Rigorous results and conjectures on stationary space-periodic 2D turbulence

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Abstract

We discuss recent results on the inviscid limits for the randomly forced 2D Navier-Stokes equation under periodic boundary conditions, their relevance for the theory of stationary space periodic 2D turbulence and some related conjectures.

0 Introduction

The stationary space-periodic 2D turbulence[1] is described by the small-viscosity 2D Navier-Stokes equation (NSE) under periodic boundary conditions, perturbed by a stationary random force:

$$\begin{aligned} &v'_\tau - \varepsilon\Delta v + (v\cdot\nabla)v + \nabla\tilde p = \varepsilon^a\,\tilde\eta(\tau,x), \quad x\in\mathbb{T}^2=\mathbb{R}^2/(2\pi\mathbb{Z}^2),\\ &\operatorname{div} v = 0,\ v=v(\tau,x)\in\mathbb{R}^2,\ \tilde p=\tilde p(\tau,x);\quad \int v\,dx \equiv \int \tilde\eta\,dx\equiv 0. \end{aligned} \tag{0.1}$$

Here $0<\varepsilon\ll 1$ and the scaling exponent a is a real number (e.g., $a=0$). In this work we assume that the force $\tilde\eta$ is a divergence-free Gaussian random field, white in time and smooth in x. The case $a\ge\frac32$ corresponds to non-interesting equations with small solutions (see [Kuk06a], Section 10.3), so we assume that $a<\frac32$. Under the mild nondegeneracy assumption on the force the Markov process which the equation defines in the function space $\mathcal{H}$,

$$\mathcal{H}=\{u(x)\in L^2(\mathbb{T}^2;\mathbb{R}^2)\mid \operatorname{div} u=0,\ \int_{\mathbb{T}^2} u\,dx=0\},$$

[1]or, at least, a natural and important type of such turbulence

has a unique stationary measure μ_ε which describes asymptotic in time statistical properties of solutions for (0.1); see below. The limit $\varepsilon \to 0$ corresponds to the transition to turbulence (cf. (1.7) below), and the limiting properties of the measure μ_ε describe the 2D turbulence. The substitution

$$v = \varepsilon^b u\,, \quad \tau = \varepsilon^{-b} t\,, \quad \nu = \varepsilon^{3/2-a}\,,$$

where $b = a - 1/2$, reduces eq. (0.1) to

$$\dot{u} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \sqrt{\nu}\, \eta(t,x), \quad \operatorname{div} u = 0, \tag{0.2}$$

where $u = u(t,x)$, $\dot{u} = u'_t$ and $\eta(t) = \varepsilon^{b/2} \tilde{\eta}(\varepsilon^{-b} t)$ is a new random field, distributed as $\tilde{\eta}$ (see [Kuk06a]).

Comparing to other equations (0.1), eq. (0.2) has the special advantage: when $\nu \to 0$ along a subsequence $\{\nu_j\}$, its stationary solution $u_{\nu_j}(t)$ converges in distribution to a stationary process $U(t) \in \mathcal{H}$, formed by solutions of the Euler equation

$$\dot{u} + (u \cdot \nabla) u + \nabla p = 0\,, \quad \operatorname{div} u = 0\,. \tag{0.3}$$

Accordingly, $\mu_{\nu_j} \rightharpoonup \mu_0$, where $\mu_0 = \mathcal{D}U(0)$ is an invariant measure for (0.3) (see below Theorem 1.1). The solutions U,[2] called the *Eulerian limits*, absorb limiting properties of the stationary solutions for (0.2) and – after re-scaling – limiting properties of stationary solutions of (0.1). So the Eulerian limits $U(t,x)$ and the limiting measures μ_0 describe the space-periodic 2D turbulence.

In this works we review recent results concerning properties of the Eulerian limits. Namely, in Section 2 we discuss disintegration of the measure μ_0 with respect to the sets, obtained by fixing values of all integrals of motion of the Euler equation (0.3), and the explicit algebraical relations, satisfied by stationary solutions of (0.2) uniformly in $\nu > 0$. In Section 3 we study distribution of the energy $\frac{1}{2}\int |U(t,x)|^2 dx$ of the Eulerian limit and of various functionals $\int f(\operatorname{rot} U(t,x))\, dx$ (corresponding to integrals of motion for eq. (0.3)). In Section 4 we consider the damped/driven KdV equation as a possible model for the 2D NSE. The inviscid limit for that equation was studied in [KP06] and is now understood much better than the Eulerian limit. We discuss the results of [KP06] and conjecture corresponding properties of the 2D NSE and the Eulerian limit.

[2]we use here plural since we do not know if U and μ_0 are unique, i.e. if they depend or not on the sequence $\nu_j \to 0$.

All results in this work use essentially the extra integrals of motion of the 2D Euler equation, so they certainly do not apply to the 3D NSE. Still the results and the methods of this work apply to other PDE of the form

$$\langle\text{Hamiltonian equation}\rangle + \nu\langle\text{dissipation}\rangle = \sqrt{\nu}\,\langle\text{random force}\rangle\,,$$

provided that the corresponding Hamiltonian PDE has at least two 'good' integrals of motion. In particular, they apply to the randomly forced complex Ginzburg-Landau equation (see [KS04] for some related results).

1 The Eulerian limit

Let us denote by $\|\cdot\|$ and by $(\cdot,\cdot)$ the L_2-norm and scalar product in the space $\mathcal{H}$. Let $(e_s, s\in\mathbb{Z}^2\setminus\{0\})$ be its standard trigonometric basis:

$$e_s(x) = \frac{\sin(s\cdot x)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix} \quad\text{or}\quad e_s(x) = \frac{\cos(s\cdot x)}{\sqrt{2}\pi|s|}\begin{bmatrix}-s_2\\ s_1\end{bmatrix}\,,$$

depending whether $s_1 + s_2\delta_{s_1,0} > 0$ or $s_1 + s_2\delta_{s_1,0} < 0$. The force η in (0.2) equals

$$\eta = \frac{d}{dt}\zeta(t,x), \quad \zeta = \sum_{s\in\mathbb{Z}^2\setminus\{0\}} b_s\beta_s(t)e_s(x)\,,$$

where $\{b_s\}$ is a set of real constants, satisfying

$$b_s = b_{-s} \neq 0 \quad \forall\, s, \qquad \sum |s|^2 b_s^2 < \infty\,,$$

and $\{\beta_s(t)\}$ are standard independent Wiener processes. Using the Leray projector $\Pi : L^2(\mathbb{T}^2;\mathbb{R}^2) \to \mathcal{H}$ we rewrite eq. (0.2) as the equation for $u(t) = u(t,\cdot)\in\mathcal{H}$:

$$\dot{u} + \nu A(u) + B(u) = \sqrt{\nu}\,\eta(t),$$

or as the diffusion equation

$$du = -(\nu Au + B(u))\,dt + \sqrt{\nu}\,d\zeta(t). \tag{1.1}$$

The equation (1.1) is known to have a unique stationary measure μ_ν (e.g., see in [Kuk06a]).[3] This is a probability Borel measure in $\mathcal{H}$ which attracts

[3]Due to results of the recent work [HM06], the stationary measure μ_ν is unique if $b_s\neq 0$ for $|s|\leq N$, where N is a ν-independent constant. Theorems 1.1 and 2.1 below remain true under this weaker assumption, but the proofs of results in Sections 3, 4 use essentially that all coefficients b_s are non-zero.

distributions of all solutions for (1.1). Let $u_\nu(t,x)$, $t \geq 0$, be a corresponding stationary solution, i.e.

$$\mathcal{D}u_\nu(t) \equiv \mu_\nu.$$

Apart from being stationary in t, this solution is known to be stationary (=homogeneous) in x.

For any $l \geq 0$ we denote by $\mathcal{H}^l, l \geq 0$, the Sobolev space $\mathcal{H} \cap H^l(\mathbb{T}^2;\mathbb{R}^2)$, given the norm

$$\|u\|_l = \Big(\int \big((-\Delta)^{l/2} u(x)\big)^2 \, dx\Big)^{1/2} \tag{1.2}$$

(so $\|u\|_0 = \|u\|$). A straightforward application of Ito's formula to $\|u_\nu(t)\|_0^2$ and $\|u_\nu(t)\|_1^2$ implies that

$$\mathbf{E}\,\|u_\nu(t)\|_1^2 \equiv \frac{1}{2}\,B_0\,, \quad \mathbf{E}\|u_\nu(t)\|_2^2 \equiv \frac{1}{2}\,B_1\,, \tag{1.3}$$

where for $l \in \mathbb{R}$ we denote $B_l = \sum |s|^{2l} b_s^2$ (note that $B_0, B_1 < \infty$ by assumption); see in [Kuk06a].

The theorem below describes what happens to the stationary solutions $u_\nu(t,x)$ as $\nu \to 0$. For the theorem's proof see [Kuk04, Kuk06a].

Theorem 1.1. *Any sequence $\tilde\nu_j \to 0$ contains a subsequence $\nu_j \to 0$ such that*

$$\mathcal{D}u_{\nu_j}(\cdot) \rightharpoonup \mathcal{D}U(\cdot) \quad in \quad \mathcal{P}\big(C(0,\infty;\mathcal{H}^1)\big)\,. \tag{1.4}$$

The limiting process $U(t) \in \mathcal{H}^1$, $U(t) = U(t,x)$, is stationary in t and in x. Moreover,

1)a) every its trajectory $U(t,x)$ is such that

$$U(\cdot) \in L_{2\,loc}(0,\infty;\mathcal{H}^2), \quad \dot U(\cdot) \in L_{1\,loc}(0\,\infty;\mathcal{H}^1)\,,$$

b) it satisfies the free Euler equation

$$\dot u + B(u) = 0\,, \tag{1.5}$$

c) $\|U(t)\|_0$ and $\|U(t)\|_1$ are time-independent quantities. If $g(\cdot)$ is a bounded continuous function, then $\int_{\mathbb{T}^2} g(\operatorname{rot} U(t,x))\,dx$ also is a time-independent quantity.

2) For each $t \geq 0$ we have

$$\mathbf{E}\|U(t)\|_1^2 = \frac{1}{2}B_0, \quad \mathbf{E}\|U(t)\|_2^2 \leq \frac{1}{2}B_1, \quad \mathbf{E}\exp\big(\sigma\|U(t)\|_1^2\big) \leq C \tag{1.6}$$

for some $\sigma > 0, C \geq 1$.

Amplification. If $B_2 < \infty$, then the convergence (1.4) holds in the space $\mathcal{P}(C(0,\infty;\mathcal{H}^\varkappa))$, for any $\varkappa < 2$.

See [Kuk06a], Remark 10.4.

Due to 1b), the measure $\mu_0 = \mathcal{D}U(0)$ is invariant for the Euler equation. By 2) it is supported by the space $\mathcal{H}^2$ and is not a δ-measure at the origin. The process U is called the *Eulerian limit* for the stationary solutions u_ν of (1.1). Note that apriori the process U and the measure μ_0 depend on the sequence ν_j.

Since $\|u\|_1^2 \le \|u\|_0\|u\|_2$, then $\mathbf{E}\,\|u\|_1^2 \le (\mathbf{E}\|u\|_0^2)^{1/2}(\mathbf{E}\|u\|_1^2)^{1/2}$ and (1.3) implies that $\frac{1}{2}B_0^2B_1^{-1} \le \mathbf{E}\|u_\nu(t)\|_0^2 \le \frac{1}{2}B_1$ for all ν. That is, the characteristic size of the solution u_ν remains ~ 1 when $\nu \to 0$. Since the characteristic space-scale also is ~ 1, then

$$\text{the Reynolds number of } u_\nu \text{ grows as } \nu^{-1} \text{ when } \nu \text{ decays to zero.} \tag{1.7}$$

Hence, Theorem 1.1 describes a transition to turbulence for space-periodic 2D flows. Since the force, proportional to the square root of the viscosity is the only way to force the 2D NSE to get a limit or order one (see Introduction and [Kuk06a], Section 10.3), then various Eulerian limits as in Theorem 1.1 with different coefficients $\{b_s\}$ (corresponding to different spectra of the applied random forces) describe all possible 2D space-periodic stationary turbulent flows.

2 Vorticity of the Eulerian limit

2.1 Disintegration of the measure $\operatorname{rot}\mu_0$

In this section and below we denote by $H^l = H^l(\mathbb{T}^d)$ the Sobolev space of functions with zero mean-value on the torus $\mathbb{T}^d, d = 1$ or 2. The norm in this space is denoted $\|\cdot\|_l$, i.e. as the norm in $\mathcal{H}^l$, and is defined as in (1.2).

Let us write the Euler equation (1.5) in terms of the vorticity $\xi(t,x) = \operatorname{rot} u(t,x) = \partial u_2/\partial x_1 - \partial u_1/\partial x_2$:

$$\dot{\xi} + (u\cdot\nabla)\xi = 0\,, \quad u = \nabla^\perp(-\Delta)^{-1}\xi\,. \tag{2.1}$$

Here $\nabla^\perp = (\partial/\partial x_2, -\partial/\partial x_1)^t$ and Δ is the Laplacian, operating on functions on $\mathbb{T}^2$ with zero mean value. By Theorem 1.1,

$$V(t,x) = \operatorname{rot} U(t,x)$$

satisfies (2.1) for every value of the random parameter. Now we show that $V(t)$ belongs to a certain functions space K where (2.1), supplemented by the initial condition $\xi(0) = \xi_0 \in K$, has a unique solution, continuously depending on ξ_0. To define this space we first set

$$\mathcal{K} = \{u \in L_{2\,loc}(\mathbb{R}; \mathcal{H}^2) \mid \dot{u} \in L_{1\,loc}(\mathbb{R}; \mathcal{H}^1)\}\,,$$

and provide $\mathcal{K}$ with the usual structure of a Fréchet space. This is a Polish space (i.e., a complete separable metric space). Next we define $\tilde{\mathcal{K}}$ as the set of solutions for (1.5), belonging to $\mathcal{K}$. This is a closed subset of $\mathcal{K}$, so also a Polish space. The group of the flow-maps of the Euler equation acts on $\tilde{\mathcal{K}}$ by time-shifts which are its continuous homeomorphisms. Now consider the continuous map

$$\pi : \tilde{\mathcal{K}} \to H^0(\mathbb{T}^2)\,, \quad u(t,x) \mapsto \operatorname{rot} u(0,x)\,.$$

Due to an uniqueness theorem of the Yudovich type (see [Kuk04], Lemma 3.5), π is an embedding. We set

$$K = \pi(\tilde{\mathcal{K}})$$

and provide K with the distance, induced from $\tilde{\mathcal{K}}$. It makes K a Polish space (we do not know if K is a linear space or not, i.e. if it is invariant with respect to the usual linear operations). It is not hard to see that $H^2(\mathbb{T}^2) \subset K$. So

$$H^2(\mathbb{T}^2) \subset K \subset H^0(\mathbb{T}^2)\,, \tag{2.2}$$

where the embeddings are continuous.

Due to what was said above, the Euler equation defines a group of continuous homeomorphisms

$$S_t : K \to K\,, \qquad t \in \mathbb{R}\,. \tag{2.3}$$

Theorem 1.1 shows that $U(\cdot) \in \tilde{\mathcal{K}}$ for each value of the random parameter. Therefore the measure

$$\theta = \mathcal{D}(V(0)) = \operatorname{rot} \circ \mu_0$$

is supported by K (i.e., $\theta(K) = 1$). Since $S_t \circ \theta = \mathcal{D}(V(t))$ and $V(t)$ is a stationary process, then θ is an invariant measure for the dynamical system

(2.3). By the estimates in item 2) of Theorem 1.1, it is supported by the space $K \cap H^1(\mathbb{T}^2)$ and $\int \exp\big(\sigma\|v\|_0^2\,\theta(dv) < \infty$.

Our next goal is to express the fact that trajectories of the process V satisfy assertion 1c) of Theorem 1.1 in terms of the measure θ. Let us denote by $\mathcal{P}(\mathbb{R})$ the set of probability Borel measures on $\mathbb{R}$, furnished with the Lipschitz-dual distance

$$\operatorname{dist}(m', m'') = \sup_{|f|\le 1,\, \mathrm{Lip}\, f \le 1} |\langle m', f\rangle - \langle m'', f\rangle|\,.$$

This is a Polish space, and convergence in the introduced distance is equivalent to the $*$-weak convergence of the measures, see [Dud02]. Due to (2.2) the map

$$M : K \to \mathcal{P}(\mathbb{R})\,, \qquad \xi \mapsto \xi \circ \big((2\pi)^{-2} dx\big)\,,$$

is continuous. Accordingly the map

$$\Psi : K \to \mathcal{P}(\mathbb{R}) \times \mathbb{R}_+ =: B\,, \quad \xi(\cdot) \mapsto (M(\xi), \|\xi\|_{-1})\,,$$

also is continuous.

Repeating (say) the arguments in [Kuk06b, Kuk06a], proving assertion 1c) of Theorem (1.1), we get that each trajectory $u(t,x)$ of (1.5), belonging to the space $\mathcal{K}$, satisfies $\Psi(u(t)) = \text{const}$. Recalling the definition of the flow-maps S_t, we get that they commute with Ψ. That is,

$$S_t : K_{\mathbf{b}} \to K_{\mathbf{b}} \qquad \forall\, t \in \mathbb{R}\,, \tag{2.4}$$

for every $\mathbf{b} \in B$, where we denoted

$$K_{\mathbf{b}} = \Psi^{-1}(\mathbf{b}) \subset K\,, \quad \mathbf{b} \in B\,.$$

Clearly $K_{\mathbf{b}}$ is a closed subset of K. We provide it with the induced topology.

Let us denote $\lambda = \Psi \circ \theta$. This is a measure on the Polish space B. Applying to the map Ψ the Disintegration Theorem (see [Par77] and [Kuk07a]), we get

Theorem 2.1. *There exists a family $\{\theta_{\mathbf{b}}, \mathbf{b} \in B\}$ of measures on Borel subsets of B such that*

1) $\theta_{\mathbf{b}}(K_{\mathbf{b}}) = 1$ for each $\mathbf{b}$, for any Borel set $A \subset K$ the function $\mathbf{b} \mapsto \theta_{\mathbf{b}}(A)$ is Borel-measurable on B, and

$$\theta(A \cap \Psi^{-1}(D)) = \int_D \theta_{\mathbf{b}}(A)\, d\lambda(\mathbf{b})\,, \tag{2.5}$$

for any Borel set $D \subset B$.

2) For λ-a.a. $\mathbf{b} \in B$ the measure $\theta_{\mathbf{b}}$ (interpreted as a measure on Borel subsets of $K_{\mathbf{b}}$) is invariant for the dynamical system (2.4).

2.2 The balance relations

Let $g(r)$ be a continuous function which has at most a polynomial growth as $r \to \infty$. Denoting by G its second integral (i.e. $G'' = g$ and $G(0) = G'(0) = 0$), applying Ito's formula to the process

$$t \mapsto \int G(\xi_\nu(t,x))\,dx\,, \quad \xi_\nu = \operatorname{rot} u_\nu\,,$$

and using that $(\operatorname{rot} e_s(x))^2 + (\operatorname{rot} e_{-s}(x))^2 \equiv |s|^2/2\pi^2$ for each s, we proved in [KP05] the following result:

Theorem 2.2. *Let $B_3 < \infty$. Then for any t and x we have*

$$\mathbf{E}\big(g(\xi_\nu(t,\,x))|\nabla\xi_\nu(t,\,x)|^2\big) = \frac{1}{2}\,(2\pi)^{-2}B_1\mathbf{E}g(\xi_\nu(t,\,x))\,. \tag{2.6}$$

We call equalities (2.6) the *balance relations.*

Let us take the balance relation (2.6), where g is a bounded continuous function such that $0 \le g \le 1$. Integrating it in dx we get:

$$\mathbf{E}\int g(\xi(x))|\nabla\xi(x)|^2\,dx = \frac{1}{2}(2\pi)^{-2}B_1\mathbf{E}\int g(\xi(x))\,dx. \tag{2.7}$$

Here we abbreviate $\xi_\nu(t,x) = \xi(x)$ ($t \ge 0$ is fixed). Assume that $B_6 < \infty$. Then $\xi(x) \in C^3$ a.s. (see in [Kuk06a]). Modifying the random field $\xi^\omega(x)$ on a null-set we achieve that $\xi^\omega(x) \in C^3$ for all ω. Denote

$$\Gamma(\tau,\omega) = \{x \in \mathbb{T}^2 \mid \xi(x) = \tau\}\,.$$

By the Sard lemma, for each ω the set $\Gamma(\tau,\omega)$ is C^3-smooth for a.e. $\tau \in \mathbb{R}$. So we may perform in the two integrals in (2.7) the co-area change of variable

$$x \mapsto (\tau,\gamma), \quad \tau = \xi^\omega(x),\ \gamma \in \Gamma(\tau,\omega)\,.$$

Transforming formally the integrals in (2.6) we find that

$$\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)} |\nabla\xi|\,d\gamma\,d\tau = \frac{1}{2}\,(2\pi)^{-2}B_1\,\mathbf{E}\int_{\mathbb{R}} g(\tau)\int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\,d\gamma\,d\tau\,,$$

where we adopted the natural convention $\int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\,d\gamma = \infty$ if τ is a critical value of ξ. Since g is arbitrary continuous function, satisfying $0 \le g \le 1$, then

$$\mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla\xi|\,d\gamma\,d\tau = \frac{1}{2}\,(2\pi)^{-2}B_1\,\mathbf{E}\int_{\Gamma(\tau,\omega)} |\nabla\xi|^{-1}\,d\gamma\,d\tau\,.$$

In [Kuk06b] we made these calculations rigorous, thus proving

Theorem 2.3. *If $B_6 < \infty$, then for any $\nu > 0$ and $t \ge 0$ the equality above holds for a.a. $\tau \in \mathbb{R}$.*

The assertions of Theorems 2.2 and 2.3 indicate that in some sense 'the periodic 2D turbulence is integrable'. Unfortunately we cannot prove that these results hold for the vorticity V of the Eulerian limit.

Choosing in (2.6) $g(v) = v^{2m}, m \in \mathbb{N}$, and using a new version of the Poincaré inequality, we in [Kuk06b] derived from Theorems 2.2 estimates for exponential moments of random variables $\xi_\nu(t,x)$. Now we *can* go to the limit as $\nu_j \to 0$:

Corollary 2.4. *Let $B_6 < \infty$. Then there exist $\sigma > 0$ and $C \ge 1$ such that for any $t \ge 0$, $x \in \mathbb{T}^2$ and $\nu \in [0,1]$ we have*

$$\mathbf{E}e^{\sigma|\xi_\nu(t,x)|} \le C, \tag{2.8}$$

where we denoted $\xi_0 = V = \operatorname{rot} U$.

This estimate with $\nu = 0$ is crucially used in the next section.

3 Distribution of energy and of functionals of vorticity

The equality in (1.6) implies that the random field $U(t,x)$ does not vanish identically, but it does not exclude that it is rather degenerate. In particular, it does not exclude that

- the vector field $U(t,\cdot) \in \mathcal{H}$ vanishes with a positive probability, or that
- its distribution $\mathcal{D}(U(t))$ is supported by a finite-dimensional subset of $\mathcal{H}$.

In this section we discuss results on the distribution of U which, in particular rule out the two possibilities above.

Let us denote by $E(u)$ the energy of a 2D vector field $u(x)$, $E(u) = \frac{1}{2}\int |u(x)|^2 dx = \frac{1}{2}\|u\|^2$. The first main difficulty in the study of distribution of $U(t) \in \mathcal{H}$ is to show that $E(U(t)) > 0$ a.s. Let us re-write the NSE (1.1), using the fast time $\tau = t/\nu$:

$$du(\tau) = -(Au + \nu^{-1}B(u))\,d\tau + d\tilde{\zeta}(\tau)\,, \tag{3.1}$$

where $\tilde{\zeta}(\tau) = \nu^{1/2}\zeta(\tau/\nu)$ is a new Wiener process in $\mathcal{H}$, distributed as $\zeta(\tau)$. If not for the term $\nu^{-1}B(u)\,d\tau$, we could apply Krylov's results (Theorem 2.3.3 from [Kry80]) to show that the process $u^{(2)}(\tau) \in \mathbb{R}^2$, formed by two first components of $u(\tau)$, is small in norm with a small probability, uniformly in $\nu > 0$. This would imply that $E(U(t)) > 0$ a.s. Motivated by this observation, we in [Kuk07b] managed to construct a new process $\tilde{u}(\tau)$ such that

- $E\tilde{u}(\tau) \equiv Eu_\nu(t)\mid_{t=\tau/\nu}$,
- $\tilde{u}(\tau)$ satisfies a ν-independent Ito equation,
- the process $\tilde{u}^{(2)}(\tau) \in \mathbb{R}^2$ has a non-degenerate diffusion.

Applying to $\tilde{u}^{(2)}(\tau)$ the Krylov result and using that $E|u(\tau)| \geq E|\tilde{u}^{(2)}(\tau)|$ we prove the inequality

$$\mathbf{P}\{E(u_\nu) < \delta\} \leq C\delta^{1/4}, \quad \forall\, \delta > 0, \tag{3.2}$$

for each $\nu > 0$, where C is independent from ν.

Due to (3.1) and Ito's formula, the process $E(\tau) = Eu_\nu(\tau)$ satisfies the Ito equation

$$dE = (-\|u\|_1^2 + \frac{1}{2}B_0)\,d\tau + \sum b_s u_s\,d\beta_s(\tau)\,.$$

Since the coefficients b_s are nonzero, then the relations (1.3) and (3.2) imply that the diffusion in the equation for E is 'non-degenerate in a qualified way'. Accordingly, the measure $\mathcal{D}E(\tau)$ is *absolutely continuous with respect to the Lebesgue measure, uniformly in ν.*[4] Now using (1.4) we get that the measure

[4]That is, for any Borel set $Q \subset [-R, R]$ we have $\mathcal{D}E(\tau)(Q) \leq p_R(|Q|)$, where p_R is ν-independent and $p_R(t) \to 0$ when $t \to 0$.

$\mathcal{D}E(U(\tau))$ is absolutely continuous with respect to the Lebesgue measure, i.e.,

$$\mathcal{D}E(U(\tau)) = e(x)\,dx\,, \quad e \in L_1(\mathbb{R}_+)\,.$$

Due to (1.6) and (3.2), (1.4),

$$\int_K^\infty e(x)\,dx \le Ce^{-\sigma K^2} \;\; \forall\, K \ge 1; \qquad \int_0^\delta e(x)\,dx \le C\delta^{1/4} \;\; \forall\, \delta > 0\,.$$

The estimate (3.2) also allows to study distributions of functionals of the vorticity $\xi_\nu = \operatorname{rot} u_\nu(t,x)$ for $\nu \ll 1$. Let us fix $m \in \mathbb{N}$ and choose any m analytic functions $f_1(\xi), \dots, f_m(\xi)$, linear independent modulo constant functions. We assume that the functions $f_j(\xi), \dots, f'''_j(\xi)$ have at most a polynomial growth as $|\xi| \to \infty$ and that $f_j''(\xi) \ge -C$ for all j and ξ (for example, each f_j is a trigonometric polynomial, or a polynomial of an even degree with a positive leading coefficient). Consider the map F,

$$F(\xi) = (F_1(\xi), \dots, F_m(\xi)) \in \mathbb{R}^m, \quad F_j = \int_{\mathbb{T}^2} f_j(\xi(x))\,dx\,,$$

and the Ito equation, satisfied by the process $F(\xi_\nu(\tau))$, $\tau = t\nu$. Using (3.2) and (2.8) we proved in [Kuk07b] that the measures $\mathcal{D}F(\xi_\nu(\tau))$ also are uniformly absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^m$. Evoking Amplification to Theorem 1.1 we get

Theorem 3.1. *If $B_6 < \infty$, then the vorticity V of the Eulerian limit U is distributed in such a way that the law of $F(V(0))$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^m$.*

Since m is arbitrary, then this result implies that the measure $\mu_0 = \mathcal{D}U(0)$ is genuinely infinite-dimensional in the sense that any compact set in $\mathcal{H} \cap C^1$ of finite Hausdorff dimension has zero μ_0-measure.

4 Model for 2D NSE (0.2) and conjectures for the Euler limit

To study further properties of the measure $\mu_0 = \mathcal{D}(U(0))$, describing the space-periodic two dimensional turbulence, is a hard task. To develop corresponding intuition, in [Kuk07a] we suggested as a model for (0.2) the equation, obtained by replacing in (0.2) the Euler equation by the KdV equation

$$\dot{u} + u_{xxx} - 6uu_x = 0\,,$$

I.e. by replacing one Hamiltonian system with infinitely many integrals of motion by another:

$$
\begin{aligned}
\dot v - \nu v_{xx} + v_{xxx} - 6 v v_x = \sqrt{\nu}\,\eta(t,x)\,, \\
x \in \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}\,, \quad \int v\,dx \equiv \int \eta\,dx \equiv 0\,.
\end{aligned}
\tag{4.1}
$$

Here $\eta(t,x)$ is a white in time nondegenerate random force, similar to that in (0.2). Equation (4.1) has a unique stationary measure m_ν. The measures m_ν and corresponding stationary solutions $v_\nu(t,x)$ posses all the properties of the stationary solutions u_ν and the measures μ_ν, discussed above in Sections 1-3. Namely,

i) along a subsequence $\nu_j \to 0$ the random fields $v_{\nu_j}(t,x)$ converge in distribution to a limiting random field $v(t,x)$, formed by solutions of the KdV equation.

ii) If $I = (I_1, I_2, \dots)$, $I_j \geq 0$, is the vector, formed by the integrals of KdV as in [MT76, KP03] (also see in [KP06]), then the limiting measure $m_0 = \mathcal{D}v(0)$ is invariant for the KdV equation and may be disintegrated with respect to the map I. Now a.e. iso-integral set $T^\infty = \{I = \text{const}\}$ is diffeomorphic to the infinite-dimensional torus $\mathbb{T}^\infty$, and the disintegration of m_0 may be written as

$$
m_0 \sim \lambda \times d\varphi, \tag{4.2}
$$

where $d\varphi$ is the Haar measure on $\mathbb{T}^\infty$ and λ is a measure on $\mathbb{R}^\infty_+$.

iii) For any n, the image of m_0 under the mapping $v(x) \mapsto (I_1(v), \dots, I_n(v))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n_+$.

Note that the disintegration (4.2) is more precise than that in Theorem 2.1, where the measures $\theta_{\mathbf{b}}$ are unknown. This happened since the iso-integral sets T^∞ for the KdV equation are much simpler than those for the 2D Euler equation, and the KdV-flow on the sets T^∞ is well understood. In particular,

iv) KdV flow on a.e. set T^∞ is ergodic.

In [KP06] we studied eq. (4.1) further and proved that

v) λ is an invariant measure for the stochastic equation for $I(\tau) \in \mathbb{R}^\infty_+$, obtained from the Ito equation for $I(\tau) = I(v(t))$, $t = \tau/\nu$, by means of the stochastic averaging.

Based on similarity between the properties i)-iii) and the corresponding features of the 2D NSE (0.2), we conjectured in [Kuk07a] that natural analogies of properties iv), v) hold for the 2D NSE and the Eulerian limit:

1. Every non-empty set $K_{\mathbf{b}}$ carries a measure $m_{\mathbf{b}}$, invariant for the Euler flow (2.4), such that for a.a. $u \in \mathcal{H}$ with respect to any stationary measure μ_ν we have

$$\lim_{T\to\infty} T^{-1}\int_0^T f(S_t u)\,dt = \langle f, m_{\mathbf{b}}\rangle\,, \quad \mathbf{b} = \Psi(u); \quad \forall f \in C_b(\mathcal{H}). \tag{4.3}$$

For λ-a.a. $\mathbf{b}$ the measure $m_{\mathbf{b}}$ coincides with $\theta_{\mathbf{b}}$ in (2.5).

Let $u(t,x)$ be a solution of (1.1). Then the vector $\Psi(u(t)) \in B$ satisfies a SDE. To describe it we introduce the space $\mathcal{C} = C_b(\mathbb{R}) \times \mathbb{R}$ and denote by $\langle\cdot,\cdot\rangle$ its natural pairing with $\mathcal{P}(\mathbb{R}) \times \mathbb{R} \supset B$. For any $\mathbf{f} \in \mathcal{C}$ we denote

$$u_{\mathbf{f}} = \langle \Psi(u), \mathbf{f}\rangle\,.$$

Applying Ito's formula to $u_{\mathbf{f}}$ and passing to the fast time $\tau = \nu t$, we get

$$d\,u_{\mathbf{f}}(\tau) = F_{\mathbf{f}}(u(\tau))\,d\tau + \sum_s \sigma_{\mathbf{f}s}(u(\tau))\,d\beta_s(\tau)\,.$$

Here $F_{\mathbf{f}}$ and $\{\sigma_{\mathbf{f}s}, s \in \mathbb{Z}^2 \setminus 0\}$ are smooth functions on $\mathcal{H}$ (depending on the coefficients b_s) and $\{\beta_s(\tau)\}$ are new standard Wiener processes.

2. Let $u_\nu(t)$ be a stationary solution of (1.1). Then along a subsequence $\nu_j \to 0$ the process $\Psi(u_\nu(\tau))$ converges in distribution to a limiting process $\mathbf{b}(\tau)$. This is a stationary Ito process in B such that for any $\mathbf{f} \in \mathcal{C}$ the process $b_{\mathbf{f}}(\tau) = \langle \mathbf{b}(\tau), \mathbf{f}\rangle$ has the drift

$$\langle F\rangle_{\mathbf{f}}(\mathbf{b}) = \langle F_{\mathbf{f}}(u), m_{\mathbf{b}}\rangle = \int_{K_{\mathbf{b}}} F_{\mathbf{f}}(u)\, m_{\mathbf{b}}(du)\,, \tag{4.4}$$

and for any $\mathbf{f}_1, \mathbf{f}_2 \in \mathcal{C}$ processes $b_{\mathbf{f}_1}$ and $b_{\mathbf{f}_2}$ have the covariation

$$\langle a\rangle_{\mathbf{f}_1\,\mathbf{f}_2}(\mathbf{b}) = \Big\langle \sum_s \sigma_{\mathbf{f}_1 s}(u)\sigma_{\mathbf{f}_2 s}(u), m_{\mathbf{b}}\Big\rangle\,. \tag{4.5}$$

I.e., the processes $\mathbf{b}_{\mathbf{f}_1}(\tau) - \int_0^\tau \langle F\rangle_{\mathbf{f}_1}(\mathbf{b}(s))\,ds$ and $\mathbf{b}_{\mathbf{f}_2}(\tau) - \int_0^\tau \langle F\rangle_{\mathbf{f}_2}(\mathbf{b}(s))\,ds$ are martingales and their bracket equals $\langle a\rangle_{\mathbf{f}_1\mathbf{f}_2}(\mathbf{b}(\tau))$.

That is to say, the limiting process $\mathbf{b}(\tau)$ is a martingale solution for a stochastic differential equation in the space B with the drift $\langle F\rangle(\mathbf{b})$ and the diffusion $\langle a\rangle(\mathbf{b})$. This is the *Whitham equation for the 2D NSE* (1.1).

In difference with the KdV case, space B does not have a natural basis, and we cannot *naturally* write this equation as a system of infinitely many

SDE. Still we can find a countable system of vectors $f_j \in \mathcal{C}$ such that their linear combinations are dense in $\mathcal{C}$, and use these vectors in (4.4), (4.5). In this way we write the Whitham equation above as an over-determined system of infinitely many SDE.

We can write down the Whitham equation, using in (4.4), (4.5) the measures $\theta_{\mathbf{b}}$ instead of the measures $m_{\mathbf{b}}$, i.e. without invoking the Ergodic Hypothesis **1**. But if the hypothesis fails, then the measures $\theta_{\mathbf{b}}$ may depend on the sequence $\nu_j \to 0$ in Theorem 1.1. In this case the Whitham equation seems to be a useless object.

3. Distribution of the limiting process $\mathbf{b}(\tau)$ is independent from the sequence $\{\nu_j \to 0\}$. Accordingly, the measure $\theta = \mathcal{D}\mathbf{b}(0)$ also is independent from the sequence, and

$$\Psi \circ \mu_\nu \rightharpoonup \theta \,, \quad \mu_\nu \rightharpoonup \mu_0 = \int m_{\mathbf{b}}\, \theta(d\mathbf{b})$$

as $\nu \to 0$.

Despite the measures $m_{\mathbf{b}}$ are unknown, the averaged drift and covariance (4.4) and (4.5) which characterise the limiting equation can be calculated by replacing the ensemble-average by the time-average (see (4.3)). So validity of the suggested scenario **1-3** may be verified numerically by comparing $\Psi(u_\nu(\tau))$ with solutions for the limiting equation.

Additional support for the conjectures above comes from the following consideration. Since the works of Moreau (1959) and Arnold (1966) on the Euler equations (see in [AK01]), it is believed that the Euler equation for rotating solid body is similar to eq. (1.5). Accordingly, let us consider the damped/driven version of the former:

$$\dot{M} + [M, A^{-1}M] + \nu M = \sqrt{\nu}\, \eta(t)\,, \tag{4.6}$$

where the random force is $\eta(t) = \frac{d}{dt}\sum_{j=1}^{3} b_j \beta_j(t) e_j$ with non-zero b_j's, and $\{e_1, e_2, e_3\}$ is the eigenbasis of the inertia operator A. It is shown in Appendix to [Kuk07b] that the unique stationary measure for (4.6) satisfies natural analogies of properties i)-iii). The methods and results of [FW98] and [KP06] leave no doubts that analogies of the assertions **1-3** hold true for eq. (4.6).

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