Stability of the Front under a Vlasov-Fokker-Planck Dynamics

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Abstract

We consider a kinetic model for a system of two species of particles interacting through a long range repulsive potential and a reservoir at given temperature. The model is described by a set of two coupled Vlasov-Fokker-Plank equations. The important front solution, which represents the phase transition, is a one-dimensional stationary solution on the real line with given asymptotic values at infinity. We prove the asymptotic stability of the front for small symmetric perturbations.

Keywords: *Vlasov, Fokker-Planck, stability, fronts*

1 Introduction and Notation

The dynamical study of phase transitions has been tackled, among the others, with an approach based on kinetic equations modeling short range and long range interactions which are responsible of critical behaviors. An example of such models has been proposed in [BL] where the authors study a system of two species of particles undergoing collisions regardless of the species and interacting via long range repulsive forces between different species. A simplification of such model has been considered in [MM] where a kinetic model has been introduced for a system of two species of particles interacting through a long range repulsive potential and with a reservoir at a given inverse temperature β. The interaction with the reservoir is modeled by a Fokker-Plank operator and the interaction between the two species by a Vlasov force. The system is described by the one-particle distribution functions $f_i(x, v, t)$, $\quad i = 1, 2$ solutions of the system of two coupled Vlasov-Fokker-Plank (VFP) equations in a domain Ω

$$\partial_t f_i + v \cdot \nabla_x f_i + F_i \cdot \nabla_v f_i = L f_i, \tag{1.1}$$

where

$$Lf_i = \nabla_v \cdot \left(M \nabla_v \left(\frac{f_i}{M} \right) \right) , \tag{1.2}$$

M is a Maxwellian with mean zero and variance β^{-1},

$$M = \left(\frac{\beta}{2\pi} \right)^{\frac{3}{2}} e^{-\frac{\beta}{2} v^2}$$

and $\beta^{-1} = T$ is the temperature of the reservoir. The self-consistent Vlasov force is

$$F_i = -\nabla_x \int_\Omega \mathrm{d}x' U(|x - x'|) \int_{\mathbb{R}^3} \mathrm{d}v f_j(x', v, t) \ .$$

The potential function U is smooth, monotone with compact support, and its integral over the whole space is equal to one. There is a natural Liapunov functional, the free energy functional, for this dynamics,

$$\begin{aligned} \mathcal{G}(f_1, f_2) := & \int_{\Omega \times \mathbb{R}^3} \mathrm{d}x\mathrm{d}v \, [(f_1 \ln f_1)(x, v) + (f_2 \ln f_2)(x, v)] + \frac{\beta}{2} \int_{\Omega \times \mathbb{R}^3} \mathrm{d}x\mathrm{d}v (f_1 + f_2) v^2 \\ & + \beta \int_{\Omega \times \Omega} \mathrm{d}x\mathrm{d}y U(|x - y|) \int_{\mathbb{R}^3} \mathrm{d}v f_1(x, v) \int_{\mathbb{R}^3} \mathrm{d}v' f_2(y, v') \ . \end{aligned}$$

In fact, we have that

$$\frac{d}{dt} \mathcal{G}(f_1, f_2) = - \sum_{i=1,2} \int_{\Omega \times \mathbb{R}^3} \mathrm{d}x\mathrm{d}v \frac{M^2}{f_i} [\nabla_v \frac{f_i}{M}]^2 \leq 0$$

and the time derivative is zero if and only if f_i are of the form $f_i = \rho_i M$, where ρ_i are functions only of the position. If we put these expressions back in the equations we get that the stationary solutions of (1.1) have densities satisfying the equations

$$\ln \rho_i(x) + \beta \int_\Omega \mathrm{d}x' U(|x - x'|) \rho_j(x') = C_i, \quad x \in \Omega, \quad i, j = 1, 2, \ i \neq j \tag{1.3}$$

and C_i are arbitrary constants, related to the total masses of the components of the mixture. Moreover, replacing f_i by $f_i = \rho_i M$ in the functional $\mathcal{G}$ and integrating out the velocity variable we get a functional on the densities ρ_i

$$\mathcal{F}(\rho_1, \rho_2) = \int_\Omega \mathrm{d}x (\rho_1 \ln \rho_1 + \rho_2 \ln \rho_2) + \beta \int_{\Omega \times \Omega} \mathrm{d}x\mathrm{d}y U(|x - y|) \rho_1(x) \rho_2(y) \tag{1.4}$$

The Euler-Lagrange equations for this functional with the constraints $\frac{1}{|\Omega|} \int_\Omega \mathrm{d}x \rho_i(x) = n_i$ are exactly (1.3).

In [CCELM1] it is proved that for $n\beta \leq 2$, with $n = n_1 + n_2$, the total mass density, equations (1.3) in a torus have a unique homogeneous solution, while for $n\beta > 2$ there are non homogeneous solutions. To explain the physical meaning of these non homogeneous solutions, we write the functional $\mathcal{F}(\rho_1, \rho_2)$ in the following equivalent form

$$\mathcal{F}(\rho_1, \rho_2) = \int_\Omega \mathrm{d}x f(\rho_1, \rho_2) + \frac{\beta}{2} \int_{\Omega\times\Omega} \mathrm{d}x\mathrm{d}y U(|x-y|)[\rho_1(x) - \rho_1(y)][\rho_2(y) - \rho_2(x)]$$

where $f(\rho_1, \rho_2)$ is the thermodynamic free energy made of the entropy and the internal energy

$$\rho_1 \log \rho_1 + \rho_2 \log \rho_2 + \beta \rho_1 \rho_2$$

The function $f(\rho_1, \rho_2)$ is not convex and has, for any given temperature T, two symmetric (under the exchange $1 \to 2$) minimizers if the total mass $\int_\Omega \mathrm{d}x[\rho_1 + \rho_2]$ is larger than a critical value $2T$. In other words, this system undergoes a first order phase transition with coexistence of two phases, one richer in the presence of species 1 and the other richer in the presence of species 2. If we look for the minimizers of the functional under the constraints on the total masses

$$\int_\Omega \mathrm{d}x \rho_i(x) = n_i |\Omega|$$

we get homogeneous minimizers if we fix (n_1, n_2) equal to one of the two minimizers of $f(n_1, n_2)$.

Otherwise, we get non homogeneous minimizers below the critical value. The structure of the minimizing profiles of density will be as close as possible to one of the two minimizers of f: they will be close to one of the minimizing values in a region B, close to the other minimizing value in the complement but for a separating interface along which the minimizing profiles will interpolate smoothly between the two values.

We can conclude then that the minimizers of $\mathcal{G}$ in a torus will be Maxwellians times densities ρ_i of the form discussed above. Since $\mathcal{G}$ is a Liapunov functional, we expect that the minimizers are related to the stable solutions of the equations. In this paper we want to study the stability of the non homogeneous stationary solutions of the equations (1.1), which are minimizers of the functional. We restrict ourselves to the one-dimensional case with

$$x = (0, 0, z) \qquad -\infty < z < \infty \ .$$

The reason for this is that in such a situation we know many properties of the minimizers. In fact, consider the following variational problem. Define the excess free energy functional in one dimension on the infinite line as

$$\hat{\mathcal{F}}(\rho_1, \rho_2) := \lim_{N\to\infty} [\mathcal{F}_N(\rho_1, \rho_2) - \mathcal{F}_N(M\rho^+, M\rho^-)] \tag{1.5}$$

where $\mathcal{F}_N$ is the free nergy associated to the interval $[-N, N]$ and (ρ^+, ρ^-) is a homogeneous minimizer of f. Note that $\mathcal{F}_N(M\rho^+, M\rho^-) = \mathcal{F}_N(M\rho^-, M\rho^+)$. Look for the minimizers of the excess free energy such that $\lim_{x\to\pm\infty} f_1(x) = M\rho^\pm, \lim_{x\to\pm\infty} f_2(x) = M\rho^\mp$. In [CCELM2] it is proved that

THEOREM 1.1. *There exists a unique C^∞ positive minimizer (front) $w = [w_1(z), w_2(z)]$, with $w_1(z) = w_2(-z)$, for the one-dimensional excess free energy $\hat{\mathcal{F}}$, defined in (1.5), in the class of functions $\rho = (\rho_1, \rho_2)$ such that*

$$\lim_{z\to\pm\infty} \rho_1 = \rho^\pm, \quad \lim_{z\to\pm\infty} \rho_2 = \rho^\mp.$$

$$\rho^- < w_i(z) < \rho^+$$

for any $z \in R$. Moreover, the front w satisfies the Euler-Lagrange equations (1.3) and its derivative w' satisfies almost everywhere the equations

$$\frac{w_1'(z)}{w_1(z)} + \beta(U * w_2')(z) = 0, \quad \frac{w_2'(z)}{w_2(z)} + \beta(U * w_1')(z) = 0 \tag{1.6}$$

The front w converges to its asymptotic values exponentially fast, in the sense that there is $\alpha > 0$ such that

$$|w_1(z) - \rho_\mp| e^{\alpha|z|} \to 0 \text{ as } z \to \mp\infty, \quad |w_2(z) - \rho_\pm| e^{\alpha|z|} \to 0 \text{ as } z \to \mp\infty.$$

All its derivatives have the same property.

Our main result is the stability of these fronts for the VFP dynamics, under suitable assumptions on the initial data. To state the result, we write f_i, solutions of (1.1), as

$$f_i = w_i M + h_i \ .$$

Then, the perturbation h_i satisfies

$$\partial_t h_i + G_i h_i = L h_i - F_i(h)\partial_{v_z} h_i, \tag{1.7}$$

where the operators G_i are defined by

$$G_i h_i = v_z \partial_z h_i - U * w_j' \partial_{v_z} h_i + \beta v_z M w_i U * \partial_z \int_{\mathbb{R}^3} \mathrm{d}v h_j(\,\cdot\,, v, t) \tag{1.8}$$

while the force $F_i(h)$ due to the perturbation is

$$F_i(h) = -\partial_z \int_{\mathbb{R}} \mathrm{d}z' U(z - z') \int_{\mathbb{R}^3} \mathrm{d}v h_j(z', v, t), \quad j \neq i\ . \tag{1.9}$$

We define $(\,\cdot\,,\cdot\,)$ as the L^2 inner product for two scalar functions (on $\mathbb{R}$ or $\mathbb{R}\times\mathbb{R}^3$ depending on the context), while $\langle\cdot,\cdot\rangle$ denotes the L^2 the inner product for vector-valued functions, and we denote $\|\cdot\|$ as their corresponding L^2 norms. Furthermore, we define the weighted L^2 norms as

$$(f_i, g_i)_M = \int_{\mathbb{R}\times\mathbb{R}^3} \mathrm{d}z\mathrm{d}v \frac{1}{w_i M} f_i g_i, \quad \langle f, g\rangle_M = \sum_{i=1,2} \int_{\mathbb{R}\times\mathbb{R}^3} \mathrm{d}z\mathrm{d}v \frac{1}{w_i M} f_i g_i,$$

with corresponding weighted L^2 norms by $\|\cdot\|_M$. We also define the dissipation rate as

$$\|g\|_D^2 = \|(I-P)g\|_M^2 + \|\nabla_v (I-P)g\|_M^2, \tag{1.10}$$

where P is the L^2 projection on the null space of $L = \{cM, c \in \mathbb{R}\}$, for any given t, z. We also define the weighted norms as

$$\|g\|_{M,\gamma} = \|z_\gamma g\|_M \qquad \|g\|_{D,\gamma} = \|z_\gamma g\|_D,$$

with the notation

$$z_\gamma = z_\gamma.$$

In the following we will also denote by $\partial_{t,z} h$ the couple of derivatives $(\partial_t h, \partial_z h)$.

THEOREM 1.2. *We assume that $h = (h_1, h_2)$ at time zero has the following symmetry property in z, v*

$$h_1(z, v, 0) = h_2(-z, Rv, 0), \quad Rv = (v_x, v_y, -v_z). \tag{1.11}$$

There is δ_0 small enough such that, if

(1)

$$\|h(0)\|_M + \|\partial_{t,z} h(0)\|_M \leq \delta_0$$

then, there is a unique global solution to (1.7) such that for some $K > 0$

$$\frac{d}{dt}\{K[\|h(t)\|_M^2 + \|\partial_{t,z} h(t)\|_M^2]\} + \|h(t)\|_D^2 + \|\partial_{t,z} h(t)\|_D^2 \leq 0. \tag{1.12}$$

(2) For $\gamma > 0$ sufficiently small, if

$$\|h(0)\|_{M,\gamma} + \|\partial_{t,z} h(0)\|_{M,\frac{1}{2}+\gamma} \leq \delta_0$$

then there is constant $C > 0$,

$$\sup_{0 \leq t \leq \infty} \|h(t)\|_{M,\gamma} + \sup_{0 \leq t \leq \infty} \|\partial_{t,z} h(t)\|_{M,\frac{1}{2}+\gamma} \leq C(\|h(0)\|_{M,\gamma} + \|\partial_{t,z} h(0)\|_{M,\frac{1}{2}+\gamma}. \tag{1.13}$$

Moreover, we have the decay estimate

$$\|h(t)\|_M^2 + \|\partial_{t,z} h(t)\|_M^2 \leq C\big[1 + \frac{t}{2\gamma}\big]^{-2\gamma} \Big[\|h(0)\|_{M,\gamma}^2 + \|\partial_{t,z} h(0)\|_{M,\frac{1}{2}+\gamma}^2\Big]. \tag{1.14}$$

Since the equation preserves the symmetry property (1.11) we have that $h_i(z, v, t)$ have the same simmetry property (1.11) at any time. The proof of the theorem is based on energy estimates and takes advantage of the fact that at time zero the perturbation is small in a norm involving also the spatial and the time derivatives. To close the energy estimates, we use the spectral gap for the Fokker-Planck operator L to control $(I-P)h$, the part of h orthogonal to the null space of L, and the conservation laws to control Ph, the component of h in the null space of L, in terms of $(I-P)h$, like the method used in [Guo].

The key difficulty in our paper is the control of the hydrodynamic part Ph, in the presence of of the Vlasov force with large amplitude. Because of the presence of the Vlasov force, the hydrodynamic equations do not give directly the control of Ph but instead of a norm involving the operator A, the second variation of the free energy $\hat{\mathcal{F}}$ at the front w, which is given by

$$\langle g, Ag\rangle := \sum_{i=1}^{2}\int_{\mathbb{R}} \mathrm{d}z g_i(z)(Ag)_i(z) = \frac{d^2}{ds^2}\hat{\mathcal{F}}(w+sg)\big|_{s=0} .$$

The operator A acts on $g = (g_1, g_2)$ as

$$(Ag)_1 = \frac{g_1}{w_1} + \beta U * g_2, \quad (Ag)_2 = \frac{g_2}{w_2} + \beta U * g_1 . \tag{1.15}$$

Since w is a minimizer of $\hat{\mathcal{F}}$ the quadratic form on the left hand side is positive and the first variation gives the Euler-Lagrange equations

$$\frac{\delta\hat{\mathcal{F}}}{\delta\rho_i}(w) = \log w_i + \beta U * w_j - C_i = 0, \quad i \neq j ,$$

Differentiating with respect to z and using the prime to denote the derivative with respect to the z variable

$$(Aw')_i = \frac{w_1'}{w_1} + \beta U * w_j' = 0, \quad i \neq j ,$$

which shows that w' is in the null space of A. Indeed, one can show (see Section 2) that w' spans the null space of A and that there exists a constant $\lambda > 0$ such that (spectral gap)

$$\langle g, Ag\rangle \geq \lambda \sum_{i=1}^{2}\int_{\mathbb{R}} \mathrm{d}z \frac{1}{w_i}|(I-\mathcal{P})g_i|^2$$

where $\mathcal{P}$ is the projector on the null space of A.

Hence, by using the spectral gap for A one can control the component of Ph on the orthogonal to the null space of A, but not the component on the null space of A. Let us write $Ph = Ma$ and $a = \alpha w' + (I-\mathcal{P})a$. What is missing at this stage is the control of $\alpha(t) = \int_{\mathbb{R}\times\mathbb{R}^3} \mathrm{d}z\mathrm{d}v Ph(z,v,t)w'(z)$ for large time. We would like to show that $\alpha(t)$ vanishes asymptotically in time, which amounts to prove that the solution of the Vlasov-Fokker-Plank equations (VFP) converges to the initial front. The existence of a Liapunov functional for this dynamics forces the system to relax to one of the stationary points for the functional, which are of the form Mw^x, with w^x any translate by x of the symmetric front w. Then, it is the conservation law, in the form

$$\int_{\mathbb{R}\times\mathbb{R}^3} \mathrm{d}z\mathrm{d}v[f(z,v,t) - M(v)w(z)] = 0$$

which should select the front the solution has to converge to. But this is a condition on the L_1 norm of the solution while the energy estimates control some L_2 norm. In the approach in [Guo] the conservation law is used in problems in finite domains or in infinite domains but in dimension greater or equal than 3. The problem we are facing here is analogous to the one in [CCO] and we refer to it for an exhaustive discussion. One can realize the connections between the problem discussed here and the one in [CCO] by looking at the hydrodynamic limit of the model. In [MM] it is proved that the diffusive limit of the VFP dynamics is

$$\partial_t \bar{\rho} = \nabla \cdot \left(\mathcal{M} \nabla \frac{\delta \mathcal{F}}{\delta \bar{\rho}} \right), \quad \mathcal{M} = \beta^{-1} \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \tag{1.16}$$

where $\bar{\rho} = (\rho_1, \rho_2)$, $\dfrac{\delta \mathcal{F}}{\delta \bar{\rho}}$ denotes the functional derivative of $\mathcal{F}$ with respect to $\bar{\rho} = (\rho_1, \rho_2)$ and $\mathcal{M}$ is the 2×2 mobility matrix. These equations are in the form of a gradient flow for the free energy functional as the equation considered in [CCO], which is an equation for a bounded magnetization $m(x,t) \in [-1,1]$:

$$\partial_t m = \nabla \cdot \left[\sigma(m) \frac{\delta \mathcal{F}}{\delta m} \right]$$

where $\sigma(m) = \beta(1-m^2)$ and $\mathcal{F}$ is a suitable non local free energy functional. In [CCO] the stability result is obtained by using suitable weighted L^2 norms, with a weight x, which allow to control the tails of the distribution and hence a control of the L^1 norm. This is possible essentially because the equation is of diffusive type.

Unfortunately, we cannot use directly the approach in [CCO] since the dissipation in the kinetic model is given by the Fokker-Plank operator and does not produces directly diffusion on the space variable. In fact, we are able to use, as explained above, weighted norms (in space) with a weight $x^\alpha, \alpha < 1$, which are not enough to control the L^1 norm. Hence, to overcome the difficulty, we consider a special initial datum. We assume, as explained before, that h is symmetric at initial time. It is easy to see that this property is conserved by the dynamics so that h is symmetric at any later time. We note that also wM is symmetric while w' is antisymmetric in the z variable. This implies the vanishing of $\sum_{i=1}^2 \int_{\mathbb{R}\times\mathbb{R}^3} \mathrm{d}z\mathrm{d}v h_i(z,v,t) M(v) w_i'(z)$, the component of a on the null space of A, which consequently is zero at any later time.

Even with such a symmetry assumption (1.11), the estimate for the hydrodynamic part Ph is delicate. Based on the precise spectral information of A, we need to further study

$$\frac{\partial}{\partial z} A g = (Ag)'.$$

To this end, we employ a crucial decompostion (2.6) for *each* component of g and a contradiction argument to establish the important lower bound for $\frac{\partial}{\partial z} A g$ (Theorem 2.4). Furthermore, in order to get the decay rate, we use polynomial additional weight function in z and a trick of interpolation to carefully derive the corresponding energy estimate in a bootstrap fashion. Once again, Theorem 2.4 is crucial to control local L^2 norm of Ph in terms of its z-derivative.

It is worth to stress that our result does not rely on a smallness assumption on the potential, like for example in [Vi], where it is proved the stability in L^1 of the constant stationary state for a one component VFP equation, on a torus, for general initial data. The assumption of small U in [Vi] guarantees the uniqueness of the stationary state, namely it means not to be in the phase transition region. On the contrary, we are working with values of the parameters (temperature and asymptotic values of densities, $\rho^{\pm}$) in the phase transition region. For values of the parameters $\rho^+ = \rho^-$, $2\beta\rho^+ \leq 2$ the minimizer is unique and we can prove that the constant solution is stable, by a simplified version of the proof given here. The critical value $\beta\rho^+ = 2$ is selected by the fact that the analogous of the operator A, that comes out from the linearization around the constant solution, is positive and has spectral gap for $\beta\rho^+ < 2$ (it coincides with the operator called L_0 in Theorem 2.2). We expect also that the constant solution will become unstable above this critical value.

Finally, we want to return to the kinetic model by [BL], mentioned at the beginning of this section and studied in a series of papers [BELM], in which the Fokker-Planck term is replaced by a Boltzmann kernel to model species blind collisions between the particles. The dynamics is described by a set of two Vlasov-Boltzmann equations, coupled through the Boltzmann collisions and the Vlasov terms and conserve not only the total masses but also energy and momentum. The stationary solutions are the same as in the previous model, Maxwellians times densities ρ_i satisfying (1.3), so that one could study the stability of these solutions with respect to the Vlasov-Boltzmann dynamics. This result is more difficult to get due to the non linearity of the Boltzmann terms. The first results on the stability of the Maxwellian are due to [Uk], [Ma]. Recently, it has been proved by energy methods in a finite domain or in $\mathcal{R}^3$ by [Guo] ([SG] for soft potentials) who has also extended the method to cover other models involving self-consistent forces and singular potentials [Guo1] and in $\mathcal{R}^d$ by [LY], who also proved the stability of a $1-d$ shock. The stability of the non homogeneous solution for a Boltzmann equation with a given small potential force has been proved in [UYZ]. We are not aware of analogous results for non small force, but a very recent one [As]. Our method is in principle suited to prove stability under Vlasov-Boltzmann dynamics on a finite interval, but what is still lacking is a detailed study of stationary solutions in a bounded domain. We plan to report on that in the future.

The paper is organized as follows. In Section 2 we collect the properties of the operators L and A and the properties of the fronts. In Section 3 we prove some Lemmas that allow to control some z-derivative of Ph in terms of $(I-P)h$. In Section 4 we give the energy estimates for the function, the time derivative and the z-derivative.

2 Spectral Gaps of L and A

In this section we collect all the relevant properties of the operators L and A and also the properties of the fronts.

LEMMA 2.1. *There is a $\nu_0 > 0$ such that for all $g = (g_1, g_2)$,*

$$\langle g, Lg\rangle_M \leq -\nu_0 \|(I-P)g\|_D^2. \tag{2.1}$$

Proof. Since w_i is bounded from below for $i = 1, 2$, we only need consider the case when g is a scalar.

Recall (1.2), the null space of L is clearly made of constants (in v) times M. Moreover, Lg is orthogonal to the null space of L in the inner product $(\,\cdot\,,\,\cdot\,)_M$. We denote by P the projector on the null space of L. Finally, the spectral gap property holds [LB]: for any g

$$(g, Lg)_M \le -\nu((I-P)g, (I-P)g)_M$$

On the other hand, a direct computation yields

$$(g, Lg)_M = -\int_{\mathbb{R}^3} \mathrm{d}v\ M^{-1}\ |\nabla_v (I-P)g|^2 + 3\beta \int_{\mathbb{R}^3} \mathrm{d}v\ M^{-1}\ |(I-P)g|^2 .$$

We thus conclude our lemma by splitting $(Lg, g) = (1-\epsilon)(Lg, g) + \epsilon(Lg, g)$ and applying above two estimates respectively, for ϵ sufficiently small. □

By (1.15), it is immediate to check that

$$\mathcal{F}(w + \epsilon u) - \mathcal{F}(w) = \epsilon^2 \langle Au, u \rangle + o(\epsilon^2).$$

THEOREM 2.2. *There exist $\nu > 0$ such that*

$$\langle u, Au \rangle \ge \nu \langle (I - \mathcal{P})u, (I - \mathcal{P})u \rangle,$$

where $\mathcal{P}$ is the projector on (Null A)*:*

$$\mathrm{Null}\, A = \{u \in L^2(\mathbb{R}) \times L^2(\mathbb{R}) \,|\, u = cw', c \in \mathbb{R}\}.$$

Proof. We first characterize Null A. We note that(1.6) imply

$$\frac{u_1^2}{w_1} = -\left(\frac{u_1}{w_1'}\right)^2 \beta w_1' U * w_2', \quad \frac{u_2^2}{w_2} = -\left(\frac{u_2}{w_2'}\right)^2 \beta w_2' U * w_1'.$$

From (1.15), (Au, u) takes the from

$$\int_{\mathbb{R}} \left[\frac{u_1^2(z)}{w_1(z)} + \frac{u_2^2(z)}{w_2(z)}\right] \mathrm{d}z + 2\beta \int_{\mathbb{R}} \int_{\mathbb{R}} u_1(z) u_2(z') U(z - z') \mathrm{d}z \mathrm{d}z' \ =$$
$$-\beta \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\frac{u_1(z)}{w_1'(z)} - \frac{u_2(z')}{w_2'(z')}\right]^2 U(z - z') w_1'(z) w_2'(z') \mathrm{d}z \mathrm{d}z' \quad . \tag{2.2}$$

But, by the monotonicity properties of w_i it follows that $-w_1'(z) w_2'(z') \mathrm{d}z \mathrm{d}z'$ is a positive measure on $\mathbb{R} \times \mathbb{R}$. Therefore the quadratic form is non negative and vanishes if and only if h is parallel to w'. In particular, this identifies the null space of the operator A.

To establish the spectral gap of A, it is sufficient to prove the lower bound for the normalized operator $\tilde{A}$: $L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that

$$(\tilde{A}u)_i = \sqrt{w_i}(A(u\sqrt{w}))_i,$$

with the abuse of notation $u\sqrt{w} = (u_1\sqrt{w_1}, u_2\sqrt{w_2})$. The explicit form is

$$(\tilde{A}u)_1 = u_1 + \beta\sqrt{w_1}U * (\sqrt{w_2}u_2), \quad (\tilde{A}u)_2 = u_2 + \beta\sqrt{w_2}U * (\sqrt{w_1}u_1).$$

The corresponding associated quadratic form is

$$\langle u, \tilde{A}u\rangle = \int_{\mathbb{R}} (u_1^2 + u_2^2) + 2\beta \int_{\mathbb{R}} \sqrt{w_1}u_1 U * (u_2\sqrt{w_2}).$$

The operator $\tilde{A}$ is a bounded symmetric operator on $\mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R})$. From the previous considerations it is also non negative and positive on the orthogonal complement of its null space. The spectral gap for $\tilde{A}$ is established in [CCELM2]. For completeness, we give a sketch of the proof below.

We decompose the operator as $\tilde{A} = \tilde{A}^0 + K$ where

$$(\tilde{A}^0u)_1 = u_1 + \beta\sqrt{\rho^+\rho^-}U * u_2, \quad (\tilde{A}^0u)_2 = u_2 + \beta\sqrt{\rho^+\rho^-}U * u_1$$

$$\begin{aligned} (Ku)_1 &= \beta\sqrt{w_1}U * (\sqrt{w_2}u_2) - \beta\sqrt{\rho^+\rho^-}U * u_2 \\ &= \beta \int \mathrm{d}z\mathrm{d}z' \left[\sqrt{w_1}(z)\sqrt{w_2}(z') - \sqrt{\rho^+\rho^-}\right] U(|z - z'|)u_2(z'), \end{aligned} \tag{2.3}$$

$$\begin{aligned} (Ku)_2 &= \beta\sqrt{w_2}U * (\sqrt{w_1}u_1) - \beta\sqrt{\rho^+\rho^-}U * u_1 \\ &= \beta \int \mathrm{d}z\mathrm{d}z' \left[\sqrt{w_2}(z)\sqrt{w_1}(z') - \sqrt{\rho^+\rho^-}\right] U(|z - z'|)u_1(z'). \end{aligned} \tag{2.4}$$

The operator $\tilde{A}^0$ has the spectral gap property. Consider the equation

$$\tilde{A}^0 u = \lambda u + f. \tag{2.5}$$

Denote by $\tilde{u}(\xi)$, $\tilde{f}(\xi)$ and $\tilde{U}(\xi)$ the Fourier transforms of u, f and U. For λ in the resolvent set of $\tilde{A}^0$ we can find a solution to (2.5) if the determinant of the matrix

$$\begin{pmatrix} 1-\lambda & \beta\tilde{U}\sqrt{\rho^+\rho^-} \\ \beta\tilde{U}\sqrt{\rho^+\rho^-} & 1-\lambda \end{pmatrix}$$

is different from zero for any $\xi \in R$. This happens if λ is such that for all $\xi \in R$

$$(1-\lambda)^2 - \beta^2(\tilde{U}(\xi))^2\rho^+\rho^- \neq 0.$$

Moreover, by the positivity of U, $\tilde{U}(\xi)| \le \tilde{U}(0) = 1$. As a consequence, the spectrum of L_0 is in the interval

$$[1 - \beta\sqrt{\rho^+\rho^-}, 1 + \beta\sqrt{\rho^+\rho^-}].$$

Now, for $\beta > \beta_c$ it is immediate to check that $\beta\sqrt{\rho^+\rho^-} < 1$ and hence the spectrum is contained in $(k, +\infty)$ for some positive k.

We claim that K is compact on $\mathcal{H}$. Indeed, uniformly for $\|u\|_{L_2} \le 1$, K satisfies

(1) $\forall \epsilon > 0 \quad \exists Z_\epsilon > 0$:

$$\int_{|z|>Z} |Ku|^2 \mathrm{d}z < \epsilon, \quad Z > Z_\varepsilon$$

(2) $\forall \epsilon > 0 \quad \exists \ell_\epsilon > 0$:

$$\int_{|z|>Z} |Ku(z+\ell) - Ku(z)|^2 \mathrm{d}z < \varepsilon, \quad \ell > \ell_\varepsilon \; .$$

These proofs follow trivially from the regularity of the convolution, the fact that U has compact support and the fact that $\lim_{x,y\to\pm\infty} \sqrt{w_1(x)w_2(y)} = \sqrt{\bar\rho^+\bar\rho^-}$. For the property (2) the boundedness of w_i' and the regularity of U are used. Hence, by Weyl's theorem we have that the spectral gap holds also for $\tilde{A}$. □

We are also interested into a lower bound on the norm of $(Au)'$. To this purpose, consider $u = (u_1, u_2) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with derivative $u' \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$. Assume u orthogonal to $w' = (w_1', w_2')$: $\langle u, w' \rangle = 0$.

We now make orthogonal decomposition of *each* component of u with respect to the corresponding $w' = (w_1', w_2')$ in the scalar L^2 inner product. In terms of the vector inner product, by a direct computation, such a process leads to

$$u = \alpha\tilde{w}' + \tilde{u} \tag{2.6}$$

where $\tilde{u}$ is such that

$$\int \mathrm{d}z \tilde{u}_1 w_1' = 0 = \int \mathrm{d}z \tilde{u}_2 w_2'$$

while $\tilde{w}' = (w_1', -w_2')$ is orthogonal to w' in the inner product $\langle \cdot\,, \cdot \rangle$ (note that $w_2'(z) = -w_1'(-z)$) with the coefficient α computed as

$$\alpha = \frac{\langle u\,,\, \tilde{w}'\,\rangle}{N},$$

$N = \langle \tilde{w}'\,,\, \tilde{w}'\,\rangle = 2\int \mathrm{d}z (w_1')^2 = 2\int \mathrm{d}z (w_2')^2$. We first prove a Lemma for $\tilde{u}$.

LEMMA 2.3. *There is a constant C such that*

$$\|(A\tilde{u})'\|^2 \ge C\|\mathcal{Q}\tilde{u}'\|^2 \; . \tag{2.7}$$

where $\mathcal{Q}$ is the orthogonal projection on the orthogonal complement of w''.

Proof. We follow the proof in [CCO]. We have

$$(A\tilde{u})'_i = \frac{d}{dz}[\frac{\tilde{u}_i}{w_i} + U * u_j] = [\frac{\tilde{u}'_i}{w_i} + U * \tilde{u}'_j] - \frac{w'_i}{w_i^2} u_i = (A\tilde{u}')_i - \frac{w'_i}{w_i^2}\tilde{u}_i \ .$$

By integrating over $\bar{z}$ after multiplication by w'_i the equation

$$\tilde{u}_i(z) = \tilde{u}_i(\bar{z}) + \int_{\bar{z}}^{z} \tilde{u}'_i(s)ds$$

we get

$$\tilde{u}_i(z) = \frac{(-1)^{i+1}}{(\rho^+ - \rho^-)} \int_{-\infty}^{+\infty} d\bar{z} w'_i(\bar{z}) \int_{\bar{z}}^{z} \tilde{u}'_i(s)ds \ .$$

We have used $\int \mathrm{d}z \tilde{u}_i w'_i = 0$.

From above, we can write $(A\tilde{u})'_i$ in terms of an operator $A+K$ acting on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that, if $h = \tilde{u}'$ then

$$(A\tilde{u})'_i = (Ah)_i + (Kh)_i \ ,$$

$$(Kh)_i(z) := \frac{(-1)^i}{(\rho^+ - \rho^-)} \frac{w'_i}{w_i^2} \int_{-\infty}^{+\infty} \mathrm{d}z' {w_i}'(z') \int_{z'}^{z} h_i(s)ds \ .$$

We prove first that

The operator K is compact on L^2. Indeed, we show that

- $\forall \varepsilon > 0 \quad \exists Z_\varepsilon > 0$:
$$\int_{|z|>Z} |K_{\bar{z}} h|^2 \mathrm{d}z < \varepsilon, \quad Z > Z_\varepsilon \ ,$$
- $\forall \varepsilon > 0 \quad \exists \ell_\varepsilon > 0$:
$$|Kh(z+\ell) - Kh(z)|^2 < \varepsilon, \quad \ell < \ell_\varepsilon \ .$$

The second is true because of the continuity of the integral. To prove the first, note that

$$\left| \int_{-\infty}^{+\infty} \mathrm{d}z' {w_i}'(z') \int_{z'}^{z} h_i(s)ds \right| \le \|h\| \int_{-\infty}^{+\infty} \mathrm{d}z |w'_i(z')| \sqrt{|z-z'|} \le C(1+|z|)\|h\| \qquad (2.8)$$

so that

$$\int_{|z|>Z} \left| \frac{w'_i}{w_i^2}(z) \right|^2 \left| \int_{-\infty}^{+\infty} \mathrm{d}z' {w_i}'(z') \int_{z'}^{z} h_i(s)ds \ \right|^2 \mathrm{d}z \le C\|h\|_{L_2} \int_{|z|>Z} \left| \frac{w'_i}{w_i^2}(z) \right|^2 |(1+|z|)^2 \ .$$

Then, by the rapid decay property of w'_i,

$$\int_{|z|>Z} |(Kh)_i|^2 \mathrm{d}z \to 0, \quad Z \to +\infty,$$

which proves (2). Now,

$$\int_{\mathbb{R}} |(A\tilde{u})'|^2 \mathrm{d}z = \int_{\mathbb{R}} \mathrm{d}z\tilde{u}' \left(A^2 + K^*A + AK^* + K^*K\right) \tilde{u}'dx \ .$$

The operator $K^*A + AK^* + K^*K$ is compact because A is bounded and K compact and its null space is spanned by w'', because by definition of $A + K$

$$0 = (Aw')' = (A + K)w'' \ .$$

But, A^2 has a strictly positive essential spectrum, hence the result follows from Weyl's theorem. Moreover

$$\int_{\mathbb{R}} |(A\tilde{w}')'|^2 = \delta > 0,$$

because $\tilde{w}'$ is orthogonal to the null space of A.

THEOREM 2.4. *For any $u \in L^2(\mathbb{R}) \times L^2(\mathbb{R}), u' \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ such that $\langle u, w' \rangle = 0$, there exists a positive constant B such that*

$$\|(Au)'\|^2 \geq B(|\alpha|^2 + \|\mathcal{Q}\tilde{u}'\|^2). \tag{2.9}$$

where $\mathcal{Q}$ is the projection on the orthogonal complement of w''. Furthermore, if $u' = \mathcal{Q}u'$, then there is a constant $k > 0$ such that

$$\|(Au)'\|^2 \geq k(|\alpha|^2 + \|\tilde{u}'\|^2)^2. \tag{2.10}$$

Proof. First, we prove that there is a constant C such that, if $u = (1 - \mathcal{P})u \neq 0$ and $\langle u, w' \rangle = 0$,

$$\|(Au)'\|^2 \geq C(\delta\alpha^2 + \|(A\tilde{u})'\|^2) \ . \tag{2.11}$$

We introduce the normalized vector ω and its decomposition along w' and the orthogonal complement by setting:

$$\omega = \frac{u}{\delta\alpha^2 + \|(A\tilde{u})'\|^2}; \qquad \omega = \eta\tilde{w} + \tilde{\omega} \ ,$$

so that equation (2.11) reads as

$$\|(A\omega)'\|^2 \geq C. \tag{2.12}$$

By the decomposition of ω we have

$$\|(A\omega)'\|^2 = \|(A\tilde{\omega})'\|^2 + \delta\eta^2 + 2\left((A\tilde{\omega})', \eta(A\tilde{w}')'\right) \ .$$

By definition, ω is such that

$$\|(A\tilde{\omega})'\|^2 + \delta\eta^2 = 1,$$

hence

$$\|(A\omega)'\|^2 = 1 + 2\langle (A\tilde{\omega})', \eta(A\tilde{w}')'\rangle \ .$$

Suppose now that the inequality (2.12) is not true. Then, for any n we can find $\tilde{\omega}_n$ and η_n such that

$$\|(A[\tilde{\omega}_n + \eta_n \tilde{w}'])'\|^2 = 1 + 2\langle (A\tilde{\omega})'_k, \eta_n(A\tilde{w}')'\rangle < \frac{1}{n} \ .$$

By weak compactness, up to subsequences, there are $\tilde{\omega}_0$ and η_0 such that $\tilde{\omega}_n$ converges weakly to $\tilde{\omega}_0$., $\eta_n \to \eta_0$. By weak convergence,

$$\langle (A\tilde{\omega}_n)', \eta_n(A\tilde{w}')'\rangle \to \langle (A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle$$

and

$$\text{liminf}[\|(A\tilde{\omega}_n)'\|^2 + \delta\eta_n^2] + 2\langle (A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle = 0$$

By lower semicontinuity,

$$\|(A\tilde{\omega}_0)'\|^2 + \delta\eta_0^2 \ \le \text{liminf}\left[\|(A\tilde{\omega}_n)'\|^2 + \delta\eta_n^2\right] = 1$$

Hence,

$$0 \le \|(A\omega_0)'\|^2 = \|(A\tilde{\omega}_0)'\|^2 + \delta\eta_0^2 + 2\langle (A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle \le 1 + 2\langle (A\tilde{\omega}_0)', \eta_0(A\tilde{w}')'\rangle \le 0 \ . \tag{2.13}$$

As a consequence,

$$\|(A\omega_0)'\|^2 = 0$$

which implies $\omega_0 = 0$: indeed $\langle \omega_0, w'\rangle = \lim_{n\to\infty}\langle \omega_n, w'\rangle = 0$ because ω_n is a sequence of vectors orthogonal to w'. Furthermore, since $\omega_n \to \omega_0 = 0$ weakly, $\eta_n \to \eta_0 = 0$. Then, $\langle (A\tilde{\omega}_0)', \eta_0(A + \tilde{w}')'\rangle = 0$ in contradiction with last inequality in (2.13). Therefore (2.12) is true and, together with (2.7), implies (2.9).

Finally, to prove (2.9), we notice that if $u' = \mathcal{Q}u'$, then by (2.6),

$$\alpha\tilde{w}'' + \tilde{u}' = u' = \mathcal{Q}u' = \alpha\mathcal{Q}\tilde{w}'' + \mathcal{Q}\tilde{u}'.$$

This implies that $\tilde{u}'$ is bounded by α and $\mathcal{Q}\tilde{u}'$, which completes the proof. □

3 Estimates of the hydrodynamic part Pf.

Decompose the solution of (1.7) in the component in the null space of L and in the one orthogonal to the null space: $h_i = Ph_i + (I - P)h_i$. Denote by a_iM the components in the null space of L: $Ph_i = \int_{\mathbb{R}^3} \mathrm{d}v h_i M = a_i M$, so that

$$h_i = a_i M + (I - P)h_i \ .$$

By using this decomposition in (1.7) we have

$$
\begin{aligned}
&M\left[\partial_t a_i + v_z\partial_z a_i - a_i U * w_j' M^{-1}\partial_{v_z} M + \beta v_z w_i U * \partial_z a_j\right] \\
&= -\partial_t (I-P)h_i - G_i(I-P)h_i - F_i(h)\partial_{v_z} h_i + L(I-P)h_i
\end{aligned}
\tag{3.1}
$$

Define

$$
\mu_i = \frac{a_i}{w_i} + \beta U * a_j := (Aa)_i
$$

so that

$$
\partial_z \mu_i = \frac{1}{w_i}\partial_z a_i - a_i\frac{w_i'}{w_i^2} + \beta U * \partial_z a_j
$$

By using the equation for the front (1.6) we can write the equation (3.1) as

$$
\begin{aligned}
&M\left[\partial_t a_i + v_z w_i \partial_z \mu_i\right] \\
&= -\partial_t (I-P)h_i - G_i(I-P)h_i - F_i(h)\partial_{v_z} h_i + L(I-P)h_i.
\end{aligned}
\tag{3.2}
$$

By integrating (3.2) over the velocity, since $-\int_{\mathbb{R}^3} \mathrm{d}v\partial_t(I-P)h_i = 0$, we have

$$
\partial_t a_i = -\int_{\mathbb{R}^3} \mathrm{d}v G_i(I-P)h_i
$$

and, by the definition of G_i,

$$
\partial_t a_i = -\partial_z \int_{\mathbb{R}^3} \mathrm{d}v v_z (I-P)h_i \tag{3.3}
$$

By integrating (3.2) over the velocity after multiplication by v_z

$$
Tw_i\partial_z\mu_i = -\int \mathrm{d}v v_z \partial_t (I-P)h_i - \int_{\mathbb{R}^3} \mathrm{d}v v_z G_i(I-P)h_i - \int_{\mathbb{R}^3} \mathrm{d}v v_z F_i(h)\partial_{v_z} h_i + \int_{\mathbb{R}^3} \mathrm{d}v v_z L(I-P)h_i\ .
$$

Moreover, by integrating by parts,

$$
\int_{\mathbb{R}^3} \mathrm{d}v v_z G_i(I-P)h_i = \int_{\mathbb{R}^3} \mathrm{d}v v_z^2 \partial_z (I-P)h_i + U * w_j' \int_{\mathbb{R}^3} \mathrm{d}v (I-P)h_i = \partial_z \int_{\mathbb{R}^3} \mathrm{d}v v_z^2 (I-P)h_i.
$$

Hence

$$
Tw_i\partial_z\mu_i = -\int_{\mathbb{R}^3} \mathrm{d}v v_z \partial_t (I-P)h_i - \partial_z \int_{\mathbb{R}^3} \mathrm{d}v v_z^2 (I-P)h_i + \int_{\mathbb{R}^3} \mathrm{d}v v_z L(I-P)h_i + F(h_i)a_i\ . \tag{3.4}
$$

Define

$$
\ell_i^a = \int_{\mathbb{R}^3} \mathrm{d}v v_z (I-P)h_i, \quad \ell_i^b = \int_{\mathbb{R}^3} \mathrm{d}v v_z^2 (I-P)h_i, \quad m_i = \int_{\mathbb{R}^3} \mathrm{d}v v_z L(I-P)h_i\ .
$$

By integrating twice by parts we get the identity:

$$
m_i = -\int_{\mathbb{R}^3} \mathrm{d}v M\partial_{v_z}\left(\frac{(I-P)h_i}{M}\right) = \beta\int_{\mathbb{R}^3} \mathrm{d}v v_z (I-P)h_i = \beta\ell_i^a\ .
$$

The following estimates are an easy consequence of (3.3) and (3.4).

$$\|\partial_t a_i\| = \|\partial_z \ell_i^a\|$$

$$\|\partial_z \mu_i\| \le \|\partial_t \ell_i^a\| + \|\partial_z \ell_i^b\| + \|m_i\| + \|\partial_z a_i\| \; \|a_i\|$$

From the definition, we have

$$\ell_i^a = \int_{\mathbb{R}^3} \mathrm{d}v v_z \sqrt{M} \frac{1}{\sqrt{M}} (I-P) h_i \le C \left[\int \mathrm{d}v \frac{|(I-P)h_i|^2}{M} \right]^{\frac{1}{2}} .$$

This and the fact that w is bounded from above and below give

$$\|\ell_i^a\|^2 \le \rho^+ \int_{\mathbb{R}} \mathrm{d}z \frac{1}{w_i} |\ell_i^a|^2 .$$

Hence,

$$\sum_{i=1}^{2} \|\ell_i^a\| \le C \|(I-P)h\|_M .$$

We will often use the notation $\dot{g}$ instead of $\partial_t g$ for the t-derivative of a function $g(t,z,v)$ and g' instead of $\partial_z g$ for its z-derivative. Set

$$\partial_z \mu_i^{(1)} = -\frac{1}{T w_i} \left[\int_{\mathbb{R}^3} \mathrm{d}v v_z (I-P) \dot{h}_i - \int_{\mathbb{R}^3} \mathrm{d}v v_z + L(I-P) h_i - F(h_i) a_i \right] \tag{3.5}$$

$$+\partial_z \left(\frac{1}{T w_i} \right) \int_{\mathbb{R}^3} \mathrm{d}v \; v_z^2 (I-P) h_i,$$

$$\partial_z \mu_i^{(2)} = -\partial_z \left(\frac{1}{T w_i} \int_{\mathbb{R}^3} \mathrm{d}v \; v_z^2 (I-P) h_i \right) \tag{3.6}$$

so that $\mu_i^{(1)} + \mu_i^{(2)} = \mu_i$. Define $a^{(j)}$ by setting $\mu_i^{(1)} = (A a^{(1)})_i$, $\mu_i^{(2)} = (A a^{(2)})_i$. Since the null space of A is given by $\alpha w' = (\alpha w_1', \alpha w_2')$ for $\alpha \in \mathbb{R}$, the equation

$$\mu_i = A g_i$$

has solutions iff

$$\sum_{i=1}^{2} \int_{\mathbb{R}} \mathrm{d}z w_i' \mu_i = 0$$

and they are of the form

$$g_i = (A^{-1}\mu)_i + \alpha w_i'$$

where $(A^{-1}\mu)$ is the unique solution orthogonal to the null space of A. Therefore, we need to show that $\mu^{(j)}$ are orthogonal to the null space of A. We shall prove it at the end of this section. Moreover, we can always choose $\alpha = 0$ since $a = a^{(1)} + a^{(2)}$ and a does not have component on the null space of A. In fact, a has by assumption at time zero the same symmetry property of w and it is preserved in time. This implies that at any time a is orthogonal to w' and hence has no component in the null space of A. This is one of the crucial points where we use the symmetry assumption on the initial perturbation.

We now estimate the L^2 norm $\|\partial_z a^{(1)}\|$. To this end, we first prove that $\mathcal{Q}\partial_z a_i^{(1)} = \partial_z a_i^{(1)}$ which is equivalent to show that $\sum_{i=1}^2 \int \partial_z a_i^{(1)} w_i'' = 0$.

LEMMA 3.1. *Assume $h_1(z,v,t) = h_2(-z,Rv,t)$ is valid. Then $\langle \partial_z a, w'' \rangle = 0$.*

Proof: We notice that this property is true for $\partial_z a$ because of the simmetry properties of the solution. In fact, $\partial_z a_1(z) = -\partial_z a_2(-z)$ and $w_1''(z) = w_2''(-z)$. We are left with proving that the same symmetry property hold for each $a^{(j)}$. It is enough to prove that for $a^{(2)}$. We have that

$$a^{(2)} = -A^{-1}\left[\frac{1}{Tw_i}\int_{\mathbb{R}^3} \mathrm{d}v\ v_z^2 (I-P)h\right]$$

and since A does not change the symmetry properties we are done if we prove that

$$\left(\frac{1}{w_1}\int_{\mathbb{R}^3} \mathrm{d}v\ v_z^2(I-P)h_1\right)(z) = \left(\frac{1}{w_2}\int_{\mathbb{R}^3} \mathrm{d}v\ v_z^2(I-P)h_2\right)(-z)\ .$$

By using the properties of h we have that the left hand side is equal to

$$\frac{1}{w_1(z)}\int_{\mathbb{R}^3_+} \mathrm{d}v\ v_z^2(I-P)[h_1(v,z)+h_2(v,-z)],$$

with $\mathbb{R}^3_+$ the set of velocities with $v_z \geq 0$, and the right hand side to

$$\frac{1}{w_2(-z)}\int_{\mathbb{R}^3_+} \mathrm{d}v\ v_z^2(I-P)[h_1(v,z)+h_2(v,-z)]\ .$$

The simmetry properties of w_i imply the result. The same argument also shows that $\mu^{(2)}$ is orthogonal to the kernel of A and hence $\mu^{(1)}$ has the same property since $\mu = Aa$ is orthogonal to the kernel of A by definition. □

THEOREM 3.2. *We have*

$$\sum_{i=1}^{2} \|a_i^{(2)}\| \leq C\|(I-P)h\|_M \leq \|h\|_D$$

Moreover, if $\|h\|_M \leq \delta_0$

$$\sum_{i=1}^{2} \|\partial_z a_i^{(1)}\| \leq C\big[\|(I-P)h\|_M + \|(I-P)\partial_t h\|_M\big]$$

Proof: From (3.6), by integration over z, since $\mu_i \to 0$ as $z \to \pm\infty$,

$$\mu_i^{(2)} = -\frac{1}{Tw_i}\int_{\mathbb{R}^3} \mathrm{d}v v_z^2 (I-P)h_i = \frac{1}{Tw_i}\ell_i^b$$

which implies

$$\sum_{i=1}^{2} \|\mu_i^{(2)}\| \le C\|(I-P)h\|_M$$

Moreover, $\mu_i^{(2)} = (Aa^{(2)})_i$ so that we have also, by Theorem 2.2,

$$\sum_{i=1}^{2} \|a_i^{(2)}\| \le C\|(I-P)h\|_M \ .$$

From (3.5) we get

$$\sum_{i=1}^{2} \|\partial_z \mu_i^{(1)}\| \le C\|(I-P)\partial_t h\|_M + C\|(I-P)h\|_M + \sup_i \|F(h_i)\|_{L_\infty} \sum_{i=1}^{2} \|a_i\| \ .$$

Now,

$$\|F(h_i)\|_{L_\infty} = \|U * \partial_z(a_j^{(1)} + a_j^{(2)})\|_{L_\infty} \le C\|\partial_z a_j^{(1)}\| + C\|a_j^{(2)}\| \ .$$

To apply Theorem 2.4 we need to show that a is orthogonal to w'. We notice that the front w is symmetric under the exchange $1 \to 2$ while the derivatives w_i' are antisymmetric. On the other hand, as already observed, a has at time zero the same symmetry properties as w and this implies that the component of a on the null space of A is zero at any time. In addition, by Lemma 3.1, we can apply Lemma 2.4 to get

$$\|\partial_z a^{(1)}\| \le C\left[\|(I-P)\partial_t h\|_M + \|(I-P)h\|_M + \|Ph\|_M[\|\partial_z a^{(1)}\| + \|a^{(2)}\|]\right.$$

$$\le C\left[\|(I-P)\dot{h}\|_M + \|(I-P)h\|_M + \|h\|_M\ \left[\|\partial_z a^{(1)}\| + \|(I-P)h\|_M\right]\right. \ .$$

To conclude the proof, using that property and the hypothesis we have that for δ_0 small enough

$$\|\partial_z a^{(1)}\| \le C[\|(I-P)\partial_t h\|_M + C\|(I-P)h\|_M + \delta_0\|(I-P)h\|_M$$

which proves Theorem 3.2. □

As a consequence of the proof we have also

$$\sum_{i=1}^{2} \|F(h_i)\|_{L_\infty} \le C\Big[\|(I-P)\partial_t h\|_M + \|(I-P)h\|_M\Big] \ . \tag{3.7}$$

From now on we use the more explicit notation $a_h M = Ph$. Moreover we use the previous decomposition: $a_h = a_h^{(1)} + a_h^{(2)}$.

LEMMA 3.3. *Let $0 \le \gamma \le \frac{1}{8}$ and $\|h\|_M$ be sufficiently small, then we have*

$$\|a_h^{(2)}\|_\gamma \le C\|h\|_{D,\gamma}, \tag{3.8}$$
$$\|\partial_z a_h^{(1)}\|_\gamma \le C\{\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma}\}, \tag{3.9}$$
$$\int \frac{z^2}{(1+z^2)^{2-2\gamma}}|a_h|^2 dz \le C\{\|h\|^2_{D,\gamma} + \|\partial_t h\|^2_{D,\gamma}\}. \tag{3.10}$$

Proof: We introduce the commutator:

$$[z_\gamma, A]a_h^{(2)} = z_\gamma A a_h^{(2)} - A(z_\gamma a_h^{(2)}),$$

Notice that

$$z_\gamma A a_h^{(2)} = -z_\gamma \frac{1}{Tw_i}\int dv v_z^2 (I-P)h_i.$$

which implies

$$\|z_\gamma A a_h^{(2)}\| \le C\|h\|_{D,\gamma} \ .$$

The commutator can be estimated by taking into account the property of the convolution and the fact that U is of finite range. We have, for a suitable z_*,

$$\begin{aligned}
[z_\gamma, A]a_h^{(2)}(z) &= \int dz' U(|z-z'|)(z_\gamma - z'_\gamma)a_h^{(2)}(z') \\
&= \int dz' U(|z-z'|)2\gamma z_*\{1+z_*^2\}^{\gamma-1}(z-z')a_h^{(2)}(z') \\
&\le C\int dz' U(|z-z'|)|z-z'|z_{\gamma-\frac{1}{2}}a_h^{(2)}(z')
\end{aligned}$$

It follows, for $0 \le \gamma \le \frac{1}{2}$,

$$\|[z_\gamma, A]a_h^{(2)}\|_{L_2} \le C\|z_{\gamma-\frac{1}{2}}a_h^{(2)}\|_{L_2} \le C\|h\|_D.$$

The last two estimates together imply

$$\|A(z_\gamma a_h^{(2)})\| \le C\|h\|_{D,\gamma} \ .$$

Since $z_\gamma a_h^{(2)}$ has the same symmetry properties of $a_h^{(2)}$, it is orthogonal to w' as well and we can use Theorem 2.2 to deduce (3.8).

To estimate $\partial_z a_h^{(1)}$, using the decomposition (2.6) we write

$$a_h^{(1)} = \alpha_h \tilde{w}' + \tilde{a}_h^{(1)}$$

Then the argument leading to (2.8) provides the estimate

$$|\tilde{a}_h^{(1)}(z)| \le \int_{-\infty}^{+\infty} d\bar{z} w'(\bar{z})\int_{\bar{z}}^{z} dy|\tilde{a}_h^{(1)}(y)| \le (1+|z|)\|\partial_z \tilde{a}_h^{(1)}\|.$$

By Theorem 2.4,

$$|\alpha_h| \le \|(Aa_h^{(1)})'\| \le C\|\partial_z a_h^{(1)}\|$$

and therefore

$$|a_h^{(1)}(z)| \le C(1+|z|)\|\partial_z a_h^{(1)}\|.$$

Hence, by Theorem 3.2, we obtain

$$|a_h^{(1)}(z)| \le (1+|z|)\left(\|h\|_D + \|\partial_t h\|_D\right).$$

But $\partial_z(Aa_h^{(1)})_i = A\{\partial_z a_h^{(1)}\}_i - \frac{w_i'}{w_i^2}(a_h^{(1)})_i$. Therefore

$$z_\gamma(A\partial_z a_h^{(1)})_i = z_\gamma\partial_z(Aa_h^{(1)})_i + \frac{z_\gamma w_i'}{w_i^2}(a_h^{(1)})_i. \tag{3.11}$$

Clearly, since w_i' decays exponentially, we have the following estimate for the second term in (3.11)

$$\|\frac{z_\gamma w_i'}{w_i^2}(a_h^{(1)})_i\| \le C\{\|h\|_D + \|\partial_t h\|_D\}$$

We examine now the first term in (3.11). We deduce from equation (3.5)

$$\|z_\gamma\partial_z(Aa_h^{(1)})\| \le C\left(\|h\|_{D,\gamma} + C\|\partial_t h\|_{D,\gamma} + \|z_\gamma F(a_h)\|_{L_\infty}\ \|a_h\|\right)\ .$$

We further split $\|z_\gamma F(a_h)\|_{L_\infty}$ in the last term as

$$\begin{aligned}
&\|z_\gamma F(a_h^{(2)})\|_{L_\infty} + \|z_\gamma U * (\partial_z a_h^{(1)})\|_{L_\infty} \le \|F(z_\gamma(a_h^{(2)}))\|_{L_\infty} + \|[z_\gamma, F]a_h^{(2)}\|_{L_\infty} \\
&\qquad + \|U * (z_\gamma\partial_z a_h^{(1)})\|_{L_\infty} + \|[z_\gamma, U]\partial_z a_h^{(1)}\|_{L_\infty} \\
\le\ & C\left(\|z_\gamma a_h^{(2)}\|_{L_2} + \|z_\gamma\partial_z a_h^{(1)}\|_{L_2}\right) \le C\left(\|h\|_{D,\gamma} + \|z_\gamma\partial_z a_h^{(1)}\|_{L_2}\right)\ .
\end{aligned}$$

The commutators are estimated as before, by using $0 \le \gamma \le \frac{1}{2}$, and we have used (3.8) in the last inequality. We hence conclude that, for $\|a\|$ small,

$$\begin{aligned}
\|A(z_\gamma\partial_z a_h^{(1)})\|_{L_2} &\le \|z_\gamma A\{\partial_z a_h^{(1)}\}\|_{L_2} + \|[z_\gamma, A]\partial_z a_h^{(1)}\|_{L_2} \\
&\le C\left(\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma} + \|z_\gamma\partial_z a_h^1\|_{L_2}\|a\|_{L_2}\right)\ .
\end{aligned}$$

Therefore, by decomposing along the null space of A, we have by the spectral gap for the operator A: Denote by $\tau = w'\|w'\|_{L_2}^{-1}$ the unit vector in the direction w'. We have

$$\begin{aligned}
\|z_\gamma\partial_z a_h^{(1)}\|_{L_2} &\le \|z_\gamma\partial_z a_h^{(1)} - \langle z_\gamma\partial_z a_h^{(1)}, \tau\rangle\tau\|_{L_2} + \|\langle z_\gamma\partial_z a_h^{(1)}, \tau\rangle\tau\|_{L_2} \\
&\le C\|A(z_\gamma\partial_z a_h^{(1)})\|_{L_2} + C\{\|h\|_D + \|\partial_t h\|_D\} \\
&\le C\{\|h\|_{D,\gamma} + \|\partial_t h\|_{D,\gamma}\} + C\|z_\gamma\partial_z a_h^{(1)}\|_{L_2}\|a\|_{L_2}
\end{aligned}$$

In conclusion, for $\|a\|_{L_2}$ small, we deduce (3.9).

To prove (3.10), since $\|\partial_z a_h^{(1)}\|_{L_2} \le C\{\|h\|_D + \|\partial_t h\|_D\}$ as well as $\|a_h^{(2)}\|_{L_2} \le C\|h\|_D$, the key is to estimate $\dfrac{z a_h^{(1)}}{(1+z^2)^{1-\gamma}}$ for z large. In fact, let $\chi(z)$ be a smooth cutoff function with $\chi(z) \equiv 1$ for $|z| \ge k$, for k large and $\chi(z) \equiv 0$ for $|z| \le k-1$. We have for the contribution due to $|z| \le k$,

$$\int \frac{z^2}{(1+z^2)^{2-\gamma}}\{1-\chi\}|a_h^{(1)}|^2 dz \le \int_{|z|\le k} |a_h^{(1)}|^2 \le C_k \int_{|z|\le k} |\partial_z a_h^{(1)}|^2 \le C_k\{\|h\|_D^2 + \|\partial_t h\|_D^2\}.$$

We now consider the contribution for $|z|$ large. Since

$$\begin{aligned}
\int \frac{\chi z^2 |a_h^{(1)}|^2}{(1+z^2)^{2-2\gamma}} dz &= -\int \frac{d}{dz}\left(\frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}}\right) z\chi|a_h^{(1)}|^2 dz \\
&= \int \frac{1}{2(1-\gamma)(1+z^2)^{1-2\gamma}} z\chi'|a_h^{(1)}|^2 + \int \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}}\chi|a_h^{(1)}|^2 \\
&\quad + \int \frac{1}{(1-2\gamma)(1+z^2)^{1-2\gamma}}\chi a_h^{(1)}\partial_z a_h^{(1)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int \left(\frac{z^2}{(1+z^2)^{2-2\gamma}} - \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}}\right)\chi|a_h^{(1)}|^2 dz \\
= &\int \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} z\chi'|a_h^{(1)}|^2 + \int \frac{1}{(1-2\gamma)(1+z^2)^{1-2\gamma}}\chi a_h^{(1)}\partial_z a_h^{(1)}.
\end{aligned}$$

For $\gamma \le \frac{1}{8}$ and $|z| > k$,

$$\frac{z^2}{(1+z^2)^{2-2\gamma}} - \frac{1}{2(1-2\gamma)(1+z^2)^{1-2\gamma}} \ge \frac{z^2}{4(1+z^2)^{2-2\gamma}}.$$

We thus have ($\chi' \equiv 0$ for $|z| \ge k$)

$$\begin{aligned}
\int \frac{z^2}{4(1+z^2)^{2-2\gamma}}\chi|a_h^{(1)}|^2 dz &\le C_k \int_{|z|\le k} |a_h^{(1)}|^2 + \frac{1}{8}\int \frac{z^2}{(1+z^2)^{2-2\gamma}}\chi|a_h^{(1)}|^2 dz \\
&\quad + C\|z_\gamma \partial_z a_h^{(1)}\|^2.
\end{aligned}$$

We thus deduce from (3.9):

$$\int \frac{z^2}{(1+z^2)^{2-2\gamma}}\chi a_h^2 dz \le C\left(\|\partial_t h\|_{D,\gamma}^2 + \|h\|_{D,\gamma}^2\right).$$

It is important to control $\partial_z[z_\gamma \partial_z a^{(1)}]$. To this end, we have

LEMMA 3.4. *Define* $\langle a_h^{(3)}, w' \rangle = 0$ *and*

$$(Aa_h^{(3)})_i \equiv -\frac{1}{Tw_i} \int v_z \partial_t (I-P) h_i.$$

Then, for $0 \le \gamma \le \frac{1}{2}$ *and* $\|a_h\|_{L_2} + \|\partial_z a_h\|_{L_2} \le \delta_0$ *sufficiently small,*

$$\begin{aligned} \|z_\gamma \partial_z a_h^{(2)}\| &\le C\left(\|\partial_z h\|_{D,\gamma} + \|h\|_D\right), &(3.12)\\ \|z_\gamma a_h^{(3)}\| &\le C\|\partial_t h\|_{D,\gamma}, &(3.13)\\ \|\partial_z\left[z_\gamma \partial_z a_h^{(1)} - z_\gamma a_h^{(3)}\right]\|_{L_2} &\le C\left(\|\partial_{t,z} h\|_{D,\gamma} + \|h\|_{D,\gamma-\frac{1}{2}}\right). &(3.14)\end{aligned}$$

Proof. For notational simplicity, we denote

$$a_g = z_\gamma \partial_z a_h.$$

We need to estimate $\|a_g^{(3)}\|_{L_2}$, $\|a_g^{(2)}\|_{L_2}$ and $\|\partial_z a_g^{(1)}\|_{L_2}$. First of all, we prove (3.13).

$$\begin{aligned} \|Az_\gamma a_h^{(3)}]\|_{L_2} &\le \|z_\gamma Aa_h^{(3)}\|_{L_2} + \|[A, z_\gamma]a_h^{(3)}\|_{L_2} \\ &\le \|\partial_t h\|_{D,\gamma} + C\|z_{\gamma-\frac{1}{2}} a_h^{(3)}\|_{L_2} \\ &\le \|\partial_t h\|_{D,\gamma} + C\|Aa_h^{(3)}\|_{L_2} \\ &\le C\|\partial_t h\|_{D,\gamma}\ . \end{aligned}$$

To get the third inequality we have used $\gamma < 1/2$ and Theorem 2.2 while in the last inequality comes from the definition of $Aa_h^{(3)}$. We thus deduce, again by Theorem 2.2,

$$\|z_\gamma a_h^{(3)}\|_{L_2} \le C\|\partial_t h\|_{D,\gamma}.$$

and hence (3.13) is proved.

We now turn to (3.12). Note that

$$\begin{aligned} Aa_g^{(2)} &= A(z_\gamma \partial_z a_h^{(2)}) \\ &= z_\gamma A\{\partial_z a_h^{(2)}\} - [z_\gamma, A]\partial_z a_h^{(2)} \\ &= z_\gamma \partial_z A\{a_h^{(2)}\} + z_\gamma \frac{w_i'}{w_i^2} a_h^{(2)} - [z_\gamma, U]\partial_z a_h^{(2)}. \end{aligned}$$

Clearly, $\|z_\gamma \frac{w_i'}{w_i^2} a_h^{(2)}\|_{L_2} \le C\|h\|_D$. Since

$$z_\gamma \partial_z A\{a_h^{(2)}\} = -z_\gamma \partial_z\{\frac{1}{Tw_i}\int dv v_z^2 (I-P)h_i\}$$

it follows that

$$\|z_\gamma \partial_z A\{a_h^{(2)}\}\|_{L_2} \le C\|\partial_z h\|_{D,\gamma}.$$

And an integration by part in $[z_\gamma, U]\partial_z a_h^{(2)}$ yields:

$$\| \int U(|z-z'|)[z_\gamma - z'_\gamma]\partial_z a_h^{(2)}(z')\|_{L_2} \leq C\|z_{\gamma-\frac{1}{2}} a_h^{(2)}\|.$$

We therefore can decompose (remind that $\tau = w'\|w'\|^{-1}$)

$$a_g^{(2)} = \langle a_g^{(2)}, \tau\rangle\tau + \{a_g^{(2)} - \langle a_g^{(2)}, \tau\rangle\tau\}.$$

Clearly,

$$\|\langle a_g^{(2)}, \tau\rangle\tau\|_{L_2} \leq \|a_h^{(2)}\|_{L_2} \leq C\|h\|_D.$$

But $\|\{a_g^{(2)} - \langle a_g^{(2)}\tau\rangle\tau\|_{L_2}$ is bounded by using the spectral gap of A :

$$C\|Aa_g^{(2)}\|_{L_2} \leq C\{\|h\|_D + \|\partial_z h\|_{D,\gamma}\},$$

Collecting terms, we deduce (3.12).

Finally, to estimate $\partial_z(a_g^{(1)} - z_\gamma a_h^{(3)})$, we use the commutation relation $(A\partial_z a_h)_i = \partial_z(Aa_h)_i + \dfrac{w_i'}{w_i^2}a_h$ to get

$$\begin{aligned}(Aa_g^{(1)})_i &= \Big(A(z_\gamma\partial_z a_h^{(1)})\Big)_i \\ &= z_\gamma\partial_z(Aa_h^{(1)})_i - ([z_\gamma, A]\partial_z a_h^{(1)})_i + \frac{z_\gamma w_i' a_h^{(1)}}{w_i^2}\end{aligned}$$

By equation (3.5) and the definition of $Aa_h^{(3)}$

$$\begin{aligned}&z_\gamma\partial_z(Aa_h^{(1)})_i = \\ &z_\gamma(Aa_h^{(3)})_i + z_\gamma\left[\frac{1}{Tw_i}\left[\int_{\mathbb{R}^3} \mathrm{d}v v_z L(I-P)h_i + F_i(h)a_i\right]\right. \\ &\left.+\partial_z\left(\frac{1}{Tw_i}\right)\int_{\mathbb{R}^3} \mathrm{d}v\ v_z^2(I-P)h_i\right].\end{aligned}$$

Therefore,

$$\begin{aligned}(Aa_g^{(1)})_i - z_\gamma(Aa_h^{(3)})_i = z_\gamma\left[\frac{1}{Tw_i}\left[\int_{\mathbb{R}^3} \mathrm{d}v v_z L(I-P)h_i + F(h_j)a_i\right]\right. \\ \left.+\partial_z\left(\frac{1}{Tw_i}\right)\int_{\mathbb{R}^3} \mathrm{d}v\ v_z^2(I-P)h_i\right] - ([z_\gamma, A]\partial_z a_h^{(1)})_i + \frac{z_\gamma w_i'(a_h^{(1)})_i}{w_i^2}.\end{aligned}$$

Using again the commutation relation $(A\partial_z a)_i = \partial_z(Aa)_i + \dfrac{w_i'}{w_i^2}a_i$, we find

$$\begin{aligned}
&\big(A\partial_z(a_g^{(1)} - z_\gamma a_h^{(3)})\big)_i \\
= \;& \partial_z\big(Aa_g^{(1)} - (A(z_\gamma a_h^{(3)})\big)_i - \frac{w_i'}{w_i^2}\{a_g^{(1)} - z_\gamma a_h^{(3)}\}_i \\
= \;& \partial_z\Big[(Aa_g^{(1)})_i - z_\gamma(Aa_h^{(3)})_i - ([z_\gamma, A]a_h^{(3)})_i\Big] - \frac{w_i'}{w_i^2}(a_g^{(1)} - z_\gamma a_h^{(3)})_i \\
= \;& \partial_z\left\{z_\gamma\left[\frac{1}{Tw_i}\left(\int_{\mathbb{R}^3}\mathrm{d}v v_z L(I-P)h_i + F_i(h)(a_h)_i\right) + \partial_z\left(\frac{1}{Tw_i}\right)\int_{\mathbb{R}^3}\mathrm{d}v\; v_z^2(I-P)h_i\right]\right. \\
& -([z_\gamma, A]\partial_z a_h^{(1)})_i - \frac{z_\gamma w_i'(a_h^{(1)})_i}{w_i^2} - ([z_\gamma, A]\partial_z a_h^{(1)})_i \\
& \left.-([A, z_\gamma]a_h^{(3)})_i\right\} + \frac{w_i'}{w_i^2}(a_g^{(1)} - z_\gamma a_h^{(3)})_i
\end{aligned}$$

The terms involving the commutator can be estimated by putting the z-derivative on the potential U in the convolution. We only need to estimate $\partial_z\left(\dfrac{z_\gamma}{Tw_i}F_i(h)(a_h)_i\right)$. We expand it as

$$\partial_z\frac{z_\gamma}{Tw_i} \times F_i(h)(a_h)_i - \frac{z_\gamma}{Tw_i}\left[(\partial_z^2 U * \partial_z(a_h)_j)(a_h)_i - (\partial_z U * \partial_z(a_h)_j)\partial_z(a_h)_i\right]$$

The first term is bounded by

$$\left(\|z_{\gamma-\frac{1}{2}}a_h^{(2)}\|_{L_2} + \|z_{\gamma-\frac{1}{2}}\partial_z a_h^{(1)}\|_{L_2}\right)\|a_h\|_{L_2}$$

We modify the second term above (up to the factor $(Tw_i)^{-1}$) as follows:

$$\begin{aligned}
&(\partial_z^2 U * z_\gamma\partial_z(a_h)_j)(a_h)_i - (\partial_z U * z_\gamma\partial_z(a_h)_j)\partial_z(a_h)_i \\
&+[\partial_z^2 U, z_\gamma]\partial_z(a_h)_j)(a_h)_i + [\partial_z U, z_\gamma]\partial_z(a_h)_j)\partial_z(a_h)_i
\end{aligned}$$

The L_2 norms of the last two terms are bounded by

$$\begin{aligned}
&\left(\|z_{\gamma-\frac{1}{2}}a_h^{(2)}\|_{L_2} + \|z_{\gamma-\frac{1}{2}}\partial_z a_h^{(1)}\|_{L_2}\right)(\|a_h\|_{L_2} + \|\partial_z a_h\|_{L_2}) \\
&\le \delta_0(\|h_{D,\gamma-\frac{1}{2}}\| + \|\partial_t h_{D,\gamma-\frac{1}{2}}\|)
\end{aligned}$$

The last inequality follows from Lemma 3.2 and the assumption. We write the first two terms as

$$\begin{aligned}
&\left(\partial_z U * \partial_z\left((a_g^{(1)})_j - z_\gamma(a_h^{(3)})_j\right) + \partial_z^2 U * z_\gamma(a_h^{(3)})_j + \partial_z U * z_\gamma\partial_z(a_h^{(2)})_j\right)\partial_z(a_h)_i \\
&+\left(U * \partial_z\left((a_g^{(1)})_j - z_\gamma(a_h^{(3)})_j\right) + \partial_z U * z_\gamma(a_h^{(3)})_j + U * z_\gamma\partial_z(a_h^{(2)})_j\right)(a_h)_i
\end{aligned}$$

Finally, we get

$$
\begin{aligned}
\|\partial_z\left(\frac{z_\gamma}{Tw_i}F_i(h)(a_h)_i\right)\|_{L_2} &\le \delta_0(\|h_{D,\gamma-\frac{1}{2}}\| + \|\partial_t h\|_{D,\gamma-\frac{1}{2}}\| \\
&+ \left(\|\partial_z(a_g^{(1)} - z_\gamma a_h^{(3)})\|_{L_2} + \|z_\gamma a_h^{(3)}\| + \|z_\gamma\partial_z a_h^{(2)}\|\right) \\
&\times (\|a_h\|_{L_2} + \|\partial_z a_h\|_{L_2})\,.
\end{aligned}
$$

We use (3.12) and (3.13) to get

$$
\|z_\gamma a_h^{(3)}\| + \|z_\gamma\partial_z a_h^{(2)}\| \le \|h\|_D + \|\partial_{t,z}h\|_{D,\gamma}\ .
$$

We therefore conclude

$$
\begin{aligned}
&\|A(\partial_z a_g^{(1)} - z_\gamma a_h^{(3)})\|_{L_2} \le C\big(\|\partial_{t,z}h\|_{D,\gamma} \\
&+\|h\|_{D,\gamma-\frac{1}{2}} + \|\partial_z(a_g^{(1)} - z_\gamma a_h^{(3)}\|_{L_2})(\|a_h\|_{L_2} + \|\partial_z a_h\|_{L_2}).
\end{aligned}
\tag{3.15}
$$

We then split $\|\partial_z(a_g^{(1)} - z_\gamma a_h^{(3)})\|_{L_2}$ into

$$
\|\langle\partial_z\{a_g^{(1)} - z_\gamma a_h^{(3)}\},\tau\rangle\tau\|_{L_2} + \|\partial_z\{a_g^{(1)} - z_\gamma a_h^{(3)}\} - \langle\partial_z(a_g^{(1)} - z_\gamma a_h^{(3)}),\tau\rangle\tau\|_{L_2}
$$

The first term is bounded by $\|h\|_D$, while the second can be absorbed in the left hand side for $\{\|a\|\|_{L_2} + \|\partial_z a\|_{L_2}\}$ small, by using the spectral gap for A. This concludes the proof of (3.14).

4 Energy Estimates and Decay

LEMMA 4.1. *Let*

$$
\partial_t g_i + G_i g_i - Lg_i = \Gamma_i,
\tag{4.1}
$$

then

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\{\int_{\mathbb{R}} \mathrm{d}z a_g A a_g \right.&+ \left.\sum_{i=1}^{2}\int_{\mathbb{R}}\mathrm{d}z\int_{\mathbb{R}^3}\mathrm{d}v\frac{1}{Mw_i}|(I-P)g_i|^2\right\} \\
&+ \sum_{i=1}^{2}\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{1}{w_i M}(I-P)g_i L(I-P)g_i \\
&= \langle Aa_g M,\Gamma\rangle_M + \langle (I-P)g,\Gamma\rangle_M.
\end{aligned}
$$

Proof: Decompose the solution of (4.1) in the component orthogonal to the null space of L and in the one in the null space: $g_i = Pg_i + (I-P)g_i$. Denote by $a_g M$ the component in the null space of L: $Pg = \int_{\mathbb{R}^3}\mathrm{d}v g M = a_g M$, so that

$$
g = a_g M + (I-P)g\ .
$$

We shall denote $a = a_g$ in the proof of this lemma. Repeating the same computation as in Section 3, we have

$$M[\partial_t a_i + v_z w_i \partial_z\{Aa\}_i] = -\partial_t(I-P)g_i - G_i(I-P)g_i + L(I-P)g_i + \Gamma_i.$$

Take the scalar product $(\,\cdot\,,\,\cdot\,)_M$ of (4.1) with $M(Aa)_i + w_i^{-1}(I-P)g_i$ to get:

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2}\frac{d}{dt}\left[\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v M a_i(Aa)_i + \int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{1}{Mw_i}|(I-P)g_i|^2\right]\\
=\;& -\sum_{i=1}^{2}\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v M w_i(Aa)_i v_z\partial_z(Aa)_i - \int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v(I-P)g_i v_z\partial_z(Aa)_i\\
& -\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v(Aa)_i G_i(I-P)g_i - \sum_{i=1}^{2}\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{1}{Mw_i}(I-P)g_i G_i(I-P)g_i\\
& +\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{1}{Mw_i}(I-P)g_i L(I-P)g_i + \langle\Gamma, Aa\rangle + \sum_{i=1}^{2}(\frac{1}{Mw_i}(I-P)g_i,\Gamma_i).
\end{aligned}$$

The first term on the right hand side vanishes since $(Aa)_i\partial_z(Aa)_i$ are functions of z, t only. By recalling the definition of G_i, (1.8),

$$G_i(I-P)g_i = v_z\partial_z(I-P)g_i + U * w_j'\partial_{v_z}(I-P)g_i,$$

we have for the third term

$$\begin{aligned}
-\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v(Aa)_i G_i(I-P)g_i &= -\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v(Aa)_i v_z\partial_z(I-P)g_i\\
&= \int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\partial_z Aa_i v_z(I-P)g_i
\end{aligned}$$

which exactly cancels with the second term $(-\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v(I-P)g_i v_z\partial_z(Aa)_i)$ in the right hand side. By using the definition of G_i we get for the fourth term

$$\begin{aligned}
&-\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{1}{Mw_i}(I-P)g_i G_i(I-P)g_i\\
=\;& -\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{1}{Mw_i}\frac{1}{2}\left[v_z\partial_z((I-P)g_i)^2 + U * w_j'\partial_{v_z}((I-P)g_i)^2\right]\\
=\;& -\int_{\mathbb{R}\times\mathbb{R}^3}\mathrm{d}z\mathrm{d}v\frac{v_z}{2Mw_i}[\frac{w_i'}{w_i} + \beta U * w_j']((I-P)g_i)^2 = 0
\end{aligned}$$

by using the equation for the front.

LEMMA 4.2. *Let $\gamma \geq 0$ be sufficiently small. Then if $\|h(t)\|_{M,\gamma} \leq \delta_0$*

$$\frac{1}{2}\frac{d}{dt}\|h(t)\|_{M,\gamma}^2 + \nu_0\|h(t)\|_{D,\gamma}^2 \leq C\{\gamma+\delta_0\}\{\|\partial_t h(t)\|_{D,\gamma}^2 + \|h(t)\|_{D,\gamma}^2\}, \tag{4.2}$$

with ν_0 given in Lemma 2.1.

Proof. Note that $g = z_\gamma h$ satisfies

$$\partial_t g_i + G_i g_i - L g_i = \frac{2zv_z\gamma g_i}{1+z^2} + \hat{G}h_i + F_i(h)\partial_{v_z} g_i \equiv \Gamma_i.$$

where

$$\hat{G}h_i = v_z M w_i \beta \int U'(z-z')\{z_\gamma - z'_\gamma\} h_j(\, z'\,, v, t) dz' dv. \tag{4.3}$$

We now apply Lemma 4.1. We first treat $F_i(h)\partial_{v_z} g_i$. Notice that $(F_i(h)\partial_{v_z} g_i, M(Ag)_i)_M = 0$, and

$$\begin{aligned}
& \sum_{i=1}^{2} (F_i(h)\partial_{v_z} g_i, \frac{1}{Mw_i}(I-P)g_i) \\
\leq\ & C\{\|F(h)\|_{L_\infty}\ \|a_g\|_{L_2} + \|F(h)\|_{L_\infty}\ \|\partial_{v_z}(I-P)g\|_M\}\|(I-P)g\|_M \\
\leq\ & C\{\|\partial_t h\|_D + \|h\|_D\}\|g\|_M\ \|g\|_D + \|h\|_M\ \|g\|_D^2 \\
\leq\ & C\delta_0\{\|h\|_D + \|\partial_t h\|_D\}^2 + C\delta_0\|g\|_D^2.
\end{aligned}$$

Next we estimate $\hat{G}h_i$. Note $\langle \hat{G}h, Aa_g\rangle = 0$. Since

$$|z_\gamma - z'_\gamma| \leq C\gamma|z-z'|$$

for γ small, recalling $a_h = a_h^{(1)} + a_h^{(2)}$, we deduce that

$$\begin{aligned}
& (\hat{G}h_i, \frac{1}{Mw_i}(I-P)g_i) \\
=\ & \beta\int v_z \left(\int U'(z-z')[z_\gamma - z'_\gamma] a_{h_j}(\, z'\,, v, t) dz' dv\right)(I-P)g_i \\
\leq\ & \beta\int v_z \left(\int U'(z-z')[z_\gamma - z'_\gamma] a_{h_j}^{(2)}(\, z'\,, v, t)] dz' dv\right)(I-P)g_i \\
& +\beta\int v_z \left(\int U(z-z')\partial_{z'}[(z_\gamma - z'_\gamma) a_{h_j}^{(1)}(\, z'\,, v, t)] dz' dv\right)(I-P)g_i \\
\leq\ & C\gamma\{\|a_h^{(2)}\| + \|\partial_z a_h^{(1)}\| + \|U * \{\frac{z a_h^{(1)}}{(1+z^2)^{1-\gamma}}\}\|\}\|(I-P)g\|_M \\
\leq\ & C\gamma\|g\|_D^2.
\end{aligned}$$

We have used (3.10) and 3.8 in Lemma 3.3. For the third term $\frac{2zv_z\gamma g_i}{1+z^2}$, we use again estimate (3.10) to get

$$\begin{aligned}
& \sum_{i=1}^{2} (\frac{2zv_z\gamma g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i) \\
=\ & \sum_{i=1}^{2} (\frac{2zv_z\gamma (I-P)g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i) + \sum_{i=1}^{2} (\frac{2zv_z\gamma P g_i}{1+z^2}, \frac{1}{Mw_i}(I-P)g_i) \\
\leq\ & C\gamma\{\|\partial_t g\|_D^2 + \|g\|_D^2\}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\langle \frac{2zv_z\gamma(I-P)g}{1+z^2}, Aa_g\rangle \\
\leq\ & C\gamma\|(I-P)g\|_M\{\|A(\frac{z}{1+|z|^2}a_g)\| + \|[A, \frac{z}{1+|z|^2}]a_g\|.
\end{aligned}
$$

Since A is bounded in L_2

$$
\|A(\frac{z}{1+|z|^2}a_g)\| \leq C\|\frac{z}{1+|z|^2}a_g\| \leq C(\|\partial_t h\|_{D,\gamma} + \|h\|_{D,\gamma})
$$

where the last inequality is true because of (3.10). The commutator term

$$
\|[A, \frac{z}{1+|z|^2}]a_g\| = \|U' * (\frac{z}{1+|z|^2} - \frac{z,}{1+|z,|^2})a_g\|
$$

can be estimated as in the previous computation as $C(\|\partial_t h\|_{D,\gamma} + \|h\|_{D,\gamma})$. Therefore $\langle \frac{2zv\gamma(I-P)g}{1+z^2}, Aa_g\rangle$ is bounded by

$$
C\gamma\|g\|_D \cdot (\|\partial_t h\|_{D,\gamma} + \|h\|_{D,\gamma}).
$$

This concludes the proof of the lemma.

LEMMA 4.3. *Recall* ν_0 *in Lemma 2.1. If* $\|\partial_{t,z}h(t)\|_{M,\gamma} + \|h(t)\|_{M,\gamma} \leq \delta_0$*, then*

$$
\frac{1}{2}\frac{d}{dt}\|\partial_t h(t)\|^2_{M,\gamma} + \nu_0\|\partial_t h(t)\|^2_{D,\gamma} \leq C\{\gamma + \delta_0\}\|\partial_{t,z}h(t)\||^2_{D,\gamma-\frac{1}{2}} + C\|h(t)\|^2_{D,\gamma-\frac{1}{2}}. \tag{4.4}
$$

Proof. Let $g = \{1+z^2\}\partial_t h$, we have

$$
[\partial_t + G_i - L]g_i = \frac{2zv_z\gamma\ g_i}{1+z^2} + \hat{G}\partial_t h_i + F_i(h)\partial_{v_z}g_i + F_i(\partial_t h)\partial_{v_z}g_i := \Gamma_i.
$$

By Lemma 4.1 we need to estimate

$$
\langle Aa_gM, \Gamma\rangle_M + \langle (I-P)g, \Gamma\rangle_M
$$

We first estimate $\frac{2\gamma zv_z\partial_t g_i}{1+z^2}$. Notice that $g = a_gM + (I-P)g$,

$$
\langle \frac{2\gamma zv_z g}{1+z^2}, Aa_g\rangle = \langle \frac{2\gamma zv_z a_g M}{1+z^2}, Aa_g\rangle + \langle \frac{2\gamma zv_z(I-P)g}{1+z^2}, Aa_g\rangle.
$$

The first term above vanishes. For the second term above, by an integration by part with the kernel, we have

$$
\begin{aligned}
&\|\frac{1}{1+|z|}Aa_g\| \leq \|A\{\frac{a_g}{1+|z|}\}\| + \|\int \frac{U(z-y)\{|z|-|y|\}}{\{1+|z|\}\{1+|y|\}}a_g(y)dy\| \\
\leq\ & C\{\|\{1+|z|^2\}^{\gamma-\frac{1}{2}}\partial_t a_h\| \leq C\|(I-P)\partial_z h\|_{M,\gamma-\frac{1}{2}}.
\end{aligned}
$$

The last inequality is due to (3.3). We therefore have

$$\langle\frac{2\gamma z v_z g}{1+z^2}, Aa_g\rangle \le \varepsilon\|\partial_t h\|^2_{D,\gamma} + C_\varepsilon\gamma\|\partial_z h\|^2_{D,\gamma-\frac{1}{2}}.$$

Now by an integration by part in the $v-$variable, we turn to

$$\begin{aligned}
&\sum_{i=1}^2 |(\frac{1}{Mw_i}(I-P)g_i, \frac{zv_z\gamma g_i}{1+z^2})| \\
\le\ & \sum_{i=1}^2 |(\frac{1}{Mw_i}(I-P)g_i, \frac{zv_z\gamma(I-P)g_i}{1+z^2})| \\
&+\sum_{i=1}^2 |(\frac{1}{Mw_i}(I-P)g_i, \frac{zv_z\gamma\{1+|z|^2\}^\gamma P\partial_t h_i}{1+z^2}) \\
\le\ & C\gamma\|\frac{1}{1+|z|}(I-P)g\cdot\nabla_{v_z}(I-P)g\|_M \\
&+C\gamma\|(I-P)g\|^2_M + C_\varepsilon\|\{1+|z|^2\}^{\gamma-1/2}\partial_t a_h\|^2 \\
\le\ & C\gamma\|\partial_{t,z}h\|^2_{D,\gamma-\frac{1}{2}}.
\end{aligned}$$

Now turn to the third term $\hat{G}\partial_t h_i$ in Γ. Since $\sum_{i=1}^2(v\hat{G}\partial_t h_i, (Aa_g)_i) = 0$,

$$\begin{aligned}
|\sum_{i=1}^2(\frac{1}{Mw_i}(I-P)g_i, \hat{G}\partial_t h_i v)| &= |\sum_{i=1}^2(\frac{1}{Mw_i}(I-P)g_i, \hat{G}\{\partial_t a_h\})| \\
&\le \varepsilon\|g\|^2_D + C_\varepsilon\gamma\|\partial_z h\|^2_{D,\gamma-\frac{1}{2}}.
\end{aligned}$$

Now for the fifth term $F_i(h)\partial_{v_z}g_i$, we note that $\sum_{i=1}^2(F_i(h)\partial_{v_z}g_i, (Aa_g)_i) = 0$, and

$$\begin{aligned}
&\sum_{i=1}^2(F_i(h)\partial_{v_z}g_i, \frac{1}{Mw_i}(I-P)g_i) \\
=\ & \sum_{i=1}^2(F_i(h)\partial_{v_z}Pg_i, \frac{1}{Mw_i}(I-P)g_i) + \sum_{i=1}^2(F_i(h)\partial_{v_z}(I-P)g_i, \frac{1}{Mw_i}(I-P)g_i) \\
\le\ & C\|F_i(h)\|_\infty \left(\|a_g\|\cdot\|(I-P)g_i\|_M + \|\partial_{v_z}(I-P)g\|^2_M\right) \\
\le\ & C\|\partial_z a_h\|\left(\|a_g\|\cdot\|g\|_D + \|g\|^2_D\right).
\end{aligned}$$

Now for the sixth term $z_\gamma F_i(\partial_t a_h)\partial_{v_z}h_i$, we note

$$\int z_\gamma F_i(\partial_t a_h)\partial_{v_z}h_i Aa_g dv = 0.$$

Since by (3.3),

$$\|z_\gamma F_i(\partial_t a_h)\| = \|F_i(z_\gamma\partial_t a_h)\| + \|[F_i, z_\gamma]\partial_t a_h\| \le C\|(I-P)\partial_z h\|_{D,\gamma}\ ,$$

we have, by using the assumption and integrating by part on v,

$$\sum_{i=1}^{2}(z_\gamma F_i(\partial_t a_h)\partial_{v_z} g_i, \frac{1}{Mw_i}(I-P)g_i) \leq C\delta_0\|\partial_z h\|_{D,\gamma} \times \|\partial_t h\|_{D,\gamma}.$$

LEMMA 4.4. *Let $0 \leq \gamma \leq \frac{1}{2}+\frac{1}{8}$. If $\|\partial_{t,z}h(t)\|_{M,\gamma} + \|h(t)\|_{M,\gamma} \leq \delta_0$, then*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_z h(t)\|^2_{M,\gamma} + \nu_0\|\partial_z h(t)\|\|^2_{D,\gamma}\\ &\leq C_\gamma\left(\|h\|^2_{D,\gamma-\frac{1}{2}} + \|\partial_{t,z}h\|^2_{D,\gamma-\frac{1}{2}} + \|h\|^2_D + \|\partial_t h\|^2_D + \delta_0\|\partial_t h\|^2_{D,\gamma}\right).\end{aligned} \quad (4.5)$$

whith ν_0 given in Lemma 2.1. .

Proof. We define $g = z_\gamma \partial_z h$ to get

$$\begin{aligned}&\partial_t g_i + G_i g_i - Lg_i = \frac{(1+2\gamma)zv_z g_i}{1+z^2} + z_\gamma \partial_z U * w_j'\partial_{v_z} h_i\\ &+\hat{G}\partial_z h_i + \partial_z U * w_j'\partial_{v_z}(z_\gamma h_i) - F_i(h)\partial_{v_z} g_i - z_\gamma F_i(\partial_z a_h)\partial_{v_z} h_i \equiv \Gamma_i.\end{aligned}$$

where $\hat{G}$ is defined in (4.3). By Lemma 4.1 we need to estimate $\langle\Gamma, Aa_g\rangle + \sum_{i=1}^{2}(\frac{1}{Mw_i}(I-P)g_i, \Gamma_i)$.

We first estimate the first term $\frac{2\gamma zv_z g_i}{1+z^2}$. Since that $g = a_g M + (I-P)g$,

$$\langle\frac{2\gamma zv_z g}{1+z^2}, Aa_g\rangle = \langle\frac{2\gamma zv_z a_g M}{1+z^2}, Aa_g\rangle + \langle\frac{2\gamma zv_z(I-P)g}{1+z^2}, Aa_g\rangle.$$

The first term above vanishes. For the second term above, we notice that by the splittng $a_h = a_h^{(1)} + a_h^{(2)}$, and by an integration by part with the kernel for $a_h^{(2)}$, we have from Lemma 3.3,

$$\begin{aligned}&\|\frac{1}{1+|z|}Aa_g\|\\ \leq\ &\|A(\frac{a_g}{1+|z|})\| + \|\int\frac{U(z-y)\{|z|-|y|\}}{\{1+|z|\}\{1+|y|\}}a_g(y)dy\|\\ \leq\ &C\left(\|z_{\gamma-\frac{1}{2}}a_h^{(2)}\| + \|\frac{a_g^{(1)}}{1+|z|}\|\right)\\ \leq\ &C\left(\|h\|\|_{D,\gamma-\frac{1}{2}} + C\|z_{\gamma-\frac{1}{2}}\partial_z a_h^{(1)}\right)\|\\ \leq\ &C\left(\|h\|_{D,\gamma-\frac{1}{2}} + \|\partial_t h\|_{D,\gamma-\frac{1}{2}}\right).\end{aligned}$$

Now by an integration by part in the $v-$variable, we turn to

$$
\begin{aligned}
& \sum_{i=1}^{2}|(\frac{1}{Mw_i}(I-P)g_i, \frac{zv_z\gamma g_i}{1+z^2})| \\
\leq & \sum_{i=1}^{2}|(\frac{1}{Mw_i}(I-P)g_i, \frac{zv_z\gamma(I-P)g_i}{1+z^2})| \\
& +\sum_{i=1}^{2}|(\frac{1}{Mw_i}(I-P)g_i, \frac{zv_z\gamma z_\gamma P\partial_z h_i}{1+z^2})| \\
\leq & C\gamma\|\frac{1}{1+|z|}(I-P)g\cdot\nabla_v(I-P)g\|_M \\
& +\varepsilon\|(I-P)g\|_M^2+C_\varepsilon\gamma\|z_\gamma\partial_z a_h\|^2 \\
\leq & \varepsilon\|(I-P)g\|_M^2 \\
& +C_\varepsilon\gamma\{\|h\|^2_{D,\gamma-\frac{1}{2}}+\|\partial_{z,t}h\|^2_{D,\gamma-\frac{1}{2}}+\|h\|_D^2+\|\partial_{z,t}h\|_D^2\}.
\end{aligned}
$$

In the last inequality we have used Lemma 3.3 and Lemma 3.4.

We now estimate the second term $z_\gamma\partial_z U * w_j'\partial_{v_z}h_i$. Notice that

$$
\sum_{i=1}^{2}(z_\gamma\partial_z U * w_j'\partial_{v_z}Ph_i, (Aa_g)_i)=0,
$$

and $z_\gamma\partial_z U * w_j'$ decays fast as $|z|\to\infty$, by (3.10). We thus split $a_g=a_g^{(1)}+a_g^{(2)}$ and use Lemma 3.3 and Lemma 3.4 to get

$$
\begin{aligned}
& \sum_{i=1}^{2}|(z_\gamma\partial_z U * w_j'\partial_{v_z}(1-P)h_i, (Aa_g)_i)| \\
\leq & C\|z_{\gamma-\frac{1}{2}}\partial_{v_z}(I-P)h_i\|_M^2+\|z_{\gamma-\frac{1}{2}}\partial_z a_h^{(1)}\|^2+\|z_{\gamma-\frac{1}{2}}a_h^{(2)}\|^2\} \\
\leq & C\{\|h\|_D+\|\partial_t h\|_D+\|h\|_{D,\gamma-\frac{1}{2}}+\|\partial_t h\|_{D,\gamma-\frac{1}{2}}\}.
\end{aligned}
$$

Notice that we are applying Lemma 3.3 for a bound on a norm with index $\gamma-\frac{1}{2}$. For that we need $\gamma\leq\frac{1}{2}+\frac{1}{8}$. On the other hand,

$$
\begin{aligned}
& \sum_{i=1}^{2}(z_\gamma\partial_z U * w_j'\partial_{v_z}h_i, \frac{1}{Mw_i}(I-P)g_i) \\
= & \sum_{i=1}^{2}(z_\gamma\partial_z U * w_j'\partial_{v_z}(I-P)h_i, \frac{1}{Mw_i}(I-P)g_i) \\
& +\sum_{i=1}^{2}(z_\gamma\partial_z U * w_j'\partial_{v_z}Ph_i, \frac{1}{Mw_i}(I-P)g_i).
\end{aligned}
$$

The first term is clearly bounded by $\varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\|h\|_D^2$. For the second term,

$$\begin{aligned} & \sum_{i=1}^{2}(z_\gamma\partial_z U * w_j'\partial_{v_z}Ph_i, \frac{1}{Mw_i}(I-P)g_i) \\ \leq \; & \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\|\frac{a_h}{1+|z|^N}\|^2 \\ \leq \; & \varepsilon\|g\|_D^2 + C_\varepsilon\{\|h\|_D^2 + \|\partial_t h\|_D^2\}, \end{aligned}$$

by (3.10) for some large N.

Now turn to the third term $\hat{G}h_i$ in Γ. Since $\langle \hat{G}h_i, Aa_g\rangle = 0$,

$$\begin{aligned} & |\sum_{i=1}^{2}(\frac{1}{Mw_i}(I-P)g_i, \hat{G}\partial_z a_h)| \\ \leq \; & \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\|\partial_t h\|^2_{D,\gamma-\frac{1}{2}} + C_\varepsilon\|h\|^2_{D,\gamma-\frac{1}{2}}. \end{aligned}$$

We now estimate the fourth term $\partial_z U * w_j'\partial_{v_z}(z_\gamma h_i)$. Since $\sum_{i=1}^2(\partial_z U * w_j'\partial_{v_z}(z_\gamma h_i), (Aa_g)_i) = 0$, we have

$$\begin{aligned} & \sum_{i=1}^{2}(\frac{1}{Mw_i}(I-P)g_i, \partial_z U * w_j'\partial_{v_z}(z_\gamma h_i)) \\ = \; & \sum_{i=1}^{2}(\frac{1}{Mw_i}(I-P)g_i, \partial_z U * w_j'\partial_{v_z}P(z_\gamma h_i)) \\ & +\sum_{i=1}^{2}(\frac{1}{Mw_i}(I-P)g_i, \partial_z U * w_j'\partial_{v_z}(I-P)(z_\gamma h_i)) \\ \leq \; & \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\left(\|\partial_z a_h^{(1)}\|^2 + \|a_h^{(2)}\|^2 + \|h\|_D^2\right). \end{aligned}$$

To get the last inequality we have used an argument similar to the one in the proof of (3.10) in Lemma 3.3. Since w' decays exponentially fast, we can divide the integration over z of the term involving $a_h^{(1)}$ in two pieces: for z small we can use Poincare' inequality to bound in terms of the z derivative. For z large we use the decay of w'. Then, by using Theorem 3.2 we get the final bound

$$\varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\left(\|\partial_z a_h^{(1)}\|^2 + \|a_h^{(2)}\|^2 + \|h\|_D^2\right) \leq \varepsilon\|(I-P)g\|_M^2 + C_\varepsilon\left(\|\partial_t h\|_D^2 + \|h\|_D^2\right)$$

Now for the fifth term $F_i(h)\partial_{v_z}g_i$, we note that $\sum_{i=1}^2(F_i(h)\partial_{v_z}g_i, Aa_g) = 0$ and

$$
\begin{aligned}
&\sum_{i=1}^{2}(F_i(h)\partial_{v_z}g_i, \frac{1}{Mw_i}(I-P)g_i) \\
= \; & \sum_{i=1}^{2}(F_i(h)\partial_{v_z}Pg_i, \frac{1}{Mw_i}(I-P)g_i) + \sum_{i=1}^{2}(F_i(h)\partial_{v_z}(I-P)g_i, \frac{1}{Mw_i}(I-P)g_i) \\
\le \; & C\|F_i(h)\|_\infty\{\|a_g\|\cdot\|(I-P)g_i\|_M + \|\nabla_v(I-P)g\|_M^2\} \\
\le \; & C\|\partial_z a_h\|\{\|a_g\|\cdot\|(I-P)g_i\|_M + \|\nabla_v(I-P)g\|_M^2\} \\
\le \; & C\delta_0\{\|g\|_D^2 + \|h\|_D^2 + \|\partial_t h\|_D^2\}.
\end{aligned}
$$

Now for the sixth term $z_\gamma F_i(\partial_z a_h)\partial_{v_z}h_i$, we note

$$
\int z_\gamma F_i(\partial_z a_h)\partial_{v_z}h_i A a_g dv = 0.
$$

We need to employ Lemma 3.4 to treat the last term as

$$
\begin{aligned}
&\sum_{i=1}^{2}(z_\gamma F_i(\partial_z a_h)\partial_{v_z}h_i, \frac{1}{Mw_i}(I-P)g_i) \\
= \; & \sum_{i=1}^{2}(F_i(a_g)\partial_{v_z}h_i, \frac{1}{Mw_i}(I-P)g_i) + \sum_{i=1}^{2}([z_\gamma, F_i](\partial_z a_h)\partial_{v_z}h_i, \frac{1}{Mw_i}(I-P)g_i) \\
\le \; & \left(\|a_g^{(2)}\|\cdot\|a_h\| + \|a_g\|\cdot\|h\|_D\right)\|(I-P)g\|_M \\
& + \Big((\|\partial_z\{a_g^{(1)} - z_\gamma a_h^{(3)}\}\| + \|z_\gamma a_h^{(3)}\|)\cdot\|a_h\| + \|a_g\|\cdot\|h\|_D\Big)\|(I-P)g\|_M \\
& + \Big(\|z_{\gamma-\frac{1}{2}}\partial_z a_h^{(1)}\|\cdot\|a_h\|\Big)\|(I-P)g\|_M \\
\le \; & C\delta_0\{\|g\|_D^2 + \|h\|_D^2 + \|\partial_{t,z}h\|_{D,\gamma}^2 + \|h\|_{D,\gamma-\frac{1}{2}}^2 + \|\partial_t h\|_{D,\gamma-\frac{1}{2}}^2\}.
\end{aligned}
$$

We deduce our lemma by letting ε small and using δ_0 small.

Proof of Theorem 1.2: To prove the first part, we start with $\gamma = 0$ in all three Lemmas 4.2, 4.3 and 4.4. We multiply by a positive number K (4.2) and (4.4) and add both to (4.5):

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\partial_z h(t)\|_M^2 + K(\|h(t)\|_M^2 + \|\partial_t h(t)\|_M^2)\Big) + K\nu_0\Big(\|h(t)\|_D^2 + \|\partial_t h(t)\|_D^2\Big) + \nu_0\|\partial_z h(t)\|_D^2 \\
&\quad \le KC\left(\delta_0\{\|\partial_t h(t)\|_D^2 + \|h(t)\|_D^2\} + \delta_0\|\partial_{t,z}h(t)\|\|_D^2 + \|h(t)\|_D^2\right) + C\|h\|_D^2 + C\|\partial_t h\|_D^2
\end{aligned}
$$

By choosing $K > \dfrac{C}{4\nu_0}$, and $\delta_0 < \dfrac{\nu_0}{4CK}$, we obtain that

$$
\frac{1}{2}\frac{d}{dt}\{\|\partial_z h\|_M^2 + K\left(\|\partial_t h\|_M^2 + \|h\|_M^2\right)\} + \frac{\nu_0}{2}\{\|\partial_z h\|_D^2 + K\left(\|\partial_t h\|_D^2 + \|h\|_D^2\right)\} \le 0. \quad (4.6)
$$

A standard continuity argument completes the first part.

To prove the second part, we first prove an inequality like (**??**) for the weighted norms with weight z_γ. Once again, we multiply (4.2) and (4.3) by K and add them to (4.4)

$$\frac{1}{2}\frac{d}{dt}\Big(K(\|h(t)\|^2_{M,\gamma_0}+\|\partial_t h(t)\|^2_{M,\gamma_0})+\|\partial_z h(t)\|^2_{M,\gamma_0}\Big)+K\nu_0\Big(\|h(t)\|^2_{D,\gamma_0}+\partial_t h(t)\|^2_{D,\gamma_0}\Big)+\nu_0\|\partial_z h(t)\|\|^2_{D,\gamma_0}$$

$$\le KC\{\gamma_0+\delta_0\}\{\|\partial_t h(t)\|^2_{D,\gamma_0}+\|h(t)\|^2_{D,\gamma_0}+\|\partial_{t,z}h(t)\|\|^2_{D,\gamma_0-\frac{1}{2}}\}+KC\|h(t)\|^2_{D,\gamma_0-\frac{1}{2}},$$

$$+C_{\gamma_0}\left(\|h\|^2_{D,\gamma_0-\frac{1}{2}}+\|\partial_{t,z}h\|^2_{D,\gamma_0-\frac{1}{2}}+\|h\|^2_D+\|\partial_t h\|^2_D+\delta_0\|\partial_t h\|^2_{D,\gamma_0}\right).$$

For γ_0 small enough and also $\gamma_0 < 1/2$, for δ_0 small enough and K sufficiently large we get

$$\frac{1}{2}\frac{d}{dt}\Big(K(\|h(t)\|^2_{M,\gamma_0}+\|\partial_t h(t)\|^2_{M,\gamma_0})+\|\partial_z h(t)\|^2_{M,\gamma_0}\Big)+K\nu_0\Big(\|h(t)\|^2_{D,\gamma_0}+\partial_t h(t)\|^2_{D,\gamma_0}\Big)+\nu_0\|\partial_z h(t)\|\|^2_{D,\gamma_0}$$

$$\le KC(\|\partial_{t,z}h(t)\|\|^2_D\}+\|h(t)\|^2_D),$$

Finally, we let $\gamma=\gamma_0$ sufficiently small in Lemma 4.2, while let $\gamma=\frac{1}{2}+\gamma_0$ in both Lemmas 4.3 and 4.4, while multiplying the first two by K. We get

$$\frac{1}{2}\frac{d}{dt}K\|h(t)\|^2_{M,\gamma_0}+K\nu_0\|h(t)\|^2_{D,\gamma_0}\le KC\{\gamma_0+\delta_0\}\{\|\partial_t h(t)\|^2_{D,\gamma_0}+\|h(t)\|^2_{D,\gamma_0}\},$$

$$\frac{1}{2}\frac{d}{dt}K\|\partial_t h(t)\|^2_{M,\gamma_0+\frac{1}{2}}+K\nu_0\|\partial_t h(t)\|^2_{D,\gamma_0+\frac{1}{2}}\le KC\{\gamma_0+\frac{1}{2}+\delta_0\}\|\partial_{t,z}h(t)\|\|^2_{D,\gamma_0}+KC\|h(t)\|^2_{D,\gamma_0}.$$

$$\frac{1}{2}\frac{d}{dt}\|\partial_z h(t)\|^2_{M,\gamma_0+\frac{1}{2}}+\nu_0\|\partial_z h(t)\|\|^2_{D,\gamma_0+\frac{1}{2}}$$

$$\le C_\gamma\left(\|h\|^2_{D,\gamma_0}+\|\partial_{t,z}h\|^2_{D,\gamma_0}+\|h\|^2_D+\|\partial_t h\|^2_D+\delta_0\|\partial_t h\|^2_{D,\gamma_0+\frac{1}{2}}\right).$$

Then, there is a large constant K such that

$$\frac{d}{dt}\{\|\partial_z h\|_{M,\frac{1}{2}+\gamma_0}+K\|\partial_t h\|^2_{M,\frac{1}{2}+\gamma_0}+K\|h\|^2_{M,\gamma_0}\}+\nu_0\|\partial_z h\|_{M,\frac{1}{2}+\gamma_0}+\nu_0 K\{\|\partial_t h\|^2_{D,\frac{1}{2}+\gamma_0}+\|h\|^2_{D,\gamma_0}\}$$

$$\le KC\{\|\partial_{t,z}h\|^2_D+\|h\|^2_D\}+K\{\|h(0)\|^2_{M,\gamma_0}+\|\partial_{t,z}h(0)\|^2_{M,\gamma_0}\}.$$

Using the first part and a standard continuity argument, we obtain:

$$\sup_{0\le t\le\infty}\{\|h(t)\|_{M,\gamma_0}+\|\partial_{t,z}h(t)\|_{M,\frac{1}{2}+\gamma_0}\}\le C\{\|h(0)\|_{M,\gamma_0}+\|\partial_{t,z}h(0)\|_{M,\frac{1}{2}+\gamma_0}\}. \tag{4.7}$$

We now turn back to (4.6). We want to control $\|h\|_M + \|\partial_{t,z}h\|_M$ but up to now we only have a uniform bound on $\|h\|_D + \|\partial_{t,z}h\|_D$. What is missing is a bound on $\|a_h\|$. But from (4.7) and an interpolation,

$$\begin{aligned}
\|a_h^{(1)}\| &\leq \|(1+|z|^2)^{1/2}\partial_z a_h^{(1)}\| \leq C\|(1+|z|^2)^{\frac{1}{2}+\gamma_0}\partial_z a_h^{(1)}\|^{\frac{1}{1+2\gamma_0}} \times \|\partial_z a_h^{(1)}\|^{\frac{2\gamma_0}{1+2\gamma_0}} \\
&\leq C\{\|h(0)\|_{\gamma_0} + \|\partial_{t,z}h(0)\|_{\frac{1}{2}+\gamma_0}\}^{\frac{1}{1+2\gamma_0}}\|h\|_D^{\frac{2\gamma_0}{1+2\gamma_0}}.
\end{aligned}$$

As for $a_h^{(2)}$, and $\partial_{t,z}a_h$, by Lemma 3.2, we conclude that they satisfy the same inequality above with $\gamma_0 = 0$. Therefore, let $E_{\gamma_0} = \{\|h(0)\|_{\gamma_0}^2 + \|\partial_{t,z}h(0)\|_{\frac{1}{2}+\gamma_0}^2\}$

$$\{\|h\|_D^2 + \|\partial_{t,z}h\|_D^2\} \geq CE_0^{-\frac{1}{2\gamma_0}}\{\|h\|^2 + \|\partial_{t,z}h\|^2\}^{\frac{1+2\gamma_0}{2\gamma_0}}.$$

We thus conclude that:

$$\frac{d}{dt}\{\|\partial_z h\|_M^2 + K(\|h\|_M^2 + \|\partial_t h\|_M^2)\} + CE_0^{-\frac{1}{2\gamma_0}}\{\|\partial_z h\|_M^2 + K(\|h\|_M^2 + \|\partial_t h\|_M^2)\}^{1+\frac{1}{2\gamma_0}} \leq 0.$$

Denoting $y(t) \equiv \|\partial_{t,z}h\|_M^2 + K\|h\|_M^2$, we have

$$y' y^{-1-\frac{1}{2\gamma_0}} \leq -CE_0^{-\frac{1}{2\gamma_0}}.$$

Integrating over 0 and t, we deduce

$$\frac{1}{2\gamma_0}\{y(0)\}^{-\frac{1}{2\gamma_0}} - \frac{1}{2\gamma_0}\{y(t)\}^{-\frac{1}{2\gamma_0}} \leq -CE_0^{-\frac{1}{2\gamma_0}}t.$$

Hence from $y(0) \leq E_0$

$$\begin{aligned}
\frac{1}{2\gamma_0}\{y(t)\}^{-\frac{1}{2\gamma_0}} &\geq t\frac{C}{2\gamma_0}E_0^{-\frac{1}{2\gamma_0}} + \{y(0)\}^{-\frac{1}{2\gamma_0}} \\
&\geq \{t\frac{C}{2\gamma_0} + 1\}E_0^{-\frac{1}{2\gamma_0}}.
\end{aligned}$$

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References

[As] private communication.

[BL] S. Bastea and J. L. Lebowitz, *Spinodal decomposition in binary gases.*, Phys. Rev. Lett. **78**, 3499-3502 (1997).

[BELM] S. Bastea, R. Esposito, J. L. Lebowitz and R.Marra, *Binary fluids with long range segregating interaction I: derivation of kinetic and hydrodynamic equations,* J. Stat. Phys. **101**, 1087–1136 (2000).

[—–] *Hydrodynamics of binary fluid phase segregation.* Phys. Rev. Letters **89** 235701-04 (2002).

[—–] *Sharp interface motion of a fluid binary mixture.* J. Stat. Phys. (2006)

[CCELM1] E. A. Carlen, C. C. Carvahlo, R. Esposito, J. L. Lebowitz, R. Marra *Free energy minimizers for a two-species model with segregation and liquid-vapor transition*, Nonlinearity **16**, 1075–1105 (2003).

[CCELM2] E. A. Carlen, C. C. Carvahlo, R. Esposito, J. L. Lebowitz, R. Marra *Displacement convexity and minimal fronts at phase boundaries*, preprint (2007)

[CCO] E. A. Carlen, , M.C. Carvalho, and E. Orlandi, *Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law, I*, J. Stat. Phys. **95**, 1069-1117 (1999).

[—-] *Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law, II.* Comm. Par. Diff. Eq. **25**, 847-886 (2000).

[Guo] Y. Guo, *The Boltzmann equation in the whole space* , Indiana Univ. Math. J. 53 10811094 (2004).

[Guo1] Y. Guo *The Vlasov-Poisson-Boltzmann system near vacuum*, Comm. Math. Phys. 218, 293313 (2001).

[—] *The Vlasov-Maxwell-Boltzmann system near Maxwellians*, Invent. math. 153, 593630 (2003).

[—] *The Vlasov-Poisson-Boltzmann system near Maxwel lians, Comm. Pure Appl. Math. 55, no. 9, 11041135 (2002).*

[SG] R. M. Strain and Y. Guo *Almost exponential decay near Maxwellians*, Comm PDE, 31, 417-429, (2006).

[LB] D. Liberzon and R.W. Brockett *Spectral analysis of Fokker-Planck and related operators arising from linear stochastic differential equations*, Siam J. Control Optim. **38**, No. 5, 1453-1467 (2000).

[LY] T.P. Liu, T. Yang and S. H. Yu - *Energy method for the Boltzmann equation* Physica D, 1–15 (2004).

[Ma] N. B. Maslova, Nonlinear Evolution Equations, Kinetic Approach, Series on Advances in Mathematics for Applied Sciences Vol. 10, World Scientific (1993).

[MM] G. Manzi and R. Marra, *Phase segregation and interface dynamics in kinetic systems,* Nonlinearity 19, 115-147 (2006).

[—–] *A kinetic model of interface motion.* International Journal of Modern Physics B **18**, 1–10 (2004).

[—–] *Kinetic Modelling of Late Stages of Phase Separation* in "Transport Phenomena and Kinetic Theory. Applications to Gases, Semiconductors, Photons, and Biological Systems", C.Cercignani and E. Gabetta Ed.s, Birkhauser (2006).

[Uk] S. Ukai, *On the existence of global solutions of a mixed problem for the nonlinear Boltzmann equation*, Proc. Japan Acad. **50**, 179–184 (1974).

[UYZ] *Global solutions to the Boltzmann equation with external forces*, S. Ukai, T. Yang and H. Zhao, Analysis and Applications, **3**, No. 2, 157193 (2005).

[Vi] C. Villani *Hypocoercive diffusion operators* in Proceedings of the 2006 International Congress of Mathematicians in Madrid.

[Yu] Shih-Hsien Yu *Hydrodynamic limits with shock waves of the Boltzmann equation*, Commun. Pure Appl. Math. **58**, 409–443 (2005).