On the random kick-forced 3D Navier-Stokes equations in a thin domain

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Abstract

We consider the Navier-Stokes equations in the thin 3D domain $\mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is a two-dimensional torus. The equation is perturbed by a non-degenerate random kick-force. We establish that, firstly, when $\varepsilon \ll 1$ the equation has a unique stationary measure and, secondly, after averaging in the thin direction this measure converges (as $\varepsilon \to 0$) to a unique stationary measure for the Navier-Stokes equation on $\mathbb{T}^2$. Thus, the 2D Navier-Stokes equations on surfaces describe asymptotic in time and limiting in ε statistical properties of 3D solutions in thin 3D domains.

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1 Introduction

In this paper we study statistical properties of the Navier-Stokes equations (NSE) perturbed by a *random* force in a thin three-dimensional domain. For the sake of definiteness and for simplicity we consider the case of the free–periodic boundary conditions. Namely, let $\mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0, \varepsilon)$, where $\mathbb{T}^2$ is the torus $\mathbb{T}^2 = \mathbb{R}^2/(l_1\mathbb{Z} \times l_2\mathbb{Z})$ and $\varepsilon \in (0, 1]$. Let $x = (x', x_3) = (x_1, x_2, x_3) \in \mathcal{O}_\varepsilon$, and let

$$u(x) = (u_1(x), u_2(x), u_3(x)), \quad x \in \mathcal{O}_\varepsilon,$$

stands for a vector field on $\mathcal{O}_\varepsilon$. We consider the NSE in $\mathcal{O}_\varepsilon$:

$$\partial_t u - \nu \triangle u + \sum_{j=1}^{3} u_j \partial_j u + \nabla p = f \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty), \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \mathcal{O}_\varepsilon \times (0, +\infty) \ \text{ and } \int_{\mathcal{O}_\varepsilon} u_j dx = 0, \ j = 1, 2\,, \tag{1.2}$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad \mathcal{O}_\varepsilon\,. \tag{1.3}$$

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For the second assumption in (1.2) to hold we assume that the force f satisfies the same relation: $\int_{\mathcal{O}_\varepsilon} f_j dx = 0,\ j = 1,2$. The equations are supplemented with the following boundary conditions:

$$\begin{cases} x' \in \mathbb{T}^2 \quad \text{(i.e., } u \text{ is } (l_1,l_2)\text{-periodic with respect to } (x_1,x_2)\,), \\ \text{and} \\ u_3|_{x_3=\varepsilon} = 0, \quad \partial_3 u_j|_{x_3=\varepsilon} = 0, \quad j=1,2, \\ u_3|_{x_3=0} = 0, \quad \partial_3 u_j|_{x_3=0} = 0, \quad j=1,2. \\ \text{(free boundary conditions in the thin direction)} \end{cases} \tag{1.4}$$

The force $f = f^\omega(x,t)$ is assumed to be a bounded random kick-force:

$$f = \sum_{k=0}^{\infty} \eta_k^{\varepsilon,\omega}(x)\delta_{kT}(t)\,, \tag{1.5}$$

where $T > 0$ is a fixed number, $\delta_{kT}(t)$ is the δ-function in t, concentrated at kT, and the kicks $\eta_k^{\varepsilon,\omega}$, $k \geq 0$, are independent identically distributed bounded random variables in V_ε. Here and below V_ε is the space of divergence-free H^1-smooth vector fields on $\mathcal{O}_\varepsilon$, satisfying (1.2) and (1.4) (see Section 2.1 for the exact definition).

The random kick-forced 2D NSE and similar to them equations, perturbed by random kick-forces or white in time forces, have been studied recently by many authors, see [8, 14, 2, 7, 13, 15, 16, 17] and references therein. Under suitable hypotheses, concerning the random force, the existence of a unique stationary measure for the corresponding Markov process has been established and properties of this measure have been studied. The results obtained allow to study ergodic properties of this equation and make it possible to justify rigorously basic hypotheses in the theory of 2D space-periodic turbulence (we refer to the survey in [13] for details). However all these results deal with *two*-dimensional models only.

Our main goal in this paper is to extend some of results, available for the 2D case, to the *three*-dimensional NSE, much more realistic from the applied point of view. By many well-known reasons we cannot do it in full generality. Our main restriction is the so-called "thin domain hypothesis" which allows us to profit from the recent developments in the theory of PDE in thin domains.

The study of global existence of smooth solutions for the NSE in thin three-dimensional domains began with the papers of Raugel and Sell [20, 21], who proved global existence of strong solutions for large initial data and forcing terms in the case of periodic conditions (PP) or mixed conditions (PD), i.e. periodic conditions in the vertical thin direction and homogeneous Dirichlet conditions on the lateral boundary. After these publications a number of papers by various authors followed, where the results for (PP) were sharpened [10, 18, 19] and extended to the cases of Dirichlet [1], and other boundary conditions [22], as well as to thin spherical domains [23] and thin two-layer domains [4]. See also [11] for some improvements of all these results for the (PP), (PD) and even for free (FF) boundary conditions. All these results deal with the force f which is L_∞ in time.

In Section 5.1 we use the results from [22] to show that there exist a large set $\mathcal{B}_\varepsilon$ of admissible initial data u_0 in V_ε and a large class of admissible kicks $\eta_k^{\varepsilon,\omega}$ for which problem (1.1)–(1.5) possesses a global unique strong solution $u^\omega(x,t;u_0,\varepsilon)$ for all ε small enough[1].

Problem (1.1)–(1.5) is closely related to the 2D NSE on $\mathbb{T}^2$ (see, e.g., [11] or [22]). To describe this relation, for any integrable vector-field $u(x)$ we define its averaging in the thin

[1] We speak about "large" data in the sense, standard for the theory of the 3D NSE in a thin domain (see, e.g., [20] or [22]). Namely, initial data u_0 and kicks $\eta_k^{\varepsilon,\omega}$ are admissible if for any $C > 0$ there exists $\varepsilon_0 > 0$ such that

$$\varepsilon^{-1}\int_{\mathcal{O}_\varepsilon} |\nabla u_0|^2\,dx \leq C \quad \text{and} \quad \varepsilon^{-1}\int_{\mathcal{O}_\varepsilon} |\nabla \eta_k^{\varepsilon,\omega}|^2\,dx \leq C \quad \text{for} \quad \varepsilon \leq \varepsilon_0\,.$$

In particular, initial data and kicks with finite C^1-norms are admissible.

direction x_3 by the formula

$$(M_\varepsilon u)_j(x) = \frac{1}{\varepsilon}\int_0^\varepsilon u_j(x',\eta)\,d\eta,\ j=1,2, \quad (M_\varepsilon u)_3(x) = 0\,, \tag{1.6}$$

where $x = (x', x_3) \in \mathcal{O}_\varepsilon$. The operator M_ε defines an orthogonal projector in V_ε. So

$$V_\varepsilon = M_\varepsilon V_\varepsilon \oplus N_\varepsilon V_\varepsilon\,, \ \text{ where } \ N_\varepsilon = I - M_\varepsilon\,. \tag{1.7}$$

Since $M_\varepsilon u$ is an x_3-independent vector function with trivial third component, then it may be identified with a 2D vector-field on $\mathbb{T}^2$. Accordingly, we identify $M_\varepsilon V_\varepsilon$ with the space

$$\widetilde{V} = \left\{ u \in H^1(\mathbb{T}^2;\mathbb{R}^2) : \operatorname{div} u = 0,\ \int_{\mathbb{T}^2} u\,dx = 0 \right\}. \tag{1.8}$$

Using the result from [22] (see Theorem 2.2 below for the exact statement) one can show that if u_0 and $\eta_k^{\varepsilon,\omega}$ are admissible and $M_\varepsilon u_0 \to \widetilde{v}_0$, $M_\varepsilon \eta_k^{\varepsilon,\omega} \to \eta_k^\omega$ as $\varepsilon \to 0$ for each k and ω, then

$$M_\varepsilon u^\omega(x,t;u_0,\varepsilon) \to v^\omega(x',t) \quad \text{as} \ \ \varepsilon \to 0\,.$$

Here $u^\omega(x,t;u_0,\varepsilon)$ is a strong solution to (1.1)-(1.5) and v^ω is a solution for the 2D NSE:

$$\partial_t v - \nu\triangle' v + \sum_{j=1}^2 v_j\partial_j v + \nabla' p = \widetilde{f} \quad \text{in} \quad \mathbb{T}^2\times(0,+\infty), \tag{1.9}$$

$$\operatorname{div}' v = 0 \quad \text{in} \quad \mathbb{T}^2\times(0,+\infty)\,; \quad \int_{\mathbb{T}^2} v(x',t)\,dx' = 0\,, \tag{1.10}$$

$$v(x',0) = \widetilde{v}_0(x') \quad \text{in} \quad \mathbb{T}^2\,, \tag{1.11}$$

where $\widetilde{f}$ is the 2D kick-force

$$\widetilde{f} = \sum_k \eta_k^\omega(x')\,\delta_{kT}(t)\,. \tag{1.12}$$

Here and below the prime indicates that we regard the differential operator as an operator with respect to the variable $x' = (x_1, x_2)$.

In this paper we are concerned with asymptotical in t statistical properties of solutions for the 3D NSE (1.1)-(1.5) with initial data in the admissible set $\mathcal{B}_\varepsilon$ and with their relations to statistical properties of solutions for the corresponding 2D problem (1.9)-(1.12).

In our first main result (see Theorem 5.1 and Corollary 5.4) we assume that the kicks satisfy some non-degeneracy conditions and the estimates [2]

$$|M_\varepsilon\eta_k^\varepsilon|_{\widetilde{V}} \le C(\log\varepsilon^{-1})^\sigma, \qquad |\nabla N_\varepsilon\eta_k^\varepsilon|_{0,\varepsilon} \le C\varepsilon^{-\gamma}\,,$$

where $\sigma < \frac{1}{2}$, $\gamma < \frac{1}{2}$ and $|\cdot|_{0,\varepsilon}$ is the L_2-norm on $\mathcal{O}_\varepsilon$ with respect to the normalised measure $\varepsilon^{-1}dx$. We prove that for any $0 \le \tau \le T$ on the set $\mathcal{B}_\varepsilon$ of admissible initial data there exists a unique Borel measure μ_ε^τ which attracts exponentially fast distribution of all (admissible) solutions for the 3D NSE (1.1)-(1.5), evaluated at time $t = kT + \tau$, $k \to \infty$. The measure μ_ε^τ is called the *stationary measure* for the process $k \to u(\cdot, kT+\tau)$. This result is an 3D analogy of the corresponding assertion for the 2D NSE.[3] Its proof is based on application of the abstract theorem from [15, 16] (theorem's statement is given in Section 3). The main difficulty in applying the theorem is to check that the flow-maps of the free 3D NSE possess

[2]Note that the second estimate means that the 'non-2D component' $N_\varepsilon\eta_k^\varepsilon(x)$ of a k-th kick is such that its gradient may be as big as $\varepsilon^{-\gamma}$, but the function itself is small and is bounded by $\varepsilon\cdot\varepsilon^{-\gamma} = \varepsilon^{1-\gamma}$.

[3]The 2D result is first proved in [14], apart from the fact that the rate of convergence is exponential. The exponential rate of convergence was established in [15, 16] and [17]. Detailed discussion and more references see in [13].

the 'squeezing property' with respect to a finite number of leading modes, see Theorem 4.2 below.

Theorem 5.1 means that any solution $u(x,t)$ of the 3D NSE defines the exponentially mixing processes

$$k \to u(\cdot, kT+\tau) \in V_\varepsilon, \quad k = 0,1,\dots,$$

parameterized by $\tau \in [0,T]$. The mixing property implies that each solution u satisfies the Strong Law of Large Numbers:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\big(u(\cdot, kT+\tau)\big) = \int f(u)\mu_\varepsilon^\tau(du) \quad \text{a.s.},$$

and

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(u(\cdot,s))\,ds = \int f(u)\langle\mu_\varepsilon\rangle(du) \quad \text{a.s.}, \qquad \langle\mu_\varepsilon\rangle = \frac{1}{T}\int_0^T \mu_\varepsilon^\tau\,d\tau\,.$$

Here f is a locally Lipschitz functional on V_ε (or on a higher order Sobolev space if the kicks are sufficiently smooth). The second convergence follows from the first one. Concerning the first convergence we note that it follows from Theorem 5.1 by exactly the same argument as in [13], Section 8.

Thus, for flows in a 3D domains $\mathcal{O}_\varepsilon$, stirred by a non-degenerate kick-force (1.5), our results justify two basic hypothesis of the statistical hydrodynamics: firstly, statistical properties of any flow $u(t,x)$ fast approach a unique statistical equilibrium, described by a stationary measure, secondly, time-averages of observable quantities coincide with their averages in ensemble. In particular, the correlation tensor of any flow converges to the correlation tensor of the stationary measure:

$$\begin{gathered}\mathbf{E}\left(u^i(x,kT+\tau)u^j(y,kT+\tau)\right) \to \int_{\mathcal{B}_\varepsilon} \left(u^i(x)u^j(y)\right)\mu_\varepsilon^\tau(du) \quad \text{as } k\to\infty\,, \\ \text{for any } \ \tau\in[0,T],\ i,j\in\{1,2,3\},\ \ x,y\in\mathcal{O}_\varepsilon\,,\end{gathered} \tag{1.13}$$

and to calculate the correlation tensor of the measure one can replace the average in ensemble by average in time.

We also note that the mixing, established in Theorem 5.1, implies that for any functional f as above the processes $\mathbb{N} \ni k \mapsto f(u(\cdot, kT+\tau))$ and $\mathbb{R} \ni t \mapsto f(u(\cdot,t))$ satisfy the Central Limit Theorem, cf. [13], Section 9. This result justifies for the 3D flows which we consider the well known property of the 3D turbulence, stating that on large time-scales observable quantities behave as Gaussian random variables.

Our second result (see Theorem 5.5 and Corollary 5.7) deals with limiting in ε properties of the stationary measures μ_ε^τ. There we assume that the kicks are nondegenerate and satisfy the estimates

$$|M_\varepsilon \eta_k^\varepsilon|_{\tilde V} \le C\,, \qquad |\nabla N_\varepsilon \eta_k^\varepsilon|_{0,\varepsilon} \le C\,,$$

where C is a fixed constant. Let us set $\vartheta_\varepsilon^\tau = M_\varepsilon \circ \mu_\varepsilon^\tau$ (this is a Borel measure on the space $\widetilde V$, defined by the relations $\vartheta_\varepsilon^\tau(Q) = \mu_\varepsilon^\tau(M_\varepsilon^{-1}(Q))$. We show that if $M_\varepsilon \eta_k^{\varepsilon,\omega}$ converge to η_k^ω as $\varepsilon \to 0$ sufficiently fast, then the measures $\vartheta_\varepsilon^\tau$ converge to the measures ϑ^τ. Here ϑ^τ is the unique stationary measure for the process $k \to v(\cdot, kT+\tau) \in \widetilde V$, where $v(x',t)$ is a solution for the kick-forced 2D NSE (1.9), (1.10) with $\widetilde f$ as in (1.12).

As a consequence of these two results we obtain in Theorem 5.8 that for any admissible solution $u(x,t)$ of the 3D NSE the distributions of $(M_\varepsilon u)(x',t)$ in $\widetilde V$ converges *uniformly* in t to the distribution of $v(x',t)$, where $v(x',t)$ solves the 2D problem (1.9)–(1.12). This result is much stronger than its deterministic counterpart (recalled in Section 2.1 as Theorem 2.2), where the convergence is uniform only on *finite* time-intervals.

The assertions of Theorems 5.1 and 5.5 jointly show that under the iterated limit first $t \to \infty$ then $\varepsilon \to 0$ the statistical properties of solutions for the 3D NSE (1.1)-(1.5) converge to those, described by the (unique) stationary measures ϑ^τ for the 2D NSE (1.9)-(1.10). For example, applying first Theorem 5.1 and next Theorem 5.5 to the energy functional $\frac{1}{2}\int |u(x)|^2\, dx$ we can prove that

$$\lim_{\varepsilon\to 0}\lim_{k\to\infty}\frac{\varepsilon^{-2}}{2}\,\mathbf{E}\sum_{j=1}^{2}\int_{\mathbb{T}^2}\Big(\int_0^\varepsilon u_j(x',x_3,kT+\tau)\,dx_3\Big)^2 dx' = \frac{1}{2}\int_{\widetilde{H}} |v|^2_{\widetilde{H}}\,\vartheta^\tau(dv)$$

for any solution u of (1.1)-(1.5) with initial data $u_0 = u_0^\varepsilon$ from the admissible set $\mathcal{B}_\varepsilon$. This means that the long-time limit of the averaged energy of the horizontal component of the 3D Navier-Stokes flow in $\mathcal{O}_\varepsilon$ can be asymptotically calculated from the corresponding 2D model by means of the ensemble averaging. Similar relations hold for full 3D energy and enstrophy. See discussion in Section 7; also see there for more examples.

Finally we note that assertions, similar to Theorems 5.1 and 5.5, remain true with the same proofs for the randomly kicked NSE in the thin spherical layer $S^2 \times (0, \varepsilon)$ (see [23] for corresponding deterministic results). This boundary-value problem may be used to model statistical behaviour a planet's atmosphere: the free boundary condition on the 'sky' $S^2 \times \{\varepsilon\}$ models the effect of gravity which keeps the atmosphere close to the planet, and the free boundary condition on $S^2 \times \{0\}$ models interaction with the surface[4].

The paper is organised as follows. In Section 2 we firstly recall the deterministic results for the 3D NSE (1.1)-(1.4) with a regular force f which we use in the further considerations. Then we define the kick-forced model, and describe our main hypotheses concerning the kicks in Assumption **(D)**. In Section 3 we quote an abstract result (see Theorem 3.2) on random kick-forced evolutions, established in [15, 16]. Section 4 contains the statement of several assertions which constitute the main ingredients in the application of Theorem 3.2 to problem (1.1)–(1.5). The proofs are rather technical and defer to Section 6. Our main results (Theorem 5.1 and Theorem 5.5) are formulated and proved in Section 5. In Section 7 we discuss some hydrodynamical consequences of our results. In Appendix we briefly describe spectral properties of the 3D Stokes operator with the boundary conditions (1.4).

Notation. We denote the integral of a function f against a measure μ as $\int f(u)\,\mu(du)$, or as $\langle f, \mu\rangle$, or as $\langle \mu, f\rangle$. The symbol $\rightharpoonup$ indicates the weak convergence of Borel measures. $\mathcal{D}\xi$ stands for the distribution of a random variable ξ. A map between Banach spaces is called locally Lipschitz if its restriction to any bounded subset of the domain of definition is Lipschitz.

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2 The model

Our main goal in this section is to describe the random kick-forced 3D model. We start with a short survey of known deterministic results.

2.1 Deterministic 3D Navier-Stokes equations on a thin domain

In this subsection we introduce the main functional spaces and collect several known results concerning the 3D NSE (1.1)-(1.4) with a regular force f. We mainly follow the approach presented in [22].

[4]If we replace the free boundary condition on $S^2 \times \{0\}$ by the non-slip condition $u_{S^2\times\{0\}} = 0$, then the analog of Theorem 5.1 remains true, while the limit in Theorem 5.5 trivialises since now the solution goes to zero with ε in an appropriate norm (by the same argument as in Theorem 5.1 in [22]).

Let W_ε be the space of divergence-free vector fields $u = (u_j)_{j=1,2,3}$ on $\mathcal{O}_\varepsilon$ such that

$$u \in \left[H^2(\mathcal{O}_\varepsilon)\right]^3, \quad \int_{\mathcal{O}_\varepsilon} u_j dx = 0, \ j = 1, 2,$$

and condition (1.4) is satisfied. Let V_ε (respectively, H_ε) be the closure of W_ε in $\left[H^1(\mathcal{O}_\varepsilon)\right]^3$ (respectively, in $\left[L^2(\mathcal{O}_\varepsilon)\right]^3$). We denote by $|\cdot|_\varepsilon$ the L_2-norm in H_ε and provide V_ε with the norm

$$\|u\|_\varepsilon \equiv |\nabla u|_\varepsilon = \left[a_\varepsilon(u, u)\right]^{1/2} ,$$

where

$$a_\varepsilon(u, v) = \sum_{j=1}^{3} \int_{\mathcal{O}_\varepsilon} \nabla u_j \cdot \nabla v_j \, dx.$$

We denote by A_ε the Stokes operator, defined as an isomorphism from V_ε onto its dual V'_ε by the relation

$$(A_\varepsilon u, v)_{V,V'} = a_\varepsilon(u, v), \quad u, v \in V_\varepsilon \, .$$

This operator extends to H_ε as a linear unbounded operator with the domain $\mathrm{D}(A_\varepsilon) = W_\varepsilon$. Let Π_ε be the Leray projector on H_ε in $(L^2(\mathcal{O}_\varepsilon))^3$. Then

$$(A_\varepsilon u)(x) = (-\Pi_\varepsilon \Delta u)(x), \quad x \in \mathcal{O}_\varepsilon,$$

for every $u \in \mathrm{D}(A_\varepsilon) = W_\varepsilon$.

Now we consider the trilinear form

$$b_\varepsilon(u, v, w) = \sum_{j,l=1}^{3} \int_{\mathcal{O}_\varepsilon} u_j \, \partial_j v_l \, w_l \, dx, \quad u, v \in \mathrm{D}(A_\varepsilon), \ w \in (L^2(\mathcal{O}_\varepsilon))^3.$$

It defines the bilinear operator $B_\varepsilon : V_\varepsilon \mapsto {V_\varepsilon}'$ by the formula

$$(B_\varepsilon(u, v), w)_{V_\varepsilon, V'_\varepsilon} = b_\varepsilon(u, v, w), \quad u, v, w \in V_\varepsilon.$$

Now the system (1.1)–(1.4) can be written in the form

$$u' + \nu A_\varepsilon u + B_\varepsilon(u, u) = f, \quad u(0) = u_0. \tag{2.1}$$

The following result concerning this system is known.

Theorem 2.1 ([22]) *Assume that* $u_0 \in V_\varepsilon$, $f \in L^\infty(\mathbb{R}_+; H_\varepsilon)$ *and*

$$\|u_0\|_\varepsilon + \sup_t |f(t, \cdot)|_\varepsilon \le R(\varepsilon) \, , \tag{2.2}$$

where $R(\varepsilon)$ *satisfies*

$$\lim_{\varepsilon \to 0} \varepsilon^{\theta_0} R(\varepsilon) = 0 \tag{2.3}$$

with some $\theta_0 \in (0, 1/2)$. *Then there exists a positive constant* $\varepsilon_0 = \varepsilon_0(R)$, *depending on the parameters of problem (1.1)–(1.4), such that for* $\varepsilon \in (0, \varepsilon_0]$, *problem (1.1)–(1.4) has a strong solution*

$$u \in C([0, T]; V_\varepsilon) \cap L^2((0, T); W_\varepsilon), \quad \forall T > 0.$$

This solution is unique in the class of weak Leray solutions.

Let us consider the 2D NSE (1.9)-(1.11). Clearly

$$\begin{aligned}&\textit{if } v(x,t) \textit{ is a solution of } (1.9)\text{-}(1.11)\textit{, then } u(x',x_3,t)=(v_1,v_2,0)^t(x',t)\\ &\textit{satisfies } (1.1)\text{-}(1.4)\textit{, where } f=(\widetilde{f},0) \textit{ and } u_0=(\widetilde{v}_0,0).\end{aligned} \tag{2.4}$$

On the contrary, let $u(x,t)$ be a solution of (1.1)-(1.3). Then the 2D vector-field $M_\varepsilon u$ (see (1.6)) converges to a solution of (1.9)-(1.10) when $\varepsilon \to 0$. To state the corresponding result we define the space $\widetilde{V}$ as in the Introduction (see (1.8)) and define the space $\widetilde{H}$ as the L_2-space of divergence-free vector functions on $\mathbb{T}^2$ with zero mean-value.

Theorem 2.2 ([22]) *Let the hypotheses of Theorem 2.1 be in force. Assume in addition that* $\|M_\varepsilon u_0\|_{\widetilde{V}}$ *and* $\sup_{t\in\mathbb{R}} |M_\varepsilon f(t)|_{\widetilde{H}}$ *are bounded uniformly in* ε *and that there exist* $\widetilde{f} \in L^\infty(\mathbb{R}_+;\widetilde{H})$ *and* $\widetilde{v}_0 \in \widetilde{V}$ *such that*

$$\lim_{\varepsilon\to 0} M_\varepsilon f(t) = \widetilde{f}(t) \ \textit{for a.e. } t\,, \quad \textit{and} \quad \lim_{\varepsilon\to 0} M_\varepsilon u_0 = \widetilde{v}_0$$

in the sense of weak convergence in $\widetilde{H}$*. Then for any* $T>0$ *we have*

$$\lim_{\varepsilon\to 0} M_\varepsilon u(\cdot) = v(\cdot) \quad \textit{in} \quad C([0,T];\widetilde{H}) \cap L^2((0,T);\widetilde{V}), \tag{2.5}$$

where $v(t)$ *solves the 2D NSE* (1.9)–(1.11)*.*

Theorem 2.1 allows to define the flow-maps S_ε^T, $T \geq 0$:

$$S_\varepsilon^T : \{\|u_0\|_\varepsilon \leq R(\varepsilon)\} \to V_\varepsilon, \quad S_\varepsilon^T u_0 = u(T),$$

where $u(t)$ solves the NSE (1.1)–(1.4) with $f \equiv 0$. Well known properties of the NSE (see [5]) imply that for any $T>0$ and $k \in \mathbb{N}$

$$\text{the map } \ S_\varepsilon^T : \{\|u_0\|_\varepsilon \leq R(\varepsilon)\} \to V_\varepsilon \cap H^k(\mathcal{O}_\varepsilon;\mathbb{R}^3) \ \text{ is Lipschitz.} \tag{2.6}$$

We denote by $\{S_0^T,\, T\geq 0\}$ the flow-maps of the 2D NSE (1.9)-(1.11) with $\widetilde{f}\equiv 0$. They are continuous in $\widetilde{V}$ and extend to continuous transformations of $\widetilde{H}$. Due to (2.4),

$$S_\varepsilon^T\,|_{\widetilde{V}} = S_0^T \quad \forall\, T\,,\ \forall\, \varepsilon\,.$$

Similar to (2.6), for any $T>0$ and $k \in \mathbb{N}$ we have

$$\text{the map } \ S_0^T : \widetilde{H} \to \widetilde{H} \cap H^k(\mathbb{T}^2;\mathbb{R}^2) \ \text{ is locally Lipschitz.} \tag{2.7}$$

2.2 Random kick-forced 3D NS model

In this subsection we describe our model. We consider problem (1.1)–(1.4) with a random external force which is a generalised vector-function of the form

$$f(x,t) = \sum_{k=1}^{\infty} \eta_k^\varepsilon(x)\,\delta_{kT}(t)\,, \quad \eta_k \in H_\varepsilon \quad \forall\, k\,, \tag{2.8}$$

where $T>0$ is fixed and $\delta_{kT}(t)$ is a δ-function concentrated at kT. Forces of this form are called kick-forces, and the functions $\eta_k^\varepsilon(x)$ are called kicks. Corresponding solutions of (1.1)–(1.4) are discontinuous in t. We normalise them to be continuous from the right. Then the solution u of the problem (1.1)-(1.4) is

- a solution of the free (unforced) NSE for $t \neq kT$, $k \in \mathbb{Z}$,

- at $t = kT$ it has the jump $\eta_k^\varepsilon(x)$.

So the dynamics of the kick-forced NS model can be described by the following relations:

$$\left.\begin{array}{rcl} u(x,0) & = & u_0(x), \\ u(x,(k+1)T) & = & S_\varepsilon^T u(x,kT) + \eta_{k+1}^\varepsilon(x) \quad \text{for } k = 0,1,2,\dots, \\ u(x,kT+\tau) & = & S_\varepsilon^\tau u(x,kT) \quad \text{if } \ 0 \le \tau < T\,, k = 0,1,2,\dots . \end{array}\right\} \tag{2.9}$$

We refer to [13, Sect.2.6] for some details concerning description of a kick model, based on the 2D NS equations.

Our main hypothesis concerning the kicks $\{\eta_k^\varepsilon\}$ is the following:

(D) The kicks $\eta_1^\varepsilon, \eta_2^\varepsilon, \dots$ depend on ε and are V_ε-valued random variables, independent and identically distributed, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Using the basis $\{e_{\lambda_j}, e_{\Lambda_j^\varepsilon}, j \ge 1\}$ (see *Appendix*) we write them in the form

$$\eta_k^\varepsilon = \eta_k^{\varepsilon,\omega} = \sum_j b_j^\varepsilon \xi_{jk}^{\varepsilon,\omega}\, e_{\lambda_j}(x) + \sum_j \hat{b}_j^\varepsilon\, \hat{\xi}_{jk}^{\varepsilon,\omega} e_{\Lambda_j^\varepsilon}(x)\,. \tag{2.10}$$

Here b_j^ε, $\hat{b}_j^\varepsilon$ are non-negative real numbers and ξ_{jk}^ε, $\hat{\xi}_{jk}^\varepsilon$ are independent (scalar) random variables such that

$$\mathcal{D}\xi_{jk}^\varepsilon = p_j^\varepsilon(x)\,dx\,, \quad \mathcal{D}\hat{\xi}_{jk}^\varepsilon = \hat{p}_j^\varepsilon(x)\,dx$$

($\mathcal{D}\xi$ stands for the distribution of a random variable ξ). The random variables and the densities satisfy the following properties:

- $p_j^\varepsilon(x) = p_j^\varepsilon(x) = 0$ for all $|x| \ge 1$ and for every ε and j;
- each p_j^ε and $\hat{p}_j^\varepsilon$ is a function of bounded total variation;
- for each $\gamma > 0$ and every j and ε we have

$$\int_{-\gamma}^{\gamma} p_j^\varepsilon(x)\,dx > 0\,, \quad \int_{-\gamma}^{\gamma} \hat{p}_j^\varepsilon(x)\,dx > 0\,.$$

Example: each ξ_{jk}^ε (each $\hat{\xi}_{jk}^\varepsilon$) is a random variable, uniformly distributed on a segment $[a_j, b_j]$ (on $[\hat{a}_j, \hat{b}_j]$), where $-1 \le a_j < 0 < b_j \le 1$ and $-1 \le \hat{a}_j < 0 < \hat{b}_j \le 1$.

Remark 2.3 For a fixed ε the hypotheses, imposed on the kicks, are exactly the same as in Condition (D) in [15, 16] but written with respect to the basis $\{e_{\lambda_j}, e_{\Lambda_j^\varepsilon}, j \ge 1\}$.

Due to the first assumption in **(D)**, concerning the densities p_j^ε and $\tilde{p}_j^\varepsilon$, without loss of generality we can assume that the random variables $|\xi_{jk}^\varepsilon|$ and $|\hat{\xi}_{jk}^\varepsilon|$ are bounded by 1 for *all* ω. Therefore relations (8.2) and (8.4) imply that

$$\|M_\varepsilon \eta_k^\varepsilon\|_\varepsilon^2 \le \varepsilon \sum (b_j^\varepsilon)^2 =: \varepsilon B^\varepsilon\,, \quad \|N_\varepsilon \eta_k^\varepsilon\|_\varepsilon^2 \le \varepsilon \sum (\hat{b}_j^\varepsilon)^2 =: \varepsilon \hat{B}^\varepsilon\,, \tag{2.11}$$

for every ε, k and ω, where M_ε is defined by (1.6) and $N_\varepsilon = I - M_\varepsilon$.

Our main goal in this paper is to show that the kick evolution in (2.9) is well defined on some large subset of V_ε and to study its statistical properties.

3 Preliminaries on random kick models

Let $\mathcal{H}$ be a separable Hilbert space with a norm $\|\cdot\|$ and an orthogonal basis $\{e_j\}$. Assume that $\mathcal{B}$ is a closed bounded subset of $\mathcal{H}$ containing the origin, and $S : \mathcal{B} \mapsto \mathcal{B}$ is a mapping, satisfying the following conditions:

(A) There exists positive constants $\gamma < 1$ and C such that

$$\|Su\| \leq \gamma \|u\| \quad \text{and} \quad \|Su_1 - Su_2\| \leq C\|Su_1 - Su_2\|$$

for all $u, u_1, u_2 \in \mathcal{B}$.

(B) There exists a sequence $\{b_j\}$ of nonnegative numbers such that

$$\sum_{j=1}^{\infty} b_j^2 \|e_j\|^2 < \infty$$

and $S\mathcal{B} + \mathcal{K} \subset \mathcal{B}$, where

$$\mathcal{K} = \left\{ u = \sum_{j=1}^{\infty} u_j e_j \ : \ |u_j| \leq b_j \quad \text{for all } j \geq 1 \right\}.$$

(C) The set $S\mathcal{B}$ is compact and there exists $N \in \mathbb{N}$ such that

$$\|(I - P_N)(Su_1 - Su_2)\| \leq \frac{1}{2}\|u_1 - u_2\| \quad \text{for all} \quad u_1, u_2 \in \mathcal{B},$$

where P_N is the orthogonal projector on the space $\text{Span}\,\{e_1, \dots, e_N\}$.

Remark 3.1 Assumption (A) is a bit stronger than (A) in [15, 16]. However it is formulated on a *bounded subset* of the space $\mathcal{H}$. The invariance property in our version of (B) is different from the corresponding requirement in [15, 16], where some kind of dissipativity is assumed. We do not need any dissipativity hypotheses because we consider dynamics in the bounded invariant set $\mathcal{B}$. As for (C), the papers [15, 16] assume that this relation holds with the factor $1/2$ replaced by q_N, where $q_N \to 0$ as $N \to \infty$. In this form the assumption also implies the compactness of the set $S\mathcal{B}$ which is needed for the existence of an invariant measure. However the analysis in [15, 16] shows that if we already know the existence of an invariant measure, then for its uniqueness it suffices to require (C) with $q_N = 1/2$ (see the proof of Lemma 3.2 in [15]).

Now we consider a sequence $\{\eta_k\}$ of i.i.d. random variables in $\mathcal{H}$, defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, having the form

$$\eta_k = \eta_k^\omega = \sum_j b_j \xi_{jk}^\omega \, e_j, \quad k = 1, 2, \dots . \tag{3.1}$$

The coefficients $b_j \geq 0$ are the same as in Assumption (B) and ξ_{jk} are independent random variables, possessing the same properties as the random variables ξ_{jk}^ε and $\hat{\xi}_{jk}^\varepsilon$ appearing in Assumption **(D)** in Subsect. 2.2.

Hypotheses (A)–(C) and the properties of $\{\eta_k\}$ allow to define a discrete-time random dynamical system (RDS) on $\mathcal{B}$ by the relation

$$u^k = Su^{k-1} + \eta_k, \quad k = 1, 2, \dots, \quad u_0 \in \mathcal{B}. \tag{3.2}$$

Let us denote by $\mathcal{M}(\mathcal{B})$ the set of probability Borel measures on $\mathcal{B}$, given the Lipschitz-dual distance

$$\operatorname{dist}(\mu,\vartheta)=\operatorname{dist}_{\mathcal{M}(\mathcal{B})}(\mu,\vartheta)=\sup_{f\in\mathcal{L}}\langle\mu-\vartheta,f\rangle\,, \tag{3.3}$$

where

$$\mathcal{L}=\mathcal{L}(\mathcal{B})=\{f:\mathcal{B}\mapsto\mathbb{R}:\ \operatorname{Lip}(f)\le 1\ \text{ and }\ |f|\le 1\}\,.$$

With this norm the set $\mathcal{M}(\mathcal{B})$ becomes a complete metric space, where the convergence in the norm is equivalent to the $*$-weak convergence of measures, see [6] and [13]. This fact holds for any set $\mathcal{B}$ which is a complete separable metric space. Since in our case $\mathcal{B}$ is bounded, then, equivalently, $\mathcal{L}$ may be replaced by the bigger (and more convenient) set $\mathcal{L}_0$, formed by all functions f on $\mathcal{B}$ such that $\operatorname{Lip} f\le 1$.

The RDS (3.2) defines transformations $\{\Psi_k,\,k\ge 0\}$ of the set $\mathcal{M}(\mathcal{B})$:

$$\Psi_k(\mu)=\vartheta,\quad \vartheta(Q)=\int \mathbf{P}\{u^k(u_0)\in Q\}\,\mu(du_0)\,, \tag{3.4}$$

where the sequence $u^k=u^k(u_0)$ is calculated according to (3.2). Clearly they can be extended to linear transformations of the space of signed Borel measures, and it is easy to see that $\Psi_k=(\Psi_1)^k$. A measure μ is called a *stationary* measure for the RDS (3.2) if $\Psi_k\mu=\mu$ for each k.

The arguments, given in the proof of Theorem 1.1 from [16], lead to the following result:

Theorem 3.2 *Let Assumptions (A)-(C) be in force and the kicks η_k be given by* (3.1) *with ξ^ω_{jk} satisfying conditions in* **(D)**, *and*

$$b_j\neq 0\quad for\quad 1\le j\le N\,,$$

where $N<\infty$ depends on the parameters of the equation and the kicks. Then the RDS (3.2) *has a unique stationary measure μ and*

$$\operatorname{dist}(\mu,\Psi_k(\vartheta))\le Ce^{-ck}\quad\forall\,\vartheta\in\mathcal{M}(\mathcal{B})\,,$$

with some $C,c>0$.

We will apply this theorem to the evolution in (2.9). To do this we need to study further the properties of solutions to problem (1.1)–(1.4) in order to verify the conditions (A)–(C) above. We do this in the next section.

4 Flow-maps

We recall that $\{S^T_\varepsilon\,,T\ge 0\}$ stand for the flow-maps of the NSE (1.1)–$(1.4)_{f\equiv 0}$, and introduce the set

$$\mathcal{B}_\varepsilon=\{u\in V_\varepsilon:\ \|M_\varepsilon u\|_\varepsilon\le a(\varepsilon),\ \|N_\varepsilon u\|_\varepsilon\le b(\varepsilon)\}\,. \tag{4.1}$$

Here $a(\varepsilon)$, $b(\varepsilon)$ are positive real numbers such that $R_*(\varepsilon)=a(\varepsilon)+b(\varepsilon)$ satisfy (2.3), so Theorem 2.1 insures that the flow-maps S^T_ε are well-defined on $\mathcal{B}_\varepsilon$. The following assertion is proved in Section 6, as well as Theorem 4.2 and Proposition 4.4 below.

Proposition 4.1 *For any $T_0>0$ we can find $\varepsilon_0>0$ and $k_*,\gamma\in(0,1)$ such that if $\eta\in V_\varepsilon$ satisfies*

$$\|M_\varepsilon\eta\|_\varepsilon\le k_*a(\varepsilon),\ \|N_\varepsilon\eta\|_\varepsilon\le k_*b(\varepsilon),$$

where

$$\lim_{\varepsilon\to 0}\frac{\sqrt{\varepsilon}\,b^2(\varepsilon)}{a(\varepsilon)}=0 \tag{4.2}$$

and (2.3) *holds with* $R_* = a + b$, *then for any* $T \geq T_0$ *and* $\varepsilon < \varepsilon_0$ *the set* $\mathcal{B}_\varepsilon$ *is invariant for the mapping*

$$u \mapsto S_\varepsilon^T u + \eta\,.$$

Besides,

$$\|S_\varepsilon^T u\|_\varepsilon \leq \gamma \|u\|_\varepsilon \ \text{ for any } \ u \in \mathcal{B}_\varepsilon\,. \tag{4.3}$$

Let $\{e_{\lambda_k}, \lambda_k\}$ be the eigenfunctions and eigenvalues of the operator A_ε, corresponding to the 2D Stokes operator (see *Appendix*). We denote by $\tilde{P}_N$ the projector of V_ε on the subspace $\mathrm{Span}\{e_{\lambda_1}, \dots, e_{\lambda_N}\}$.

Theorem 4.2 *Assume that* $a(\varepsilon)$ *and* $b(\varepsilon)$ *satisfy* (4.2) *and*

$$a(\varepsilon) \leq C_1 \sqrt{\varepsilon} \left[\log \frac{1}{\varepsilon}\right]^\sigma, \quad b(\varepsilon) \leq C_2 \left[\log \frac{1}{\varepsilon}\right]^{\sigma/2}, \tag{4.4}$$

where $0 \leq \sigma < 1/2$. *Take any* $u_1, u_2 \in \mathcal{B}_\varepsilon$. *Then for each* $T > 0$ *we have*

- *if* $\sigma > 0$, *then there exists* $\varepsilon_0 \in (0, 1]$ *and for every* $\delta > 0$ *there is* $C_\delta > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0]$ *we have*

  $$\|(S_\varepsilon^T u_1 - S_\varepsilon^T u_2)\|_\varepsilon \leq C_\delta \left(\frac{1}{\varepsilon}\right)^\delta e^{-\frac{\lambda_1 \nu}{2} T} \|u_1 - u_2\|_\varepsilon\,. \tag{4.5}$$

  If $\sigma = 0$, *then this estimate holds with* $\delta = 0$.

- *For any* $q < 1$ *there exists* $\varepsilon_0 > 0$, *and for* $\varepsilon \in (0, \varepsilon_0]$ *there exists* $N = N(\varepsilon, T)$ *such that*

  $$\|(I - \tilde{P}_N)(S_\varepsilon^T u_1 - S_\varepsilon^T u_2)\|_\varepsilon \leq q \|u_1 - u_2\|_\varepsilon\,. \tag{4.6}$$

  Moreover, if $\sigma = 0$, *then* N *may be chosen independent of* ε.

Remark 4.3 The assumptions in Theorem 4.2 allow the initial data and the vectors η to be large. For instance, if both $u_0 = (u_{01}, u_{02}, u_{03})$ and $\eta = (\eta_1, \eta_2, \eta_3)$ are restrictions on $\mathcal{O}_\varepsilon$ of $C^1(\overline{\mathcal{O}}_1)$-functions, then

$$\|M_\varepsilon u_0\|_\varepsilon^2 + \|N_\varepsilon u_0\|_\varepsilon^2 \leq \varepsilon c_0 |u_0\|_{C^1(\mathcal{O}_1)}^2,$$

and

$$\|M_\varepsilon \eta\|_\varepsilon^2 + \|N_\varepsilon \eta\|_\varepsilon^2 \leq \varepsilon c_0 \|\eta\|_{C^1(\mathcal{O}_1)}^2.$$

Therefore choosing $a^2(\varepsilon) = c_0 \varepsilon \left(\log \frac{1}{\varepsilon}\right)^{3/4}$ and $b^2(\varepsilon) = c_0 \left(\log \frac{1}{\varepsilon}\right)^{1/4}$ satisfying (4.2) and (4.4) we can see that large values of u_0 and η are possible, when ε is small. In general, with this choice of $a^2(\varepsilon)$ and $b^2(\varepsilon)$, C^1-norms of u_0 and η may be of order $\left(\log \frac{1}{\varepsilon}\right)^{3/4}$.

The following assertion is useful in the study of limit behaviour as $\varepsilon \to 0$ of random kick evolution in (2.9).

Proposition 4.4 *Under the conditions of Theorem 4.2 the set* $A_\varepsilon S_\varepsilon^T \mathcal{B}_e$ *is bounded in* H_ε *for every* ε. *Moreover, if* ε_0 *is sufficiently small, then for any* $\rho > 0$ *there exist* $C(\rho)$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$ *and* $0 < \tau \leq T$ *we have*

$$\sup\left\{|A_\varepsilon S_\varepsilon^\tau u|_\varepsilon \,:\, u \in \mathcal{B}_\varepsilon,\ \|u\|_\varepsilon \leq \rho\sqrt{\varepsilon}\right\} \leq C(\rho) \left(1 + \frac{1}{\sqrt{\tau}}\right) \sqrt{\varepsilon}\,. \tag{4.7}$$

5 Main results

Now we are in position to state and prove our main results.

5.1 Well-definiteness of RDS

We return to the formal evolutions described in (2.9) and assume that the quantities B^ε and $\hat{B}^\varepsilon$, defined in (2.11), satisfy the inequalities

$$\sqrt{\varepsilon B^\varepsilon} \le a_*(\varepsilon) \ \text{ and } \ \sqrt{\varepsilon \hat{B}^\varepsilon} \le b_*(\varepsilon), \tag{5.1}$$

where $a_*(\varepsilon)$ and $b_*(\varepsilon)$ meet (4.2) and (4.4) with some $\sigma \in [0, 1/2)$. Let us set

$$a(\varepsilon) = Ca_*(\varepsilon)\,, \quad b(\varepsilon) = Cb_*(\varepsilon)\,.$$

Choosing C sufficiently large we achieve that $a(\varepsilon)$ and $b(\varepsilon)$ satisfy assumptions of Proposition 4.1 and Theorem 4.2. In particular, the set $\mathcal{B}_\varepsilon$ (see (4.1)) is invariant for the RDS (3.2) with $S = S_\varepsilon^T$ and $\eta_k = \eta_k^{\varepsilon,\omega}$:

$$u^k = S_\varepsilon^T(u^{k-1}) + \eta_k^{\varepsilon,\omega}, \quad k = 1, 2, \dots \tag{5.2}$$

Accordingly, the dynamics in (2.9) is globally well-defined for $T \ge T_0$ and $\varepsilon \le \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(T_0)$ is the same as in Proposition 4.1.

Thus, equations (1.1)–(1.4), where f is the random kick-force given by (2.8), defines in $\mathcal{B}_\varepsilon$ the dynamics

$$u_0 \longmapsto u(k; u_0)\,, \quad k \ge 0\,,$$

where $u(k; u_0) = u(\cdot, kT)$ and $u(x,t)$ is calculated according to (2.9) (that is, according to (5.2) with $u^0 = u_0$).

Let us take any $\tau \in [0, T]$. If $u(x,t)$ is calculated using (2.9), then $u^{k,\tau} := u(kT + \tau, \cdot)$, $k \ge 0$,[5] is a trajectory of the RDS

$$u^{k,\tau} = S_\varepsilon^\tau\big(S_\varepsilon^{T-\tau}(u^{k-1,\tau}) + \eta_k^{\varepsilon,\omega}\big) \tag{5.3}$$

(for $\tau = 0$ it coincides with the system (5.2)). For the same reasons as before (see also the argument given in the proof of Proposition 4.1 in Section 6), the set $\mathcal{B}_\varepsilon$ is invariant for this system for any τ if $\varepsilon \ll 1$.

Our goal is to study asymptotical properties of the RDS's (5.3) with $0 \le \tau \le T$ as $k \to \infty$ and their limiting properties as $\varepsilon \to 0$. We pay the main attention to the case $\tau = 0$, i.e., to RDS (5.2).

5.2 Asymptotical behaviour of solutions

For the purposes of this subsection and of the next one it is convenient to provide the space V_ε with the scaled norm

$$\|\cdot\|_{0,\varepsilon} = \varepsilon^{-1/2}\|\cdot\|_\varepsilon\,.$$

Note that the basis $\{e_{\lambda_j}, e_{\Lambda_j^\varepsilon}, j \ge 1\}$ is a Hilbert basis of the space $(V_\varepsilon, \|\cdot\|_{0,\varepsilon})$ and that the projection $M_\varepsilon : (V_\varepsilon, \|\cdot\|_{0,\varepsilon}) \to \tilde{V}$ has unit norm. We furnish the set $\mathcal{B}_\varepsilon$ with the distance, corresponding to this norm. As in Section 3 we denote by $\mathcal{M}(\mathcal{B}_\varepsilon)$ the set of probability Borel measures on $\mathcal{B}_\varepsilon$, given the Lipschitz-dual distance (3.3) (with $\mathcal{B} := \mathcal{B}_\varepsilon$). The RDS (5.2) defines transformations $\{\Psi_k^\varepsilon, k \ge 0\}$ of the set $\mathcal{M}(\mathcal{B}_\varepsilon)$ (see (3.4)). Due to the relation

$$\langle \Psi_k^\varepsilon(\vartheta), g\rangle = \int_{\mathcal{B}_\varepsilon} \mathbf{E} g\big(u(k; u_0)\big)\, \vartheta(du_0)$$

[5] For $\tau = T$ we define by continuity $u^{k,T} = \lim_{\tau \to T-0} u(kT + \tau, \cdot)$.

it follows from (4.5) that the transformations Ψ_k^ε are continuous in the $*$-weak topology. We note that $\Psi_k^\varepsilon(\vartheta) = \mathcal{D}u(\cdot, kT)$, where $u(x,t)$ is a solution (in the sense (2.9)) for the problem (1.1)–(1.4) with $f = f^\varepsilon$ given by (2.8), and u_0 is a random vector, independent of the kicks $\eta_k^{\varepsilon,\omega}$, $k \geq 1$, and such that $\mathcal{D}(u_0) = \vartheta$.

The main result in this subsection is the following assertion.

Theorem 5.1 *Let Assumption* **(D)** *and condition* (5.1) *be in force. Then for any $T_0 > 0$ there exist $\varepsilon_0 \in (0,1)$ and $c_0 > 0$ such that for $T \geq T_0$ and $0 < \varepsilon \leq \varepsilon_0$ we have:*

1) the set

$$\mathcal{B}_\varepsilon = \{u \in V_\varepsilon : \|M_\varepsilon u\|_\varepsilon \leq c_0 a_*(\varepsilon), \ \|N_\varepsilon u\|_\varepsilon \leq c_0 b_*(\varepsilon)\}$$

is invariant with respect to the RDS (5.2)*.*

2) There exists $N = N(\varepsilon) \in \mathbb{N}$ such that under the condition

$$b_j^\varepsilon > 0 \quad \forall\, 1 \leq j \leq N\,, \tag{5.4}$$

imposed on the kicks amplitudes in (2.10)*, the system* (5.2)*, interpreted as an RDS in $\mathcal{B}_\varepsilon$, has a unique stationary measure μ_ε, and*

$$\mathrm{dist}_{\mathcal{M}(V_\varepsilon)}(\mu_\varepsilon, \Psi_{\varepsilon k}(\vartheta)) \leq C_\varepsilon e^{-c_\varepsilon k} \tag{5.5}$$

for every $\vartheta \in \mathcal{M}(\mathcal{B}_\varepsilon)$, with some $C_\varepsilon, c_\varepsilon > 0$.

3) Under the condition

$${C'}^{-1} \leq B_\varepsilon \leq C'\,, \ \ \hat{B}_\varepsilon \leq C' \quad \text{for all}\, \varepsilon \ \text{ and for some } C' > 1\,, \tag{5.6}$$

where the values B^ε and $\hat{B}^\varepsilon$ defined in (2.11)*, the number N in* (5.4) *does not depend on ε. Moreover, if, in addition, the random variables ξ_{jk}^ε and $\hat{\xi}_{jk}^\varepsilon$ do not depend on ε, and* (5.4) *strengthens as $\inf_{\varepsilon>0} \min_{1\leq j\leq N} b_j^\varepsilon > 0$, then the constants C_ε and c_ε in* (5.5) *can be chosen independent of ε, provided the initial measure ϑ satisfies the relation*

$$\mathrm{supp}\,\vartheta \subset \mathcal{B}_\varepsilon \cap \{u \in V_\varepsilon \ : \ \|u\|_{0,\varepsilon} \leq c_1\}\,.$$

In this case the stationary measure μ_ε is supported by the set

$$\mathcal{B}_\varepsilon^0 = \{u \in V_\varepsilon : \|M_\varepsilon u\|_{0,\varepsilon} \leq C, \ \|N_\varepsilon u\|_{0,\varepsilon} \leq C\} \tag{5.7}$$

for some constant $C > 0$.

We recall that estimates (5.5) means the following: if $u^\omega(x,t)$, $t \geq 0$, is a random solution of the kick-forced NSE (1.1)-(1.4) such that $u_0 = u_0^\omega \in \mathcal{B}_\varepsilon$ for every ω and $g \in \mathcal{L}_0(\mathcal{B}_\varepsilon)$ (see the notations in Section 3), then

$$|\mathbf{E}\, g(u(\cdot, kT)) - \langle g, \mu_\varepsilon\rangle| \leq C_\varepsilon e^{-c_\varepsilon k} \quad \text{for } \ k \geq 0\,. \tag{5.8}$$

Proof. The invariance of the set $\mathcal{B}_\varepsilon$ follows from argument given in Subsection 5.1. To prove the existence of a stationary measure for (5.2) we note that due to (2.6) $S_\varepsilon^T(\mathcal{B}_\varepsilon)$ is a compact subset in $\mathcal{B}_\varepsilon$ and we can use the standard Krylov-Bogolyubov procedure to prove that a stationary measure exists (see, e.g., [13, Section 3.3] for some details).

The uniqueness of a stationary measure follows from Theorem 3.2 since the assumption (5.4) with a suitable $N = N(\varepsilon)$ jointly with the established properties of the system (5.2) imply that it satisfies the assumptions (A)-(C) from Section 3 and **(D)** from Subsection 2.2, so Theorem 3.2 applies. Indeed, in the assumption (A) the first relation follows from (4.3) and the second one follows from (4.5). Assumption (B) holds trivially by the statement of Proposition 4.1. As for Assumption (C), the compactness of the set $S_\varepsilon^T(\mathcal{B}_\varepsilon)$ is established above and the squeezing relation follows from (4.6). Finally, **(D)** is the set of assumptions which we have imposed on the densities p_j^ε and $\hat{p}_j^\varepsilon$. Consequently Theorem 3.2 implies the uniqueness of a stationary measure and relation (5.5).

Under condition (5.6) we can assume that $a_*(\varepsilon)$ and $b_*(\varepsilon)$ in (5.1) satisfy (4.2) and (4.4) with $\sigma = 0$. Then the set $\mathcal{B}_\varepsilon$ has a diameter of order one (with respect to the norm $\|\cdot\|_{0,\varepsilon}$) uniformly in ε and the r.h.s. in (4.5) is independent of ε, as well as the constant N in (4.6). Moreover, the Lipschitz constant C in (A) is ε-independent. That is, all the data, needed to apply Theorem 3.2, are independent from ε. Thus $N(\varepsilon)$, C_ε and c_ε in Theorem 5.1 can be chosen independent of ε. •

Remark 5.2 Since the set $\mathcal{B}_\varepsilon$, supporting all relevant measures, is bounded, then the convergence holds for locally Lipschitz functions g on V_ε (i.e., for functions, which are Lipschitz on bounded subsets of V_ε).

Remark 5.3 If g is a locally Lipschitz function on a Sobolev space $H^n(\mathcal{O}_\varepsilon)$, $n \geq 1$, then the convergence still holds, provided that the b-coefficients b_j^ε and $\hat{b}_j^\varepsilon$ decay with j sufficiently fast. Indeed, if

$$\sum_j \left(\lambda_j^n \left|b_j^\varepsilon\right|^2 + \left[\Lambda_j^\varepsilon\right]^n \left|\hat{b}_j^\varepsilon\right|^2 \right) \leq C_\varepsilon < \infty \tag{5.9}$$

for some $n \geq 1$, then $\|\eta_k^\varepsilon\|_{\mathcal{H}_n(\mathcal{O}_\varepsilon)} \leq C'_\varepsilon$ for each $\omega \in \Omega$, where

$$\mathcal{H}_n(\mathcal{O}_\varepsilon) := D(A_\varepsilon^{n/2}) \subset H^n(\mathcal{O}_\varepsilon).$$

Since $\|S_\varepsilon^T(u)\|_{\mathcal{H}_n(\mathcal{O}_\varepsilon)} \leq C'_\varepsilon$ by (2.6), then now the stationary measure μ_ε is supported by a bounded set in $\mathcal{H}_n(\mathcal{O}_\varepsilon)$. Due to the arguments in [13], Section 6.4, we have that under the condition in (5.9) the convergence in (5.8) holds for any measurable function g which is a locally Lipschitz function on $\mathcal{H}_{n-1}(\mathcal{O}_\varepsilon)$. The corresponding constant C_ε depends on g. In particular, if (5.9) holds with $n = 3$, then we can take $g(u) = u^i(x)u^j(y)$, where $i, j \in \{1, 2, 3\}$ and any $x, y \in \mathcal{O}_\varepsilon$ are fixed. Since $H^2(\mathcal{O}_\varepsilon) \subset C(\overline{\mathcal{O}}_\varepsilon)$, then g is a locally Lipschitz function on $H^2(\mathcal{O}_\varepsilon)$. Thus we obtain relation (1.13) claimed in the Introduction.

Corollary 5.4 *Let the assumptions of Theorem 5.1 be in force and h be a locally Lipschitz function on the space $H^l(\mathcal{O}_\varepsilon; \mathbb{R}^3)$ for some $l \in \mathbb{N}$. Then for any $0 < \tau \leq T$ we have*

$$|\mathbf{E}\, h(u(\cdot, kT + \tau)) - \langle h, \mu_\varepsilon^\tau \rangle| \leq C_{\varepsilon,h,\tau} e^{-c_\varepsilon k} \quad \text{for } k = 0, 1, \ldots . \tag{5.10}$$

Here $\mu_\varepsilon^\tau = S_\varepsilon^\tau \circ \mu_\varepsilon$ and u is a solution of (1.1)-(1.3), where $u_0 = u_0^\omega \in \mathcal{B}_\varepsilon$ for every ω. Moreover, if the assumptions of item 3) of Theorem 5.1 are in force and $h \in \mathcal{L}(\mathcal{B}_\varepsilon)$, then the constants C and c may be chosen independent from ε, τ and h. So in this case

$$\mathrm{dist}_{\mathcal{M}(V_\varepsilon)}(\mu_\varepsilon^\tau, \mathcal{D}u(\cdot, kT + \tau)) \leq Ce^{-ck} \quad \forall\, k \tag{5.11}$$

for every $\tau \in [0, T]$ and every $\varepsilon \leq \varepsilon_0$ provided that $u(\cdot, 0) \in \mathcal{B}_\varepsilon^0$.

Proof. Due to (2.9) the l.h.s. of (5.10) equals the l.h.s. of (5.8) with $g = h \circ S_\varepsilon^\tau$. Since $u \in \mathcal{B}_\varepsilon$, then by (2.6) $(h \circ S_\varepsilon^\tau)(u)$ is a bounded Lipschitz function on $\mathcal{B}_\varepsilon$. Now the estimate (5.10) follows from (5.8).

Under the assumptions of the second assertion, we use (4.5) to get that the function $C^{-1} h \circ S_\varepsilon^\tau \in \mathcal{L}(\mathcal{B}_\varepsilon^0)$ for some C, independent from ε, h and τ. Therefore the desired result follows from item 3) of Theorem 5.1. •

Since $\{u(kT + \tau, \cdot), k \geq 0\}$ is a trajectory of the RDS (5.3) in $\mathcal{B}_\varepsilon$, then by (5.10) the latter has the unique stationary measure μ_ε^τ which attracts exponentially fast distributions of all trajectories of the system.

Let the initial condition u_0 in (1.1)-(1.3) be such that $\mathcal{D}(u_0) = \mu_\varepsilon$, and $u(x,t)$ be a corresponding solution. Then

$$\mathcal{D}u(kT+\tau) = \mu_\varepsilon^\tau \quad \forall \tau \in [0,T]\,, \quad k = 0,1,\dots\,.$$

Abusing language, we call such solutions *stationary* (in fact, they are T-periodic in distribution).

5.3 Limit $\varepsilon \to 0$.

Theorem 2.2 suggests that statistical properties of solutions $u(x,t)$, averaged in x_3, are close to those of solutions for the 2D NSE. To prove corresponding results we have to strengthen assumption (5.1) on the kicks η_k^ε. Namely, we assume the following:

(L)
- The random variables $\xi_{jk} \equiv \xi_{jk}^\varepsilon$ and $\hat{\xi}_{jk} \equiv \hat{\xi}_{jk}^\varepsilon$ in (2.10) are independent of ε;
- $b_j^\varepsilon \to b_j$ as $\varepsilon \to 0$ in the sense that

$$\Big|\sum (b_j^\varepsilon - b_j) e_{\lambda_j}\Big|_{\tilde{H}}^2 \equiv \sum_j \frac{1}{\lambda_j}(b_j^\varepsilon - b_j)^2 \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0; \tag{5.12}$$

- relations (5.6) are in force.

Let us define 2D kicks $\tilde{\eta}_k = \sum b_j \xi_{jk} e_{\lambda_j}(x')$ and consider the 2D kick-force

$$\widetilde{f} = \sum \delta_{kT}(t)\, \tilde{\eta}_k^\omega(x')\,. \tag{5.13}$$

Clearly $\widetilde{f} = \lim M_\varepsilon f^\varepsilon$, where $f = f^\varepsilon$ is given by (2.8). Similar to the 3D case, the corresponding kick-forced 2D NSE (1.9)-(1.10) defines a continuous discrete-time RDS in the space $\widetilde{V}$, and defines the semigroup $\{\Psi_k, k \geq 0\}$ of continuous transformations of the space of Borel measures on $\widetilde{V}$. Moreover, this system extends to a continuous RDS in the space $\widetilde{H}$, and the transformations Ψ_k extend to continuous (in the $*$-weak topology) transformations of the space of Borel measures in $\tilde{H}$, see [13]. If

$$b_j > 0 \quad \forall j \leq N \tag{5.14}$$

with a suitable $N < \infty$, then by the same reasons as above this system has a unique stationary measure ϑ. This is a Borel measure in $\widetilde{V}$, supported by a ball of finite radius.

Due to (5.12), assumption (5.14) implies (5.4) if $0 < \varepsilon \leq \varepsilon_0 \ll 1$, and Theorem 5.1 applies. For such ε let us denote $\vartheta_\varepsilon = M_\varepsilon \circ \mu_\varepsilon$, i.e.

$$\vartheta_\varepsilon(Q) = \mu_\varepsilon \{u \in \mathcal{B}_\varepsilon \,:\, M_\varepsilon u \in Q\}\,.$$

Theorem 5.5 *Let Assumptions* **(D)** *and* **(L)** *be in force and* (5.14) *holds with a sufficiently large N. Then*

$$\vartheta_\varepsilon \rightharpoonup \vartheta \quad \text{as} \quad \varepsilon \to 0\,, \tag{5.15}$$

where $\rightharpoonup$ stands for the $*$-weak convergence of measures in $\widetilde{H}$ and ϑ is the unique stationary measure of the kick-forced 2D NSE (1.9)-(1.10) *with $\tilde{f}$ defined in* (5.13)*. Moreover, if in addition we assume that*

$$\sum_j j^2 \left|b_j^\varepsilon\right|^2 \leq C\,, \tag{5.16}$$

then the convergence in (5.15) *holds true in $*$-weak sense of measures on the space $H^{2-\delta} := H^{2-\delta}(\mathbb{T}^2) \cap \widetilde{V}$, for every $\delta > 0$. In particular, if g is a continuous functional on $C(\mathbb{T}^2;\mathbb{R}^2)$, then*

$$\langle g, \vartheta_\varepsilon \rangle \to \langle g, \vartheta \rangle \quad \text{as} \quad \varepsilon \to 0\,. \tag{5.17}$$

Proof. By Theorem 5.1 under condition (5.6) the stationary measure μ_ε has its support in the set $\mathcal{B}^0_\varepsilon$ given by (5.7). Hence, $\operatorname{supp}\vartheta_\varepsilon \subset \{\|v\|_{\widetilde V} \le C\}$ for each ε. So by the Prokhorov theorem the family of measures $\{\vartheta_\varepsilon, 0<\varepsilon\le\varepsilon_0\}$ is precompact in the set of measures in $\widetilde H$, given the $*$-weak topology. It remains to prove that any limiting measure $\tilde\vartheta$ of this family equals ϑ. Let us take a sequence $\varepsilon_j\to 0$ such that $\vartheta_{\varepsilon_j} \rightharpoonup \tilde\vartheta$. By the Skorokhod representation theorem (e.g., see [12]), on a probability space, for which we take the segment [0,1] given the Borel sigma-algebra and Lebesgue measure, we can construct $\widetilde V$-valued random variables $\tilde v$ and $\{v_{\varepsilon_j}\}$, such that

$$\mathcal{D}(v_{\varepsilon_j}) = \vartheta_{\varepsilon_j}\,,\ \mathcal{D}(\tilde v) = \tilde\vartheta \ \text{ and } v_{\varepsilon_j}\to \tilde v \text{ in } \widetilde V \text{ a.s.}$$

We view them as random variables on the probability space $\Omega_{\rm new} = [0,1]\times[0,1]$, depending only on the first factor, and for each j find a $\mathcal{B}_{\varepsilon_j}$-valued random variable u_{ε_j} on $\Omega_{\rm new}$ such that $M_\varepsilon u_{\varepsilon_j} = v_{\varepsilon_j}$ and $\mathcal{D}(u_{\varepsilon_j}) = \mu_{\varepsilon_j}$, see below Lemma 5.9.

Next we construct on $\Omega_{\rm new}$ random variables $\xi_{jk\,new}$ etc, distributed as ξ_{jk} etc and independent from the previously constructed random variables. We define the random vectors $\tilde\eta_{1\,new}$ and $\tilde\eta^{\varepsilon_j}_{1\,new}$, using these "new" random variables. Then

$$\Psi_1^{\varepsilon_j}(\mu_{\varepsilon_j}) = \mathcal{D}(S^T_{\varepsilon_j}u_{\varepsilon_j} + \eta^{\varepsilon_j}_{1\,new}) \ \text{ and } \ \Psi_1(\vartheta_{\varepsilon_j}) = \mathcal{D}(S^T_0 v_{\varepsilon_j} + \tilde\eta_{1\,new})\,.$$

Let g be any continuous function on $\widetilde H$ such that $\operatorname{Lip}(g)\le 1$ and $|g|\le 1$. Then

$$\begin{aligned}
&|\langle g, \Psi_1(\vartheta_{\varepsilon_j})\rangle - \langle g, M_{\varepsilon_j}\circ\Psi_1^{\varepsilon_j}(\mu_{\varepsilon_j})\rangle| \\
&= |\mathbf{E}\big(g(S^T_0(v_{\varepsilon_j}) + \eta_{1\,new}) - g\big(M_{\varepsilon_j}(S^T_{\varepsilon_j}(u_{\varepsilon_j})) + M_{\varepsilon_j}(\eta^{\varepsilon_j}_{1\,new})\big)| \\
&\le \mathbf{E}\big(|S^T_0(v_{\varepsilon_j}) - M_{\varepsilon_j}(S^T_{\varepsilon_j}(u_{\varepsilon_j}))|\wedge 2\big) + \mathbf{E}|\eta_{1\,new} - M_{\varepsilon_j}(\eta^{\varepsilon_j}_{1\,new})|\,.
\end{aligned}$$

Since $M_\varepsilon u_{\varepsilon_j} = v_{\varepsilon_j}\to\tilde v$ a.s., then by Theorem 2.2 the random variable in the first expectation in the r.h.s. goes to zero with ε_j for each ω. By (5.12) the random variable in the second expectation goes to zero with ε_j uniformly in ω. So the r.h.s. goes to zero with ε_j and the rate of convergence is independent of g as above. Since $M_{\varepsilon_j}\circ\Psi_1^{\varepsilon_j}(\mu_{\varepsilon_j}) = M_{\varepsilon_j}\circ\mu_{\varepsilon_j} = \vartheta_{\varepsilon_j}$, then we have seen that

$$\operatorname{dist}_{\mathcal{M}(\widetilde H)}(\vartheta_{\varepsilon_j}, \Psi_1(\vartheta_{\varepsilon_j}))\to 0 \ \text{ as } \ \varepsilon_j\to 0\,. \tag{5.18}$$

As the transformation Ψ_1 is continuous in $\mathcal{M}(\widetilde H)$, then by (5.18) $\tilde\vartheta$ is a stationary measure of the 2D NSE. So it equals ϑ.

To prove the second part of the theorem, we note that

$$\begin{aligned}
\int_{\widetilde H}|A_0u|^2_{\widetilde H}\vartheta_\varepsilon(du) =& \frac{1}{\varepsilon}\int_{\mathcal{B}_\varepsilon}|A_\varepsilon M_\varepsilon u|^2_\varepsilon\mu_\varepsilon(du) = \frac{1}{\varepsilon}\mathbf{E}\int_{\mathcal{B}_\varepsilon}|A_\varepsilon M_\varepsilon(S^T_\varepsilon u + \eta^\omega_1)|^2_\varepsilon\mu_\varepsilon(du)\\
&\le \frac{2}{\varepsilon}\int_{\mathcal{B}_\varepsilon}|A_\varepsilon S^T_\varepsilon u|^2_\varepsilon\mu_\varepsilon(du) + \frac{2}{\varepsilon}\mathbf{E}\int_{\mathcal{B}_\varepsilon}|A_\varepsilon M_\varepsilon\eta^\omega_1|^2_\varepsilon\mu_\varepsilon(du)\,,
\end{aligned}$$

where the second equality holds since μ_ε is a stationary measure. Since $M_\varepsilon\eta^\varepsilon_1 = \sum b^\varepsilon_j\xi^\omega_{j1}e_{\lambda_j}$ (see (8.2)), then due to (5.16) and (8.4) the second term in the r.h.s. is bounded by $\sum(b^\varepsilon_j)^2\lambda_j^2\le C_1$. Since μ_ε is supported by (5.7), due to Proposition 4.4 the first term is $\le C_2$. Hence, $\int_{\widetilde H}|A_0u|^2_{\widetilde H}\vartheta_\varepsilon(du)\le C_3$ for all ε, and by Prokhorov's theorem we conclude that the family $\{\vartheta_\varepsilon\}$ is precompact in the $*$-weak topology of the space of measures on $H^{2-\delta}$. This implies the desired convergence.

The convergence in (5.17) holds since for $\delta<1/2$ the space $H^{2-\delta}$ is continuously embedded in $C(\mathbb{T}^2;\mathbb{R}^2)$. •

Remark 5.6 Note that the proof implies that the measures ϑ_ε and ϑ are supported by the same ball $\{\|v\|_{\widetilde V}\le C\}$.

The following assertion shows that the convergence of $M_\varepsilon \circ \mu_\varepsilon^\tau$ in the space $H^{2-\delta}$ takes place for $\tau > 0$ *without* condition (5.16).

Corollary 5.7 *Let the assumptions of the first assertion of Theorem 5.5 be in force. Then for any* $\tau > 0$

$$M_\varepsilon \circ \mu_\varepsilon^\tau \rightharpoonup S_0^\tau \circ \vartheta =: \vartheta^\tau \quad as \ \ \varepsilon \to 0 \tag{5.19}$$

in Borel measures in the space $H^{2-\delta} = H^{2-\delta}(\mathbb{T}^2) \cap \widetilde{V}$ *for every* $\delta > 0$*. Moreover, the rate of convergence* (5.19) *in the space of measures in* $\widetilde{H}$ *is independent from* $\tau \in [0, T]$.

Proof. Recall that $\mu_\varepsilon^\tau = S_\varepsilon^\tau \circ \mu_\varepsilon$ is the unique stationary measure of the RDS (5.3). Similarly, ϑ^τ is the unique stationary measure for the 2D RDS

$$v^{k,\tau} = S_0^\tau (S_0^{T-\tau}(v^{k-1,\tau}) + \tilde{\eta}_k^\omega)\,.$$

Arguing as in the proof of Theorem 5.5 we see that the convergence (5.19) holds in the space $\widetilde{H}$, and its rate is independent from τ. Since the measures μ_ε with $\varepsilon \ll 1$ are supported by the ball (5.7), then by Proposition 4.4 the measures $M_\varepsilon \circ \mu_\varepsilon^\tau$ are supported by a ball in the space $H^2(\mathbb{T}^2) \cap \widetilde{V}$ with the radius independent of ε. So they form a precompact family of Borel measures on $H^{2-\delta}$, and the assertion follows. •

In our last theorem we obtain an analogy of the assertion of Theorem 2.2 for distributions of solutions to the random NSE. In difference with the deterministic situation, now an analogy of convergence (2.5) holds *uniformly* in $t \geq 0$.

Theorem 5.8 *Let us assume that condition* (5.1) *and Assumptions* **(D)** *and* **(L)** *hold. Let* $T \geq T_0$ *and* $N \in \mathbb{N}$ *(independent of* ε*) be as in Theorem 5.1, and* $b_j > 0$ *for* $j \leq N$*. Let* $u^{\varepsilon,\omega}(t,x)$ *be a solution of the random kick-forced NSE* (1.1)*-*(1.3)*, where* $u_0 = u_0^{\varepsilon,\omega}$ *is a random vector independent of the kicks and such that* $u_0^{\varepsilon,\omega} \in \mathcal{B}_\varepsilon^0$ *for each* ω*. Assume that* $M u_0^{\varepsilon,\omega} \to v_0^\omega \in \widetilde{V}$ *weakly in* $\widetilde{H}$ *for each* ω*, and denote by* $v^\omega(t,x')$ *a solution of the 2D NSE*(1.9), (1.10), (5.13)*, equal* v_0^ω *at* $t = 0$*. Then*

$$\mathrm{dist}_{\mathcal{M}(\widetilde{H})}\big(\mathcal{D}M_\varepsilon u^\varepsilon(t), \mathcal{D}v(t)\big) \to 0 \quad as \ \ \varepsilon \to 0\,, \tag{5.20}$$

uniformly in $t \geq 0$*. Moreover, if* (5.16) *holds, then also*

$$\mathrm{dist}_{\mathcal{M}(H^{2-\delta})}\big(\mathcal{D}M_\varepsilon u^\varepsilon(kT), \mathcal{D}v(kT)\big) \to 0 \quad as \ \ \varepsilon \to 0$$

for any $\delta > 0$*, uniformly in* $k \geq 1$*, where as above* $H^{2-\delta} = H^{2-\delta}(\mathbb{T}^2) \cap \widetilde{V}$.

Proof. Let us fix any $\Theta \geq 1$. Applying iteratively Theorem 2.2 on the time-segments $[0, T]$, $[T, 2T]$, ... we get that

$$\sup_{0 \leq t \leq \Theta} |M_\varepsilon u^{\varepsilon,\omega}(t) - v^\omega(t)|_{\widetilde{H}} =: \varkappa^\omega(\varepsilon, \Theta) \to 0 \ \text{ as } \ \varepsilon \to 0 \tag{5.21}$$

for each ω. Accordingly, for any $g \in \mathcal{L}(\widetilde{H})$ we have

$$\sup_{0 \leq t \leq \Theta} |\mathbf{E}\, g\big(M_\varepsilon u^\varepsilon(t)\big) - \mathbf{E}\, g(v(t))| \leq \mathbf{E} \min\big(2, \varkappa^\omega(\varepsilon, \Theta)\big) =: \varkappa_1(\varepsilon, \Theta),$$

where $\varkappa_1(\varepsilon, \Theta) \to 0$ as $\varepsilon \to 0$, for each Θ. So

$$\mathrm{dist}_{\mathcal{M}(\widetilde{H})}\big(\mathcal{D}M_\varepsilon u^\varepsilon(t), \mathcal{D}v(t)\big) \leq \varkappa_1(\varepsilon, \Theta) \quad \text{for } 0 \leq t \leq \Theta. \tag{5.22}$$

If $t = kT + \tau \geq \Theta$, then using (5.11) we get

$$\mathrm{dist}_{\mathcal{M}(\widetilde{H})}\big(\mathcal{D}M_\varepsilon u^\varepsilon(t), M_\varepsilon \circ \mu_\varepsilon^\tau\big) \leq C e^{-c\Theta/T}\,.$$

Similarly the solution $v(t)$ satisfies

$$\operatorname{dist}_{\mathcal{M}(\widetilde{H})}\big(\mathcal{D}v(t),\vartheta^\tau\big)\le Ce^{-c\Theta/T}$$

(see [13] and the beginning of this subsection). Using Corollary 5.7 we get from the last two estimates that

$$\operatorname{dist}_{\mathcal{M}(\widetilde{H})}\big(\mathcal{D}M_\varepsilon u^\varepsilon(t),\mathcal{D}v(t)\big)\le \varkappa_2(\Theta)+\varkappa_3(\varepsilon)\ \text{ for } t\ge\Theta\ ,$$

where $\varkappa_2\to 0$ as $\Theta\to\infty$ and $\varkappa_3\to 0$ as $\varepsilon\to 0$. Jointly with (5.22) this relation implies the first assertion of the theorem.

Let us assume (5.16). Then due to relation (4.7) applied to each interval $[kT,(k+1)T]$, for $t\ge T$, $\varepsilon\le\varepsilon_0$ and all ω we have $|M_\varepsilon A_\varepsilon u^\varepsilon(t)|_{\widetilde{H}}\le C$. Similar using (2.7) we find that $|A_0 v(t)|_{\widetilde{H}}\le C_1$. Interpolating these inequalities with (5.21) we get

$$\sup_{0\le t\le\Theta}|M_\varepsilon u^{\varepsilon,\omega}(t)-v^\omega(t)|_{H^{2-\delta}}=:\varkappa_\delta^\omega(\varepsilon,\Theta)\to 0\ \text{ as }\ \varepsilon\to 0$$

for each ω. Now arguing as above and using the second assertion of Theorem 5.5 we complete the proof. •

In the lemma below the segment [0,1] is given the Borel sigma algebra and Lebesgue measure.

Lemma 5.9 *Let A and B be complete metric spaces (infinite and non-countable), given Borel σ-algebras, μ be a Borel measure on $A\times B$ and ξ be a measurable map $[0,1]\to A$ such that $\mathcal{D}(\xi)=\mu_1$, where μ_1 is the projection of the measure μ to A. Then there exists a measurable map $\eta:[0,1]^2\to B$ such that the distribution of the map $\xi\times\eta:[0,1]^2\to A\times B$ is μ. Here we extended ξ to a function on $[0,1]^2$, depending on the first factor only.*

Proof. Since both spaces A and B are measurably isomorphic to the segment [0,1], given the Borel σ-algebra ([6], Section 13.1), then without loss of generality we may assume that $A=B=[0,1]$. By the theorem on the conditional distribution ([6], Section 10.2) we can write μ as $\mu(da\,db)=\mu_1(da)\mu_2(a;db)$. Here μ_2 is measurable in the sense that the function $F(a;b)=\mu(a;[0,b])$ is measurable both in a and b. Since F as a function of b is continuous from the right, then it is measurable as a map of Borel spaces $[0,1]^2\to[0,1]$. Let us define the function $\rho(a,y):[0,1]^2\to[0,1]$ by the relation

$$\rho(a,y)=\inf\{\tau:\ F(a,\tau)\ge y\}\,.$$

It is measurable, monotonic in y and continuous from the right. The mapping $[0,1]\ni y\mapsto\rho(a,y)$ transforms the Lebesgue measure dy to the measure $\mu(a;db)$ ([6], Section 9.1). Now we set $\eta(x,y)=\rho(\xi(x),y)$. The mapping $\xi\times\eta:[0,1]^2\to A\times B$ possesses the desired properties. •

6 Proofs of results stated in Section 4.

6.1 Preliminaries

The following properties established in [22] are important in the further considerations.

(i) $M_\varepsilon\nabla'=\nabla' M_\varepsilon$ and $N_\varepsilon\nabla'=\nabla' N_\varepsilon$, where $\nabla'=(\partial_{x_1},\partial_{x_2},0)$.

(ii) For all $u,v\in H^1(\mathcal{O}_\varepsilon)^3$ we have $\int_{\mathcal{O}_\varepsilon}\nabla N_\varepsilon u\nabla M_\varepsilon v dx=0$ and

$$|u|_\varepsilon^2=|N_\varepsilon u|_\varepsilon^2+|M_\varepsilon u|_\varepsilon^2,\quad \|u\|_\varepsilon^2=\|N_\varepsilon u\|_\varepsilon^2+\|M_\varepsilon u\|_\varepsilon^2\,. \tag{6.1}$$

(iii) For all $u, w, v \in V_\varepsilon$ we have

$$b_\varepsilon(u, w, M_\varepsilon v) = b_\varepsilon(M_\varepsilon u, M_\varepsilon w, M_\varepsilon v) + b_\varepsilon(N_\varepsilon u, N_\varepsilon w, M_\varepsilon v) \tag{6.2}$$

and

$$b_\varepsilon(u, w, N_\varepsilon v) = b_\varepsilon(N_\varepsilon u, w, N_\varepsilon v) + b_\varepsilon(M_\varepsilon u, N_\varepsilon w, N_\varepsilon v). \tag{6.3}$$

(iv) If $u \in D(A_\varepsilon)$, then $M_\varepsilon u \in D(A_\varepsilon)$ and

$$\Delta N_\varepsilon u = N_\varepsilon \Delta u, \quad \Delta M_\varepsilon u = M_\varepsilon \Delta u\,.$$

(v) If $u \in D(A_\varepsilon)$, then

$$b_\varepsilon(M_\varepsilon u, M_\varepsilon u, A_\varepsilon M_\varepsilon v) = b_0(M_\varepsilon u, M_\varepsilon u, A_0 M_\varepsilon v) = 0, \tag{6.4}$$

where b_0 is the 2D trilinear form and A_0 is the 2D Stokes operator on $\mathbb{T}^2$ (i.e., in the space $\widetilde{H}$).

(vi) If $u \in D(A_\varepsilon)$, then $A_\varepsilon u = -\Delta u$ and (see Lemma 2.5 [22])

$$\sum_{i,j}^{3} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2_{L_2(\mathcal{O}_\varepsilon)} \le |A_\varepsilon u|^2_{L_2(\mathcal{O}_\varepsilon)}\,, \quad \forall u \in \mathrm{D}(A_\varepsilon). \tag{6.5}$$

As in [22] we also use the "thin domain" analogues of the classical Poincaré, Agmon and Ladyzhenskaya inequalities given in the following assertion.

Lemma 6.1 ([22]) *There exist positive constants c and c_q, $2 \le q \le 6$, such that for all $\varepsilon \in (0,1]$ the following inequalities hold true:*

$$|N_\varepsilon u|_\varepsilon \le \varepsilon \, |\partial_3 N_\varepsilon u|_\varepsilon\,, \quad \textit{for all } u \in V_\varepsilon \tag{6.6}$$

(Poincaré's inequality);

$$|N_\varepsilon u|_{(L^\infty(\mathcal{O}_\varepsilon))^3} \le c |N_\varepsilon u|^{1/4}_{(L_2(\mathcal{O}_\varepsilon))^3} \left(\sum_{i,j}^{3} \left| \frac{\partial^2 N_\varepsilon u}{\partial x_i \partial x_j} \right|^2_{L_2(\mathcal{O}_\varepsilon)} \right)^{3/4} \tag{6.7}$$

for all $u \in D(A_\varepsilon)$ (Agmon's inequality);

$$|N_\varepsilon u|^2_{(L^q(\mathcal{O}_\varepsilon))^3} \le c_q \varepsilon^{(6-q)/q} \|N_\varepsilon u\|^2_\varepsilon \quad \textit{for all } u \in V_\varepsilon,\ 2 \le q \le 6, \tag{6.8}$$

(Ladyzhenskaya's inequality).

We will also use the following version of Lemmas 3.1 and 3.2 of [22] (in the case when the external force is absent, $f \equiv 0$).

Lemma 6.2 ([22]) *Let the hypotheses of Theorem 2.1 be in force and $u(t)$ be a solution to (1.1)–(1.4) with $f \equiv 0$. Assume that*

$$\|M_\varepsilon u_0\|_\varepsilon \le a(\varepsilon),\ \|N_\varepsilon u_0\|_\varepsilon \le b(\varepsilon),$$

where $R_*(\varepsilon) \equiv a(\varepsilon) + b(\varepsilon)$ satisfy (2.3). Then there exists $T(\varepsilon) > 0$ such that $T(\varepsilon) \to +\infty$ and for all $0 < t < T(\varepsilon)$ the following inequalities hold:

$$\|N_\varepsilon u(t)\|^2_\varepsilon \le b^2(\varepsilon) \exp\left\{ -\frac{\nu t}{2\varepsilon^2} \right\}, \tag{6.9}$$

$$\int_0^t |A_\varepsilon N_\varepsilon u(s)|^2_\varepsilon ds \le \frac{2}{\nu}\, b^2(\varepsilon), \tag{6.10}$$

$$\|M_\varepsilon u(t)\|^2_\varepsilon \le a^2(\varepsilon) \exp\{-\nu\lambda_1 t\} + c_1(\nu)\varepsilon b^4(\varepsilon), \tag{6.11}$$

$$\int_0^t |A_\varepsilon M_\varepsilon u(s)|^2_\varepsilon ds \le \frac{2}{\nu} a^2(\varepsilon) + c_1(\nu)\varepsilon b^4(\varepsilon). \tag{6.12}$$

Here above λ_1 is the first eigenvalue of 2D Stokes operator A_0.

6.2 Estimates for the trilinear form

The following estimates are proved in [22] in the case when $u = w$. The proof given below is a slight modification of the argument from [22].

Lemma 6.3 *For every $\theta < 1/2$, there exist positive constants ε_0, c such that for all $\varepsilon \in (0, \varepsilon_0)$, $u, w \in \mathrm{D}(A_\varepsilon)$, $v \in (L^2(\mathcal{O}_\varepsilon))^3$, we have*

$$|b_\varepsilon(M_\varepsilon u, N_\varepsilon w, v)| \le c\varepsilon^\theta \|M_\varepsilon u\|_\varepsilon \cdot |A_\varepsilon N_\varepsilon w|_\varepsilon \cdot |v|_\varepsilon; \tag{6.13}$$

$$|b_\varepsilon(N_\varepsilon u, w, v)| \le c\varepsilon^{1/2} |A_\varepsilon N_\varepsilon u|_\varepsilon \cdot \|w\|_\varepsilon \cdot |v|_\varepsilon; \tag{6.14}$$

$$|b_\varepsilon(N_\varepsilon u, N_\varepsilon w, v)| \quad \le \quad c\varepsilon^{1/2} \|N_\varepsilon u\|_\varepsilon \cdot |A_\varepsilon N_\varepsilon w|_\varepsilon \cdot |v|_\varepsilon. \tag{6.15}$$

Proof. Estimate (6.13): Since $(M_\varepsilon u)_3 = 0$, we obviously have that

$$|b_\varepsilon(M_\varepsilon u, N_\varepsilon w, v)| \le \sum_{j=1}^{2} \sum_{l=1}^{3} \int_{\mathcal{O}_\varepsilon} |(M_\varepsilon u)_j| \, |(\partial_j \, N_\varepsilon u)_l| \, |v_l| \; dx.$$

Applying Hölder's inequality, we obtain

$$|b_\varepsilon(M_\varepsilon u, N_\varepsilon w, v)| \le \sum_{j=1}^{2} \sum_{l=1}^{3} |(M_\varepsilon u)_j|_{L^{p_1}(\mathcal{O}_\varepsilon)} \, |(\partial_j \, N_\varepsilon w)_l|_{L^{p_2}(\mathcal{O}_\varepsilon)} \, |v_l|_{L^2(\mathcal{O}_\varepsilon)} \,,$$

where $p_1^{-1} + p_2^{-1} = 1/2$, $2 < p_2 \le 6$. Since $\partial_j N_\varepsilon w = N_\varepsilon \partial_j w$ for all $w \in D(A_\varepsilon)$ and $j = 1, 2$, we can use (6.5) and (6.8) to write

$$|\partial_j (N_\varepsilon w)_l|_{L^{p_2}(\mathcal{O}_\varepsilon)} \le c\varepsilon^{\frac{6-p_2}{2p_2}} \, \|\partial_j N_\varepsilon w\|_\varepsilon \le c\varepsilon^{\frac{6-p_2}{2p_2}} \, |A_\varepsilon N_\varepsilon w|_\varepsilon \, .$$

for any $l = 1, 2, 3$. One can also see that

$$\begin{aligned} |(M_\varepsilon u)_j|_{L^{p_1}(\mathcal{O}_\varepsilon)} \quad &= \quad \varepsilon^{1/p_1} \, |(M_\varepsilon u)_j|_{L^{p_1}(\mathbb{T}^2)} \le c\varepsilon^{1/p_1} \, |(M_\varepsilon u)_j|_{H^1(\mathbb{T}^2)} \\ &\le c\varepsilon^{1/p_1 - 1/2} \, \|(M_\varepsilon u)_j\|_\varepsilon \, . \end{aligned}$$

Therefore inequality (6.13) with $\theta = 2/p_2 - 1/2$ follows.

Estimate (6.14): Using (6.7) we get that

$$\begin{aligned} |b_\varepsilon(N_\varepsilon u, w, v)| \quad &\le \quad \sum_{j=1}^{3} \sum_{l=1}^{3} \int_{\mathcal{O}} |(N_\varepsilon u)_j| \cdot |(\partial_j w)_l| \, |v_l| \; dx \\ &\le \quad \sum_{j=1}^{3} \sum_{l=1}^{3} |(N_\varepsilon u)_j|_{L^\infty(\mathcal{O}_\varepsilon)} \, |(\partial_j w)_l|_{L^2(\mathcal{O}_\varepsilon)} \, |v_l|_{L^2(\mathcal{O}_\varepsilon)} \\ &\le \quad c \, |N_\varepsilon u|_\varepsilon^{1/4} \, |A_\varepsilon N_\varepsilon u|_\varepsilon^{3/4} \, \|w\|_\varepsilon \, |v|_\varepsilon \, . \end{aligned}$$

By (6.6) we have that $|N_\varepsilon u|_\varepsilon \le \varepsilon \left| A_\varepsilon^{1/2} N_\varepsilon u \right|_\varepsilon \le \varepsilon^2 \, |A_\varepsilon N_\varepsilon u|_\varepsilon$, and (6.14) follows.

Estimate (6.15): We obviously have that

$$\begin{aligned} |b_\varepsilon(N_\varepsilon u, N_\varepsilon w, v)| \quad &\le \quad \sum_{j=1}^{3} \sum_{l=1}^{3} \int_{\mathcal{O}_\varepsilon} |(N_\varepsilon u)_j| \cdot |(\partial_j \, N_\varepsilon w)_l| \, |v_l| \; dx \\ &\le \quad \sum_{j=1}^{3} \sum_{l=1}^{3} |(N_\varepsilon u)_j|_{L^3(\mathcal{O}_\varepsilon)} \, |(\partial_j N_\varepsilon w)_l|_{L^6(\mathcal{O}_\varepsilon)} \, |v_l|_{L^2(\mathcal{O}_\varepsilon)} \\ &\le \quad c \, |N_\varepsilon u|_{L^3(\mathcal{O}_\varepsilon)} \left(\sum_{j=1}^{3} |\partial_j N_\varepsilon w|_{L^6(\mathcal{O}_\varepsilon)} \right) |v|_\varepsilon \, . \end{aligned}$$

Thus by (6.8) we obtain (6.15). •

6.3 Proof of Proposition 4.1.

Applying (6.9) and (6.11) with $\tilde{a}(\varepsilon) = \|M_\varepsilon u_\varepsilon(t)\|_\varepsilon$ and $\tilde{b}(\varepsilon) = \|N_\varepsilon u_\varepsilon(t)\|_\varepsilon$ instead of $a(\varepsilon)$ and $b(\varepsilon)$ yields

$$\begin{aligned}
\|u_\varepsilon(t)\|_\varepsilon^2 &= \|M_\varepsilon u_\varepsilon(t)\|_\varepsilon^2 + \|N_\varepsilon u_\varepsilon(t)\|_\varepsilon^2 \\
&\le \tilde{a}^2(\varepsilon)\exp\{-\nu\lambda_1 t\} + c_1\varepsilon\tilde{b}^4(\varepsilon) + \tilde{b}^2(\varepsilon)\exp\left\{-\frac{\nu t}{2\varepsilon^2}\right\} \\
&\le \left[\tilde{a}^2(\varepsilon) + \tilde{b}^2(\varepsilon)\right]\max\left\{\exp\{-\nu\lambda_1 t\}, c_1\varepsilon\tilde{b}^2(\varepsilon) + \exp\left\{-\frac{\nu t}{2\varepsilon^2}\right\}\right\}.
\end{aligned}$$

Thus

$$\|S_\varepsilon^T u_0\|_\varepsilon^2 \le \|u_0\|_\varepsilon^2 \max\left\{\exp\{-\nu\lambda_1 T_0\}, c_1\varepsilon R^2(\varepsilon) + \exp\left\{-\frac{\nu T_0}{2\varepsilon^2}\right\}\right\}.$$

for all $T \ge T_0$. Therefore we can find ε_0 and $0 < \gamma < 1$ such that (4.3) holds.

Now we prove the invariance of the set $\mathcal{B}_\varepsilon$. From (6.9) and (6.11) we have that

$$|A_\varepsilon^{1/2} N_\varepsilon[u(T)+\eta]|_\varepsilon \le b(\varepsilon)\exp\left\{-\frac{\nu T}{4\varepsilon^2}\right\} + k_* b(\varepsilon)$$

and

$$|A_\varepsilon^{1/2} M_\varepsilon[u(T)+\eta]|_\varepsilon \le a(\varepsilon)\exp\left\{-\frac{\nu\lambda_1}{2}T\right\} + c_1\sqrt{\varepsilon}b^2(\varepsilon) + k_* a(\varepsilon),$$

Thus the set $\mathcal{B}_\varepsilon$ given by (4.1) is invariant with respect mapping $u \mapsto S_\varepsilon^T u + \eta$ if

$$\exp\left\{-\frac{\nu T_0}{4\varepsilon^2}\right\} + k_* \le 1$$

and

$$c_1\sqrt{\varepsilon}b^2(\varepsilon) \le a(\varepsilon)\left[1 - \exp\left\{-\frac{\nu\lambda_1}{2}T_0\right\} - k_*\right].$$

Now we can choose $k_* = \frac{1}{2}\left[1 - \exp\left\{-\frac{\nu\lambda_1}{2}T_0\right\}\right]$ and, due to (4.2), find $\varepsilon_0 = \varepsilon_0(T_0)$ such that the inequalities above hold for all $\varepsilon \in (0, \varepsilon_0)$.

6.4 Proof of Theorem 4.2.

Let $u_1(t)$ and $u_2(t)$ be two strong solutions to 3D Navier-Stokes problem (2.1) with $f \equiv 0$. Then the difference $u(t) = u_1(t) - u_2(t)$ satisfies the equation

$$u' + \nu A_\varepsilon u + B_\varepsilon(u, u_1) + B_\varepsilon(u_2, u) = 0. \tag{6.16}$$

Step 1: Preliminary estimate for N_ε-component.

Multiplying (6.16) in H_ε by $A_\varepsilon N_\varepsilon u$ we get

$$\frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2}N_\varepsilon u|_\varepsilon^2 + \nu|A_\varepsilon N_\varepsilon u|_\varepsilon^2 + b_\varepsilon(u, u_1, A_\varepsilon N_\varepsilon u) + b_\varepsilon(u_2, u, A_\varepsilon N_\varepsilon u) = 0. \tag{6.17}$$

Now we estimate the trilinear terms in (6.17). By (6.3) we have

$$b_\varepsilon(u, u_1, A_\varepsilon N_\varepsilon u) = b_\varepsilon(N_\varepsilon u, u_1, A_\varepsilon N_\varepsilon u) + b_\varepsilon(M_\varepsilon u, N_\varepsilon u_1, A_\varepsilon N_\varepsilon u)$$

By (6.14) and (6.13) we obtain

$$|b_\varepsilon(N_\varepsilon u, u_1, A_\varepsilon N_\varepsilon u)| \le c\varepsilon^{1/2}\|u_1\|_\varepsilon |A_\varepsilon N_\varepsilon u|_\varepsilon^2$$

and

$$|b_\varepsilon(M_\varepsilon u, N_\varepsilon u_1, A_\varepsilon N_\varepsilon u)| \le \delta |A_\varepsilon N_\varepsilon u|_\varepsilon^2 + \frac{c}{\delta}\,\varepsilon^{2\theta}\|M_\varepsilon u\|_\varepsilon^2 |A_\varepsilon N_\varepsilon u_1|_\varepsilon^2$$

for any $\delta > 0$, where $0 < \theta < 1/2$ can be chosen in arbitrary way.

Similarly

$$b_\varepsilon(u_2, u, A_\varepsilon N_\varepsilon u) = b_\varepsilon(N_\varepsilon u_2, u, A_\varepsilon N_\varepsilon u) + b_\varepsilon(M_\varepsilon u_2, N_\varepsilon u, A_\varepsilon N_\varepsilon u),$$

where

$$|b_\varepsilon(N_\varepsilon u_2, u, A_\varepsilon N_\varepsilon u)| \le \delta |A_\varepsilon N_\varepsilon u|_\varepsilon^2 + \frac{c}{\delta}\,\varepsilon \|u\|_\varepsilon^2 |A_\varepsilon N_\varepsilon u_2|_\varepsilon^2$$

for any $\delta > 0$, and

$$|b_\varepsilon(M_\varepsilon u_2, N_\varepsilon u, A_\varepsilon N_\varepsilon u)| \le c\varepsilon^\theta \|M_\varepsilon u_2\|_\varepsilon |A_\varepsilon N_\varepsilon u|_\varepsilon^2 \le c\varepsilon^\theta \|u_2\|_\varepsilon |A_\varepsilon N_\varepsilon u|_\varepsilon^2 .$$

Using in (6.17) these inequalities with suitable $\delta > 0$ we get that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|N_\varepsilon u\|_\varepsilon^2 &+ \left(\nu - c_0\left[\varepsilon^{1/2}\|u_1\|_\varepsilon + \varepsilon^\theta \|u_2\|_\varepsilon\right]\right) |A_\varepsilon N_\varepsilon u|_\varepsilon^2 \\ &\le c_1 \varepsilon \|u\|_\varepsilon^2 |A_\varepsilon N_\varepsilon u_2|_\varepsilon^2 + c_2 \varepsilon^{2\theta} \|M_\varepsilon u\|_\varepsilon^2 |A_\varepsilon N_\varepsilon u_1|_\varepsilon^2 \end{aligned} \tag{6.18}$$

for every $\theta \in [0, 1/2)$. Under the hypotheses concerning $a(\varepsilon)$ and $b(\varepsilon)$, we have from relations (6.9) and (6.11) in Lemma 6.2 that

$$\varepsilon^{1/2}\|u_1(t)\|_\varepsilon + \varepsilon^\theta \|u_2(t)\|_\varepsilon \le c\varepsilon^\theta (\log \varepsilon^{-1})^\sigma,$$

for any pair of initial data $u_1(0)$ and $u_2(0)$ from $\mathcal{B}_\varepsilon$. Choosing $0 < \varepsilon_0 \ll 1$ we get that

$$\begin{aligned} &\frac{d}{dt}\|N_\varepsilon u\|_\varepsilon^2 + \nu |A_\varepsilon N_\varepsilon u|_\varepsilon^2 \\ &\quad \le c_1 \varepsilon \|N_\varepsilon u\|_\varepsilon^2 \, |A_\varepsilon N_\varepsilon u_2|_\varepsilon^2 + c_2 \varepsilon^{2\theta} \|M_\varepsilon u\|_\varepsilon^2 \psi_N(t, u_1, u_2)\, , \end{aligned} \tag{6.19}$$

for all $0 < \varepsilon \le \varepsilon_0$, where

$$\psi_N(t, u_1, u_2) = |A_\varepsilon N_\varepsilon u_1(t)|_\varepsilon^2 + |A_\varepsilon N_\varepsilon u_2(t)|_\varepsilon^2. \tag{6.20}$$

Step 2: Preliminary estimate for M_ε-component.

Multiplying (6.16) in H_ε by $A_\varepsilon M_\varepsilon u$ we get

$$\frac{1}{2}\frac{d}{dt}|A_\varepsilon^{1/2} M_\varepsilon u|_\varepsilon^2 + \nu |A_\varepsilon M_\varepsilon u|_\varepsilon^2 + b_\varepsilon(u, u_1, A_\varepsilon M_\varepsilon u) + b_\varepsilon(u_2, u, A_\varepsilon M_\varepsilon u) = 0. \tag{6.21}$$

As above, we estimate trilinear terms in (6.21). By (6.2) we have

$$b_\varepsilon(u, u_1, A_\varepsilon M_\varepsilon u) = b_\varepsilon(M_\varepsilon u, M_\varepsilon u_1, A_\varepsilon M_\varepsilon u) + b_\varepsilon(N_\varepsilon u, N_\varepsilon u_1, A_\varepsilon M_\varepsilon u).$$

It is clear that

$$\begin{aligned} |b_\varepsilon(M_\varepsilon u, M_\varepsilon u_1, A_\varepsilon M_\varepsilon u)| &\le c\sqrt{\varepsilon}\|M_\varepsilon u\|_{L^4(\mathbb{T}^2)} \sum_{j=1,2} \|\partial_j M_\varepsilon u_1\|_{L^4(\mathbb{T}^2)} |A_\varepsilon M_\varepsilon u|_\varepsilon \\ &\le c\sqrt{\varepsilon}\|M_\varepsilon u\|_{H^1(\mathbb{T}^2)} \|M_\varepsilon u_1\|_{H^2(\mathbb{T}^2)} |A_\varepsilon M_\varepsilon u|_\varepsilon \\ &\le \frac{c}{\sqrt{\varepsilon}}\|M_\varepsilon u\|_\varepsilon |A_\varepsilon M_\varepsilon u_1|_\varepsilon |A_\varepsilon M_\varepsilon u|_\varepsilon \\ &\le \delta |A_\varepsilon M_\varepsilon u|_\varepsilon^2 + \frac{c}{\delta\varepsilon}\|M_\varepsilon u\|_\varepsilon^2 |A_\varepsilon M_\varepsilon u_1|_\varepsilon^2 \end{aligned}$$

for every $\delta > 0$. As for the second term, by (6.15) we have that

$$|b_\varepsilon(N_\varepsilon u, N_\varepsilon u_1, A_\varepsilon M_\varepsilon u)| \le \delta |A_\varepsilon M_\varepsilon u|_\varepsilon^2 + \frac{c\varepsilon}{\delta} \|N_\varepsilon u\|_\varepsilon^2 |A_\varepsilon N_\varepsilon u_1|_\varepsilon^2$$

for every $\delta > 0$.

Now we estimate the term $b_\varepsilon(u_2, u, A_\varepsilon M_\varepsilon u)$. As above we have that

$$b_\varepsilon(u_2, u, A_\varepsilon M_\varepsilon u) = b_\varepsilon(M_\varepsilon u_2, M_\varepsilon u, A_\varepsilon M_\varepsilon u) + b_\varepsilon(N_\varepsilon u_2, N_\varepsilon u, A_\varepsilon M_\varepsilon u).$$

It is obvious that

$$\begin{aligned} |b_\varepsilon(M_\varepsilon u_2, M_\varepsilon u, A_\varepsilon M_\varepsilon u)| &\le c\|M_\varepsilon u_2\|_{L^\infty(\mathbb{T}^2)} \|M_\varepsilon u\|_\varepsilon |A_\varepsilon M_\varepsilon u|_\varepsilon \\ &\le \frac{c}{\sqrt{\varepsilon}} |A_\varepsilon M_\varepsilon u_2|_\varepsilon \|M_\varepsilon u\|_\varepsilon |A_\varepsilon M_\varepsilon u|_\varepsilon \\ &\le \delta |A_\varepsilon M_\varepsilon u|_\varepsilon^2 + \frac{c}{\delta\varepsilon} \|M_\varepsilon u\|_\varepsilon^2 |A_\varepsilon M_\varepsilon u_2|_\varepsilon^2 \end{aligned}$$

for every $\delta > 0$. It also follows from (6.14) that

$$|b_\varepsilon(N_\varepsilon u_2, N_\varepsilon u, A_\varepsilon M_\varepsilon u)| \le \delta |A_\varepsilon M_\varepsilon u|_\varepsilon^2 + \frac{c\varepsilon}{\delta} \|N_\varepsilon u\|_\varepsilon^2 |A_\varepsilon N_\varepsilon u_2|_\varepsilon^2, \quad \forall\, \delta > 0.$$

Using in (6.21) the inequalities above with appropriate $\delta > 0$ we get

$$\begin{aligned} &\frac{d}{dt} \|M_\varepsilon u\|_\varepsilon^2 + \nu |A_\varepsilon M_\varepsilon u|_\varepsilon^2 \\ &\qquad \le c_1 \varepsilon \|N_\varepsilon u\|_\varepsilon^2 \psi_N(t, u_1, u_2) + \frac{c_2}{\varepsilon} \|M_\varepsilon u\|_\varepsilon^2 \psi_M(t, u_1, u_2), \end{aligned} \tag{6.22}$$

where $\psi_N(t, u_1, u_2)$ is defined above in (6.20) and

$$\psi_M(t, u_1, u_2) = |A_\varepsilon M_\varepsilon u_1(t)|_\varepsilon^2 + |A_\varepsilon M_\varepsilon u_2(t)|_\varepsilon^2.$$

Step 3: Proof of (4.5)**.**

It follows from (6.19) and (6.22) that

$$\frac{d}{dt} \|u\|_\varepsilon^2 + \lambda_1 \nu \|u\|_\varepsilon^2 \le c\|u\|_\varepsilon^2 \psi_\varepsilon(t, u_1, u_2) \tag{6.23}$$

where $\psi_\varepsilon(t, u_1, u_2) = \varepsilon^{2\theta} \psi_N(t, u_1, u_2) + \varepsilon^{-1} \psi_M(t, u_1, u_2)$. By Lemma 6.2 we have that

$$\int_0^t \psi_\varepsilon(\tau, u_1, u_2) d\tau \le c_0 \varepsilon^{2\theta} b^2(\varepsilon) + \frac{c_1 a^2(\varepsilon)}{\varepsilon} + c_2 b^4(\varepsilon) \le c_3 + c_4 \left[\log \frac{1}{\varepsilon}\right]^{2\sigma}$$

for all $0 < \varepsilon \le \varepsilon_0$. Thus by Gronwall's lemma from (6.23) we have that

$$\begin{aligned} \|u(t)\|_\varepsilon^2 &\le \|u(0)\|_\varepsilon^2 \exp\left\{ -\lambda_1 \nu t + c \int_0^t \psi_\varepsilon(\tau, u_1, u_2) d\tau \right\} \\ &\le \|u(0)\|_\varepsilon^2 \exp\left\{ -\lambda_1 \nu t + c_1 + c_2 \left[\log \frac{1}{\varepsilon}\right]^{2\sigma} \right\}. \end{aligned} \tag{6.24}$$

Since $0 < \sigma < 1/2$, we have that $c_2 \left[\log \frac{1}{\varepsilon}\right]^{2\sigma} \le c_\delta + 2\delta \log \frac{1}{\varepsilon}$. Hence we arrived at the relation

$$\|u(t)\|_\varepsilon^2 \le C_\delta \left[\frac{1}{\varepsilon}\right]^{2\delta} \|u(0)\|_\varepsilon^2 e^{-\lambda_1 \nu t}, \quad t \in [0, T],\ \forall \delta > 0, \tag{6.25}$$

which implies (4.5). If $\sigma = 0$, then we recover from (6.24) estimate (4.5) with $\delta = 0$.

Step 4: Main estimate for N_ε-component.

Since $\|u(t)\|_\varepsilon^2 = \|N_\varepsilon u\|_\varepsilon^2 + \|M_\varepsilon u\|_\varepsilon^2$, inserting estimate (6.25) in (6.19) yields

$$\frac{d}{dt}\|N_\varepsilon u\|_\varepsilon^2 + \nu |A_\varepsilon N_\varepsilon u|_\varepsilon^2 \le c_\delta \varepsilon^{2\theta-\delta}\|u(0)\|_\varepsilon^2 \psi_N(t, u_1, u_2),$$

where $\psi_N(t, u_1, u_2)$ is given by (6.20). Since by (6.6)

$$|A_\varepsilon N_\varepsilon u|_\varepsilon^2 \ge \varepsilon^{-2}|A_\varepsilon^{1/2} N_\varepsilon u|_\varepsilon^2 = \varepsilon^{-2}\|N_\varepsilon u\|_\varepsilon^2,$$

we obtain that

$$\|N_\varepsilon u(t)\|_\varepsilon^2 \le \|N_\varepsilon u(0)\|^2 \exp\left\{-\frac{\nu}{\varepsilon^2}\, t\right\} + c_\delta \varepsilon^{2\theta-\delta}\|u(0)\|_\varepsilon^2 \int_0^t \psi_N(\tau, u_1, u_2) d\tau.$$

Now application of estimate (6.10) yields

$$\|N_\varepsilon u(t)\|_\varepsilon^2 \le \|N_\varepsilon u(0)\|^2 \exp\left\{-\frac{\nu}{\varepsilon^2}\, t\right\} + c_\delta \varepsilon^{2\theta-\delta} b^2(\varepsilon)\|u(0)\|_\varepsilon^2.$$

Thus under the hypotheses of Theorem 4.2 we have that

$$\|N_\varepsilon u(t)\|_\varepsilon \le \|N_\varepsilon u(0)\| \exp\left\{-\frac{\nu}{2\varepsilon^2}\, t\right\} + c_\varrho \varepsilon^\varrho \|u(0)\|_\varepsilon, \quad 0 \le t \le T, \tag{6.26}$$

for every $0 < \varrho < 1/2$, provided that $u_1(0), u_2(0) \in \mathcal{B}_\varepsilon$.

Step 5: Estimate for $(I - P_N)M_\varepsilon$-component.

Let $1/6 \le \gamma < 1/2$ and P_N be the (spectral) orthonormal projector on the first N eigenvectors of the 2D Stokes operator A_0 and $0 < \lambda_1 \le \lambda_2, \ldots$ be the eigenvalues of A_0. Let $w = (I - P_N)M_\varepsilon u$. Then w solves the following 2D problem in $\mathbb{T}^2$:

$$w' + \nu A_0 w + (I - P_N)M_\varepsilon \left[B_\varepsilon(u, u_1) + B_\varepsilon(u_2, u)\right] = 0.$$

Therefore

$$w(t) = e^{-\nu A_0 t} w(0) - \int_0^t e^{-\nu A_0(t-\tau)}(I - P_N)M_\varepsilon \left[B_\varepsilon(u, u_1) + B_\varepsilon(u_2, u)\right] d\tau\,.$$

Thus

$$\begin{aligned}\|A_0^{1/2} w(t)\|_{\widetilde{H}} \quad &\le \quad \|A_0^{1/2} w(0)\|_{\widetilde{H}} e^{-\nu\lambda_{N+1} t} \\ &\quad + \Psi_\gamma(t, u_1, u_2) \cdot \int_0^t \|A_0^{1/2+\gamma} e^{-\nu A_0(t-\tau)}(I - P_N)\|_{\mathcal{L}(\widetilde{H})} d\tau,\end{aligned}$$

where

$$\Psi_\gamma(t, u_1, u_2) = \sup_{\tau\in[0,t]} \left\|A_0^{-\gamma} M_\varepsilon \left[B_\varepsilon(u(\tau), u_1(\tau)) + B_\varepsilon(u_2(\tau), u(\tau))\right]\right\|_{\widetilde{H}}.$$

It is well-known (see, e.g. [3, Chap.2] that a positive operator A_0 with a compact resolvent satisfies

$$\int_0^t \left\|A_0^\beta e^{-\nu A_0(t-\tau)}(I - P_N)\right\|_{\widetilde{H}} d\tau \le \frac{C_\beta}{\lambda_{N+1}^{1-\beta}}$$

for every $0 \le \beta < 1$. Therefore,

$$\|A_0^{1/2} w(t)\|_{\widetilde{H}} \le \|A_0^{1/2} w(0)\|_{\widetilde{H}} e^{-\nu\lambda_{N+1} t} + \frac{C_\gamma}{\lambda_{N+1}^{1/2-\gamma}} \Psi_\gamma(t, u_1, u_2)\,. \tag{6.27}$$

Now we estimate $\Psi_\gamma(t,u_1,u_2)$. We obviously have that

$$\left\|A_0^{-\gamma}M_\varepsilon B_\varepsilon(u,u_1)\right\|_{\widetilde{H}} = \sup\left\{(M_\varepsilon B_\varepsilon(u,u_1), A_0^{-\gamma}v)_{\widetilde{H}} :\ v\in\widetilde{H}, \|v\|_{\widetilde{H}}=1\right\}$$
$$= \frac{1}{\varepsilon}\sup\left\{b_\varepsilon(u,u_1,M_\varepsilon A_0^{-\gamma}v) :\ v\in\widetilde{H}, \|v\|_{\widetilde{H}}=1\right\}. \tag{6.28}$$

It is clear that

$$b_\varepsilon(u,u_1,M_\varepsilon A_0^{-\gamma}v) \quad = \quad b_\varepsilon(u,M_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v)+b_\varepsilon(N_\varepsilon u,N_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v).$$

To estimate the first term we note that

$$|b_\varepsilon(u,M_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v)| \quad\le\quad C\|M_\varepsilon u_1\|_\varepsilon \sum_{j,l=1}^{3}\left[\int_{\mathcal{O}_\varepsilon}|u_j|^2|(A_0^{-\gamma}v)_l|^2\,dx\right]^{1/2}$$
$$\le\quad C\|M_\varepsilon u_1\|_\varepsilon |u|_{L^6(\mathcal{O}_\varepsilon)}\left[\varepsilon\int_{\mathbb{T}^2}|(A_0^{-\gamma}v)|^3\,dx\right]^{1/3}.$$

Since $H^{1/3}(\mathbb{T}^2)\subset L_3(\mathbb{T}^2)$ and $\gamma\ge 1/6$, we have that

$$\left[\int_{\mathbb{T}^2}|(A_0^{-\gamma}v)|^3\,dx\right]^{1/3} \le C|(A_0^{-\gamma}v)|_{H^{1/3}(\mathbb{T}^2)}\le C\|v\|_{\widetilde{H}}.$$

We also have[6] that $|u|_{L_6(\mathcal{O}_\varepsilon)}\le C\varepsilon^{-1/3}\|u\|_\varepsilon$. Therefore, applying (6.11) to the term $\|M_\varepsilon u_1\|_\varepsilon$, we have that

$$|b_\varepsilon(u,M_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v)|\le C\|u\|_\varepsilon\|M_\varepsilon u_1\|_\varepsilon\|v\|_{\widetilde{H}}\le C\left(a(\varepsilon)+\sqrt{\varepsilon}b^2(\varepsilon)\right)\|u\|_\varepsilon\|v\|_{\widetilde{H}}. \tag{6.29}$$

As for the second term, we obviously have that

$$|b_\varepsilon(N_\varepsilon u,N_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v)|\le C\varepsilon^{1/p_2}\|N_\varepsilon u_1\|_\varepsilon|N_\varepsilon u|_{L_{p_1}(\mathcal{O}_\varepsilon)}|M_\varepsilon A_0^{-\gamma}v|_{L_{p_2}(\mathbb{T}^2)},$$

where $p_1^{-1}+p_2^{-1}=1/2$, $2<p_1\le 6$. Since $H^{1-2/p_2}(\mathbb{T}^2)\subset L_{p_2}(\mathbb{T}^2)$, from (6.8) we obtain that

$$|b_\varepsilon(N_\varepsilon u,N_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v)|\le C\varepsilon^\theta\|N_\varepsilon u_1\|_\varepsilon\|N_\varepsilon u\|_\varepsilon\|A_0^{1/p_1-\gamma}v\|_{\widetilde{H}},$$

where $\theta=1/p_2+(6-p_1)/(2p_1)=2/p_1$. We can choose $p_1=\gamma^{-1}$ and apply (6.9) to $\|N_\varepsilon u_1\|_\varepsilon$ to obtain that

$$|b_\varepsilon(N_\varepsilon u,N_\varepsilon u_1,M_\varepsilon A_0^{-\gamma}v)|\le C\varepsilon^{2\gamma}b(\varepsilon)\|N_\varepsilon u\|_\varepsilon\|v\|_{\widetilde{H}}.$$

Using this estimate and also (6.29) in (6.28) we find that

$$\left\|A_0^{-\gamma}M_\varepsilon B_\varepsilon(u,u_1)\right\|_{\widetilde{H}}\le C\varepsilon^{-1}d(\varepsilon)\|u\|_\varepsilon,$$

where

$$d(\varepsilon)=a(\varepsilon)+\sqrt{\varepsilon}b^2(\varepsilon)+\varepsilon^{2\gamma}b(\varepsilon) \tag{6.30}$$

for every $1/6\le\gamma<1/2$. A similar argument yields

$$\left\|A_0^{-\gamma}M_\varepsilon B_\varepsilon(u_2,u)\right\|_{\widetilde{H}}\le C\varepsilon^{-1}d(\varepsilon)\|u\|_\varepsilon.$$

Therefore using (6.25) we obtain that

$$\Psi_\gamma(t,u_1,u_2)\le C_\delta d(\varepsilon)\left(\frac{1}{\varepsilon}\right)^{1+\delta}\|u(0)\|_\varepsilon.$$

Since $\|A_0^{1/2}w(t)\|_{\widetilde{H}}=\varepsilon^{-1/2}\|w(t)\|_\varepsilon$, then now (6.27) yields the estimate

$$\|w(t)\|_\varepsilon\le\|w(0)\|_\varepsilon e^{-\nu\lambda_{N+1}t}+C_\delta d(\varepsilon)\left(\frac{1}{\varepsilon}\right)^{1/2+\delta}\lambda_{N+1}^{-1/2+\gamma}\|u(0)\|_\varepsilon, \tag{6.31}$$

where $w(t)=(I-P_N)M_\varepsilon u(t)$, $d(\varepsilon)$ is given by (6.30), and $\delta=0$ if $\sigma=0$ in (4.4).

[6]The estimate follows from the Ladyzhenskaya inequality in the domain $\mathbb{T}^2\times(0,1)$ which transforms to $\mathcal{O}_\varepsilon$ by the scaling transformation $(x',x_3)\mapsto(x',\varepsilon x_3)$.

Concluding step: Proof of (4.6)**.**

Since $(I-\tilde{P}_N)u(t) = (I-P_N)M_\varepsilon u(t) + N_\varepsilon u(t)$, we get from (6.26) and (6.31) that

$$\|(I-\tilde{P}_N)u(t)\|_\varepsilon \le q(N,\varepsilon,t)\|u(0)\|_\varepsilon, \tag{6.32}$$

where

$$q(N,\varepsilon,t) = \exp\left\{-\frac{\nu}{2\varepsilon^2}\,t\right\} + \exp\left\{-\nu\lambda_{N+1}\,t\right\} + c_\varrho\varepsilon^\varrho + \frac{C_\delta d(\varepsilon)}{\lambda_{N+1}^{1/2-\gamma}}\left(\frac{1}{\varepsilon}\right)^{1/2+\delta}.$$

Clearly for any $T>0$ we can choose $\varepsilon_0>0$ and for $\varepsilon\le\varepsilon_0$ find $N=N(\varepsilon,T)$ such that $q(N,\varepsilon,T)\le q$. Therefore (6.32) implies (4.6).

If $\sigma=0$ in relations (4.4), then $\delta=0$ and, after choosing $\gamma=1/4$, we have $d(\varepsilon)\le C\sqrt{\varepsilon}$. Thus the coefficient in the last term of the expression for $q(N,\varepsilon,t)$ does not depend on ε and hence N may be chosen independent of ε.

The proof of Theorem 4.2 is complete.

6.5 Proof of Proposition 4.4

The first part follows from (2.6). To prove estimate (4.7) we start with an assertion which is well-known for every fixed ε (see, e.g., [5]). We repeat the corresponding argument because we need to control dependence of the constants on ε.

Lemma 6.4 *There exist positive constants ε_0, c such that for all $\varepsilon\in(0,\varepsilon_0)$, $u\in V_\varepsilon$, $w\in \mathrm{D}(A_\varepsilon)$, $v\in H_\varepsilon$, we have*

$$|b_\varepsilon(u,w,v)| \le c\varepsilon^{-1/2}|u|_\varepsilon^{1/4}\|u\|_\varepsilon^{3/4}\|w\|_\varepsilon^{1/4}|A_\varepsilon w|_\varepsilon^{3/4}\cdot|v|_\varepsilon. \tag{6.33}$$

Proof. We obviously have that

$$|b_\varepsilon(u,w,v)| \le \sum_{j=1}^{3}\sum_{l=1}^{3}|u_j|_{L^4(\mathcal{O}_\varepsilon)}\,|\partial_j w_l|_{L^4(\mathcal{O}_\varepsilon)}\,|v_l|_{L^2(\mathcal{O}_\varepsilon)}\,.$$

Transforming $\mathcal{O}_\varepsilon$ into $\mathcal{O}_1$ by the scaling transformation $(x',x_3)\mapsto(x',\varepsilon^{-1}x_3)$ and applying the well-known estimates for L_4-norm, we obtain that

$$|u_j|_{L^4(\mathcal{O}_\varepsilon)} \le C\varepsilon^{-1/4}|u|_\varepsilon^{1/4}\|u\|_\varepsilon^{3/4}. \tag{6.34}$$

Using a similar estimate for $|\partial_j w_l|_{L^4(\mathcal{O}_\varepsilon)}$ we arrive at (6.33). •

To continue with the proof of Proposition 4.4 we note that application of Lemma 6.2 with $a(\varepsilon)=b(\varepsilon)=\rho\sqrt{\varepsilon}$ yields

$$\|u(t)\|_\varepsilon^2 + \int_0^t |A_\varepsilon u(s)|_\varepsilon^2 ds \le c(\rho)\varepsilon, \quad t\in[0,T]. \tag{6.35}$$

It follows from (6.33) that

$$|B_\varepsilon(u,u)|_\varepsilon \le c\varepsilon^{-1/2}\|u\|_\varepsilon^{5/4}|A_\varepsilon u|_\varepsilon^{3/4} \le c\varepsilon^{1/8}|A_\varepsilon u|_\varepsilon^{3/4}, \quad t\in[0,T].$$

Since $u(t)$ satisfies (2.1) with $f=0$, then

$$|u' + \nu A_\varepsilon u|_\varepsilon \le c\varepsilon^{1/8}|A_\varepsilon u|_\varepsilon^{3/4} \le \delta|A_\varepsilon u|_\varepsilon + C_\delta\sqrt{\varepsilon}, \quad t\in[0,T]\,. \tag{6.36}$$

for any $\delta > 0$. This inequality with $\delta = 1$ and (6.35) give us that

$$\int_0^t |u'(s)|_\varepsilon^2 ds \le c(\rho)\varepsilon, \quad t \in [0,T]. \tag{6.37}$$

The equation for $w := u'$

$$w' + \nu A_\varepsilon w + B_\varepsilon(w,u) + B_\varepsilon(u,w) = 0$$

implies that

$$\frac{1}{2}\frac{d}{dt}|w|_\varepsilon^2 + \nu\|w\|_\varepsilon^2 \le |b_\varepsilon(w,u,w)|. \tag{6.38}$$

Since

$$|b_\varepsilon(w,u,w)| \le C\|u\|_\varepsilon\, |w|^2_{L^4(\mathcal{O}_\varepsilon)} \le C\sqrt{\varepsilon}\,|w|^2_{L^4(\mathcal{O}_\varepsilon)}\ ,$$

then it follows from (6.34) that

$$|b_\varepsilon(w,u,w)| \le C|w|_\varepsilon^{1/2}\|w\|_\varepsilon^{3/2} \le \nu\|w\|_\varepsilon^2 + C|w|_\varepsilon^2.$$

Consequently (6.38) yields

$$|w(t)|_\varepsilon^2 \le |w(s)|_\varepsilon^2 + C\int_s^t |w(\tau)|_\varepsilon^2 d\tau, \quad 0 \le s \le t \le T$$

Integration of this relation with respect to s over the interval $[0,t]$ yields

$$t|w(t)|_\varepsilon^2 \le C(1+t)\int_0^t |w(\tau)|_\varepsilon^2 d\tau, \quad 0 \le t \le T$$

Since $w = u'$, then evoking (6.37) we get that

$$|u'(t)|_\varepsilon^2 \le C_\rho(1+t^{-1})\varepsilon, \quad 0 \le t \le T,$$

which, due to (6.36) with $\delta = \nu/2$ implies (4.7). Thus the proof of Proposition 4.4 is complete.

7 Some hydrodynamical consequences

In this section we use our results to study asymptotic properties of some quantities, characterising fluid's motion. We assume that the hypotheses of the first assertion in Theorem 5.5 are in force.

1. Energy. Let $e(u)$ be the normalised energy of a flow u in $\mathcal{O}_\varepsilon$:

$$e(u) = \frac{1}{2\varepsilon}\int_{\mathcal{O}_\varepsilon} |u(x)|^2\, dx = \frac{1}{2\varepsilon}|u|_\varepsilon^2$$

(note that $e(u) = \frac{1}{2}|u|_{\widetilde{H}}^2$ if $M_\varepsilon u = u$). Then

a) The averaged energy $\mathbf{E}e(u(kT+\tau))$ of any solution u for (1.1)-(1.5) at time $kT+\tau$ ($0 \le \tau \le T$) with the initial data from $\mathcal{B}_\varepsilon$ converges as $k \to \infty$ to the energy of the stationary measure $\int e(u)\,\mu_\varepsilon^\tau(du)$ (we use Theorem 5.1 and Remark 5.2). Furthermore, calculating the latter quantity one can replace the average in ensemble by the average in time (cf. Introduction).

b) The following relation holds

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}_\varepsilon} e(u)\mu_\varepsilon^\tau(du) = \int_{\widetilde{H}} e(v)\,\vartheta^\tau(dv)\,. \tag{7.1}$$

To prove (7.1) we note that by (6.1) we have that

$$\int_{\mathcal{B}_\varepsilon} e(u)\mu_\varepsilon^\tau(du) = \frac{1}{2\varepsilon}\int_{\widetilde V} |M_\varepsilon u|_\varepsilon^2\, \mu_\varepsilon^\tau(du) + \frac{1}{2\varepsilon}\int_{\mathcal{B}_\varepsilon^0} |N_\varepsilon u|_\varepsilon^2 \mu_\varepsilon(du), \tag{7.2}$$

where $\mathcal{B}_\varepsilon^0$ is given by (5.7). However, under the conditions of Theorem 5.5 by (6.6) we have that

$$|N_\varepsilon u|_\varepsilon^2 \le \varepsilon^2 \|N_\varepsilon u\|_\varepsilon^2 \le C\varepsilon^3, \quad u \in \mathcal{B}_\varepsilon^0. \tag{7.3}$$

So when $\varepsilon \to 0$ the second term in the r.h.s. of (7.2) disappears. Applying Theorem 5.5 to the first term we get (7.1).

The obtained relation means that under the limit $\varepsilon \to 0$ the averaged 3D energy per unit volume of a stationary flow in $\mathcal{O}_\varepsilon$ converges to the averaged 2D energy per unit area of a stationary 2D flow in $\mathbb{T}^2$.

c) Let the random force satisfies the hypothesis of Theorem 5.8, and $u^\varepsilon(x,t)$ be a solution of (1.1)-(1.5) such that $u_0 = u_0^\varepsilon \in \mathcal{B}_\varepsilon^0$ and $Mu_0^\varepsilon = \widetilde v_0$. Let $v(x',t)$ be a solution of problem (1.9)–(1.11) with the force $\widetilde f$ given by (5.13). Using Theorem 5.8 jointly with (7.3) we find that the averaged energy of the 3D solution $\mathbf{E}e(u(t))$ converges to the averaged energy of the 2D solution $\mathbf{E}e(v(t))$ as $\varepsilon \to 0$, uniformly in time t.

2. Enstrophy. Consider the normalised enstrophy functional

$$\Omega(u) = \frac{1}{2\varepsilon}\|u\|_\varepsilon^2 = \frac{1}{2\varepsilon}\int_{\mathcal{O}_\varepsilon} |\mathrm{curl}\, u(x)|^2\, dx\,.$$

a) Let the assumptions of item 2) of Theorem 5.1 hold, as well as (5.9) with $n = 2$. Then the averaged enstrophy of any admissible solution u converges to the averaged enstrophy of the stationary measure: $\mathbf{E}\,\Omega(u(kT+\tau)) \to \int \Omega(u)\,\mu_\varepsilon^\tau(du)$ as $k \to \infty$, for any τ. Assume now that the kicks have zero mean-value:

$$\mathbf{E}\eta_k^\varepsilon = 0 \quad \forall k\,.$$

Then $\mathbf{E}|u(T)|_\varepsilon^2 = \mathbf{E}|u(T-0)|_\varepsilon^2 + \mathbf{E}|\eta_1^\varepsilon|_\varepsilon^2$, so the energy balance relation for a solution u on the segment $[0,T]$ takes the form

$$\mathbf{E}|u(T)|_\varepsilon^2 + 2\nu\mathbf{E}\int_0^T \|u(\tau)\|_\varepsilon^2 d\tau = \mathbf{E}|u(0)|_\varepsilon^2 + \varepsilon D_\varepsilon,$$

where

$$D_\varepsilon = \varepsilon^{-1}\mathbf{E}|\eta_1^\varepsilon|_\varepsilon^2 = \sum (b_j^\varepsilon)^2 \lambda_j^{-1}\mathbf{E}(\xi_{j1}^\varepsilon)^2 + \sum (\hat b_j^\varepsilon)^2 (\Lambda_j^\varepsilon)^{-1}\mathbf{E}(\hat\xi_{j1}^\varepsilon)^2\,. \tag{7.4}$$

Applying the energy balance relation to a stationary solution u we get that

$$\int_0^T d\tau \int_{\mathcal{B}_\varepsilon} \Omega(u)\,\mu_\varepsilon^\tau(du) = \frac{1}{4\nu} D_\varepsilon\,. \tag{7.5}$$

Accordingly, the enstrophy of any solution satisfies

$$\frac{1}{T}\int_t^{t+T} \mathbf{E}\Omega(u(s))\,ds \to \frac{1}{4\nu T} D_\varepsilon \ \text{ as } \ t \to \infty\,.$$

b) Assume assumptions of Theorem 5.8 (including (5.16)). If $\tau > 0$, then (4.7) implies that $\Omega\big(Nu(kT+\tau)\big) \le C\tau^{-1}\varepsilon$, so by Theorem 5.8 for any $\tau > 0$ we have

$$\lim_{\varepsilon\to 0} \int_{\mathcal{B}_\varepsilon} \Omega(u)\,\mu_\varepsilon^\tau(du) = \int_{\widetilde H} \Omega(v)\,\vartheta^\tau(dv)\,. \tag{7.6}$$

Assume that the second term in the r.h.s. of (7.4) goes to zero with ε (this is a mild assumption since $(\Lambda_j^\varepsilon)^{-1} \le \varepsilon^2$ for each j). Then due to (5.12) $D_\varepsilon \to \sum b_j^2 \lambda_j^{-1} \mathbf{E}\xi_{j1}^2$, so by (7.5) we obtain that

$$\int_0^T d\tau \int_{\mathcal{B}_\varepsilon} \Omega(u)\, \mu_\varepsilon^\tau(du) \to \frac{1}{2\nu} \sum b_j^2 \lambda_j^{-1} \mathbf{E}\xi_{j1}^2 \,.$$

But the r.h.s. of (7.6), integrated in $d\tau$ over $[0,T]$, also equals to the limiting value above (see (3.12) in [13]). So the limiting relation (7.6), valid for $\tau > 0$, remains true after integrating in $d\tau$ over $[0,T]$.

3. Correlation tensor. Assume that (5.16) holds. We have from (5.17) in Theorem 5.5 and from Corollary 5.7 that

$$\lim_{\varepsilon\to 0} \int_{\mathcal{B}_\varepsilon} \langle u_i\rangle(x')\langle u_j\rangle(y')\mu_\varepsilon^\tau(du) = \int_{\widetilde{H}} v_i(x')v_j(y')\, \vartheta^\tau(dv), \quad i,j \in \{1,2\}\,, \tag{7.7}$$

for any $\tau \in [0,T]$. Here x', y' are any points in $\mathbb{T}^2$, for a function $f(x)$ on $\mathcal{O}_\varepsilon$ we denote $\langle f\rangle(x') = \varepsilon^{-1}\int f(x',x_3)\,dx_3$ and $\vartheta^\tau = S_0^\tau \circ \vartheta$. So the 'horizontal' components of the correlation tensor for the stationary 3D solution, averaged in the thin direction, converge to the correlation tensor for the stationary 2D solution.

If $\tau > 0$, then for the same reasons as above

$$\lim_{\varepsilon\to 0} \int_{\mathcal{B}_\varepsilon} u_i(x_\varepsilon)u_j(y_\varepsilon)\mu_\varepsilon^\tau(du) = \int_{\widetilde{H}} v_i(x')v_j(y')\, \vartheta^\tau(dv) \;\text{ for }\; i,j \in \{1,2,3\}\,.$$

Here $x_\varepsilon \in \mathcal{O}_\varepsilon$ is any point of the form $x_\varepsilon = (x', x_3(\varepsilon))$, similar with y_ε, and the r.h.s. vanishes if $i = 3$ or $j = 3$.

3. Enstrophy production. The averaged enstrophy production for a 3D flow u in $\mathcal{O}_\varepsilon$ equals

$$\mathbf{E}\varepsilon_\Omega(u), \quad \text{where} \quad \varepsilon_\Omega(u) = \sum_{i,j} \int_{\mathcal{O}_\varepsilon} \omega_i(x)\omega_j(x)s_{ij}dx.$$

Here $\omega = \operatorname{curl} u$ and $s_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ For a 2D flow we have $\varepsilon_\Omega = 0$. If u is a 3D flow, satisfying the boundary conditions in (1.4) then integrating by parts we find that $\varepsilon_\Omega(u) = b_\varepsilon(u,u,A_\varepsilon u)$. Therefore

$$\frac{1}{\varepsilon}\int_0^T d\tau \int_{\mathcal{B}_\varepsilon^0} \varepsilon_\Omega(u)\mu_\varepsilon^\tau(du) = \frac{1}{\varepsilon}\int_0^T d\tau \int_{\mathcal{B}_\varepsilon^0} b_\varepsilon(u(\tau),u(\tau),A_\varepsilon u(\tau))\mu_\varepsilon(du)$$

However, by the symmetry relation in (6.4) we can write $b_\varepsilon(u,u,A_\varepsilon u)$ as follows

$$\begin{aligned} b_\varepsilon(u,u,A_\varepsilon u) \;=\;& b_\varepsilon(M_\varepsilon u, N_\varepsilon u, A_\varepsilon N_\varepsilon u) + b_\varepsilon(N_\varepsilon u, M_\varepsilon u, A_\varepsilon N_\varepsilon u) \\ &+ b_\varepsilon(N_\varepsilon u, N_\varepsilon u, A_\varepsilon M_\varepsilon u) + b_\varepsilon(N_\varepsilon u, N_\varepsilon u, A_\varepsilon N_\varepsilon u). \end{aligned}$$

Now we use Lemma 6.3 and estimate (6.35) to obtain that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^T d\tau \int_{\mathcal{B}_\varepsilon^0} \varepsilon_\Omega(u)\mu_\varepsilon^\tau(du) = 0$$

under condition (5.6), in agreement with the fact that for the limiting 2D flow we have $\varepsilon_\Omega = 0$.

8 Appendix: spectral problem for the Stokes operator

The spectral boundary value problem which corresponds to operator A_ε has the form

$$\begin{cases} -\Delta w = \lambda w, \quad \operatorname{div} w = 0 \quad \text{in } \mathcal{O}_\varepsilon = \mathbb{T}^2 \times (0,\varepsilon), \\ \\ w(x', x_3) \text{ is } (l_1, l_2)\text{-periodic with respect to } x', \\ \\ w_3|_{x_3=\varepsilon} = 0, \quad \partial_3 w_j|_{x_3=\varepsilon} = 0, \quad j = 1,2, \\ w_3|_{x_3=0} = 0, \quad \partial_3 w_j|_{x_3=0} = 0, \quad j = 1,2. \\ \\ \int_{\mathcal{O}_\varepsilon} u_j dx = 0, \; j = 1,2. \end{cases} \tag{8.1}$$

Using the decomposition (1.7), where the spaces $M_\varepsilon V_\varepsilon \approx \widetilde{V}$ and $N_\varepsilon V_\varepsilon$ are invariant for Δ by iv) in Section 6.1, we see that the spectrum consists of two branches. Recalling estimate (6.6) we find that these branches are: (i) the spectrum of the 2D Stokes operator A_0, $0 < \lambda_1 \le \lambda_2 \le \ldots$, and (ii) series of eigenvalues $0 < \Lambda_1^\varepsilon \le \Lambda_2^\varepsilon \le \ldots$, depending on ε and greater than ε^{-2} . We denote the corresponding eigenfunctions e_{λ_j} and $e_{\Lambda_j^\varepsilon}$. We have

$$M_\varepsilon e_{\lambda_j} = e_{\lambda_j}\,, \quad M_\varepsilon e_{\Lambda_j^\varepsilon} = 0\,. \tag{8.2}$$

The eigenvalues λ_j are properly ordered numbers $\left(s_1 \frac{2\pi}{l_1}\right)^2 + \left(s_2 \frac{2\pi}{l_2}\right)^2$, $s = (s_1, s_2) \in \mathbb{Z}^2 \backslash \{0\}$, and

$$C^{-1} j \le \lambda_j \le Cj \quad \text{for all } j, \tag{8.3}$$

with some $C > 1$ (see, e.g., [5]). We normalise the eigenfunctions as follows:

$$\|e_{\lambda_j}\|_\varepsilon = \|e_{\Lambda_j^\varepsilon}\|_\varepsilon = \sqrt{\varepsilon} \quad \forall\, j\,. \tag{8.4}$$

Then $\|e_{\lambda_j}\|_{\tilde{V}} = 1$ for all j.

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