

Approximate resonance states in the semigroup decomposition of resonance evolution

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Abstract

The semigroup decomposition formalism makes use of the functional model for C_0 class contractive semigroups for the description of the time evolution of resonances. For a given scattering problem the formalism allows for the association of a definite Hilbert space state with a scattering resonance. This state defines a decomposition of matrix elements of the evolution into a term evolving according to a semigroup law and a background term. We discuss the case of multiple resonances and give a bound on the size of the background term. As an example we treat a simple problem of scattering from a square barrier potential on the half-line.

1 Introduction

Originally formulated for the analysis of scattering problems involving solution of hyperbolic wave equations in the exterior domain of compactly supported obstacles, the Lax-Phillips scattering theory¹ was developed as a tool most suitable for dealing with resonances in the scattering of electromagnetic or acoustic waves. Subsequent to its introduction by Lax and Phillips, various authors have contributed to further development of the theory^{2,3,4,5,6}. Notable recent additions were made by Sjöstrand and Sworski⁷ who extended the scope of the theory to include general classes of semibounded, compactly supported perturbations of the

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Laplacian in the wave equation, and by Kuzhell, via the development of a formalism providing conditions for the application of the Lax-Phillips structure to an abstract form of the wave equation⁸ and to certain classes of Schrödinger operators⁹. In addition, Kuzhell and Moskalyova¹⁰ applied the Lax-Phillips theory in the analysis of scattering systems involving singular perturbations of the Laplacian.

Several recent papers have dealt with the adaptation of the Lax-Phillips theory to quantum mechanical scattering problems. An early work in this direction is Ref. 11,12,13. A general formalism was developed in Ref. 14 and subsequently applied to several physical models in Ref. 15,16,17. Such efforts to adapt the Lax-Phillips formalism to the framework of quantum mechanics are motivated by certain appealing features of the Lax-Phillips theory. One of these features is the fact that the time evolution of resonances in this theory is given in terms of a continuous, one parameter, strongly contractive semigroup $\{\mathbf{Z}(t)\}_{t \geq 0}$

$$\mathbf{Z}(t_1)\mathbf{Z}(t_2) = \mathbf{Z}(t_1 + t_2), \quad t_1, t_2 \geq 0.$$

If \mathcal{H} is a (separable) Hilbert space corresponding to a particular scattering system and $\{\mathbf{U}(t)\}_{t \in \mathbb{R}}$ is a unitary group defined on \mathcal{H} describing the evolution of the system, the basic premises of the Lax-Phillips theory include the assumption of the existence of an *incoming subspace* \mathcal{D}_- and an *outgoing subspace* \mathcal{D}_+ with respect to $\{\mathbf{U}(t)\}_{t \in \mathbb{R}}$ which are assumed furthermore to be orthogonal to each other. Denoting by \mathbf{P}_- and \mathbf{P}_+ respectively the projections on the orthogonal complements of \mathcal{D}_- and \mathcal{D}_+ in \mathcal{H} , and letting $\mathcal{K} = \mathcal{H} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+)$, the Lax-Phillips semigroup $\{\mathbf{Z}(t)\}_{t \geq 0}$ defined by

$$\mathbf{Z}(t) = \mathbf{P}_+\mathbf{U}(t)\mathbf{P}_- = \mathbf{P}_\mathcal{K}\mathbf{U}(t)\mathbf{P}_\mathcal{K}, \quad t \geq 0, \quad (1)$$

annihilates \mathcal{D}_\pm and maps \mathcal{K} into itself. The subspace \mathcal{K} contains the scattering resonances and the Lax-Phillips semigroup $\{\mathbf{Z}(t)\}_{t \geq 0}$ describes their time evolution. In the Lax-Phillips framework resonances are associated with pure states in the Hilbert space \mathcal{H} .

A basic difficulty encountered in the work on application of the Lax-Phillips theory in quantum mechanics originates from the fact that in this theory the continuous spectrum of the generator of evolution is required to be unbounded from below as well as from above. Hence a formalism utilizing the original structure of the theory, such as in Ref. 14, is not suitable for application to large classes of scattering problems in quantum mechanics (except for limited types of problems, such as the Stark effect Hamiltonian¹⁷, or problems in a relativistically covariant framework^{15,16}, which can be analyzed by direct mapping to the Lax-Phillips structure. The case of a Schrödinger equation with compactly supported potential may also be analyzed within the Lax-Phillips framework through the use of the invariance principle of wave operators¹⁸). The subject of the present paper is a theoretical framework, termed the *semigroup decomposition* of resonance evolution, developed with the goal of overcoming such difficulties. Proposed by one of the authors (Y.S.) of the present article^{19,20}, this formalism makes use of the Sz.-Nagy-Foias theory of contraction operators and contractive semigroups on Hilbert space²¹ which, from the mathematical point of view, is the fundamental theory underlying the Lax-Phillips construction through the notion of model operators for C_0 class semigroups (see Section 2).

The presentation of the semigroup decomposition formalism in Ref. 20 is based on the following assumptions:

- (i) We are considering a scattering system consisting of a “free” unperturbed Hamiltonian \mathbf{H}_0 and a perturbed Hamiltonian \mathbf{H} , both defined on a Hilbert space \mathcal{H} .

- (ii) $\text{ess sup} \sigma_{ac}(\mathbf{H}_0) = \text{ess sup} \sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$. For simplicity it is assumed further that the multiplicity of the a.c. spectrum is one.
- (iii) The Møller wave operators $\Omega^\pm \equiv \Omega^\pm(\mathbf{H}_0, \mathbf{H})$ exist and are complete.
- (iv) The S -matrix in the *energy representation* (the spectral representation for \mathbf{H}_0), denoted by $\hat{S}(\cdot)$ has an extension to a meromorphic function $\mathcal{S}(\cdot)$ in an open, simply connected, region $\Sigma \subset \mathbb{C}$ such that $\Sigma \cap \mathbb{R}$ is an open interval in \mathbb{R} . The operator valued function $\mathcal{S}(\cdot)$ is holomorphic in $\Sigma \cap \mathbb{C}^+$ and has a simple pole (we generalize to the case of multiple poles in Section 3 below) at a point $z = \mu \in \Sigma \cap \mathbb{C}^-$ and no other singularity in $\bar{\Sigma}$, the closure of Σ .

It is shown in Ref. 20 that there exists a dense set $\Lambda \subset \mathcal{H}_{ac}(\mathbf{H})$ and a well defined state $\psi_\mu \in \mathcal{H}_{ac}(\mathbf{H})$ such that for any $g \in \Lambda$ and any $f \in \mathcal{H}_{ac}$ the properties (i)-(iv) above induce, for positive times, a decomposition of matrix elements of the evolution $\mathbf{U}(t)$ in the form

$$(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}(\mathbf{H})} = R(g, f; t) + \alpha(g, \mu)(\psi_\mu, f)_{\mathcal{H}_{ac}(\mathbf{H})} e^{-i\mu t}, \quad t \geq 0. \quad (2)$$

In a sense to be made precise in the next section the second term on the right hand side of Eq. (2) originates from an evolution semigroup of Lax-Phillips type and the eigenvalue of the generator of this semigroup is exactly μ , i.e., the point of singularity of the S -matrix. The quantity $R(g, f; t)$ on the right hand side of Eq. (2) is what we shall call a *background* term. We note that if in Eq. (2) we choose f to be orthogonal to ψ_μ then the exponentially decaying semigroup term (second term on the r.h.s. of Eq. (2)) vanishes. We call ψ_μ an *approximate resonance state* and note that the characterization of ψ_μ as an approximate resonance state rather than as an exact resonance state stems from the fact that one can show (see Ref. 20) that there is no choice of g and f that makes the background term $R(g, f; t)$ vanish.

An explicit expression for the approximate resonance state ψ_μ is provided in Ref. 20. It is shown there that, if we denote by $\{|E^-\rangle\}_{E \in \mathbb{R}^+}$ the set of outgoing solutions of the Lippmann-Schwinger equation (using Dirac's notation), then ψ_μ is given by

$$\psi_\mu = \frac{1}{2\pi i} \int_{\mathbb{R}^+} dE \frac{1}{E - \mu} |E^-\rangle. \quad (3)$$

Following the introduction of approximate resonance states, the present paper discusses some generalizations. Thus, in Section 3 we assume that the region $\Sigma \cap \mathbb{C}^-$ contains multiple resonance poles of the S -matrix $\mathcal{S}(\cdot)$, say at $z = \mu_1, \dots, \mu_n$ and obtain the form of the expression for the approximate resonance states and semigroup decomposition of evolution matrix elements in this case. In particular, we apply the semigroup decomposition to the *survival amplitude*, a central notion in the characterization of the time evolution of resonances. Theorem 5 below then provides an *a priori* upper bound on the size of the background term in this case.

As a final remark we note that a modification of the Lax-Phillips theory was recently used by H. Baumgartel for the description of scattering resonances in certain quantum mechanical problems²² (see also Ref. 23). In particular, the assumption of orthogonality of \mathcal{D}_\pm , essential in the context of the original Lax-Phillips formalism, is replaced in Ref. 22 by the requirement that an incoming subspace \mathcal{D}_- and an outgoing subspace \mathcal{D}_+ exist and the respective projections commute. The modified assumptions on \mathcal{D}_\pm , accompanied by certain assumptions on S -matrix analyticity properties, result in a modified Lax-Phillips structure which

is then applied to the Friedrichs model, leading to the construction of appropriate Gamow type vectors²⁴ associated with scattering resonances. The framework presented in Ref. 22 has several points of intersection with the semigroup decomposition formalism discussed in the present paper. The nature of these relationships will be discussed elsewhere.

The rest of the paper is organized as follows: In Section 2 we describe the formalism providing the semigroup decomposition of resonance evolution starting with a short discussion of the functional model for C_0 continuous contractive semigroups followed by a description of the semigroup decomposition formalism introduced in Ref. 19,20. In Section 3 we extend the framework of Ref. 19,20 to the case of multiple resonances and, furthermore, find an estimate on the size of the background term in the expression for the time evolution of the survival probability of a resonance. In Section 4 we analyze a simple but illuminating example involving a one dimensional model of scattering from a square barrier potential. Section 5 contains a short summary of the contents of the paper and some indication on further possible courses of investigation.

2 The semigroup decomposition for resonance evolution

2.1 Classification of contractive semigroups

Several distinct classes of contractive semigroups are identified within the framework of the Sz.-Nagy-Foias theory. Let $\{\mathbf{T}(t)\}_{t \geq 0}$ be a strongly contractive semigroup defined on a Hilbert space \mathcal{H} . The classes C_0 , $C_{0.}$, C_1 , $C_{1.}$ are defined by

$$\begin{aligned} \{\mathbf{T}(t)\}_{t \in \mathbb{R}^+} \in C_0. & \quad \text{if} \quad \mathbf{T}(t)h \rightarrow 0, \quad \forall h \in \mathcal{H} \\ \{\mathbf{T}(t)\}_{t \in \mathbb{R}^+} \in C_0 & \quad \text{if} \quad \mathbf{T}^*(t)h \rightarrow 0, \quad \forall h \in \mathcal{H} \\ \{\mathbf{T}(t)\}_{t \in \mathbb{R}^+} \in C_1. & \quad \text{if} \quad \mathbf{T}(t)h \not\rightarrow 0, \quad \forall h \in \mathcal{H}, \quad h \neq 0 \\ \{\mathbf{T}(t)\}_{t \in \mathbb{R}^+} \in C_1 & \quad \text{if} \quad \mathbf{T}^*(t)h \not\rightarrow 0, \quad \forall h \in \mathcal{H}, \quad h \neq 0 \end{aligned}$$

The classes $C_{\alpha\beta}$ with $\alpha, \beta = 0, 1$ are then defined by

$$C_{\alpha\beta} = C_\alpha \cap C_\beta, \quad \alpha, \beta = 0, 1.$$

The semigroup $\{\mathbf{Z}(t)\}_{t \in \mathbb{R}^+}$ describing the time evolution of resonances in the Lax-Phillips theory is readily characterized by the fact that $\{\mathbf{Z}^*(t)\}_{t \in \mathbb{R}^+}$ belongs to the class C_0 . The structure of the Lax-Phillips outgoing spectral (and translation) representation is then determined by that of the *functional model*^{21,25} for C_0 class semigroups provided by the Sz.-Nagy-Foias theory. We say an operator \mathbf{A} is a *model operator*²⁵ for a given class C of operators if every operator in C is similar to a multiple of a part of \mathbf{A} (a part of an operator \mathbf{A} is a restriction of \mathbf{A} to one of its invariant subspaces). By a functional model we mean that the model operator for a given class C has a canonical representation on suitable function spaces. For a C_0 class semigroup $\{\mathbf{T}(t)\}_{t \geq 0}$ the associated functional model is essentially obtained through a procedure of *isometric dilation* of the *cogenerator* of $\{\mathbf{T}(t)\}_{t \geq 0}$ and the similarity mapping to the functional model is in fact a unitary transformation.

2.2 The functional model for C_0 semigroups

We turn now to a brief description of the functional model for semigroups in the class C_0 . Denote by \mathbb{C}^+ the upper half of the complex plane and let $H_{\mathcal{N}}^2(\mathbb{C}^+)$ be the Hardy space

of vector valued functions analytic in the upper half-plane and taking values in a separable Hilbert space \mathcal{N} . The set of boundary values on \mathbb{R} of functions in $H_{\mathcal{N}}^2(\mathbb{C}^+)$, denoted below by $H_{\mathcal{N}^+}^2(\mathbb{R})$, is a Hilbert space isomorphic to $H_{\mathcal{N}}^2(\mathbb{C}^+)$. In a similar manner the Hardy space of \mathcal{N} valued functions analytic in the lower half-plane is denoted by $H_{\mathcal{N}}^2(\mathbb{C}^-)$ and $H_{\mathcal{N}^-}^2(\mathbb{R})$ is the isomorphic Hilbert space consisting of boundary values on \mathbb{R} of functions in $H_{\mathcal{N}}^2(\mathbb{C}^-)$. Define $\{u(t)\}_{t \in \mathbb{R}}$, a family of unitary, multiplicative operators $u(t) : L_{\mathcal{N}}^2(\mathbb{R}) \mapsto L_{\mathcal{N}}^2(\mathbb{R})$ by

$$[u(t)f](\sigma) = e^{-i\sigma t}f(\sigma), \quad f \in L_{\mathcal{N}}^2(\mathbb{R}), \quad \sigma \in \mathbb{R}. \quad (4)$$

Assume that $\{\mathbf{T}(t)\}_{t \geq 0}$ is a C_0 class semigroup defined on a Hilbert space \mathcal{K} . Let the semigroup $\{\hat{T}(t)\}_{t \geq 0}$, defined on a Hilbert space $\hat{\mathcal{K}}$, be the functional model for $\{\mathbf{T}(t)\}_{t \geq 0}$ and let $W : \mathcal{K} \mapsto \hat{\mathcal{K}}$ be the similarity transforming $\{\mathbf{T}(t)\}_{t \geq 0}$ into its functional model $\{\hat{T}(t)\}_{t \geq 0}$ i.e., $\hat{T}(t) = W\mathbf{T}(t)W^{-1}$. Then there exists a Hilbert space \mathcal{N} such that $\hat{\mathcal{K}}$ is a closed subspace of $H_{\mathcal{N}^+}^2(\mathbb{R})$, W is unitary, and the functional model is given by

$$\hat{T}(t) = W\mathbf{T}(t)W^* = P_{\hat{\mathcal{K}}}u^*(t)|_{\hat{\mathcal{K}}}, \quad t \geq 0. \quad (5)$$

Here $P_{\hat{\mathcal{K}}}$ is the orthogonal projection from $H_{\mathcal{N}^+}^2(\mathbb{R})$ onto $\hat{\mathcal{K}}$, the subspace $\hat{\mathcal{K}}$ is given by

$$\hat{\mathcal{K}} = H_{\mathcal{N}^+}^2(\mathbb{R}) \ominus \Theta_T(\cdot)H_{\mathcal{N}^+}^2(\mathbb{R}), \quad (6)$$

and $\Theta_T(\cdot) : H_{\mathcal{N}^+}^2(\mathbb{R}) \mapsto H_{\mathcal{N}^+}^2(\mathbb{R})$ is an *inner function*^{26,29,30} for $H_{\mathcal{N}^+}^2(\mathbb{R})$ (depending, of course, on $\{\mathbf{T}(t)\}_{t \geq 0}$) i.e., an operator valued function with the properties:

1. For each $\sigma \in \mathbb{R}$ the operator $\Theta_T(\sigma) : \mathcal{N} \mapsto \mathcal{N}$ is the boundary value at σ of an operator valued function $\Theta_T(\cdot)$ analytic in the upper half-plane.
2. $\|\Theta_T(z)\|_{\mathcal{N}} \leq 1$ for $\text{Im } z > 0$.
3. $\Theta_T(\sigma)$, $\sigma \in \mathbb{R}$ is, pointwise, a unitary operator on \mathcal{N} .

The operator valued function $\Theta_T(\cdot)$ is, in fact, the *characteristic function*²¹ of the cogenerator of the semigroup $\{\hat{T}(t)\}_{t \geq 0}$ (or $\{\mathbf{T}(t)\}_{t \geq 0}$).

Let P_+ be the orthogonal projection of $L_{\mathcal{N}}^2(\mathbb{R})$ on $H_{\mathcal{N}^+}^2(\mathbb{R})$. The *Toeplitz operator* with symbol $u(t)$ (see, for example, Ref. 26,27 and references therein), is an operator $T_{u(t)} : H_{\mathcal{N}^+}^2(\mathbb{R}) \mapsto H_{\mathcal{N}^+}^2(\mathbb{R})$ defined by

$$T_{u(t)}f \stackrel{\text{def}}{=} P_+u(t)f, \quad f \in H_{\mathcal{N}^+}^2(\mathbb{R}). \quad (7)$$

We note that $\{T_{u(t)}\}_{t \geq 0}$ is a strongly contractive semigroup on $H_{\mathcal{N}^+}^2(\mathbb{R})$ (see, for example, Ref. 1,19,21,28. Taking the conjugate of $\hat{T}(t)$ in $H_{\mathcal{N}^+}^2(\mathbb{R})$ and using Eq. (5) one finds that

$$\hat{T}^*(t) = W\mathbf{T}^*(t)W^* = T_{u(t)}|_{\hat{\mathcal{K}}}, \quad t \geq 0. \quad (8)$$

It follows from the discussion above that the Lax-Phillips semigroup $\{\mathbf{Z}(t)\}_{t \geq 0}$ has a functional model in the form of Eq. (8) (recall that $\{\mathbf{Z}^*(t)\}_{t \geq 0}$ is a C_0 class semigroup), i.e., if we denote the functional model for $\{\mathbf{Z}(t)\}_{t \geq 0}$ by $\{\hat{Z}(t)\}_{t \geq 0}$ then we have

$$\hat{Z}(t) = W\mathbf{Z}(t)W^* = T_{u(t)}|_{\hat{\mathcal{K}}}, \quad t \geq 0 \quad (9)$$

where $\hat{\mathcal{K}} \subset H_{\mathcal{N}^+}^2(\mathbb{R})$ is an invariant subspace for $\{T_{u(t)}\}_{t \geq 0}$ given by

$$\hat{\mathcal{K}} = H_{\mathcal{N}^+}^2(\mathbb{R}) \ominus \Theta_Z(\cdot)H_{\mathcal{N}^+}^2(\mathbb{R}) \quad (10)$$

and the inner function $\Theta_Z(\cdot)$ and the Hilbert space \mathcal{N} are determined by $\{\mathbf{Z}(t)\}_{t \geq 0}$. A semigroup $\{\hat{Z}(t)\}_{t \geq 0}$ of the form given by Eq. (9) and Eq. (10) is referred to in Ref. 20 as a *Lax-Phillips type semigroup*.

A central theorem of the Lax-Phillips theory, corresponding to an important result in the Sz.-Nagy-Foias theory relating the spectrum of a *completely non-unitary (cnu)* contraction to points of singularity of the characteristic function states the following

Theorem 1 *Denote by \hat{B} the generator of a Lax-Phillips type semigroup $\{\hat{Z}(t)\}_{t \geq 0}$. If $\text{Im } \mu < 0$, then μ belongs to the point spectrum of \hat{B} if and only if $\Theta_Z^*(\bar{\mu})$ has a nontrivial null space.*

We note that the analytic continuation of $\Theta_Z(z)$ to the lower half-plane is given by

$$\Theta_Z(z) \stackrel{\text{def}}{=} (\Theta_Z^*(\bar{z}))^{-1}, \quad \text{Im } z < 0$$

and so a null space for $\Theta_Z^*(\bar{\mu})$ implies the existence of a pole for $\Theta_Z(z)$ at $z = \mu$. In the case of the Lax-Phillips theory the characteristic function $\Theta_Z(\cdot)$ for the Lax-Phillips semigroup is identical to the Lax-Phillips S -matrix and its poles are the scattering resonances. As will be seen below, the situation is a bit more involved in the semigroup decomposition formalism.

We do not elaborate here further on the relations between the functional model for C_0 semigroups discussed above and the full structure of the Lax-Phillips spectral representations and wave operators. The reader is referred to Ref. 1,21.

2.3 The semigroup decomposition

In order to apply the functional model for C_0 semigroups, which is at the heart of the Lax-Phillips structure, to the description of resonance evolution it is necessary to relate, for $t \geq 0$, the evolution $\mathbf{U}(t)$ defined on the Hilbert space \mathcal{H} of the scattering problem to the Toeplitz evolution semigroup $T_{u(t)}$ of Eq. (7) defined on $H_{\mathcal{N}^+}^2(\mathbb{R})$ and then restrict the latter, according to Eq. (9), to a subspace $\hat{\mathcal{K}}$ of $H_{\mathcal{N}^+}^2(\mathbb{R})$ associated with an appropriate inner function $\Theta_{\hat{Z}}(\cdot)$. In the framework of the Lax-Phillips theory this relation is guaranteed by the special properties of the Lax-Phillips incoming and outgoing subspaces \mathcal{D}_{\pm} (with \mathcal{D}_- and \mathcal{D}_+ denoting, respectively, the incoming and outgoing subspace), since in this case the Lax-Phillips semigroup is a C_0 semigroup. However, for many quantum mechanical scattering problems one usually cannot find subspaces with the properties of \mathcal{D}_{\pm} . A way of overcoming this difficulty, proposed in Ref. 19 is to combine the standard functional model for C_0 semigroups with the notion of a *quasi-affine mapping* (see, for example, Ref. 21, Pg. 70):

Definition 1 (Quasi-affine mapping) *A quasi-affine map from a Hilbert space \mathcal{H}_1 into a Hilbert space \mathcal{H}_0 is a linear, one to one continuous mapping of \mathcal{H}_1 into a dense linear manifold in \mathcal{H}_0 . If $\mathbf{A} \in \mathcal{B}(\mathcal{H}_1)$ and $\mathbf{B} \in \mathcal{B}(\mathcal{H}_0)$ then \mathbf{A} is a quasi-affine transform of \mathbf{B} if there is a quasi-affine map $\theta : \mathcal{H}_1 \mapsto \mathcal{H}_0$ such that $\theta \mathbf{A} = \mathbf{B} \theta$.*

The following theorem is proved in Ref. 19 for a scattering system consisting of unperturbed and perturbed Hamiltonians, respectively \mathbf{H}_0 and \mathbf{H} , having semibounded continuous spectrum:

Theorem 2 (Outgoing/Incoming contractive nesting) *Let \mathbf{H}_0 and \mathbf{H} be self-adjoint operators on a Hilbert space \mathcal{H} . Let $\{\mathbf{U}(t)\}_{t \in \mathbb{R}}$ be the unitary evolution group on \mathcal{H} generated by \mathbf{H} [i.e, $\mathbf{U}(t) = \exp(-i\mathbf{H}t)$]. Denote by $\mathcal{H}_{ac}(\mathbf{H}_0)$ and $\mathcal{H}_{ac}(\mathbf{H})$, respectively, the absolutely continuous subspaces of \mathbf{H}_0 and \mathbf{H} . Assume that the absolutely continuous spectrum of \mathbf{H}_0 and \mathbf{H} has multiplicity one and that $\text{ess Supp } \sigma_{ac}(\mathbf{H}_0) = \text{ess Supp } \sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$. Assume furthermore that the Møller wave operators $\hat{\Omega}^\pm \equiv \hat{\Omega}^\pm(\mathbf{H}_0, \mathbf{H}) : \mathcal{H}_{ac}(\mathbf{H}_0) \mapsto \mathcal{H}_{ac}(\mathbf{H})$ exist and are complete. Then there are mappings $\hat{\Omega}_\pm : \mathcal{H}_{ac}(\mathbf{H}) \mapsto H_+^2(\mathbb{R})$ such that*

- (i) $\hat{\Omega}_\pm$ are contractive quasi-affine mappings of $\mathcal{H}_{ac}(\mathbf{H})$ into $H_+^2(\mathbb{R})$.
- (ii) For every $t \geq 0$ the evolution $\mathbf{U}(t)$ is a quasi-affine transform of the Toeplitz operator $T_{u(t)}$ via the mapping $\hat{\Omega}_\pm$ i.e., for every $f \in \mathcal{H}_{ac}(\mathbf{H})$ we have

$$\hat{\Omega}_\pm \mathbf{U}(t)f = T_{u(t)} \hat{\Omega}_\pm f \quad t \geq 0. \quad (11)$$

□

We call the triplet $(\mathcal{H}_{ac}(\mathbf{H}), H_+^2(\mathbb{R}), \hat{\Omega}_-)$ the *incoming contractive nesting* of $\mathcal{H}_{ac}(\mathbf{H})$ into $H_+^2(\mathbb{R})$ and denote $f_{in} = \hat{\Omega}_- f$. Similarly, the triplet $(\mathcal{H}_{ac}(\mathbf{H}), H_+^2(\mathbb{R}), \hat{\Omega}_+)$ is the *outgoing contractive nesting* of $\mathcal{H}_{ac}(\mathbf{H})$ into $H_+^2(\mathbb{R})$ and we denote $f_{out} = \hat{\Omega}_+ f$.

Define

$$\Xi_{\hat{\Omega}_+} \stackrel{\text{def}}{=} \hat{\Omega}_+^* H_+^2(\mathbb{R}).$$

Then, since $\hat{\Omega}_+^*$ is quasi-affine, the linear space $\Xi_{\hat{\Omega}_+} \subset \mathcal{H}_{ac}(\mathbf{H})$ is dense in $\mathcal{H}_{ac}(\mathbf{H})$. Moreover, since $\hat{\Omega}_+^*$ is one to one, for each $g \in \Xi_{\hat{\Omega}_+}$ there is a unique $\tilde{g} \in H_+^2(\mathbb{R})$ such that $g = \hat{\Omega}_+^* \tilde{g}$. We note that in Ref. 20 a dense set $\Lambda_{\hat{\Omega}_+}$, analogous to $\Xi_{\hat{\Omega}_+}$, is defined somewhat differently, i.e., $\Lambda_{\hat{\Omega}_+} \stackrel{\text{def}}{=} \hat{\Omega}_+^* \hat{\Omega}_+ \mathcal{H}_{ac}(\mathbf{H})$. However, it will be seen below that the definition of $\Xi_{\hat{\Omega}_+}$ above, unlike that of $\Lambda_{\hat{\Omega}_+}$, allows for a full characterization of approximate resonance states. Using Theorem 2 we have, for every $g \in \Xi_{\hat{\Omega}_+}$ and $f \in \mathcal{H}_{ac}(\mathbf{H})$ and for $t \geq 0$

$$\begin{aligned} (g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}(\mathbf{H})} &= (\hat{\Omega}_+^* \tilde{g}, \mathbf{U}(t)f)_{\mathcal{H}_{ac}(\mathbf{H})} = \\ &= (\tilde{g}, T_{u(t)} \hat{\Omega}_+ f)_{H_+^2(\mathbb{R})} = (\tilde{g}, T_{u(t)} f_{out})_{H_+^2(\mathbb{R})}, \quad t \geq 0 \end{aligned} \quad (12)$$

Following the definitions of the incoming and outgoing nestings of $\mathcal{H}_{ac}(\mathbf{H})$ into $H_+^2(\mathbb{R})$ it is natural to define the *nested S-matrix*

$$S_{nest} \stackrel{\text{def}}{=} \hat{\Omega}_+ \hat{\Omega}_-^{-1}.$$

Let $U : \mathcal{H}_{ac}(\mathbf{H}_0) \mapsto L^2(\mathbb{R}^+)$ be the unitary transformation of $\mathcal{H}_{ac}(\mathbf{H}_0)$ onto the spectral representation for \mathbf{H}_0 (also called the energy representation for \mathbf{H}_0). If $\mathbf{S} = (\hat{\Omega}_-^*)^{-1} \hat{\Omega}_+^*$ is the scattering operator associated with \mathbf{H}_0 and \mathbf{H} then $\tilde{S}(\cdot) : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ defined by

$$\tilde{S}(\cdot) \stackrel{\text{def}}{=} U \mathbf{S} U^* \quad (13)$$

is the energy representation of the S-matrix. Let $P_{\mathbb{R}^+} : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ be the orthogonal projection in $L^2(\mathbb{R})$ on the subspace of functions supported on \mathbb{R}^+ and define the inclusion map $I : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R})$ by

$$(If)(\sigma) = \begin{cases} f(\sigma), & \sigma \geq 0 \\ 0, & \sigma < 0. \end{cases} \quad (14)$$

Then the inverse $I^{-1} : P_{\mathbb{R}^+}L^2(\mathbb{R}) \mapsto L^2(\mathbb{R}^+)$ is, of course, one to one on $P_{\mathbb{R}^+}L^2(\mathbb{R})$. Let $\theta : H_+^2(\mathbb{R}) \mapsto L^2(\mathbb{R}^+)$ be a map given by

$$\theta f = I^{-1}P_{\mathbb{R}^+}f, \quad f \in H_+^2(\mathbb{R}). \quad (15)$$

By a theorem of Van Winter³¹, θ is a quasi-affine transform mapping $H_+^2(\mathbb{R})$ into $L^2(\mathbb{R}^+)$. The adjoint map $\theta^* : L^2(\mathbb{R}^+) \mapsto H_+^2(\mathbb{R})$ is then also a contractive quasi-affine map. An explicit expression for θ^* is provided by the following lemma¹⁹:

Lemma 1 *Let $I : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R})$ be the inclusion map defined in Eq. (14). Let P_+ be the orthogonal projection of $L^2(\mathbb{R})$ onto $H_+^2(\mathbb{R})$. Then for every $f \in L^2(\mathbb{R}^+)$ we have*

$$\theta^*f = P_+If, \quad f \in L^2(\mathbb{R}^+). \quad (16)$$

□

It is shown in Ref. 19 that the nested S-matrix can be expressed in the form

$$S_{nest} = \theta^* \tilde{S}(\cdot) (\theta^*)^{-1}. \quad (17)$$

Following Ref. 20 we now use assumption (iv) in Section 1. The S-matrix $\tilde{S}(\cdot)$ is then the restriction of its extension $\mathcal{S}(\cdot)$ on \mathbb{R}^+ . Under these assumptions $\mathcal{S}(\cdot)$ has, in the region Σ , a representation of the form (see Ref. 20)

$$\mathcal{S}(z) = \mathcal{B}_\mu(z)\mathcal{S}'(z), \quad z \in \Sigma$$

where

$$\mathcal{B}_\mu(z) = \frac{z - \bar{\mu}}{z - \mu}, \quad z \in \mathbb{C} \setminus \{\mu\} \quad (18)$$

and $\mathcal{S}'(\cdot)$ is analytic and has no zeros in Σ . Restricting $\mathcal{S}(\cdot)$ to the positive real axis we obtain

$$\tilde{S}(E) = \tilde{B}_\mu(E)\tilde{S}'(E), \quad E \geq 0 \quad (19)$$

where by definition $\tilde{B}_\mu(E) \stackrel{def}{=} \mathcal{B}_\mu(E)$ and $\tilde{S}'(E) \stackrel{def}{=} \mathcal{S}'(E)$ for $E \geq 0$. We note that both $\tilde{S}'(\cdot)$ and $\tilde{B}_\mu(\cdot)$ are considered here as multiplicative unitary operators on $L^2(\mathbb{R}^+)$ (moreover, they are pointwise unitary a.e. for $E \geq 0$). Moreover, $\mathcal{B}_\mu(\cdot)$ can be regarded as a multiplicative operator on $L^2(\mathbb{R})$. In fact, considered as a multiplicative operator on $H_+^2(\mathbb{R}) \subset L^2(\mathbb{R})$, \mathcal{B}_μ is a *Blaschke* factor (the definition of Blaschke products and Blaschke factors can be found, for example in Ref. 29,30, see e.g., Eq. (29) below). Such a factor is the simplest example of an inner function for $H_+^2(\mathbb{R})$. We make use of this fact through the following proposition, not stated as such, but implicitly used in Ref. 20:

Proposition 1 *Let $\tilde{B}_\mu(\cdot) : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ be defined by $\tilde{B}_\mu(E) = \mathcal{B}_\mu(E)$, $E \geq 0$ where $\mathcal{B}_\mu(\cdot)$ is defined in Eq. (18). Let $\theta^* : L^2(\mathbb{R}^+) \mapsto H_+^2(\mathbb{R})$ be the adjoint of the map θ defined in Eq. (15). Let $\hat{\mathcal{K}}_\mu \subset H_+^2(\mathbb{R})$ and $\hat{\mathcal{K}}_{\bar{\mu}} \subset H_-^2(\mathbb{R})$ be subspaces defined by*

$$\hat{\mathcal{K}}_\mu \stackrel{def}{=} H_+^2(\mathbb{R}) \ominus \mathcal{B}_\mu(\cdot)H_+^2(\mathbb{R}), \quad \hat{\mathcal{K}}_{\bar{\mu}} \stackrel{def}{=} H_-^2(\mathbb{R}) \ominus \mathcal{B}_{\bar{\mu}}(\cdot)H_-^2(\mathbb{R}),$$

where $\mathcal{B}_{\bar{\mu}}(z) = (z - \mu)(z - \bar{\mu})^{-1}$ and denote by $P_{\hat{\mathcal{K}}_\mu}$ and $P_{\hat{\mathcal{K}}_{\bar{\mu}}}$ the orthogonal projections of $L^2(\mathbb{R})$ on $\hat{\mathcal{K}}_\mu$ and $\hat{\mathcal{K}}_{\bar{\mu}}$ respectively. For every $f \in L^2(\mathbb{R}^+)$ we then have

$$\theta^* \tilde{B}_\mu f = \mathcal{B}_\mu \theta^* f + P_{\hat{\mathcal{K}}_\mu} \mathcal{B}_\mu P_{\hat{\mathcal{K}}_{\bar{\mu}}} \bar{\theta}^* f. \quad (20)$$

here $\bar{\theta}^* : L^2(\mathbb{R}^+) \mapsto H_-^2(\mathbb{R})$ and $\bar{\theta}^* f = P_- If$ with I defined in Eq. (14) and P_- the orthogonal projection of $L^2(\mathbb{R})$ onto $H_-^2(\mathbb{R})$. □

Proof: Using Eq. (16) in Lemma 1 we get

$$\theta^* \tilde{B}_\mu f = P_+ I \tilde{B}_\mu f = P_+ \mathcal{B}_\mu I f = P_+ \mathcal{B}_\mu (P_+ + P_-) I f = P_+ \mathcal{B}_\mu \theta^* f + P_+ \mathcal{B}_\mu P_- \bar{\theta}^* f$$

Eq. (20) then follows from the fact, proved in Ref. 20, that $P_+ \mathcal{B}_\mu P_- = P_{\hat{\mathcal{K}}_\mu} \mathcal{B}_\mu P_{\hat{\mathcal{K}}_\mu}$ and from the property of $\mathcal{B}_\mu(\cdot)$ of being an inner function for $H_+^2(\mathbb{R})$. ■

We note that since $\mathcal{B}_\mu(\cdot)$ is an inner function Eq. (9), (10) and Theorem 1 imply that

$$T_{u(t)} P_{\hat{\mathcal{K}}_\mu} = \hat{Z}(t) P_{\hat{\mathcal{K}}_\mu} = e^{-i\mu t} P_{\hat{\mathcal{K}}_\mu}, \quad t \geq 0. \quad (21)$$

Combining Eq. (17), Eq. (19) and Eq. (20) we obtain

$$\begin{aligned} f_{out} &= S_{nest} f_{in} = \theta^* \tilde{S}(\theta^*)^{-1} f_{in} = \theta^* \tilde{B}_\mu \tilde{S}'(\theta^*)^{-1} f_{in} = \\ &= \mathcal{B}_\mu \theta^* \tilde{S}'(\theta^*)^{-1} f_{in} + P_{\hat{\mathcal{K}}_\mu} \mathcal{B}_\mu P_{\hat{\mathcal{K}}_\mu} \bar{\theta}^* \tilde{S}'(\theta^*)^{-1} f_{in}. \end{aligned} \quad (22)$$

Using the decomposition of f_{out} from Eq. (22) in the r.h.s. of Eq. (12) and applying Eq. (21) we obtain the *semigroup decomposition* for $t \geq 0$ of the time evolution corresponding to the resonance at $z = \mu$

$$\begin{aligned} (g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}(\mathbf{H})} &= (\tilde{g}, T_{u(t)} f_{out})_{H_+^2(\mathbb{R})} = \\ &= (\tilde{g}, u(t) \mathcal{B}_\mu \theta^* \tilde{S}'(\theta^*)^{-1} f_{in})_{H_+^2(\mathbb{R})} + e^{-i\mu t} (\tilde{g}, \mathcal{B}_\mu \bar{\theta}^* \tilde{S}'(\theta^*)^{-1} f_{in})_{H_+^2(\mathbb{R})}. \end{aligned} \quad (23)$$

As is seen above in Eq. (21), the exponential decay in the second term on the r.h.s of Eq. (23) originates with the semigroup $\hat{Z}(t)$. The first term on the r.h.s. of Eq. (23) is the *background term* and is responsible for deviations from a purely exponential decay law.

3 Approximate resonance states

It is an interesting fact that the semigroup decomposition described in the previous section associates a unique state in $\mathcal{H}_{ac}(\mathbf{H})$ with a resonance pole at $z = \mu$ ($\text{Im } \mu < 0$). The following theorem is proved in Ref. 20

Theorem 3 (approximate resonance state) *Under the assumptions of Theorem 2, let $\tilde{S} : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ be the S -matrix in the energy representation defined in Eq. (13). Assume that $\tilde{S}(\cdot)$ is the restriction to \mathbb{R}^+ of a function $\mathcal{S}(\cdot)$ meromorphic in an open region Σ with a single, simple pole at a point $z = \mu$, $\mu \in \Sigma \cap \mathbb{C}^-$. For any $f \in \mathcal{H}_{ac}(\mathbf{H})$ define $f_{out} = \hat{\Omega}_+ f$ and $f_{in} = \hat{\Omega}_- f$. There exists a unique state $\psi_\mu \in \mathcal{H}_{ac}(\mathbf{H})$ such that*

$$f_{out} = \mathcal{B}_\mu \theta^* \tilde{S}'(\theta^*)^{-1} f_{in} + \frac{|\text{Im } \mu|}{\pi} (\psi_\mu, f)_{\mathcal{H}_{ac}(\mathbf{H})} x_\mu \quad (24)$$

where θ^* is the map given by lemma 1, \mathcal{B}_μ is given in Eq. (18), \tilde{S}' is defined by Eq. (19) and $x_\mu \in H_+^2(\mathbb{R})$ is given by $x_\mu(\sigma) = (\sigma - \mu)^{-1}$, $\sigma \in \mathbb{R}$. □

Combining Eq. (24) and Eq. (23) we can write the semigroup decomposition in the form

$$(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}(\mathbf{H})} = (\tilde{g}, u(t)\mathcal{B}_\mu\theta^*\tilde{S}'(\theta^*)^{-1}f_{in})_{H_+^2(\mathbb{R})} + \frac{\text{Im } \mu}{\pi} e^{-i\mu t} (\tilde{g}, x_\mu)_{H_+^2(\mathbb{R})} (\psi_\mu, f)_{\mathcal{H}_{ac}(\mathbf{H})}, \quad t \geq 0 \quad (25)$$

where $g \in \Xi_{\hat{\Omega}_+}$ and $\tilde{g} = (\hat{\Omega}_+^*)^{-1}g$. The eigenstate $x_\mu \in H_+^2(\mathbb{R})$ of the semigroup $\hat{Z}(t)$ providing the exponential decay of the second term on the r.h.s. of Eq. (25) is called below the *Hardy space resonance state*. The state $\psi_\mu \in \mathcal{H}_{ac}(\mathbf{H})$ whose existence is implied by Theorem 3 is called *approximate resonance state*. We observe that if in Eq. (25) we choose $f \in \mathcal{H}_{ac}(\mathbf{H})$ orthogonal to ψ_μ then the second term on the r.h.s. of that equation is identically zero.

Denote by $\{|E^-\rangle\}_{E \in \mathbb{R}^+}$ the set of outgoing solutions of the Lippmann-Schwinger equation. For every $f \in \mathcal{H}_{ac}(\mathbf{H})$ we have

$$(U(\Omega^-)^*f)(E) = \langle E^- | f \rangle, \quad E \in \mathbb{R}^+ \quad (26)$$

It is shown in Ref. 20 that an explicit expression for the approximate resonance state ψ_μ is given by

$$\psi_\mu = \frac{1}{2\pi i} \int_{\mathbb{R}^+} dE \frac{1}{E - \mu} |E^-\rangle. \quad (27)$$

In this section we explore several properties of approximate resonance states ψ_μ . Our first step is to extend the discussion above to the case of multiple resonances:

Theorem 4 (multiple resonance case) *Under the assumptions of Theorem 2, let $\tilde{S} : L^2(\mathbb{R}^+) \mapsto L^2(\mathbb{R}^+)$ be the S -matrix in the energy representation defined in Eq. (13). Assume that $\tilde{S}(\cdot)$ is the restriction to \mathbb{R}^+ of a function $\mathcal{S}(\cdot)$ meromorphic in the open region Σ with n simple poles at points $z = \mu_i$, $i = 1, \dots, n$, $\mu_i \in \Sigma \cap \mathbb{C}^-$. Then there exist n distinct states $\{\psi_{\mu_i}^\Sigma\}_{i=1, \dots, n}$, $\psi_{\mu_i}^\Sigma \in \mathcal{H}_{ac}(\mathbf{H})$, such that for every $f \in \mathcal{H}_{ac}(\mathbf{H})$ we have*

$$f_{out} = \mathcal{B}_{\mu_1 \dots \mu_n} \theta^* \tilde{S}'(\theta^*)^{-1} f_{in} + \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} (\psi_{\mu_j}^\Sigma, f)_{\mathcal{H}_{ac}(\mathbf{H})} x_{\mu_j} \quad (28)$$

where

$$\mathcal{B}_{\mu_1 \dots \mu_n}(z) \stackrel{\text{def}}{=} \prod_{i=1}^n \frac{z - \bar{\mu}_i}{z - \mu_i}. \quad (29)$$

In Eq. (28) $\tilde{S}'(\cdot)$ is the restriction to \mathbb{R}^+ of a function $\mathcal{S}'(\cdot)$ analytic in Σ and having no poles in $\bar{\Sigma}$ and $x_{\mu_j}(\sigma) = (\sigma - \mu_j)^{-1}$, $\sigma \in \mathbb{R}$. The states $\psi_{\mu_j}^\Sigma$, $j = 1, \dots, n$ are given by

$$\psi_{\mu_j}^\Sigma = \int_{\mathbb{R}^+} dE \prod_{\substack{i=1 \\ i \neq j}}^n \frac{E - \bar{\mu}_i}{E - \mu_i} \frac{1}{E - \mu_j} |E^-\rangle. \quad (30)$$

□

Proof: Assume that $\mathcal{S}(\cdot)$, the extension of $\tilde{S}(\cdot)$ from \mathbb{R}^+ into $\Sigma \cup \mathbb{R}^+$ has n simple poles in $\Sigma \cap \mathbb{C}^-$. Then, applying the same arguments as in Ref. 20, we find that $\mathcal{S}(\cdot)$ can be factorized in Σ in the form

$$\mathcal{S}(z) = \mathcal{B}_{\mu_1 \dots \mu_n}(z) \mathcal{S}'(z)$$

where $\mathcal{B}_{\mu_1 \dots \mu_n}$, defined in Eq. (29), is a finite Blaschke product and $\mathcal{S}'(\cdot)$ has no poles in Σ . In addition we have, of course

$$\tilde{S}(E) = \tilde{B}_{\mu_1 \dots \mu_n}(E) \tilde{S}'(E), \quad E \geq 0.$$

The semigroup decomposition then follows exactly as in Section 1 with $\mathcal{B}_{\mu_1 \dots \mu_n}$ and $\tilde{B}_{\mu_1 \dots \mu_n}$ replacing \mathcal{B}_μ and \tilde{B}_μ respectively. For the resonance term in Eq. (22) we get in this case

$$P_+ \mathcal{B}_{\mu_1 \dots \mu_n} P_- \bar{\theta}^* \tilde{S}'(\theta^*)^{-1} f_{in}, \quad f \in \mathcal{H}_{ac}(\mathbf{H})$$

Recalling that

$$(P_+ f)(\sigma) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\sigma' \frac{1}{\sigma - \sigma' + i0} f(\sigma'), \quad f \in L^2(\mathbb{R}), \quad \sigma \in \mathbb{R}$$

and

$$(\bar{\theta}^* f)(\sigma) = \frac{1}{2\pi i} \int_0^{\infty} dE \frac{1}{E - \sigma + i0} f(E), \quad f \in L^2(\mathbb{R}^+)$$

(see Ref. 19) we obtain

$$\begin{aligned} (P_+ \mathcal{B}_{\mu_1 \dots \mu_n} P_- \bar{\theta}^* \tilde{S}'(\theta^*)^{-1} f_{in})(\sigma) &= \\ &= \frac{-1}{4\pi^2} \int_0^{\infty} dE \int_{-\infty}^{\infty} d\sigma' \frac{1}{\sigma - \sigma' + i0} \prod_{i=1}^n \frac{\sigma' - \bar{\mu}_i}{\sigma' - \mu_i} \frac{1}{E - \sigma' + i0} \tilde{S}'(E) ((\theta^*)^{-1} f_{in})(E) = \\ &= \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} \frac{1}{\sigma - \mu_j} \int_0^{\infty} dE \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} \frac{1}{E - \mu_j} \tilde{S}'(E) ((\theta^*)^{-1} f_{in})(E) = \\ &= \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} x_{\mu_j}(\sigma) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} \int_0^{\infty} dE \frac{1}{E - \bar{\mu}_j} \prod_{\substack{i=1 \\ i \neq j}}^n \frac{E - \mu_i}{E - \bar{\mu}_i} \tilde{S}(E) ((\theta^*)^{-1} f_{in})(E) = \\ &= \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} x_{\mu_j}(\sigma) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} \int_0^{\infty} dE \frac{1}{E - \bar{\mu}_j} \prod_{\substack{i=1 \\ i \neq j}}^n \frac{E - \mu_i}{E - \bar{\mu}_i} (U(\mathbf{\Omega}^-)^* f)(E) = \\ &= \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} x_{\mu_j}(\sigma) \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} \int_0^{\infty} dE \frac{1}{E - \bar{\mu}_j} \prod_{\substack{i=1 \\ i \neq j}}^n \frac{E - \mu_i}{E - \bar{\mu}_i} \langle E^- | f \rangle \end{aligned}$$

where $x_{\mu_j} \in H_+^2(\mathbb{R})$ is the Hardy space resonance state corresponding to μ_j i.e., $x_{\mu_j}(\sigma) = (\sigma - \mu_j)^{-1}$. Defining the states $\psi_{\mu_j}^\Sigma$, $j = 1, \dots, n$ according to Eq. (30) we obtain

$$(P_+ \mathcal{B}_{\mu_1 \dots \mu_n} P_- \bar{\theta}^* \tilde{S}'(\theta^*)^{-1} f_{in})(\sigma) = \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} (\psi_{\mu_j}^\Sigma, f)_{\mathcal{H}_{ac}(\mathbf{H})} x_{\mu_j}(\sigma) \quad (31)$$

This proves Theorem 4. ■

We observe that Eq. (30) is a generalization of Eq. (27). Hence $\psi_{\mu_j}^\Sigma$ is the approximate resonance state corresponding to the pole of $\mathcal{S}(\cdot)$ at $z = \mu_j$. Combining Eq. (31) and Eq. (23) we get the semigroup decomposition for the multi-resonance case

$$(g, \mathbf{U}(t)f)_{\mathcal{H}_{ac}(\mathbf{H})} = (\tilde{g}, u(t)\mathcal{B}_{\mu_1 \dots \mu_n} \theta^* \tilde{S}'(\theta^*)^{-1} f_{in})_{H_+^2(\mathbb{R})} + \sum_{j=1}^n \frac{|\text{Im } \mu_j|}{\pi} \prod_{\substack{i=1 \\ i \neq j}}^n \frac{\mu_j - \bar{\mu}_i}{\mu_j - \mu_i} (\psi_{\mu_j}^\Sigma, f)_{\mathcal{H}_{ac}(\mathbf{H})} (\tilde{g}, x_{\mu_j})_{H_+^2(\mathbb{R})} e^{-i\mu_j t}, \quad t \geq 0. \quad (32)$$

The approximate resonance states in Eq. (30) and semigroup decomposition of Eq. (28) and Eq. (32) depend, of course, on the region Σ . If $\{\mu_j\}_{j=1, \dots, n}$ are the poles in $\Sigma \cap \mathbb{C}^-$ of the meromorphic extension $\mathcal{S}(\cdot)$ of the S -matrix $\tilde{S}(\cdot)$, the approximate resonance state defined in Eq. (30) for a resonance at $z = \mu_j$ is therefore denoted by $\psi_{\mu_j}^\Sigma$. However, for certain arguments the exact form of Σ is irrelevant and it is useful to define the notion of an n 'th order approximate resonance state:

Definition 2 (n'th order approximate resonance state) *If the number of poles of $\mathcal{S}(\cdot)$ entering into the definition of the approximate resonance state $\psi_{\mu_j}^\Sigma$ in Eq. (30), not including μ_j itself, is n we say that $\psi_{\mu_j}^\Sigma$ is an n 'th order approximate resonance state for the resonance at $z = \mu_j$. In particular, regardless of the exact nature of the region Σ , the zero'th order approximate resonance state is always defined to be given by Eq. (27) with $\mu = \mu_j$ and is denoted by $\psi_{\mu_j}^{(0)}$.*

Remark: Note that in general there are many choices of the n resonance poles (different than μ_j) included in the construction of what we call an n 'th order approximation $\psi_{\mu_j}^{(n)}$. In cases that the nature of the region Σ is irrelevant and only the order of the approximate resonance state is significant we replace the notation $\psi_{\mu_j}^\Sigma$ by $\psi_{\mu_j}^{(n)}$, where n is the order of the approximate resonance state considered.

The semigroup decomposition and approximate resonance states for the multi-resonance case possess some interesting properties. For example, we have

$$(\psi_{\mu_j}^\Sigma, \psi_{\mu_k}^\Sigma)_{\mathcal{H}_{ac}(\mathbf{H})} = (\psi_{\mu_j}^{(0)}, \psi_{\mu_k}^{(0)})_{\mathcal{H}_{ac}(\mathbf{H})} = \int_{\mathbb{R}^+} dE \frac{1}{E - \mu_j} \frac{1}{E - \bar{\mu}_k}$$

and, in particular

$$\|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})}^2 = \|\psi_{\mu_j}^{(n)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^2 = \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^2 = \int_{\mathbb{R}^+} dE \frac{1}{|E - \mu_j|^2}. \quad (33)$$

We see that, although the definition of $\psi_{\mu_i}^\Sigma$ in Eq. (30) depends on all of the poles $\{\mu_j\}_{j=1, \dots, n} \subset \Sigma \cap \mathbb{C}^-$, the scalar product of $\psi_{\mu_i}^\Sigma$ and $\psi_{\mu_j}^\Sigma$ depends only on μ_i and μ_j . In fact, if $\mathcal{S}(\cdot)$ can be extended to a meromorphic function in a region $\Sigma' \supset \Sigma$ (we keep the notation $\mathcal{S}(\cdot)$ for the extended function) and $\mathcal{S}(\cdot)$ has now $m > n$ simple poles in $\Sigma' \cap \mathbb{C}^-$ we may calculate approximate resonance states of order $m - 1$ for all resonances in Σ' according to Eq. (30). However, for $\mu_j, \mu_k \in \Sigma$ we would still have $(\psi_{\mu_j}^{\Sigma'}, \psi_{\mu_k}^{\Sigma'})_{\mathcal{H}_{ac}(\mathbf{H})} = (\psi_{\mu_j}^\Sigma, \psi_{\mu_k}^\Sigma)_{\mathcal{H}_{ac}(\mathbf{H})}$ i.e., scalar products (and norms) are independent of the order of the approximate states when we enlarge the region Σ . In particular we have $\|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})} = \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}$ for every region Σ containing μ_j .

An interesting question is whether the peculiar properties of scalar products and norms of the approximate resonance states mentioned above characterize also the time evolution of these states. We shall see below that, at least partially, the answer to this question is positive. For this we consider one of the basic notions associated with resonance evolution, i.e., that of the survival amplitude

$$A_{\psi_{\mu_j}^{\Sigma}}(t) \stackrel{\text{def}}{=} \frac{(\psi_{\mu_j}^{\Sigma}, \mathbf{U}(t)\psi_{\mu_j}^{\Sigma})_{\mathcal{H}_{ac}(\mathbf{H})}}{(\psi_{\mu_j}^{\Sigma}, \psi_{\mu_j}^{\Sigma})_{\mathcal{H}_{ac}(\mathbf{H})}}. \quad (34)$$

Making use of Eq. (30) and (33) we get a simple expression for this quantity

$$A_{\psi_{\mu_j}^{(0)}}(t) = \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^{-2} \int_{\mathbb{R}^+} \frac{1}{|E - \mu_j|^2} e^{-iEt}, \quad t \geq 0$$

where $\|\psi_{\mu_j}^{(0)}\|$ is given in Eq. (33). Again, we see that the expression for the survival amplitude for the approximate resonance state $\psi_{\mu_j}^{\Sigma}$ depends only on the pole at $z = \mu_j$ and has the same form as for a single resonance. This suggests that the semigroup decomposition of the survival amplitude for the multiple resonance case is similar to that of a single resonance. When combined with an important characterization of approximate resonance states in the form of Lemma 2 below, such considerations lead to the following useful *a priori* estimate on the size of the background term in the semigroup decomposition of the survival amplitude:

Theorem 5 *Let $A_{\psi_{\mu_j}^{\Sigma}}(t)$, $j = 1, \dots, n$ be the survival amplitude defined in Eq. (34) and let the background term $R_{\mu_j}(t)$ be defined by the relation*

$$A_{\psi_{\mu_j}^{\Sigma}}(t) = R_{\mu_j}(t) + e^{-i\mu_j t}, \quad t \geq 0. \quad (35)$$

Then we have

$$|R_{\mu_j}(t)| \leq \left(\frac{\|x_{\mu_j}\|_{H_+^2(\mathbb{R})}^4}{\|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^4} - 1 \right)^{1/2}, \quad t \geq 0. \quad (36)$$

where $x_{\mu_j}(\sigma) = (\sigma - \mu_j)^{-1}$ is the Hardy space resonance state and $\psi_{\mu_j}^{(0)} \in \mathcal{H}_{ac}(\mathbf{H})$ is the zero'th order approximate resonance state corresponding to the resonance at $z = \mu_j$. \square

Proof: We first have

Proposition 2 *For $j = 1, \dots, n$, let $\psi_{\mu_j}^{\Sigma}$ be defined by Eq. (30) and let $A_{\psi_{\mu_j}^{\Sigma}}(t)$ be the survival amplitude defined in Eq. (34). Then*

$$A_{\psi_{\mu_j}^{\Sigma}}(t) = \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^{-2} (x_{\mu_j}, u(t)\mathcal{B}_{\mu_j}\theta^* \tilde{S}'_{\mu_j}(\theta^*)^{-1} \psi_{\mu_j, in}^{(0)})_{H_+^2(\mathbb{R})} + e^{-i\mu_j t}, \quad t \geq 0. \quad (37)$$

where $x_{\mu_j} \in H_+^2(\mathbb{R})$ is the Hardy space resonance state and $\psi_{\mu_j}^{(0)} \in \mathcal{H}_{ac}(\mathbf{H})$ is the zero'th order approximate resonance state corresponding to the pole at μ_j , and $\tilde{S}'_{\mu_j}(\cdot)$ is defined as in Eq. (19), i.e.,

$$\tilde{S}'_{\mu_j}(E) = \frac{E - \mu_j}{E - \bar{\mu}_j} \tilde{S}(E) = \tilde{B}_{\bar{\mu}_j}(E) \tilde{S}(E).$$

\square

Proof of Proposition 2: We need first the following easily proved, but important, lemma

Lemma 2 For $j = 1, \dots, n$, let $\psi_{\mu_j}^\Sigma$ be defined by Eq. (30). Define

$$\mathcal{B}_{\mu_1 \dots \hat{\mu}_k \dots \mu_n}(z) \stackrel{\text{def}}{=} \prod_{\substack{i=1 \\ i \neq k}}^n \frac{z - \bar{\mu}_i}{z - \mu_i}$$

where $\{\mu_j\}_{j=1, \dots, n}$ are the poles of $\mathcal{S}(\cdot)$ in Σ , and let $x_{\mu_j}(\sigma) = (\sigma - \mu_j)^{-1}$. Then we have

$$\psi_{\mu_j}^\Sigma = \hat{\Omega}_+^* \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}.$$

In particular $\psi_{\mu_j}^{\Sigma(0)} = \hat{\Omega}_+^* x_{\mu_j}$. □

Proof of Lemma 2: It is proved in Ref. 19 that, if Ω^\pm are the Møller wave operators, $U : \mathcal{H}_{ac}(\mathbf{H}_0) \mapsto L^2(\mathbb{R}^+)$ the mapping to the energy representation for \mathbf{H}_0 (see Eq. (13) above) and θ^* the map given in Lemma 1, then the quasi-affine nesting maps $\hat{\Omega}_\pm$ are given by $\hat{\Omega}_\pm = \theta^* U(\Omega^\mp)^*$, hence we have $\hat{\Omega}_+^* = \Omega^- U^* \theta$. Furthermore, by the definition of θ we have

$$(\theta \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j})(E) = \prod_{\substack{i=1 \\ i \neq j}}^n \frac{E - \bar{\mu}_i}{E - \mu_i} \frac{1}{E - \mu_j}, \quad E \in \mathbb{R}^+. \quad (38)$$

Moreover, according to Eq. (26) for every $g \in L^2(\mathbb{R}^+)$ we have

$$\Omega^- U^* g = \int_{\mathbb{R}^+} dE |E^- \rangle g(E). \quad (39)$$

Applying Eq. (39) with $g = \theta \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}$ and comparing with Eq. (30) proves the lemma ■

Note that by Lemma 2 we have $(\hat{\Omega}_+^*)^{-1} \psi_{\mu_j}^\Sigma = \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}$. Hence by Eq. (32) we get

$$\begin{aligned} A_{\psi_{\mu_j}^\Sigma}(t) &= \|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})}^{-2} (\psi_{\mu_j}^\Sigma, \mathbf{U}(t) \psi_{\mu_j}^\Sigma)_{\mathcal{H}_{ac}(\mathbf{H})} = \\ &= \|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})}^{-2} (\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, u(t) \mathcal{B}_{\mu_1 \dots \mu_n} \theta^* \tilde{S}'(\theta^*)^{-1} \psi_{\mu_j, in}^\Sigma)_{H_+^2(\mathbb{R})} \\ &+ \sum_{k=1}^n \frac{|\text{Im } \mu_k|}{\pi} \prod_{\substack{i=1 \\ i \neq k}}^n \frac{\mu_k - \bar{\mu}_i}{\mu_k - \mu_i} \frac{(\psi_{\mu_k}^\Sigma, \psi_{\mu_j}^\Sigma)_{\mathcal{H}_{ac}(\mathbf{H})}}{\|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})}^2} (\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, x_{\mu_k})_{H_+^2(\mathbb{R})} e^{-i\mu_k t} \end{aligned} \quad (40)$$

In the second term on the r.h.s. of Eq. (40) we first separate the term with $k = j$ and get

$$\begin{aligned} &\sum_{k=1}^n \frac{|\text{Im } \mu_k|}{\pi} \prod_{\substack{i=1 \\ i \neq k}}^n \frac{\mu_k - \bar{\mu}_i}{\mu_k - \mu_i} \frac{(\psi_{\mu_k}^\Sigma, \psi_{\mu_j}^\Sigma)_{\mathcal{H}_{ac}(\mathbf{H})}}{\|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})}^2} (\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, x_{\mu_k})_{H_+^2(\mathbb{R})} e^{-i\mu_k t} = \\ &= \sum_{\substack{k=1 \\ k \neq j}}^n \frac{|\text{Im } \mu_k|}{\pi} \prod_{\substack{i=1 \\ i \neq k}}^n \frac{\mu_k - \bar{\mu}_i}{\mu_k - \mu_i} \frac{(\psi_{\mu_k}^\Sigma, \psi_{\mu_j}^\Sigma)_{\mathcal{H}_{ac}(\mathbf{H})}}{\|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})}^2} (\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, x_{\mu_k})_{H_+^2(\mathbb{R})} e^{-i\mu_k t} + e^{-i\mu_j t} \end{aligned}, \quad t \geq 0$$

Here, use has been made of Eq. (42) below. The above expression can be further simplified since $x_{\mu_k} \in \hat{\mathcal{K}}_{\mu_k} = H_+^2(\mathbb{R}) \ominus \mathcal{B}_{\mu_k}(\cdot)H_+^2(\mathbb{R})$ implies that for $k \neq j$ we have

$$(\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, x_{\mu_k})_{H_+^2(\mathbb{R})} = (\mathcal{B}_{\mu_k} \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \hat{\mu}_k \dots \mu_n} x_{\mu_j}, x_{\mu_k})_{H_+^2(\mathbb{R})} = 0$$

where

$$\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \hat{\mu}_k \dots \mu_n}(z) \stackrel{\text{def}}{=} \prod_{\substack{i=1 \\ i \neq k, j}}^n \frac{z - \bar{\mu}_i}{z - \mu_i}.$$

The first term on the r.h.s. of Eq. (40) can also be simplified. We have

$$\begin{aligned} & (\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, u(t) \mathcal{B}_{\mu_1 \dots \mu_n} \theta^* \tilde{S}'(\theta^*)^{-1} \psi_{\mu_j, in}^\Sigma)_{H_+^2(\mathbb{R})} = \\ & = (u(-t) \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j}, \mathcal{B}_{\mu_1 \dots \mu_n} \theta^* \tilde{S}'(\theta^*)^{-1} \psi_{\mu_j, in}^\Sigma)_{H_+^2(\mathbb{R})} = \\ & = (\mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} u(-t) x_{\mu_j}, \mathcal{B}_{\mu_1 \dots \mu_n} \theta^* \tilde{S}'(\theta^*)^{-1} \psi_{\mu_j, in}^\Sigma)_{H_+^2(\mathbb{R})} = \\ & = (x_{\mu_j}, u(t) \mathcal{B}_{\mu_j} \theta^* \tilde{S}'(\theta^*)^{-1} \psi_{\mu_j, in}^\Sigma)_{H_+^2(\mathbb{R})}, \quad t \geq 0 \end{aligned}$$

Moreover, using Lemma 2 we find that

$$\begin{aligned} \theta^* \tilde{S}'(\theta^*)^{-1} \psi_{\mu_j, in}^\Sigma &= \theta^* \tilde{S}'(\theta^*)^{-1} \hat{\Omega}_- \hat{\Omega}_+^* \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j} = \\ &= \theta^* \tilde{S}'(\theta^*)^{-1} \theta^* \tilde{S}^* \theta \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j} = \theta^* \overline{\tilde{B}_{\mu_1 \dots \mu_n}} \theta \mathcal{B}_{\mu_1 \dots \hat{\mu}_j \dots \mu_n} x_{\mu_j} = \\ &= \theta^* \overline{\tilde{B}_{\mu_j}} \theta x_{\mu_j} = \theta^* \tilde{B}_{\mu_j} \tilde{S}(\theta^*)^{-1} \hat{\Omega}_- \hat{\Omega}_+^* x_{\mu_j} = \theta^* \tilde{S}'_{\mu_j}(\theta^*)^{-1} \psi_{\mu_j, in}^0. \end{aligned}$$

Recalling that Eq. (33) implies that $\|\psi_{\mu_j}^\Sigma\|_{\mathcal{H}_{ac}(\mathbf{H})} = \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}$ the proof of Proposition 2 is complete. \blacksquare

From Proposition 2 we see that, independent of the region Σ , the semigroup decomposition of the survival amplitude depends only on the zero'th order approximate resonance state. Comparison of Eq. (37) and Eq. (35) gives

$$R_{\mu_j}(t) = \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^{-2} (x_{\mu_j}, u(t) \mathcal{B}_{\mu_j} \theta^* \tilde{S}'_{\mu_j}(\theta^*)^{-1} \psi_{\mu_j, in}^{(0)})_{H_+^2(\mathbb{R})}, \quad t \geq 0, \quad (41)$$

This expression for $R_{\mu_j}(t)$ is identical to the zero'th order background term we would get from Eq. (25) with $f = g = \psi_{\mu_j}^{(0)}$. We now exploit this fact to obtain the desired estimate in Theorem 5. Applying Theorem 3 to the zero'th order approximate resonance state $\psi_{\mu_j}^{(0)}$ we obtain

$$\psi_{\mu_j, out}^{(0)} = \hat{\Omega}_+ \psi_{\mu_j}^{(0)} = \mathcal{B}_{\mu_j} \theta^* \tilde{S}'_{\mu_j}(\theta^*)^{-1} \psi_{\mu_j, in}^{(0)} + \frac{|\text{Im } \mu_j|}{\pi} \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^2 x_{\mu_j}.$$

Now, since both $\hat{\Omega}_+$ and $\hat{\Omega}_+^*$ are contractive we note that Lemma 2 implies that $\|\psi_{\mu_j, out}^{(0)}\|_{H_+^2(\mathbb{R})} = \|\hat{\Omega}_+ \hat{\Omega}_+^* x_{\mu_j}\|_{H_+^2(\mathbb{R})} \leq \|x_{\mu_j}\|_{H_+^2(\mathbb{R})}$. In addition in $H_+^2(\mathbb{R})$ we have $x_{\mu_j} \perp \mathcal{B}_{\mu_j} H_+^2(\mathbb{R})$. Therefore,

$$\|\mathcal{B}_{\mu_j} \theta^* \tilde{S}'_{\mu_j}(\theta^*)^{-1} \psi_{\mu_j, in}^{(0)}\|_{H_+^2(\mathbb{R})}^2 + \frac{|\text{Im } \mu_j|^2}{\pi^2} \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^4 \|x_{\mu_j}\|_{H_+^2(\mathbb{R})}^2 \leq \|x_{\mu_j}\|_{H_+^2(\mathbb{R})}^2.$$

It is easy to verify that

$$\|x_{\mu_j}\|_{H_+^2(\mathbb{R})}^2 = \int_{-\infty}^{\infty} \frac{1}{|\sigma - \mu_j|^2} d\sigma = \frac{\pi}{|\operatorname{Im} \mu_j|}, \quad (42)$$

hence the inequality above can be written in the form

$$\|\mathcal{B}_{\mu_j} \theta^* \tilde{S}'_{\mu_j}(\theta^*)^{-1} \psi_{\mu_j, in}^{(0)}\|_{H_+^2(\mathbb{R})}^2 \leq \left(\|x_{\mu_j}\|_{H_+^2(\mathbb{R})}^4 - \|\psi_{\mu_j}^{(0)}\|_{\mathcal{H}_{ac}(\mathbf{H})}^4 \right) \|x_{\mu_j}\|_{H_+^2(\mathbb{R})}^{-2}. \quad (43)$$

Applying the Schwartz inequality to the r.h.s. of Eq. (41) and using the bound from Eq. (43) we get the estimate in Eq. (36). \blacksquare

As mentioned above the background term cannot be identically zero. Hence deviations from exponential decay of the survival probability are to be expected. In fact, it is easy to verify that the survival probability behaves for short times as $|A_{\psi_{\mu_j}}(t)|^2 = 1 - O(t^2)$. Note that Eq. (35) implies that at $t = 0$ we must have $R_{\mu_j}(0) = 0$. This is also seen from Eq. (41), since for $t = 0$ we have $u(0) = 1$ and $x_{\mu_j} \perp \mathcal{B}_{\mu_j}(\cdot)H_+^2(\mathbb{R})$. Deviations from exponential decay are then due to the fact that $x_{\mu_j} \not\perp u(t)\mathcal{B}_{\mu_j}H_+^2(\mathbb{R})$ for $t > 0$.

4 Example: Scattering from square barrier potential

In this section we apply the results of the previous two sections to a simple one dimensional model with a square barrier potential. Although simple, this model provides a good illustration for the various results obtained above. In particular, we present numerical calculations of approximate resonance states of various orders accompanied with plots of the time evolution of the corresponding survival amplitudes and estimates of the size of the background term following from Theorem 5.

The model we consider is a Schrödinger equation in one spatial dimension on the half-line \mathbb{R}^+ with a square barrier potential. Thus we consider the free Hamiltonian $\mathbf{H}_0 = -\partial_x^2$ acting on $L^2(\mathbb{R}^+)$ (where \mathbf{H}_0 is defined as a self-adjoint extension to $L^2(\mathbb{R}^+)$ from the original domain of definition $D(-\partial_x^2) = \{\phi(x) \in W_2^2(\mathbb{R}^+) \mid \phi(x) = 0\}$) and the full Hamiltonian is given by $\mathbf{H} = \mathbf{H}_0 + \mathbf{V}$ where \mathbf{V} is a multiplicative operator $(\mathbf{V}f)(x) = V(x)f(x)$ with

$$V(x) = \begin{cases} 0, & 0 < x < a \\ V_0, & a \leq x \leq b \\ 0, & b < x, \end{cases}$$

where $b > a > 0$ and we take $V_0 > 0$. In this case there are no bound state solutions of the eigenvalue problem for \mathbf{H} and we have $\sigma(\mathbf{H}) = \sigma_{ac}(\mathbf{H}) = \mathbb{R}^+$. In order to find the scattering states, calculate the S -matrix and finally the approximate resonance states for this problem one solves the eigenvalue problem

$$-\partial_x^2 \psi_E(x) + V(x)\psi_E(x) = E\psi_E(x), \quad E \in \mathbb{R}^+$$

for the continuous spectrum generalized eigenfunctions $\psi_E(x)$. Imposing boundary conditions one finds that

$$\psi_E(x) = \begin{cases} \alpha_1(k) \sin kx, & 0 < x \leq a \\ \alpha_2(k)e^{ik'x} + \beta_2(k)e^{-ik'x}, & a < x < b \\ \alpha_3(k)e^{ikx} + \beta_3(k)e^{-ikx}, & b \leq x \end{cases} \quad (44)$$

where $k = E^{1/2}$ and $k' = \sqrt{E - V_0}$ for $E \geq V_0 > 0$ or $k' = i\sqrt{V_0 - E}$ for $V_0 > E > 0$. The coefficients in Eq. (44) are given by³²

$$\begin{aligned}
\alpha_2(k) &= \frac{1}{2}e^{-ik'a} \left[\sin ka + \frac{k}{ik'} \cos ka \right] \alpha_1(k) \\
\beta_2(k) &= \frac{1}{2}e^{ik'a} \left[\sin ka - \frac{k}{ik'} \cos ka \right] \alpha_1(k) \\
\alpha_3(k) &= \frac{1}{4}e^{-ikb} \left[(1 + k'/k)e^{ik'(b-a)}(\sin ka + \frac{k}{ik'} \cos ka) \right. \\
&\quad \left. + (1 - k'/k)e^{-ik'(b-a)}(\sin ka - \frac{k}{ik'} \cos ka) \right] \alpha_1(k) \\
\beta_3(k) &= \frac{1}{4}e^{ikb} \left[(1 - k'/k)e^{ik'(b-a)}(\sin ka + \frac{k}{ik'} \cos ka) \right. \\
&\quad \left. + (1 + k'/k)e^{-ik'(b-a)}(\sin ka - \frac{k}{ik'} \cos ka) \right] \alpha_1(k)
\end{aligned} \tag{45}$$

with $\alpha_1(k)$ to be determined by normalization conditions (see below).

Given the full set of solutions $\{\psi_E(x)\}_{E \in \mathbb{R}^+}$ for the continuous spectrum it is easy to find the sets $\{\psi_E^\pm(x)\}_{E \in \mathbb{R}^+}$ of solutions of the Lippmann-Schwinger equation corresponding to incoming and outgoing asymptotic conditions. Using Dirac's notation we have³²

$$\begin{aligned}
\langle x|E^+ \rangle &\equiv \psi_E^+(x) = \frac{-1}{2i} \frac{\psi_E(x)}{\beta_3(k)} \\
\langle x|E^- \rangle &\equiv \psi_E^-(x) = \frac{1}{2i} \frac{\psi_E(x)}{\alpha_3(k)}.
\end{aligned} \tag{46}$$

The normalization conditions for the Lippmann-Schwinger states in Eq. (46) determines $\alpha_1(k)$ with the result $\alpha_1(k) = (2\pi k)^{-1/2}$. In the energy representation the S -matrix is then given by³²

$$\tilde{S}(E) = -\frac{\alpha_3(k)}{\beta_3(k)}, \quad k = E^{1/2}$$

We now have the ingredients for the calculation of the scattering resonances and the corresponding approximate resonance states. Note first that $\alpha_3(k)$ and $\beta_3(k)$ can be extended to analytic functions in the complex k plane and as a result the poles of the analytic continuation of $\tilde{S}(E)$ to the lower half-plane (i.e., across the square root cut along the positive real axis) are identified with zeros of the function $\beta_3(k)$. For a resonance at a point $z = \mu_j$ in the lower half-plane below the positive real axis we set $\mu_j = E_{\mu_j} - i\Gamma_{\mu_j}/2$, with E_{μ_j} being the resonance energy and Γ_{μ_j} its width.

Using Eq. (27) and the expression for the outgoing Lippmann-Schwinger eigenfunctions $\langle x|E^- \rangle$, Eq. (44), (45) and (46), the zero'th order approximate resonance states for the square barrier problem can be calculated numerically. Considering a larger number of resonance poles we are able to calculate higher order approximate resonance states using Eq. (30). As an example we consider the three lowest energy resonance poles for barrier parameters $a = 2$, $b = 3$, $V_0 = 10$. These poles are located at $\mu_1 = 1.8213 - i0.0023$, $\mu_2 = 7.0237 - i0.0564$ and $\mu_3 = 14.2336 - i0.8923$. The zero'th order probability densities $|\psi_{\mu_j}^{(0)}(x)|^2$, $j = 1, 2, 3$ for these poles are shown as dashed lines in Fig. 2 while the solid lines on the same figure correspond to the

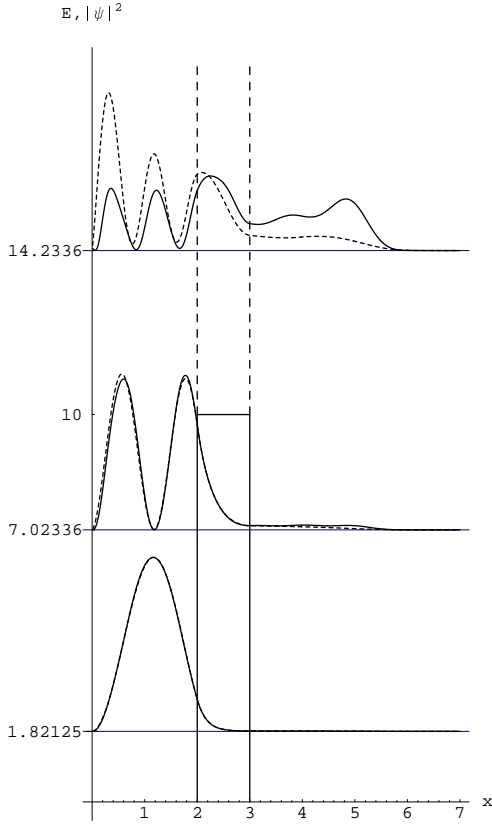


Figure 2: Probability densities of approximate resonance states $\psi_{\mu_j}^{(n)}$, $j = 1, 2, 3$ for $a = 2$, $b = 3$, $V_0 = 10$. Dashed lines - $|\psi_{\mu_j}^{(0)}(x)|^2$, $j = 1, 2, 3$. Solid lines - $|\psi_{\mu_j}^{(9)}(x)|^2$, $j = 1, 2, 3$.

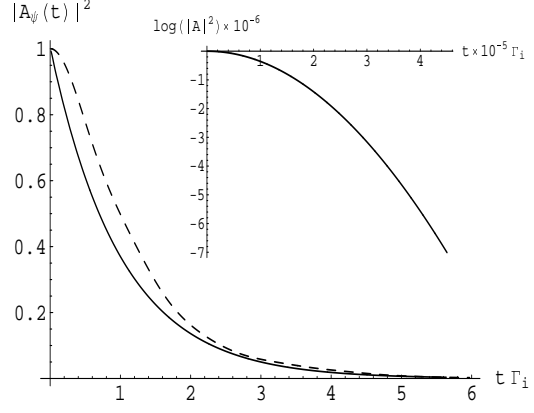


Figure 1: survival amplitudes: Solid line - $|A_{\psi_{\mu_1}}(t)|^2$, dashed line - $|A_{\psi_{\mu_3}}(t)|^2$. Insert shows short time behaviour of $|A_{\psi_{\mu_1}}(t)|^2$.

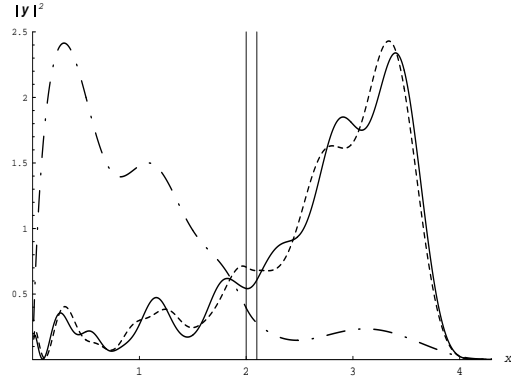


Figure 3: Probability densities of the approximate resonance states for $\mu'_3 = 17.4652 - i4.4029$. Dot dashed line - $|\psi_{\mu'_3}^{(0)}(x)|^2$, dashed line - $|\psi_{\mu'_3}^{(8)}(x)|^2$, solid line - $|\psi_{\mu'_3}^{(9)}(x)|^2$.

9'th order probability densities $|\psi_{\mu_j}^{(9)}(x)|^2$, $j = 1, 2, 3$, where the ten lowest energy resonances μ_j , $j = 1, \dots, 10$ are taken into account in Eq. (30). We observe the significant change in the probability density profile between the zero'th and 9'th order approximate resonance states for the resonance μ_3 , whose energy is higher than the barrier's energy, while the lower two states are essentially unchanged. Numerical calculations show that approximate resonance states converge in L^2 norm to a limiting state as a function of the order of approximation. An example is provided in Fig. 3 which shows the probability density $|\psi_{\mu'_3}^{(n)}(x)|^2$ for the resonance $\mu'_3 = 17.4652 - i4.4029$, at barrier parameters $a = 2$, $b = 2.1$, $V_0 = 10$, and for the orders $n = 0, 8, 9$. At present a rigorous criterion for the rate of convergence of approximate resonance states as a function of order is not yet established.

Turning to a consideration of the time evolution of survival probabilities for resonances of the square barrier model, we first recall the fact that the time evolution of the survival probabilities of higher order approximate resonance states corresponding to the same resonance

pole is independent of the order and is, in fact, identical to that of the zero'th order state. Bearing this in mind we may omit in our notation any indication of the region Σ or the order n and set $A_{\psi_{\mu_j}}(t) \equiv A_{\psi_{\mu_j}^\Sigma}(t) = A_{\psi_{\mu_j}^{(n)}}(t)$. The time dependence of the survival probability $|A_{\psi_{\mu_1}}(t)|^2$ for the states corresponding to the lower resonance in Fig. 2 is shown as a solid line in Fig. 1. The time evolution of $|A_{\psi_{\mu_1}}(t)|^2$ follows closely an exact exponential decay law with a decay constant $\Gamma_{\mu_1} = 2|\Im(\mu_1)|$. This behaviour is reflected in the bound $|R_{\mu_1}(t)| \leq 0.028$ on the size of the background term calculated using Theorem 5. The time development of $|A_{\psi_{\mu_1}}(t)|^2$ deviates from the exponential law at a very short time scale, as is clearly seen in the insert in Fig. 1. The behaviour of the survival probability for the other resonances in Fig. 2 (not shown in Fig. 1) is similar. The short time deviations from exponential decay are related to the known Zeno effect.

The nearly exact exponential decay law of the survival probability $|A_{\psi_{\mu_1}}(t)|^2$ is to be contrasted with the time development of $|A_{\psi_{\mu'_3}}(t)|^2$ for the states in Fig. 3. The survival probability $|A_{\psi_{\mu'_3}}(t)|^2$ is described by the dashed line in Fig. 1. Deviations from an exponential decay law in this case are evidently larger. This conforms with the results of Theorem 5 which produces the larger bound $|R_{\mu'_3}(t)| \leq 0.422$.

5 Summary

The semigroup decomposition formalism makes use of the fundamental mathematical theory underlying the structure of the Lax-Phillips scattering theory, i.e., the functional model for C_0 contractive semigroups, for the description of the time evolution of resonances. If the S -matrix is meromorphic in a region Σ and is known to have resonance poles there at points $z = \mu_1, \dots, \mu_n \in \Sigma$, the semigroup formalism allows for the association of a unique Hilbert space state $\psi_{\mu_j}^\Sigma(x) \in \mathcal{H}_{ac}(\mathbf{H})$, $j = 1, \dots, n$ with each resonance. The states $\psi_{\mu_j}^\Sigma(x)$, called approximate resonance states, define the decomposition of matrix elements of the evolution and are associated with its semigroup part. Theorem 5 provides an upper bound on the size of the remaining background term. Depending on one's knowledge of the location of the resonance poles it is possible to calculate approximate resonance states of different orders. Numerical calculations show that the sequence of approximate resonance states appear to converge in L^2 norm to a limiting function as a function of the order. However, rigorous criteria for the rate of convergence are needed. Another possible course of further investigation involves the study of relations between known frameworks for the treatment of the problem of resonances, such as the rigged Hilbert space method and the use of dilation analyticity and the formalism discussed in the present paper.

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¹P.D. Lax and R.S. Phillips, *Scattering Theory* (Academic Press, New York, 1967).

²V.M. Adamjan, *Funkts. Anal. Prilozh.* **10**, 1 (1976).

- ³J. Cooper and W. Strauss, *Indiana Univ. Math. J.* **34**, 33 (1985).
- ⁴R. Phillips, *Indiana Univ. Math. J.* **33**, 832 (1984).
- ⁵C. Foias, *J. Funct. Anal.* **19**, 273 (1975).
- ⁶D.Z. Arov, *Soviet Math. Dokl.* **15**, 848 (1974).
- ⁷J. Sjöstrand and M. Sworski, *J. Funct. Anal.* **123**, 336 (1994).
- ⁸S. Kuzhel, *Ukrainian Math. J.* **55**, no. 5, 621 (2003).
- ⁹S. Kuzhel, *Methods Funct, Anal. Topology* **7**, 13 (2001).
- ¹⁰S. Kuzhel and U. Moskalyova, *J. Math. Kyoto Univ.* **45**, no. 2, 265 (2005).
- ¹¹C. Flesia and C. Piron, *Helv. Phys. Acta* **57**, 697 (1984).
- ¹²L.P. Horwitz and C. Piron, *Helv. Phys. Acta* **66**, 693 (1993).
- ¹³E. Eisenberg and L.P. Horwitz, "Time, irreversibility, and unstable systems in quantum mechanics", in *Advances in Chemical Physics*, edited by I. Prigogine and S. Rice (Wiley, New York, 1997), Vol. XCIX.
- ¹⁴Y. Strauss, L.P. Horwitz, and E. Eisenberg, *J. Math. Phys.* **41**, 8050 (2000).
- ¹⁵Y. Strauss and L.P. Horwitz, *Found. Phys.* **30**, 653 (2000).
- ¹⁶Y. Strauss and L.P. Horwitz, *J. Math. Phys.*, **43**, 2394 (2002).
- ¹⁷T. Ben Ari and L.P. Horwitz, *Phys. Lett. A* **332**, 168 (2004).
- ¹⁸M. Reed and B. Simon, *methods of modern mathematical physics, Vol. 3, Scattering theory* (Academic Press, New York, 1979).
- ¹⁹Y. Strauss, *J. Math. Phys.* **46**, 32104 (2004).
- ²⁰Y. Strauss, *J. Math. Phys.* **46**, 102109 (2005).
- ²¹B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space* (North-Holland, Amsterdam, London, 1970).
- ²²H. Baumgärtel, *Rep. Math. Phys.* **52**, 295 (2003) and errata, *Rep. Math. Phys.* **53**, 329 (2004).
- ²³H. Baumgärtel, *Rev. Math. Phys.* **18**, 61 (2006).
- ²⁴G. Gamow, *Z. Phys.* **51**, 204 (1928).
- ²⁵C.S. Kubrusly, *An introduction to Models and Decompositions in Operator Theory* (Birkhauser, Boston, 1997).
- ²⁶M. Rosenblum and J. Rovnyak, *Hardy classes and Operator Theory* (Oxford University Press, New York, 1985).
- ²⁷N.K. Nikolskii, *Treatise on the Shift Operator* (Springer-Verlag, New York, 1986).
- ²⁸Y. Strauss, *Int. J. Theo. Phys.* **42**, 2285 (2003).
- ²⁹K. Hoffman, *Banach Spaces of Analytic Functions* (Prentice Hall, Englewood Cliffs, NJ, 1962).

³⁰P.L. Duren, *Theory of \mathcal{H}^p Spaces* (Academic, New York, London, 1970).

³¹C. Van Winter, *Trans. Am. Mat. Soc.* **162**, 103 (1971).

³²R. de la Madrid and M. Gadella, *Amer. J. Phys.* **70**, 626 (2002).