QUASI-PERIODIC SOLUTIONS OF NONLINEAR WAVE EQUATIONS WITH A PRESCRIBED POTENTIAL

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Abstract. It is proved that for a prescribed potential V there are many quasi-periodic solutions of nonlinear wave equations $u_{tt} - u_{xx} + V(x)u \pm u^3 + O(|u|^5) = 0$ subject to Dirichlet boundary conditions.

1. Introduction and main results.

In this paper we deal with the existence of the quasi-periodic (or, equivalently, invariant tori) of the nonlinear wave equation

$$u_{tt} = u_{xx} - V(x)u - u^3 \tag{1.1}$$

subject to Dirichlet boundary conditions

$$u(t,0) = 0 = u(t,\pi), \qquad -\infty < t < +\infty, \tag{1.2}$$

where the potential V is in the square-integrable function space $L^2[0,\pi]$.

The existence of solutions, periodic in time, for non-linear wave (NLW) equations has been studied by many authors. See [2, 3, 8] and the references theirin, for example. There are, however, relatively less methods to find a quasi-periodic solutions of NLW. The KAM (Kolmogorov-Arnold-Moser) theory is a very powerful tool in constructing families of quasi-periodic solutions for some nearly integrable Hamiltonian systems of finitely or infinitely many degrees of freedom. Some partial differential equations such as (1.1) may be viewed as an infinitely dimensional Hamiltonian system. On this line Wayne[12] obtained the time-quasi-periodic solutions of (1.1), when the potential V is lying on the outside of the set of some "bad" potentials. In [12] the set of all potentials is given some Gaussian measure and then the set of "bad" potentials is of small measure. At almost the same time, similar result was obtained by Kuksin[5] provided that the potential V depends on an n-dimensional external parameter in "a non-degenerate way". In a word, the works of Wayne[12] and Kuksin[5] tell us that there are many quasi-periodic solutions of (1.1) for "most" potential $V(x)$. It does not, however, follows that there is any quasi-periodic solution of (1.1) for a prescribed potential V.

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Bobenko & Kuksin[1] and Pöschel[10] (in alphabetical order) investigated the case $V(x) \equiv m \in \mathbb{R}$. In order to use an infinitely dimensional version of KAM theorem developed by Kuksin[5] and Pöschel[9], it is necessary to assume that there are some parameters in the Hamiltonian corresponding to (1.1). When $V(x) \equiv m > 0$, these parameters can be extracted from the nonlinear term u^3 by Birkhoff normal form[10], or by regarding (1.1) as a perturbation of sine-Gordon/sinh-Gordon equation[1]. And it was then shown that, for a prescribed potential $V(x) \equiv m > 0$, there are many elliptic invariant tori which are the closure of some quasi-periodic solutions of (1.1). See [10] for the details. By Remark 7 in [10, p.274], the same result holds also true for the parameter values $-1 < m < 0$. When $m \in (-\infty, -1) \setminus \mathbb{Z}$, it is shown in [13] that there are many hyperbolic-elliptic invariant tori for (1.1).

Naturally, we should ask that *whether or not there is any quasi-periodic solution of (1.1) for a prescribed (not random) potential $V(x)$ which is not necessary to be constant , for example, $V = m + \cos x$.*

In this paper, we will answer this question. To give the statement of our results, we need to introduce some notations. We study equation (1.1) as an infinitely dimensional Hamiltonian system. Following Pöschel[10], the phase space one may take, for example, the product of the usual Sobolev spaces $\mathcal{W} = H_0^1([0,\pi]) \times L^2([0,\pi])$ with coordinates u and $v = u_t$. The Hamiltonian is then

$$H = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle Au, u\rangle + \frac{1}{4}u^4$$

where $A = d^2/dx^2 - V(x)$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in L^2. The Hamiltonian equation of motions are

$$u_t = \frac{\partial H}{\partial v} = v, \qquad -v_t = \frac{\partial H}{\partial u} = Au + u^3.$$

Our aim is to construct time-quasi-periodic solutions of small amplitude. Such quasi-periodic solutions can be written in the form

$$u(t,x) = U(\omega_1 t, \cdots, \omega_n t, x),$$

where $\omega_1, \cdots, \omega_n$ are rationally independent real numbers which are called the basic frequency of u, and U is an analytic function of period 2π in the first n arguments. Thus, u admits a Fourier series expansion

$$u(t,x) = \sum_{k\in\mathbb{Z}^n} e^{\sqrt{-1}\langle k,\omega\rangle t} U_k(x),$$

where $\langle k, \omega\rangle = \sum_j k_j\omega_j$ and $U_k \in L^2[0,\pi]$ with $U_k(0) = U_k(\pi)$. Since the quasi-periodic solutions to be constructed are of small amplitude, (1.1) may be considered as the linear equation $u_{tt} = u_{xx} - V(x)u$ with a small nonlinear perturbation u^3. Let $\phi_j(x)$ and λ_j $(j = 1, 2, ...)$ be the eigenfunctions and eigenvalues of the Sturm-Liouville problem $-Ay = \lambda y$ subject to Dirichlet boundary conditions $y(0) = y(\pi) = 0$, respectively. Then every solution of the linear system is the superposition of their harmonic oscillations and of the form

$$u(t,x) = \sum_{j\geq 1} q_j(t)\phi_j(x), \qquad q_j(t) = y_j \cos(\sqrt{\lambda_j}t + \phi_j^0)$$

with amplitude $y_j \geq 0$ and initial phase ϕ_j^0. The solution $u(t,x)$ is periodic, quasi-periodic or almost periodic depending on whether one, finitely many or infinitely many modes are excited, respectively. In particular, for the choice

$$N_d = \{j_1, j_2, \cdots, j_d\} \subset \mathbb{N},,$$

of finitely many modes there is an invariant $2d$-dimensional linear subspace E_{N_d} that is completely foliated into rational tori with frequencies $\lambda_{j_1}, \cdots, \lambda_{j_d}$:

$$\begin{aligned} E_{N_d} &= \{(u,v) = (q_{j_1}\phi_{j_1} + \cdots + q_{j_d}\phi_{j_d}, \dot{q}_{j_1}\phi_{j_1} + \cdots + \dot{q}_{j_d}\phi_{j_d})\} \\ &= \bigcup_{y \in \bar{\mathbb{P}}^d} \mathcal{T}_j(y), \end{aligned}$$

where $\mathbb{P}^d = \{y \in \mathbb{R}^d : y_j > 0 \quad \text{for} \quad 1 \leq j \leq d\}$ is the positive quadrant in $\mathbb{R}^d$ and

$$\mathcal{T}_{N_d}(y) = \{(u,v) : q_{j_k}^2 + \lambda_{j_k}^{-2}\dot{q}_{j_k}^2 = y_k, \quad \text{for} \quad 1 \leq k \leq d\}.$$

Upon restoring the nonlinearity u^3 the invariant manifold E_{N_d} with their quasi-periodic solutions will not persist in their entirety due to resonance among the modes and the strong perturbing effect of u^3 for large amplitudes. In a sufficiently small neighborhood of the origin, however, there does persist a large Cantor subfamily of rotational d-tori which are only slightly deformed. More exactly, we have the following theorem:

Theorem 1.1. *Assume that $V(x)$ is sufficiently smooth in the interval $[0,\pi]$, and $\int_0^\pi V(x)\, dx \neq 0$. Let K and N be positive constants large enough. Let $N_d = \{i_p \in \mathbb{N} : p = 1, 2, \cdots, d\}$ with*

$$\min N_d > NK, \max N_d \leq C_0 dNK, \ \text{and}\ \mathcal{K}_1 \leq |i_p - i_q| \leq \mathcal{K}_2, \quad \text{for} \quad p \neq q,$$

where $C_0 > 1$ is an absolute constant and $\mathcal{K}_1, \mathcal{K}_2$, positive constants large enough, depending on K instead of N.[1] *Then, for given compact set $\mathcal{C}^*$ in $\mathbb{P}^d$ with positive Lebesgue measure, there is a set $\mathcal{C} \subset \mathcal{C}^*$ with meas $\mathcal{C} > 0$, a family of d-tori*

$$\mathcal{T}_{N_d}(\mathcal{C}) = \bigcup_{y \in \mathcal{C}} \mathcal{T}_{N_d}(y) \subset E_{N_d}$$

over $\mathcal{C}$, and a Lipschitz continuous embedding

$$\Phi : \mathcal{T}_{N_d}[\mathcal{C}] \hookrightarrow H_0^1([0,\pi]) \times L^2([0,\pi]) = \mathcal{W},$$

which is a higher order perturbation of the inclusion map $\Phi_0 : E_{N_d} \hookrightarrow \mathcal{W}$ restricted to $\mathcal{T}_{N_d}[\mathcal{C}]$, such that the restriction of Φ to each $\mathcal{T}_{N_d}(y)$ in the family is an embedding of a rotational invariant d-torus for the nonlinear equation (1.1).

Remarks 1. Since Φ is a higher order perturbation of the inclusion map Φ_0, using Theorem 5.1 below we find that the obtained quasi-periodic solution of (1.1) reads as

$$u(t,x) = \sum_{j \in N_d} \sqrt{\frac{\xi_j}{\lambda_j}} \cos(\tilde{\omega}_j t) \cdot \phi_j(x) + O(\epsilon^2), \qquad \text{for any } \xi = (\xi_j)_{j \in N_d} \in \mathcal{C},$$

[1]Therefore, we still have freedom to let N large enough after fixing $\mathcal{K}_1$ and $\mathcal{K}_2$.

where

$$\tilde{\omega}_j = \sqrt{\lambda_j} + \epsilon^{5/4} \sum_{k_l \in N_d} A_{k_j k_l} \xi_l + O(\epsilon^2), \quad j, k_j \in N_d,$$

$$A_{mn} = \frac{1}{2\sqrt{\lambda_m \lambda_n}} \int_0^\pi \phi_m^2(x) \phi_n^2(x)\ dx.$$

And it follows from Lemma 1 of [10, p.278] that the solution $u(t,x)$ is a classical smooth solution.

2. The assumption $\int_0^\pi V(x)\ dx \neq 0$ is not essential. Using Titchmarsh's method[11] we can write

$$\sqrt{\lambda_j} = j + \frac{c_1}{j} + \frac{c_2}{j^2} + \cdots + \frac{c_n}{j^n} + O\left(\frac{1}{j^{n+1}}\right),$$

where c_j's are some constants depending on V, in particular, $c_1 = -\frac{1}{2\pi} \int_0^\pi V(x)\ dx$. Then the assumption $\int_0^\pi V(x)\ dx \neq 0$ is equivalent to $c_1 \neq 0$. The assumption $c_1 \neq 0$ is used just only in the proof Lemma 2.3. By overcoming more technical trouble we can show that Lemma 2.3 still holds true under conditions $c_1 = 0, \cdots, c_{k-1} = 0$ and $c_k \neq 0$ for some $1 \leq k \leq n$. Therefore the assumption $\int_0^\pi V(x)\ dx \neq 0$ can be nearly replaced by $V(x) \neq 0$ in the Theorem 1.1. If $V(x) \equiv 0$, it is well-known open problem that whether or not there is any quasi-periodic solution of (1.1). Recently the problem has been answered positively in [14].

3. Theorem 1.1 still holds true for the following equation

$$u_{tt} = u_{xx} - V(x)u \pm u^3 + \sum_{m \geq k \geq 2} a_k u^{2k+1}$$

where $m \geq$ is a positive integer and a_k's are some real numbers.

4. The method proving Theorem 1.1 can be applied to NLS equation:

$$\sqrt{-1} u_t - u_{xx} + V(x)u \pm u^3 = 0$$

subject to Dirichlet boundary conditions.

5. If $\lambda_1 > 0$, then the obtained invariant tori are elliptic. If $\lambda_1 < 0$, then the tori are hyperbolic-elliptic. When $\lambda_1 > 0$, one can use the KAM theorem by Pöschel[9] to prove Theorem 1.1. When $\lambda_1 < 0$, one can use a variant version of the KAM theorem by Pöschel. See [13] for the variant. For convenience we assume $\lambda_1 > 0$ in the following argument.

6. We can give the measure estimate of the set $\mathcal{C}$:

$$\text{meas } \mathcal{C} \geq \text{ meas } \mathcal{C}^* \cdot (1 - O(\epsilon^{1/13})).$$

2. Sturm-Liouville problems.

Consider the Sturm-Liouville (S-L) problems

$$\begin{cases} l(y) := -\frac{d^2 y}{dx^2} + V(x)y = \lambda y, \\ y(0) = y(\pi) = 0. \end{cases} \tag{2.0}$$

It is well known that the S-L problems possess infinite many simple eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to +\infty. \tag{2.1}$$

Denotes by ϕ_n the normalized eigenfunctions corresponding to λ_n. Expand the eigenfunctions $\phi_n(x)$ into odd 2π-periodic functions, and hence their Fourier series expansions are of the form

$$\phi_n(x) = \sum_{m=1}^{\infty} \hat{\phi}_n(m) \sin(mx).$$

It is well known that the functions ϕ_n's are a normalized orthogonal basis of the space consisting of the square integrable and odd functions in $[-\pi, \pi]$.

Lemma 2.2. *For the eigenvalues λ_n's and eigenfunctions ϕ_n's we have the following asymptotic formulae*

$$\mu_n := \sqrt{\lambda_n} = n + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + O(1/n^4) \tag{2.2}$$

and

$$\phi_n(x) = \kappa_n^{-1} \left(\sin nx - \frac{\cos nx}{2n} \int_0^x V(s)ds + \tilde{\phi}_n(x) \right), \tag{2.3}$$

where $\kappa_n > 0$ is a constant depending on n such that $||\phi_n||_{L^2} = 1$, and

$$\tilde{\phi}_n(x) = O\left(\frac{1}{n^2}\right), \quad \tilde{\phi}'_n(x) = O\left(\frac{1}{n}\right), \quad \tilde{\phi}''_n(x) = O(1), \quad \prime = \frac{d}{dx} \tag{2.4}$$

uniformly for $x \in [0, 2\pi]$, and

$$c_1 = -\frac{1}{2\pi} \int_0^{\pi} V(x)dx.$$

Proof. The proof can be found in [11] and many text books. □

Claim. We claim that

$$\kappa_n^2 = \pi + O(1/n^2). \tag{2.5}$$

In fact, by (2.3) and (2.4), we get

$$\begin{aligned}
\kappa_n^2 &= \kappa_n^2 \int_{-\pi}^{\pi} \phi_n^2(x)\ dx = \int_{-\pi}^{\pi} (\sin nx + n^{-1} \cos nx \mathcal{V}(x))^2\ dx + O(n^{-2}) \\
&= \int_{-\pi}^{\pi} \sin^2 nx\ dx + n^{-1} \int_{-\pi}^{\pi} \mathcal{V}(x) \sin 2nx\ dx + O(n^{-2}) \\
&= \int_{-\pi}^{\pi} \sin^2 nx\ dx - (\mathcal{V}(\pi) - \mathcal{V}(-\pi))/2n^2 + \frac{1}{2n^2} \int_{-\pi}^{\pi} \mathcal{V}'(x) \cos 2nx\ dx \\
&= \pi + O(n^{-2}).
\end{aligned}$$

Assumption. *We assume $c_1 \neq 0$ in this paper.*

Let K and N be two sufficiently large positive integers which will be specified later.

Lemma 2.3. *For any $i, j, n, l \in \mathbb{N}$, Assume $K < i \leq j \leq n \leq l$. Then for μ_i's we have*

$$|\pm\mu_i \pm \mu_j \pm \mu_n \pm \mu_l| > \frac{1}{(i+2)^3},$$

unless $\{\pm i, \pm j, \pm n, \pm l\} \subset \{p, -p, q, -q : p, q \in \mathbb{Z}\}$.

Proof. Recall we have assumed $1 \ll K < i \leq j \leq n \leq l$. If there are three plus and one minus among $\Upsilon := \pm\mu_i \pm \mu_j \pm \mu_n \pm \mu_l$, then, without loss of generality, we can write

$$\Upsilon = i + j + n - l + c_1\left(\frac{1}{i} + \frac{1}{j} + \frac{1}{n} - \frac{1}{l}\right) + O(i^{-2}) + \cdots + O(l^{-2}).$$

If $i + j + n - l \neq 0$, then $|\Upsilon| \geq 1 - 4c_1K^{-1} + O(K^{-2}) > 1/2$ when $K \gg 1$. And if $i + j + n - l = 0$, then

$$\begin{aligned}
|\Upsilon| &= c_1\left(\frac{1}{i} + \frac{1}{j} + \frac{1}{n} - \frac{1}{l}\right) - |O(i^{-2})| \\
&\geq c_1\left(\frac{1}{i} + \frac{1}{j} + \frac{1}{n} - \frac{1}{i+j+n}\right) - |O(i^{-2})| \\
&\geq \frac{c_1}{i} - |O(i^{-2})| \\
&> \frac{1}{(i+2)^3}.
\end{aligned}$$

Now let us assume there are two plus and two minus among Υ, say $\Upsilon = \mu_i + \mu_j - \mu_n - \mu_l$. If $i + j - n - l \neq 0$, then, obviously, $|\Upsilon| \geq 1 - |O(i^{-1})| > 1/2$. Now we assume $i + j - n - l = 0$. Then, we can write

$$\begin{aligned}
\Upsilon =& c_1\left(\frac{1}{i} + \frac{1}{j} - \frac{1}{n} - \frac{1}{l}\right) + c_2\left(\frac{1}{i^2} + \frac{1}{j^2} - \frac{1}{n^2} - \frac{1}{l^2}\right) \\
&+ c_3\left(\frac{1}{i^3} + \frac{1}{j^3} - \frac{1}{n^3} - \frac{1}{l^3}\right) + O(i^{-4}) + \cdots + O(l^{-4}).
\end{aligned}$$

We assume $c_1 = 1$, otherwise we consider $c^{-1}\Upsilon$. Since $i + j = n + l$, we can set $n - i = j - l := p$. Then $p \geq 1$. Thus,

$$\begin{aligned}
\Upsilon =& \frac{1}{i} - \frac{1}{l} - \left(\frac{1}{i+p} - \frac{1}{l+p}\right) + c_2\left(\frac{1}{i^2} - \frac{1}{l^2} - \left(\frac{1}{(i+p)^2} - \frac{1}{(l+p)^2}\right)\right) \\
&+ c_3\left(\frac{1}{i^3} - \frac{1}{l^3} - \left(\frac{1}{(i+p)^3} - \frac{1}{(l+p)^3}\right)\right) + O(i^{-4}) + \cdots + O(l^{-4}).
\end{aligned}$$

Let

$$f(t) := \frac{1}{i+t} - \frac{1}{l+t} + c_2\left(\frac{1}{(i+t)^2} - \frac{1}{(l+t)^2}\right) + c_3\left(\frac{1}{(i+t)^3} - \frac{1}{(l+t)^3}\right).$$

It is easy to verify that for $t \geq 0$

$$-\frac{\partial_t f(t)}{l-i} = \frac{i+l+2t}{(i+t)^2(l+t)^2}(1 + O((i+t)^{-1}) > 0.$$

Thus $\partial_t f(t) < 0$. This implies that

$$f(0) - f(p) \geq f(0) - f(1).$$

Write $l = i + q$ with $q \geq 1$. Let

$$g(t) = \frac{1}{i+t} + \frac{c_2}{(i+t)^2} + \frac{c_3}{(i+t)^3} - \frac{1}{i+t+1} - \frac{c_2}{(i+t+1)^2} - \frac{c_3}{(i+t+1)^3}.$$

Then $f(0) - f(1) = g(0) - g(q)$. It is easy to check that $\partial_t g(t) < 0$ for $t > 0$, and thus $f(0) - f(1) \geq g(0) - g(1)$. A simple calculation shows that

$$g(0) - g(1) = \frac{2}{i(i+1)(i+2)} + O(i^{-4}).$$

Then

$$\Upsilon = f(0) - f(p) + O(i^{-4}) \geq \frac{2}{i(i+1)(i+2)} + O(i^{-4}) > \frac{1}{(i+2)^3}.$$

So the proof of this lemma is complete. □

Now let us pick d positive integers. Let

$$N_d = \{i_p \in \mathbb{N} : p = 1, 2, \cdots, d\}.$$

Assume

$$\min N_d > NK, \max N_d \leq C_0 dNK, \text{ and } \mathcal{K}_1 \leq |i_p - i_q| \leq \mathcal{K}_2, \quad \text{for} \quad p \neq q, \tag{2.6}$$

where $C_0 > 1$ is an absolute constant and $\mathcal{K}_1, \mathcal{K}_2$, large enough, positive constants depending on K.

Lemma 2.4. *For any set $S = \{i, j, n, l\} \subset \mathbb{N}$, assume $S \cap N_d$ possesses at least two elements. Then there is a positive constant C depending on $\max N_d$, N and K such that*

$$|\Upsilon| := |\mu_i \pm \mu_j \pm \mu_n \pm \mu_l| > C, \tag{2.7}$$

unless $\{\mu_i, \mu_j, \mu_n, \mu_l\} \subset \{p, -p, q, -q : p, q \in \mathbb{R}\}$.

Proof. Assume $i \leq j \leq n \leq l$ without loss of generality.

Case 1. $i > K$. In this case, all of i, j, n and l are larger than K. Since $S \cap N_d \neq \emptyset$, we have $\min N_d \geq i$. By Lemma 2.3, we have

$$|\Upsilon| > \frac{1}{(1+i)^3} \geq \frac{1}{(1 + \min N_d)^3}.$$

Case 2. $i \leq K$ and $j \leq K$. In this case, $n, l \in N_d$. According to the construction N_d, we have that

$$|\mu_i \pm \mu_j \pm \mu_n \pm \mu_l| \geq \mathcal{K}_1 - C_K \geq 1, \quad \text{if } \mathcal{K}_1 \text{ large enough},$$

where $C_K > 0$ is a constant depending on K. (The constant C_K may take different values in different places in the following argument.)

Case 3. $i \le K$ *and* $K < j \le \mathcal{K}_1/2$. Then

$$|\mu_i \pm \mu_j \pm \mu_n \pm \mu_l| \ge \mathcal{K}_1/2 - C_K \ge 1 \quad \text{if } \mathcal{K}_1 \gg 1.$$

Case 4. $i \le K$ *and* $j > \mathcal{K}_1/2$. Since $S \cap N_d$ possesses at least two elements, we get that $j < n \le C_0 dKN$ in view of $\max N_d \le C_0 dKN$. Observe that

$$c_4 := \inf_{i \le K; j,n,l \in \mathbb{N}} \{|\mu_i \pm j \pm n \pm l| : \mu_i \pm j \pm n \pm l \ne 0\} > 0$$

which depends on K only, independent of N. Thus, if $\mu_i \pm j \pm n \pm l \ne 0$, then

$$\begin{aligned} |\Upsilon| &\ge |\mu_i \pm j \pm n \pm l| - O(\mathcal{K}_1^{-1}) \\ &\ge c_4 - O(\mathcal{K}_1^{-1}) > c_4/2 \end{aligned}$$

if $\mathcal{K}_1 \gg 1$. Now we are in position to consider $\mu_i \pm j \pm n \pm l = 0$. For convenience we assume $c_1 = 1$ without loss of generality. In this case, we have

$$|\Upsilon| = \left| \pm \left(\frac{1}{j} + O(j^{-2}) \right) \pm \left(\frac{1}{n} + O(n^{-2}) \right) \pm \left(\frac{1}{l} + O(l^{-2}) \right) \right|.$$

For cases $(+,+,+)$, $(-,-,-)$, $(+,+,-)$, clearly we have

$$|\Upsilon| \ge \frac{1}{j} + O(j^{-2}) \ge \frac{1}{2C_0 dKN}.$$

For case $(+,-,+)$, we get that $j < n$. (Otherwise, if $n = j$, we get that $-\mu_i = l < 0$ by $\mu_i + j - n + l = 0$. It contradicts to $l > 0$.)

$$\begin{aligned} |\Upsilon| &\ge \left(\frac{1}{j} + O(j^{-2}) \right) - \left(\frac{1}{n} + O(n^{-2}) \right) \\ &= \frac{n(1 + O(j^{-1})) - j(1 + O(n^{-1}))}{nj} \\ &\ge \frac{1}{2nj} \ge \frac{1}{2(C_0 dKN)^2}. \end{aligned}$$

Finally, let us consider case $(+,-,-)$. In this case we still get that $j < n$. Otherwise, if $n = j$, we get that $\mu_i = l$ by $\mu_i + j - n - l = 0$. It contradicts to $l \ge n \ge j > \mathcal{K}_1/2 > \max\{\mu_i : i \le K\}$ if $\mathcal{K}_1 \gg 1$. Since $\mathcal{K}_1/2 < j < n \le l$ and $i \le K$, we write $O(j^{-1}), O(n^{-1}), O(l^{-1}) = o(1)$ and $\mu_i/l = o(1)$ for $\mathcal{K}_1 \gg 1$. By $\mu_i + j - n - l = 0$, we get

$$\frac{n}{l} = \frac{\mu_i + j}{l} - 1.$$

We have now that

$$\begin{aligned} |\Upsilon| &= \left| \frac{1 + o(1)}{j} - \frac{1 + o(1)}{n} - \frac{1 + o(1)}{l} \right| \\ &= \left| \frac{1 + o(1)}{j} - \frac{1}{n} \left(1 + o(1) + \frac{(1 + o(1))n}{l} \right) \right| \\ &= \left| \frac{1 + o(1)}{j} - \frac{1}{n} \left(1 + o(1) + (1 + o(1)) \left(\frac{\mu_i + j}{l} - 1 \right) \right) \right| \\ &= \left| \frac{1 + o(1)}{j} - \frac{1}{n} \left(\frac{j}{l} + o(1) \right) \right| \\ &\ge \frac{1 + o(1)}{j} - \frac{1}{n}(1 + o(1)) \\ &\ge \frac{1}{2nj} \ge \frac{1}{2(C_0 dKN)^2}. \end{aligned}$$

This completes the proof. □

3. Hamiltonian for NLW.

From now on we focus our attention on the nonlinearity u^3, since terms of order five or more will not make any difference. Letting

$$u = \sum_n q_n(t)\phi_n(x) \tag{3.1}$$

and inserting it into (1.1) we get

$$\frac{d^2 q_n}{dt^2} + \mu_n^2 q_n + \epsilon\langle \tilde{u}^3, \phi_n\rangle_{L^2} = 0, \quad n = 1, 2, \cdots \tag{3.2}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ is the usual inner product of $L^2[0, 2\pi]$. Let

$$\tilde{q}_n = \mu_n^{1/2} q_n, \quad \tilde{p}_n = \mu_n^{-1/2}\frac{dq_n}{dt}, \quad n \geq 1. \tag{3.3}$$

Then we get a Hamiltonian system

$$\dot{\tilde{p}}_n = -\frac{\partial H}{\partial \tilde{q}_n}, \quad \dot{\tilde{q}}_n = \frac{\partial H}{\partial \tilde{p}_n}, \quad n = 1, 2, ... \tag{3.4}$$

where

$$H(\tilde{p}, \tilde{q}) = \frac{1}{2}\sum_{n\geq 1}\mu_n(\tilde{p}_n^2 + \tilde{q}_n^2) + \epsilon G(\tilde{q}) \tag{3.5}$$

with

$$G(\tilde{q}) = \sum_{i,j,n,l} G_{ijkl}\tilde{q}_i\tilde{q}_j\tilde{q}_n\tilde{q}_l \tag{3.6}$$

$$G_{ijnl} = \frac{1}{4}(\mu_i\mu_j\mu_k\mu_l)^{-1/2}\int_{-\pi}^{\pi}\phi_i\phi_j\phi_n\phi_l dx. \tag{3.7}$$

Let us introduce the complex coordinate change

$$\tilde{q}_n = \frac{z_n + \bar{z}_n}{\sqrt{2}}, \quad \tilde{p}_n = \frac{z_n - \bar{z}_n}{\sqrt{-2}}, \quad n \geq 1,$$

Then

$$\sum_{n\geq 1} d\tilde{q}_n \wedge d\tilde{p}_n = \sum_{n\geq 1}\sqrt{-1}dz_n \wedge d\bar{z}_n.$$

Thus the Hamiltonian (3.5) is changed into

$$H(z, \bar{z}) = \sum_{n\geq 1}\mu_n\ z_n\bar{z}_n + \epsilon G(z, \bar{z}) \tag{3.8}$$

where

$$G(z, \bar{z}) = \frac{1}{4}\sum_{i,j,n,l} G_{ijnl}(z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_n + \bar{z}_n)(z_l + \bar{z}_l), \tag{3.9}$$

where $z = (z_j)_{j\in\mathbb{N}} \in \ell^{a,s}$.

By an argument similar to that of Lemma 3.1 in [4, p.506], we can get the regularity of the vector fields of G:

Lemma 3.1. *For $a \geq 0$ and $s > 0$, the vector field X_G is analytic as a map from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$, with*

$$||X_G||_{a,s+1} = O(||z||^3_{a,s}).$$

Remark. If $V(x)$ is analytic in the strip domain $|\Im x| \leq \sigma$ with $\sigma > 0$, then $a = \sigma$. If $V(x)$ is in the Sobolev space $H^s(0,\pi)$ with $s > 0$, then $a = 0$.

Following Pöschel's[10,p.282] notation, we introduce another set of coordinates $(..., w_2, w_1, w_1, w_2, ...)$ in $\ell_b^{a,s}$ by setting $z_j = w_j$, $\bar{z}_j = w_j$ where $\ell_b^{a,s}$ consists of all bi-infinite sequence with finite norm

$$||w||^2_{a,s} = \sum_{|j|=1}^{\infty} |w_j|^2 |j|^{2s} e^{2a|j|}.$$

Set $G_{ijnl} = G_{|i||j||n||l|}$.Then

$$H(w) = \sum_{n\geq 1} \mu_n w_n w_{-n} + \epsilon G(w) \tag{3.10}$$

where

$$G(w) = \sum_{i,j,n,l} G_{ijnl} w_i w_j w_n w_l, \tag{3.11}$$

Lemma 3.2. *We have that*

$$\kappa_n^2 \kappa_m^2 \int_{-\pi}^{\pi} \phi_n^2(x)\phi_m^2(x)dx = \begin{cases} \frac{\kappa_n^2}{2} + O\left(\frac{1}{m^2}\right) + O\left(\frac{1}{mn|m-n|}\right), & m \neq n \\ \frac{\kappa_n^2}{2} + \frac{\pi}{4} + O\left(\frac{1}{m^2}\right), & m = n. \end{cases}$$

Proof. Set $\mathcal{V}(x) = (-1/2)\int_0^x V(x)\ dx$. By (2.3),

$$\begin{aligned} &\kappa_n^2\kappa_m^2 \int_{-\pi}^{\pi} \phi_n^2(x)\phi_m^2(x)dx = \int_{-\pi}^{\pi} \phi_n^2(x)(\sin mx + m^{-1}\mathcal{V}(x)\cos mx)^2\ dx + O(m^{-2}) \\ &= \frac{\kappa_n^2}{2}\int_{-\pi}^{\pi} \phi_n^2(x)\ dx + \kappa_n^2 \int_{-\pi}^{\pi} \phi_n^2(x)(\frac{1}{m}\mathcal{V}(x)\sin 2mx - \frac{1}{2}\cos 2mx)\ dx + O(m^{-2}) \\ &= \frac{\kappa_n^2}{2} + \kappa_n^2 \int_{-\pi}^{\pi} \phi_n^2(x)(\frac{1}{m}\mathcal{V}(x)\sin 2mx - \frac{1}{2}\cos 2mx)\ dx + O(m^{-2}) \end{aligned} \tag{3.12}$$

where we have used the fact $\int_{-\pi}^{\pi} \phi_n^2(x)\ dx = 1$. Using (2.4) we get

$$\begin{aligned} &\kappa_n^2 \int_{-\pi}^{\pi} \phi_n^2(x)\cos 2mx\ dx \\ &= \int_{-\pi}^{\pi} (\sin nx + \frac{1}{n}\cos nx\mathcal{V}(x)\ \ + \tilde{\phi}_n(x))^2 \cos 2mx\ dx \\ &= \int_{-\pi}^{\pi} (\sin^2 nx + \frac{1}{n}\sin 2nx\mathcal{V}(x) + f_n(x) + g_n(x))\cos 2mx\ dx \end{aligned}$$

where

$$f_n(x) = (\tilde{\phi}_n(x) + 2\sin nx + \frac{2}{n}\cos nx\mathcal{V}(x))\tilde{\phi}_n(x)$$

and

$$g_n(x) = \frac{1}{n^2}\cos^2 nx\mathcal{V}^2(x).$$

Using (2.4) we get that $d^2f_n(x)/dx^2$ is bounded on $-\pi,\pi]$. Clearly, $d^2g_n(x)/dx^2$ is bounded on $-\pi,\pi]$, too. By part integrating we get

$$\int_{-\pi}^{\pi}(f_n(x)+g_n(x))\cos 2mx\;dx = -\frac{1}{4m^2}\int_{-\pi}^{\pi}(f_n''(x)+g_n''(x))\cos 2mx\;dx = O(m^{-2}).$$

Therefore,

$$\kappa_n^2\int_{-\pi}^{\pi}\phi_n^2(x)\cos 2mx\;dx = \int_{-\pi}^{\pi}(\sin^2 nx + \frac{1}{n}\sin 2nx\mathcal{V}(x))\cos 2mx\;dx + O(m^{-2}). \tag{3.13}$$

Similarly, we have

$$\frac{\kappa_n^2}{m}\int_{-\pi}^{\pi}\phi_n^2(x)\mathcal{V}(x)\sin 2mx\;dx = \frac{-1}{2m}\int_{-\pi}^{\pi}\cos 2nx\sin 2mx\mathcal{V}(x)\;dx+O(m^{-2}). \tag{3.14}$$

Observe that

$$\begin{aligned}
&\int_{-\pi}^{\pi}\left(\frac{1}{n}\sin 2nx\cos 2mx + \frac{1}{m}\cos 2nx\sin 2mx\right)\mathcal{V}(x)\;dx\\
&= \int_{-\pi}^{\pi}\left(\frac{1}{2n}+\frac{1}{2m}\right)\mathcal{V}(x)\sin 2(m+n)x\;dx\\
&\quad + \int_{-\pi}^{\pi}\left(\frac{1}{2m}-\frac{1}{2n}\right)\mathcal{V}(x)\sin 2(m-n)x\;dx\\
&= \frac{1}{8mn(m+n)}\int_{-\pi}^{\pi}\mathcal{V}''(x)\sin 2(m+n)x\;dx\\
&\quad - \frac{1}{8mn(m-n)}\int_{-\pi}^{\pi}\mathcal{V}''(x)\sin 2(m-n)x\;dx\\
&= O(m^{-2}) + O(1/mn|m-n|).
\end{aligned} \tag{3.15}$$

Using (3.12-15), we get

$$\begin{aligned}
&\kappa_n^2\kappa_m^2\int_{-\pi}^{\pi}\phi_n^2(x)\phi_m^2(x)dx\\
&= \frac{\kappa_n^2}{2} - \frac{1}{2}\int_{-\pi}^{\pi}\sin^2 nx\cos 2mx\;dx + O(m^{-2}) + O(1/mn|m-n|).
\end{aligned} \tag{3.16}$$

Notice that

$$-\frac{1}{2}\int_{-\pi}^{\pi}\sin^2 nx\cos 2mx\;dx = \begin{cases} 0, & m\neq n,\\ \pi/4, & m=n.\end{cases} \tag{3.17}$$

By (3.16) and (3.17), the proof is complete. $\square$

4. Birkhoff Normal Form.

Let

$$\mu_i' = \sqrt{-1}(\mathrm{sgn} i)\mu_{|i|}.$$

Consider a Hamilton function

$$F = \sum_{i,j,n,l} F_{ijnl} w_i w_j w_n w_l$$

with coefficients

$$F_{ijnl} = \begin{cases} \frac{G_{ijnl}}{\mu_i'+\mu_j'+\mu_n'+\mu_l'}, & \text{for} \quad (\{|i|,|j|,|n|,|l|\}\cap N_d)^\sharp \geq 2 \\ 0, & \text{otherwise,} \end{cases}$$

where $S^\sharp$ denotes the number of the elements of the set S. By Lemma 2.4, the vector fields X_F is analytic as a map from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$ with $||X_F||_{a,s+1} = O(||z||^3_{a,s})$. Let X_F^1 be the time-1 map of the flow of the Hamiltonian vector field ϵF. Then the Hamiltonian (2.15) is changed by X_F^1 into

$$H = H \circ X_F^1 = \sum_{n\geq 1} \tilde{\mu}_n w_n w_{-n} + \epsilon \tilde{G} + R \tag{4.1}$$

where

$$\tilde{G} = \sum_{i\in N_d; j\geq 1} G_{iijj} z_i \bar{z}_i z_j \bar{z}_j, \tag{4.2}$$

$$R = \epsilon O(||\hat{w}||^4_{a,s}) + \epsilon O(||\hat{w}||^3_{a,s}||z||_{a,s}) + \epsilon^2 O(||w||^6_{a,s}) \tag{4.3}$$

and $\hat{w} = (w_j)_{|j|\notin N_d}$. For $j\in N_d$, we introduce the action-angle variables as follows:

$$z_j = \sqrt{I_j} e^{\sqrt{-1}\theta_j}, \quad z_{-j} = \sqrt{I_j} e^{-\sqrt{-1}\theta_j}, \quad j\in N_d. \tag{4.4}$$

Then (3.1) can be written as

$$H = \sum_{n\in N_d} \mu_n I_n + \sum_{n\notin N_d} \mu_n z_n \bar{z}_n + \epsilon\langle AI, I\rangle + \epsilon \sum_{j\notin N_d} (BI)_j z_j \bar{z}_j + R, \tag{4.5}$$

where

$$R = R(I,\theta,\hat{w}) := R((\sqrt{I_j} e^{\pm\sqrt{-1}\theta_j})_{j\in N_d}, \hat{w}) \tag{4.6}$$

$$A = (G_{iijj})_{i,j\in N_d}, \quad B = (G_{jjii})_{i\in N_d; j\notin N_d}. \tag{4.7}$$

Remark that A is a matrix of order $d\times d$, B a matrix of order $\infty\times d$. Now let us introduce the parameter vector ξ and the new action variable ρ as follows

$$I_j = \epsilon^{1/4}\xi_j + \rho_j, \quad j\in N_d, \xi_j\in[1,2], |\rho_j| < \epsilon. \tag{4.8}$$

Then

$$H = \sum_{j\in N_d} \omega_j \rho_j + \sum_{j\notin N_d} \Omega_j z_j \bar{z}_j + R^* \tag{4.9}$$

where we omit a constant which does not affect the dynamics in the Hamiltonian above, and

$$\omega_j = \mu_j + \epsilon^{5/4}(A\xi)_j, \quad j\in N_d \tag{4.10}$$

$$\Omega_j = \mu_j + \epsilon^{5/4}(B\xi)_j, \quad j\notin N_d \tag{4.11}$$

$$R^* = R(\xi+\rho,\theta,w) + \epsilon \sum_{j\notin N_d} (B\rho)_j z_j \bar{z}_j + \epsilon\langle A\rho,\rho\rangle. \tag{4.12}$$

Lemma 4.1. *The matrix A is non-singular, and $||A^{-1}||_{\max} \leq CN^2$.*

Proof. The proof is from Lemma 3.2.

Lemma 4.2. *Denote by ω^* the d vector $(\omega_j)_{j\in N_d}$ and Ω the vector of infinite dimension $(\Omega_j)_{j\notin N_d}$. Then*

$$meas\{\xi \in [1,2]^d : \langle k, \omega^*(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\} = 0 \tag{4.13}$$

and

$$\langle l, \Omega(\xi)\rangle \neq 0 \quad on \quad [1,2]^d, \tag{4.14}$$

for all integer vectors $(k,l) \in \mathbb{Z}^d \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$ and where "meas " $\equiv$ Lebesgue measure for sets, $|l| = \sum_j |l_j|$ for integer vectors, and $\langle \cdot, \cdot\rangle$ is usual real (or complex) scalar product.

Proof. Recall that all eigenvalues λ_n's of the Sturm-Liouville problem (2.1) are simple. It follows that (4.14) holds true. Let $\alpha = (\mu_j)_{j\in N_d}$ and $\beta = (\mu_j)_{j\notin N_d}$. In order to show (4.13) we have to show that $\langle \alpha, k\rangle \neq \langle \beta, l\rangle$ or $Ak \neq B^T l$ where T means the transpose of the matrix. Suppose to the contrary that $\langle \alpha, k\rangle = \langle \beta, l\rangle$ and $Ak = B^T l$.

Case 1. Assume $|l| = 1$. In view of (2.5) we have $\kappa_n^2 = \pi + O(1/n)$. When $n \gg 1$, we can write $O(1/n) = o(1)$. By Lemma 3.2, we can write $\mu_i\mu_j A_{ij} = 1/2\pi + a_{ij}$ for $i \neq j$ and $\mu_i^2 A_{ii} = 3/4\pi + a_{ii}$, and $a_{ij} = o(1)$ for $i, j \in N_d$. By Lemma 3.2 again, we can write $\mu_i\mu_j B_{ij} = 1/\pi + b_{ij}$ for $i \neq j$ and $b_{ij} = o(1)$ for $i \in N_d$ and $j \notin N_d$. Let $v = (\mu_j^{-1})_{j\in N_d}$, $w = (\mu_j^{-1})_{j\notin N_d}$ and $k = (k_1, ..., k_p, ...k_d)$. Multiplying both of sides of $Ak = B^T l$ by 2π, we then have

$$\frac{1}{\mu_{i_p}}\langle k, v\rangle + \frac{1}{2\mu_{i_p}^2}k_p + \sum_{q=1}^{d}\frac{a_{i_p i_q}}{\mu_{i_p}\mu_{i_q}}k_q = \frac{1}{\mu_{i_p}}\langle l, w\rangle + \frac{1}{\mu_{i_p}}\sum_{j\notin N_d}\frac{b_{i_p j}}{\mu_j}l_j, \tag{4.15}$$

where $l = (l_j)_{j\notin N_d}$. Multiplying both of sides of the equality above by μ_{i_p} and making sum from $p = 1$ to d, we get

$$(d + 1/2)\langle k, v\rangle = d\langle l, w\rangle + \sum_{1\leq p\leq d; j\notin N_d}\frac{b_{i_p j}}{\mu_j}l_j - \sum_{1\leq p,q\leq d} a_{i_p i_q}\frac{k_q}{\mu_{i_q}} \tag{4.16}$$

Recall that we have assumed $\min N_d \geq NK$ and $\max N_d \leq C_0 dNK$. It follows that $||A^{-1}||_{\max} \leq C(KN)^2$ and $\mu_i = O(KN)$ for $i \in N_d$. Then

$$||k||_{\max} = ||A^{-1}B^T l||_{\max} \leq C||A^{-1}||_{\max}||B^T l||_{\max} \leq CNK|\langle l, w\rangle|,$$

where C is a constant depending on d only. Since $\mu_{i_q} \geq NK$ for $i_q \in N_d$, we get $k_q/\mu_{i_q} = O(1)\langle w, l\rangle$. Note that $a_{i_p i_q} = o(1)$. We then get

$$a_{i_p i_q}\frac{k_q}{\mu_{i_q}} = o(1)\langle w, l\rangle. \tag{4.17}$$

By $|l| = 1$ and $b_{ij} = o(1)$, we have

$$\sum_{j\notin N_d}\frac{b_{i_p j}}{\mu_j}l_j = b_{i_p j_0}l_{j_0}\mu_{j_0}^{-1} = \langle w, l\rangle o(1). \tag{4.18}$$

Therefore, by (4.16-18) we get

$$\langle k,v\rangle = s_0\langle w,l\rangle(1+o(1)), \tag{4.19}$$

with $s_0 = d/(d+1/2)$. In view of (4.15) and (4.19), we get

$$k_p = \mu_{i_p} s_1\langle w,l\rangle(1+o(1)) \tag{4.20}$$

where $s_1 = 2-2s_0 = \frac{2}{2d+1}$. Multiplying the both sides of (4.20) by μ_{i_p} and recalling that $\langle \alpha,k\rangle = \langle \beta,l\rangle$, we get

$$\langle \beta,l\rangle = \sum_{p=1}^{d}\mu_{i_p}^2 s_1\langle w,l\rangle(1+o(1)).$$

Observe that $\langle \beta,l\rangle = \langle w,l\rangle^{-1}$ for $|l|=1$. We get

$$\langle w,l\rangle^{-2} = (\sum_{p=1}^{d}\mu_{i_p}^2)s_1(1+o(1)). \tag{4.21}$$

By (4.20,21) and in view of $0<s_1<1$, we get

$$k_{i_p}^2 = \frac{\mu_{i_p}^2}{\sum_{j\in N_d}\mu_j^2}s_1(1+o(1))<1,\quad p=1,...,d.$$

This is absurd.

Case 2. Assume $|l|=2$. In this case, we can write $l=(\cdots,l_j,\cdots,l_{j_0},\cdots,l_{j_0'}\cdots)$ where $l_j=0$ for $j\notin\{j_0,j_0'\}$, and $l_{j_0}=\pm1$, $l_{j_0'}=\pm1$, and $j_0\le j_0'$. For convenience we set $\tilde\mu_i := \kappa_i^2\mu_i$. By Lemma 3.2, we can write $\tilde\mu_i\tilde\mu_j A_{ij} = \kappa_j^2/2 + a_{ij}/2$ for $i\ne j$ and $\tilde\mu_i^2 A_{ii} = \kappa_i^2/2+\pi/4+a_{ii}/2$, and $a_{ij}=O(N^{-2})$ for $i,j\in N_d$. By B_{ij} denote the matrix elements of B^T. Note that for B_{ij} we have $i\in N_d$ and $j\notin N_d$. By Lemma 3.2 again, we can write $\tilde\mu_i\tilde\mu_j B_{ij} = \kappa_j^2/2+b_{ij}/2$ and

$$b_{ij} = O\left(\frac{1}{i^2}\right)+O\left(\frac{1}{ij|i-j|}\right). \tag{4.22}$$

Let $v=(\mu_j^{-1})_{j\in N_d}$, $w=(\mu_j^{-1})_{j\notin N_d}$ and $k=(k_1,...,k_p,...k_d)$. Multiplying both of sides of $Ak=B^Tl$ by 2, we then have

$$\frac{1}{\tilde\mu_{i_p}}\langle k,v\rangle + \frac{1}{2\tilde\mu_{i_p}\mu_{i_p}}k_p + \sum_{q=1}^{d}\frac{a_{i_pi_q}}{\tilde\mu_{i_p}\tilde\mu_{i_q}}k_q = \frac{1}{\tilde\mu_{i_p}}\langle l,w\rangle + \frac{1}{\tilde\mu_{i_p}}\sum_{j\notin N_d}\frac{b_{i_pj}}{\tilde\mu_j}l_j + \frac{\kappa_{i_p}^2-\pi}{2\tilde\mu_{i_p}\kappa_{i_p}^2}\cdot\frac{k_p}{\mu_{i_p}}. \tag{4.23}$$

Multiplying both of sides of the equality above by $\tilde\mu_{i_p}$ and making sum from $p=1$ to d, we get

$$\begin{aligned}(d+1/2)\langle k,v\rangle =& d\langle l,w\rangle + \sum_{p=1}^{d}\left(\frac{b_{i_pj_0}}{\tilde\mu_{j_0}}l_{j_0} + \frac{b_{i_pj_0'}}{\tilde\mu_{j_0'}}l_{j_0'}\right) - \sum_{1\le p,q\le d}a_{i_pi_q}\frac{k_q}{\tilde\mu_{i_q}}\\ &+\sum_{p=1}^{d}\frac{\kappa_{i_p}^2-\pi}{2\kappa_{i_p}^2}\cdot\frac{k_p}{\mu_{i_p}}.\end{aligned} \tag{4.24}$$

Recall that we have assumed $\min N_d \geq NK$ and $\max N_d \leq C_0 dNK$. In the following argument we assume that K is fixed and $N \gg 1$.
Sub-case 2.1. Suppose $j_0 \geq N^{2/3}$. Then $\tilde{\mu}_{j_0}, \tilde{\mu}_{j_0'} \geq N^{2/3}$. Since $i_p \in N_d$, we get $i_p \geq KN$, and thus we get $\tilde{\mu}_{i_p} \geq KN/2$. Then

$$||k||_{\max} = ||A^{-1}B^T l||_{\max} \leq C||A^{-1}||_{\max}||B^T l||_{\max} \leq CN^{1/3},$$

where C is a constant depending on d and K only. Since $\tilde{\mu}_{i_q} \geq NK$ for $i_q \in N_d$, we get $k_q/\tilde{\mu}_{i_q} = O(N^{-2/3})$. Recall that $a_{i_p i_q} = O(N^{-2})$. By (2.5), we have $(\kappa_{i_p}^2 - \pi)/(2\kappa_{i_p}^2) = O(N^{-2})$. We then get

$$\frac{\kappa_{i_p}^2 - \pi}{2\kappa_{i_p}^2} \cdot \frac{k_p}{\mu_{i_p}} = O(N^{-8/3}), \quad a_{i_p i_q} \frac{k_q}{\tilde{\mu}_{i_q}} = O(N^{-8/3}). \tag{4.25}$$

By (4.22) we have $b_{i_p j_0}, b_{i_p j_0'} = O(N^{-5/3})$. Moreover,

$$b_{i_p j_0}/\tilde{\mu}_{j_0}, b_{i_p j_0'}/\tilde{\mu}_{j_0'} = O(N^{-8/3}). \tag{4.26}$$

By (4.24,25,26) we get

$$(d + 1/2)\langle k, v\rangle = d\langle l, w\rangle + O(N^{-8/3}). \tag{4.27}$$

That is,

$$\langle k, v\rangle = s_0\langle l, w\rangle + O(N^{-8/3}). \tag{4.28}$$

By (4.23) and (4.28) we get

$$k_p = s_1\langle l, w\rangle\mu_{i_p} + O(N^{-8/3})\mu_{i_p}.$$

Note $\tilde{\mu}_{i_p} = O(N)$. Thus,

$$k_p = s_1\langle l, w\rangle\mu_{i_p} + O(N^{-5/3}). \tag{4.29}$$

It follows that

$$k_{p+1} - k_p = (\mu_{i_{p+1}} - \mu_{i_p})s_1\langle l, w\rangle + O(N^{-5/3}). \tag{4.30}$$

Recall that we have chosen N_d with restriction $|i_{p+1} - i_p| \leq \mathcal{K}_2$. Thus, $\mathcal{K}_1/2 \leq |\mu_{i_{p+1}} - \mu_{i_p}| \leq 2\mathcal{K}_2$. Besides, there is a absolute constant $\mathcal{K}_3$ depending on the potential function $V(x)$, such that $|\langle l, w\rangle| \leq \mathcal{K}_3$ for $|l| \leq 2$. Thus, if there is a $\bar{p} \in \{1, \cdots, d-1\}$ such that $k_{\bar{p}+1} \neq k_{\bar{p}}$, then we have

$$1 \leq |k_{\bar{p}+1} - k_{\bar{p}}| \leq 2s_1\mathcal{K}_2\mathcal{K}_3 = \frac{4\mathcal{K}_2\mathcal{K}_3}{2d+1}.$$

This is impossible if $d \geq 2\mathcal{K}_2\mathcal{K}_3$. If $k_{p+1} = k_p$ for any $p \in \{1, 2, \cdots, d-1\}$, then by (4.30) we have that $\langle l, w\rangle = O(N^{-5/3})$. In view of (4.29), we get $|k_p| = O(N^{-2/3})$ for any $p = 1, ..., d$. Thus, $|k_p|$ is small when $N \gg 1$. Since k_p is an integer, we get $k_p = 0$ for all $p \in \{1, \cdots, d-1\}$. It implies that $k = 0$. Recall that we have

supposed $\langle\alpha,k\rangle=\langle\beta,l\rangle$ in the beginning of the proof. Thus, $\langle\beta,l\rangle=0$. It is impossible for $|l|=2$.
Sub-case 2.2. Suppose $j_0\le j_0'\le N^\iota$ with $2/3<\iota<1$. Since $i_p\ge KN$, we have $|i_p-j_0|,|i_p-j_0'|\ge N/2$. In view of (4.22) we get $b_{i_pj_0}=O(N^{-2})$ and $b_{i_pj_0'}=O(N^{-2})$. We further suppose that $l_{j_0}\cdot l_{j_0'}<0$, say $l_{j_0}=1$, $l_{j_0'}=-1$ with $j_0<j_0'$. Since the eigenvalues λ_n's is simple[2], using (2.2) we get that there is a constant $C>0$ such that $|\mu_{j_0}-\mu_{j_0'}|\ge 1/C$. Now we have the following estimate:

$$\left|\frac{1}{\langle l,w\rangle}\left(\frac{b_{i_pj_0}}{\tilde\mu_{j_0}}-\frac{b_{i_pj_0'}}{\tilde\mu_{j_0'}}\right)\right|=\left|\frac{\kappa_{j_0}^{-2}b_{i_pj_0}\tilde\mu_{j_0'}-\kappa_{j_0'}^{-2}b_{i_pj_0'}\tilde\mu_{j_0}}{\mu_{j_0}-\mu_{j_0'}}\right|\le CN^{-(2-\iota)}. \tag{4.31}$$

This implies

$$\frac{b_{i_pj_0}}{\tilde\mu_{j_0}}-\frac{b_{i_pj_0'}}{\tilde\mu_{j_0'}}=o(1)\langle w,l\rangle. \tag{4.32}$$

Moreover,

$$\tilde\mu_{i_p}B^Tl=\langle w,l\rangle+\frac{b_{i_pj_0}}{\tilde\mu_{j_0}}-\frac{b_{i_pj_0'}}{\tilde\mu_{j_0'}}=\langle w,l\rangle(1+o(1)). \tag{4.33}$$

And then

$$||k||=||A^{-1}B^Tl||\le d||A^{-1}||||B^Tl||\le CN\langle w,l\rangle(1+o(1)), \tag{4.34}$$

where $||\cdot||$ is max-norm. By (4.23) we have

$$\langle k,v\rangle+\frac{1}{2\mu_{i_p}}k_p=\langle l,w\rangle(1+o(1)). \tag{4.35}$$

Making sum from $p=1$ to $p=d$ we get

$$\langle k,v\rangle=s_0\langle l,w\rangle(1+o(1)). \tag{4.36}$$

By (4.35,36), we get

$$k_p=\mu_{i_p}s_1\langle w,l\rangle(1+o(1)). \tag{4.37}$$

It follows that

$$k_{p+1}-k_p=(\mu_{i_{p+1}}-\mu_{i_p})s_1\langle l,w\rangle(1+o(1)). \tag{4.38}$$

Clearly, the left hand of (4.38) is non-zero, when $N\gg1$. Thus,

$$1\le|k_{p+1}-k_p|\le 2s_1\mathcal{K}_2\mathcal{K}_3=\frac{4\mathcal{K}_2\mathcal{K}_3}{2d+1}.$$

This is impossible if $d\ge 2\mathcal{K}_2\mathcal{K}_3$.
Sub-case 2.3. Suppose that $j_0<N^{2/3}$*,* $j_0'>N^{2/3}$ *with* $j_0'-j_0<(N^{\iota'}-N^{2/3})/2$ *and* $2/3<\iota'<\iota<1$. It follows that $j_0<j_0'<N^\iota$. Thus, this case is reduced to the sub-case 2.2.

[2] Recall $\mu_n=\sqrt{\lambda_n}$.

Sub-case 2.4. Suppose that $j_0 < N^{2/3}$, $j_0' > N^{2/3}$ *with* $j_0' - j_0 \geq (N^{\iota'} - N^{2/3})/2$. It follows that $j_0' - j_0 \geq N^{\iota'}/4$. By (4.22) we get $b_{i_p j_0}, b_{i_p j_0'} = O(N^{-1})$. Thus

$$\left|\frac{1}{\langle l, w\rangle}\left(\frac{b_{i_p j_0}}{\tilde{\mu}_{j_0}} - \frac{b_{i_p j_0'}}{\tilde{\mu}_{j_0'}}\right)\right| = \left|\frac{\kappa_{j_0}^{-2} b_{i_p j_0}\tilde{\mu}_{j_0'} - \kappa_{j_0'}^{-2} b_{i_p j_0'}\tilde{\mu}_{j_0}}{\mu_{j_0} - \mu_{j_0'}}\right|$$
$$\leq \frac{|O(N^{-1})|(j_0' + j_0)}{j_0' - j_0} \leq |O(N^{-1})| \sup_{p \geq N^{\iota'}/4} \frac{p + 2j_0}{p} = |o(1)|$$

This implies that (4.32) hold true. Thus the further proof is the same as that of Sub-case 2.2. $\square$

Finally we will give out the estimates of the perturbed term R^* in (4.12). To this end we need some notations which are taken from [9]. Let $\ell^{a,s}$ is the Hilbert space of all complex sequence $w = (\cdots, w_1, w_2, \cdots)$ with

$$||w||_{a,s}^2 = \sum_{j \notin N_d} |w_j|^2 |j|^{2s} e^{2a|j|} < \infty, \quad a, s > 0.$$

Let $\theta = (\theta_j)_{j \in N_d}$ and $\rho = (\rho_j)_{j \in N_d}$, $Z = (z_j)_{j \notin N_d}$, and $\xi = (\xi_j)_{j \in N_d}$. Let us introduce the phase space

$$\mathcal{P}^{a,s} = \hat{\mathbb{T}}^d \times \mathbb{C}^d \times \ell^{a,s} \times \ell^{a,s} \ni (x, y, Z, \bar{Z}),$$

where $\hat{\mathbb{T}}^d$ is the complexification of the usual d-torus $\mathbb{T}^d$. Set

$$D(s, r) := \{(x, y, Z, \bar{Z},) \in \mathcal{P}^{a,s} : |\mathrm{Im} x| < s, |y| < r^2, ||Z||_{a,s} + ||\bar{Z}||_{a,s} < r\},$$

where $|\cdot|$ denotes the sup-norm for complex vectors and $||\cdot||_{a,s}$ is the norm in the space $\ell^{a,s}$. We define the weighted phase norms

$$|W|_r = |W|_{\bar{s},r} = |x| + \frac{1}{r^2}|y| + \frac{1}{r}||Z||_{a,\bar{s}} + \frac{1}{r}||\bar{Z}||_{a,\bar{s}}$$

for $W = (x, y, z, \bar{z}) \in \mathcal{P}^{a,\bar{s}}$ with $\bar{s} = s + 1$. Denote by Σ the parameter set $[1, 2]^d$. For a map $U : D(s, r) \times \Sigma \to \mathcal{P}^{a,\bar{s}}$, define its Lipschitz semi-norm $|U|_r^{\mathcal{L}}$:

$$|U|_r^{\mathcal{L}} = \sup_{\xi \neq \zeta} \frac{|\Delta_{\xi\zeta} U|_r}{|\xi - \zeta|},$$

where $\Delta_{\xi\zeta} U = U(\cdot, \xi) - U(\cdot, \zeta)$, and where the supremum is taken over Σ. Denote by X_{R^*} the vector field corresponding the Hamiltonian R^* with respect to the symplectic structure $d\theta \wedge d\rho + \sqrt{-1} dZ \wedge \bar{Z}$, namely,

$$X_{R^*} = (\partial_\rho R^*, -\partial_\theta R^*, \nabla_{\bar{Z}} R^*, -\nabla_Z R^*).$$

Lemma 4.3. *The perturbation $R^*(\theta,\rho,Z,\bar{Z};\xi)$ is real analytic for real argument $(\theta,\rho,Z,\bar{Z})\in D(s,r)$ for given $s,r>0$, and Lipschitz in the parameters $\xi\in\Sigma$, and for each $\xi\in\Sigma$ its gradients with respect to $Z,\bar{Z}$ satisfy*

$$\partial_Z R^*, \partial_{\bar{Z}} R^* \in \mathcal{A}(\ell^{a,s},\ell^{a,s+1}),$$

where $\mathcal{A}(\ell^{a,s},\ell^{a,s+1})$ denotes the class of all maps from some neighborhood of the origin in $\ell^{a,s}$ into $\ell^{a,s+1}$, which is real analytic in the real and imaginary parts of the complex coordinate Z. In addition, for the perturbed term R^* we have the following estimates

$$\sup_{D(s,r)\times\Sigma} |X_{R^*}|_r \le C\epsilon^2,$$

$$\sup_{D(s,r)\times\Sigma} |\partial_\zeta X_{R^*}|_r \le C\epsilon^2,$$

where $r=\epsilon^{1/2}$.

Proof. By (4.3) and (4.12) the proof is immediately completed. □

5. A KAM theorem with application to Hamiltonian (4.9).

In this section we state the KAM theorem. This theorem was first proved by Kuksin[5,6]. Also see [9]. Here we recite the theorem from [9]. The KAM theorem was used to show there are plenty of quasi-periodic solutions of some nonlinear partial differential equations. See [7, 10], for example. Let us consider the perturbations of a family of linear integrable Hamiltonian

$$H_0=\sum_{j=1}^{d}\omega_j(\xi)y_j+\frac{1}{2}\sum_{j=d+1}^{\infty}\breve{\lambda}_j(\xi)(u_j^2+v_j^2), \tag{5.1}$$

in d-dimensional angle-action coordinates (x,y) and infinite-dimensional Cartesian coordinates (u,v) with symplectic structure

$$\sum_{j=1}^{d} dx_j\wedge dy_j+\sum_{j=d+1}^{\infty} du_j\wedge dv_j. \tag{5.2}$$

The tangent frequencies $\omega^*=(\omega_1,...,\omega_d)$ and normal ones $\breve{\lambda}=(\breve{\lambda}_{d+1},\breve{\lambda}_{d+2},...)$ depend on d parameters

$$\xi\in\Pi\subset\mathbb{R}^d, \tag{5.3}$$

with Π a closed bounded set of positive Lebesgue measure.

For each ξ there is an invariant d-torus $\mathcal{T}_0^d=\mathbb{T}^d\times\{0,0,0\}$ with frequencies $\omega^*(\xi)$. In its normal space described by the uv-coordinates the origin is an elliptic fixed point with characteristic frequencies $\lambda(\xi)$. The KAM theorem by Pöschel will show that the persistence of a large portion of this family of linearly stable rotational tori under small perturbations $H=H_0+P$ of H_0. to this end the following assumptions are made.

Assumption A: *Non-degeneracy.* The real map $\xi\mapsto\omega^*(\xi)$ is a lipeomorphism between Π and its image, that is, a homomorphism which is Lipschitz continuous in both directions. Moreover,

$$\text{meas}\{\xi:\langle k,\omega^*(\xi)\rangle+\langle l,\breve{\lambda}(\xi)\rangle=0\}=0 \tag{5.4}$$

and

$$\langle l, \breve{\lambda}(\xi)\rangle \neq 0 \quad \text{on} \quad \Pi, \tag{5.5}$$

for all integer vectors $(k,l) \in \mathbb{Z}^d \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$ and where "meas " $\equiv$ Lebesgue measure for sets, $|l| = \sum_j |l_j|$ for integer vectors, and $\langle\cdot,\cdot\rangle$ is usual real (or complex) scalar product.

Assumption B *Spectral Asymptotic and the Lipschitz property.* There exist $\varrho \geq 1$ and $\delta < \tau - 1$ such that

$$\breve{\lambda}_j = j^\varrho + \cdots + O(j^\delta), \tag{5.6}$$

where the dots stands for fixed lower order term in j, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\hat{\lambda}$ with $\hat{\lambda}_j = j^\varrho + \cdots$ such that the tails $\breve{\lambda}_j - \hat{\lambda}_j$ give rise to a Lipschitz map

$$\breve{\lambda}_j - \hat{\lambda}_j : \Pi \to \ell_\infty^{-\delta}, \tag{5.7}$$

where ℓ_∞^p is the space of all real sequences with finite norm $|w|_p = \sup_j |w_j| j^p$.

Assumption C: *Regularity.* The perturbation $P(x,y,z,\bar{z};\xi)$ is real analytic for real argument $(x,y,z,\bar{z}) \in D(s,r)$ for given $s,r>0$, and Lipschitz in the parameters $\xi \in \Pi$, and for each $\xi \in \Pi$ its gradients with respect to $z,\bar{z}$ satisfy

$$P_z, P_{\bar{z}} \in \mathcal{A}(\ell^{a,p},\ell^{a,\bar{p}}), \quad \begin{cases} \bar{p} \geq p & \text{for} \quad \varrho > 1, \\ \bar{p} > p & \text{for} \quad \varrho = 1, \end{cases} \tag{5.8}$$

where $\mathcal{A}(\ell^{a,p},\ell^{a,\bar{p}})$ denotes the class of all maps from some neighborhood of the origin in $\ell^{a,p}$ into $\ell^{a,\bar{p}}$, which is real analytic in the real and imaginary parts of the complex coordinate z.

In order to state Pöschel's theorem, we assume that

$$|\omega^*|^{\mathcal{L}}_{\Pi} + |\breve{\lambda}|^{\mathcal{L}}_{-\delta,\Pi} \leq M < \infty, \quad |(\omega^*)^{-1}|^{\mathcal{L}}_{\omega^*(\Pi)} \leq L < \infty. \tag{5.9}$$

In addition,we introduce the notations

$$\langle l\rangle_\varrho = \max(1, |\sum_j j^\varrho l_j|), \quad A_k = 1 + |k|^\tau,$$

where $\tau > d+1$ is fixed later. Finally, let $\mathcal{Z} = \{(k,l) \neq 0, |l| \leq 2\} \subset \mathbb{Z}^d \times \mathbb{Z}^\infty$.

We can now state the basic KAM Theorem which is recited from Pöschel[9]. The same theorem is also proven by Kuksin[5,6].

Theorem 5.1. (Theorem A in [9].) *Suppose $H = N + P$ satisfies assumptions A, B, and C, and*

$$\varepsilon = \sup_{D(s,r)\times\Pi} |X_P|_r + \sup_{D(s,r)\times\Pi} \frac{\alpha}{M} |X_P|^{\mathcal{L}}_r \leq \gamma\alpha, \tag{5.10}$$

where $0 < \alpha \leq 1$ is a parameter, and γ depends on the parameters described below. Then there is a Cantor set $\Pi_\alpha \subset \Pi$ with Meas $(\Pi_\alpha \setminus \Pi) \to 0$ as $\alpha \to 0$, a Lipschitz continuous family of torus embedding $\Phi : \mathbb{T}^d \times \Pi_\alpha \to \mathcal{P}^{a,\bar{p}}$, and a Lipschitz continuous map $\tilde{\omega} : \Pi_\alpha \to \mathbb{R}^d$, such that for each $\xi \in \Pi_\alpha$ the map Φ restricted to

$\mathbb{T}^d \times \{\xi\}$ is a real analytic embedding of an elliptic rotational torus with frequencies $\tilde{\omega}(\xi)$ for the Hamiltonian H at ξ.

Each embedding is analytic on $|\Im x| < s/2$, and

$$|\Phi - \Phi_0|_r + \frac{\alpha}{M}|\Phi - \Phi_0|_r^{\mathcal{L}} \leq c\varepsilon/\alpha, \tag{5.11}$$

$$|\tilde{\omega} - \omega^*| + \frac{\alpha}{M}|\tilde{\omega} - \omega^*|^{\mathcal{L}} \leq c\varepsilon, \tag{5.12}$$

uniformly on that domain and Π_α, where $\Phi_0 : \mathbb{T}^d \times \Pi \to \mathcal{T}_0^d$ is the trivial embedding, and $c \leq \gamma^{-1}$ depends on the same parameters as γ.
Moreover, there exist a family of Lipschitz maps ω_j^* and Λ_j on Π for $0 \leq j \in \mathbb{Z}$ satisfying $\omega_0^* = \omega^*$, $\Lambda_0 = \Omega$ and

$$|\omega_j^* - \omega^*| + \frac{\alpha}{M}|\omega_j^* - \omega^*|^{\mathcal{L}} \leq c\varepsilon, \tag{5.13}$$

$$|\Lambda_j - \breve{\lambda}|_{-\delta} + \frac{\alpha}{M}|\Lambda_j - \breve{\lambda}|_{-\delta}^{\mathcal{L}} \leq c\varepsilon, \tag{5.14}$$

such that $\Pi \setminus \Pi_\alpha \subset \bigcup \mathcal{R}_{k,l}^j(\alpha)$, where

$$R_{k,l}^j(\alpha) = \left\{ \xi \in \Sigma : |\langle k, \omega_j^*(\xi)\rangle + \langle l, \Lambda_j\rangle| \leq \alpha \frac{\langle l\rangle_d}{A_k} \right\}, \tag{5.15}$$

and the union is taken over all $j \geq 0$ and $(k,l) \in \mathcal{Z}$ such that $|k| > K_0 2^{j-1}$ for $j \geq 1$ with a constant $K_0 \geq 1$ depending only on d and τ.

Concerning the measure of the "bad" frequency set $\Pi \setminus \Pi_\alpha$, we recite Pöschel's Theorem D in [9].

Theorem 5.2. (Theorem D in [9].)*Suppose that in Theorem 5.1 the unperturbed frequencies are affine functions of the parameters. Then there is a constant c such that*

$$meas(\Pi \setminus \Pi_\alpha) \leq c(diam\Pi)^{d-1}\alpha^\mu, \quad \mu = \begin{cases} 1, & for \quad \varrho > 1, \\ \frac{\kappa}{\kappa+1-(\varpi/4)}, & for \quad \varrho = 1, \end{cases} \tag{5.16}$$

for all sufficiently small α, where ϖ is any number in $[0, \min(\bar{p} - p, 1))$, and where, in the case $\varrho = 1$, κ is a positive constant such that

$$\frac{\breve{\lambda}_i - \breve{\lambda}_j}{i - j} = 1 + O(j^{-k}), \quad i > j \tag{5.17}$$

uniformly on Π.

Remark. By checking Pöschel's proof in [9], we find the constant c in Theorem 5.2 equals to $L^d M^{d-1}$.

In order to apply Theorems 5.1 and 5.2 to the Hamiltonian (4.9), we let $\Pi = \Sigma = [1,2]^d$ and

$$\omega_j(\xi) = \mu_{i_j} + \epsilon^{5/4}(A\xi)_{i_j}, \quad i_j \in N_d, j = 1, ..., d, \tag{5.18}$$

and, for $j = d+1, d+2, ...,$

$$\breve{\lambda}_j = \begin{cases} \mu_j + \epsilon^{5/4}(B\xi)_j, & j \geq \max N_d \\ \mu_{j-d} + \epsilon^{5/4}(B\xi)_{j-d}, & j-d \notin N_d, j-d \leq \max N_d \end{cases} \tag{5.19}$$

In view of Lemmas 4.1 and 4.2, the **Assumption A** is verified. Set $\varrho = 1$ and $\delta = -1$. By (2.2), $\mu_j = j + O(1/j)$. Thus, the **Assumption B** can be verified easily, and we see that $\kappa = 2$ in Theorem 5.2. The **Assumption C** can be verified easily by Lemma 4.3, letting $\bar{p} = p+1, p = s$. Using (5.18) and (5.19) we find that (5.9) is satisfied with $M = C_1\epsilon^{5/4}$ and $L = C_2\epsilon^{-5/4}$. Let $P = R^*$. (See (4.12) for R^*.) Set $x = \theta$, $y = \rho$, $z = Z$, $r = \epsilon^{1/2}$. By Lemma 4.3, the smallness condition (5.10) is verified by letting $\alpha = \epsilon^{2-\iota}$ with $\iota \ll 1$ fixed. Since $\kappa = 2$, we can let $\mu = 2/3 - \iota$ in Theorem 5.2.

Finally, we finish the proof by using the theorems 5.1 and 5.2.

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