvalue E, then the first summand on the right hand side of (1.19) vanishes. The normalization of u leads therefore to (1.20).

□

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