Destruction of Absolutely Continuous Spectrum by Perturbation Potentials of Bounded Variation

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Abstract

We show that absolutely continuous spectrum of one-dimensional Schrödinger operators may be destroyed by adding to them decaying perturbation potentials of bounded variation.

1 Introduction

In this paper we study discrete one-dimensional Schrödinger operators on $\ell^2(\mathbb{N})$ of the form

$$H_V = H_0 + V, \tag{1.1}$$

where $H_0 = \Delta + V_0$ is a discrete Schrödinger operator and V is a decaying perturbation potential. More explicitly, Δ is the discrete Dirichlet Laplacian on $\ell^2(\mathbb{N})$, defined by

$$(\Delta\psi)(n) = \psi(n+1) + \psi(n-1) \tag{1.2}$$

for $n > 1$ and $(\Delta\psi)(1) = \psi(2)$. V_0 and V are discrete potentials, that is,

$$(V_0\psi)(n) = V_0(n)\psi(n), \qquad (V\psi)(n) = V(n)\psi(n), \qquad n = 1, 2, \dots, \quad (1.3)$$

where $\{V_0(n)\}$ and $\{V(n)\}$ are sequences of real numbers. We say that a potential V is *decaying* if $V(n) \to 0$ as $n \to \infty$.

For operators $H = H_V$ of the form (1.1), we define $\Sigma_{\text{ac}}(H)$, *the essential support of the absolutely continuous spectrum of* H, as the equivalence class, up to sets of zero Lebesgue measure, of the set

$$\Big\{ E \in \mathbb{R} \;\Big|\; \lim_{\epsilon \downarrow 0} \operatorname{Im} \langle \delta_1, (H - E - i\epsilon)^{-1} \delta_1 \rangle \text{ exists and is finite and non-zero} \Big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\ell^2(\mathbb{N})$ and $\delta_j(n)$ is 1 if $j = n$ and 0 otherwise. The absolutely continuous spectrum of H, $\sigma_{\text{ac}}(H)$, coincides with the essential closure of $\Sigma_{\text{ac}}(H)$, namely,

$$\sigma_{\text{ac}}(H) = \big\{ E \in \mathbb{R} \;\big|\; |\Sigma_{\text{ac}}(H) \cap (E - \epsilon, E + \epsilon)| > 0 \;\; \forall \epsilon > 0 \big\},$$

where $|\cdot|$ denotes Lebesgue measure. In what follows we may write equalities of the form $\Sigma_{\text{ac}}(H) = S$, where S may be a concrete subset of $\mathbb{R}$. Since $\Sigma_{\text{ac}}(\cdot)$ is an equivalence class of sets rather than a concrete subset of $\mathbb{R}$, such equalities should be understood as equalities up to sets of zero Lebesgue measure, or more precisely, as saying $S \in \Sigma_{\text{ac}}(H)$.

We say that a potential V is *of bounded variation* if

$$\sum_{n=1}^{\infty} |V(n+1) - V(n)| < \infty.$$

Recall Weidmann's classical result [23]:

Theorem 1.1 (Weidmann's theorem) *If $V_0 = 0$ and V is a decaying potential of bounded variation, then $\Sigma_{\text{ac}}(H_V) = \Sigma_{\text{ac}}(\Delta) = (-2, 2)$.*

Remark. Weidmann [23] actually proves an analog of this for continuous Schrödinger operators on $L^2([0, \infty), dx)$. For a proof of the discrete case, see Dombrowski-Nevai [7] or Simon [19].

Weidmann's theorem bears some similarity to the following well-known consequence of the Birman-Kato theory (see [18, Chapter XI.3]):

Theorem 1.2 *For any V_0, if $\sum_{n=1}^{\infty} |V(n)| < \infty$, then $\Sigma_{\text{ac}}(H_V) = \Sigma_{\text{ac}}(H_0)$.*

A notable difference, though, is that Theorem 1.2 allows an arbitrary V_0, while Weidmann's theorem applies only to $V_0 = 0$.

The purpose of this paper is to answer the following natural question: Does Weidmann's theorem extends to the general case of $V_0 \neq 0$? Namely, is it true that for a general V_0 and a decaying V of bounded variation, $\Sigma_{\mathrm{ac}}(H_V) = \Sigma_{\mathrm{ac}}(H_0)$? The answer turns out to be negative, as indicated by our first result:

Theorem 1.3 *If H_0 has nowhere dense spectrum, then there exists a decaying potential V of bounded variation for which $\Sigma_{\mathrm{ac}}(H_V) = \emptyset$.*

The point here is that there are known examples of V_0's for which the spectrum of H_0 is both nowhere dense and absolutely continuous. Such examples, with limit periodic V_0's, have been constructed by Avron-Simon [1]. Another example is the weakly coupled irrational almost Mathieu operator, namely, the case $V_0(n) = \lambda \cos(2\pi\alpha n + \theta)$, where $\lambda, \alpha, \theta \in \mathbb{R}$, $|\lambda| < 2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ (see, e.g., [16] and references therein). Thus, Theorem 1.3 provides a fairly broad class of H_0's whose absolutely continuous spectrum can always be destroyed by adding to them a decaying perturbation potential of bounded variation. However, the requirement for H_0 to have nowhere dense spectrum along with the fact that $\Sigma_{\mathrm{ac}}(\Delta)$ is an interval raises again a natural question: Does Weidmann's theorem extends at least to the case of V_0's for which $\Sigma_{\mathrm{ac}}(H_0)$ is made of intervals? The answer turns out to be negative, again, as indicated by our second result:

Theorem 1.4 *There exist a decaying potential V_0 and a decaying potential of bounded variation V, such that $\Sigma_{\mathrm{ac}}(H_0) = \Sigma_{\mathrm{ac}}(\Delta) = (-2, 2)$, but $\Sigma_{\mathrm{ac}}(H_V) = \emptyset$.*

We note that in spite of Theorems 1.3 and 1.4, Weidmann's theorem does have natural extensions and generalizations. In particular, we note the following result of Golinskii-Nevai [8]:

Theorem 1.5 *If $V_0 = 0$, V is decaying, and for some $q \in \mathbb{N}$,*

$$\sum_{n=1}^{\infty} |V(n+q) - V(n)| < \infty, \tag{1.4}$$

then $\Sigma_{\mathrm{ac}}(H_V) = \Sigma_{\mathrm{ac}}(\Delta) = (-2, 2)$.

Remarks. 1. The result is actually more general. In particular, it covers cases where V isn't decaying, so that (1.4) implies its convergence to a periodic potential of period q. (This is equivalent, of course, to the case where V is decaying and obeys (1.4) and V_0 is some periodic potential of period q.) The result says, in this case, that $\Sigma_{\rm ac}(H_V)$ coincides with the spectrum of the limiting periodic operator, and thus, in particular, it extends Theorem 1.1 to the case of any periodic V_0.

2. Before [8], some related results were obtained by Stolz [21, 22].

We believe that the following stronger statement should be true:

Conjecture 1.6 *If $V_0 = 0$, V is decaying, and for some $q \in \mathbb{N}$,*

$$\sum_{n=1}^{\infty} |V(n+q) - V(n)|^2 < \infty, \tag{1.5}$$

then $\Sigma_{\rm ac}(H_V) = \Sigma_{\rm ac}(\Delta) = (-2, 2)$.

We note that this conjecture for the special case $q = 1$ has also been made by Simon [20, Chapter 12]. Recently Kupin [13] came quite close to establishing it by showing that if $\sum_{n=1}^{\infty} |V(n+1) - V(n)|^2 < \infty$ and in addition, there exists some $p > 0$ for which $\sum_{n=1}^{\infty} |V(n)|^p < \infty$, then $\Sigma_{\rm ac}(H_V) = \Sigma_{\rm ac}(\Delta) = (-2, 2)$. Kupin's result is among the strongest of several recent results (see [12, 14, 24]) in which an ℓ^p requirement for V itself and an ℓ^2 requirement for some type of variation of V combine to ensure that $\Sigma_{\rm ac}(H_V) = \Sigma_{\rm ac}(\Delta) = (-2, 2)$. In most other results of this genre, p is a concrete number, such as 3 or 4. We consider the proving (or disproving) of the full Conjecture 1.6 to be an interesting (and potentially hard) open problem. It may have some connection to the results of Christ-Kiselev [4] for continuous Schrödinger operators, although it isn't fully clear what should be considered a discrete analog of their results and how connected to Conjecture 1.6 it may be.

We also recall the following conjecture of Kiselev-Last-Simon [11]:

Conjecture 1.7 *For any V_0, if $\sum_{n=1}^{\infty} |V(n)|^2 < \infty$, then $\Sigma_{\rm ac}(H_V) = \Sigma_{\rm ac}(H_0)$.*

For the special case $V_0 = 0$, it has been established by Deift-Killip [5], and for periodic V_0, by Killip [10]. A related result concerning general V_0 has been recently obtained by Breuer-Last [2], who show stability of absolutely continuous spectrum associated with bounded generalized eigenfunctions under random decaying ℓ^2 perturbation potentials (also see a related result by Kaluzhny-Last [9]).

We believe that Theorems 1.3 and 1.4 elucidate a fundamental difference between spectral stability under perturbation potentials decaying sufficiently fast (namely, statements like Theorem 1.2 and Conjecture 1.7) and "Weidmann type" statements like Theorem 1.5 and Conjecture 1.6, which appear not to extend much beyond cases where the unperturbed operator is periodic.

The rest of this paper is organized as follows. In Section 2 we filter some results from [17] that we use later. In Section 3 we prove Theorem 1.3 and in Section 4 we prove Theorem 1.4.

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2 Preliminaries

Given a discrete Schrödinger operator of the form $\tilde{H} = \Delta + \tilde{V}$, we denote its associated 2×2 transfer matrices as follows:

$$T_n(E) = \begin{pmatrix} E - \tilde{V}(n) & -1 \\ 1 & 0 \end{pmatrix},$$

$$\Phi_{m,n}(E) = T_n(E) T_{n-1}(E) \cdots T_m(E),$$

$$\Phi_n(E) = \Phi_{1,n}(E) = T_n(E) T_{n-1}(E) \cdots T_1(E).$$

The following is an immediate consequence of [17, Theorem 3.10]:

Proposition 2.1 *For Lebesgue a.e. $E \in \Sigma_{\mathrm{ac}}(\tilde{H})$,*

$$\limsup_{N \to \infty} \frac{1}{N (\log N)^2} \sum_{n=1}^{N} \|\Phi_n(E)\|^2 < \infty.$$

The following is an easy consequence of [17, Theorem 4.2] and its proof:

Proposition 2.2 *Let $H_1 = \Delta + V_1$ be another discrete Schrödinger operator on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$. Suppose that for some $m, k \in \mathbb{N}$, $k \geq 4$, we have $\tilde{V}(n) = V_1(n)$ for every $n \in \{m, m+1, \ldots, k\}$, and that for some $E \in \mathbb{R}$ and $\delta > 0$, $\sigma(H_1) \cap (E - \delta, E + \delta) = \emptyset$, where $\sigma(\,\cdot\,)$ denotes the spectrum of the operator. Then for every $\ell \in \{4, 5, \ldots, k\}$,*

$$\|\Phi_{m,m+\ell}(E)\| \geq \frac{1}{2}\delta^2(1+\delta^2)^{\frac{\ell-3}{2}}\,.$$

3 Proof of Theorem 1.3

Proof of Theorem 1.3. If V_0 is unbounded, then so is $V_0 + V$ for any decaying V, and thus, by well known results (see, e.g., [17, Theorem 4.1]), we would get $\Sigma_{\mathrm{ac}}(H_V) = \emptyset$ for any decaying V. We may thus assume, without loss of generality, that V_0 is bounded, so that the spectrum of H_0, $\sigma(H_0)$, is a compact set. Its complement, $\mathbb{R} \setminus \sigma(H_0)$, is an open set and thus a union of countably many disjoint open intervals: $\mathbb{R} \setminus \sigma(H_0) = \bigcup_{\nu=1}^{\infty} I_\nu$, $I_\nu = (a_\nu, b_\nu)$. We call the I_ν's *gaps* in $\sigma(H_0)$, and denote the collection of all these gaps by $\mathcal{G}$, that is, $\mathcal{G} = \{I_\nu\}_{\nu=1}^{\infty}$. Since $\sigma(H_0)$ is compact, $\mathcal{G}$ contains exactly two elements of infinite length, which are $(-\infty, \min\sigma(H_0))$ and $(\max\sigma(H_0), \infty)$. The other elements of $\mathcal{G}$ are open intervals of finite length and contained in $(\min\sigma(H_0), \max\sigma(H_0))$.

For every $\delta > 0$, we let $\mathcal{G}_\delta$ denote the collection of elements in $\mathcal{G}$ of length larger than δ, that is, $\mathcal{G}_\delta = \{I_\nu \in \mathcal{G} \mid |I_\nu| > \delta\}$, where $|I_\nu| = b_\nu - a_\nu$. We note that for each $\delta > 0$, $\mathcal{G}_\delta$ is a finite set, and denote $m(\delta) = \#\mathcal{G}_\delta$. We reorder the elements of $\mathcal{G}_\delta$, so that they are ordered by their order of occurrence on the real line. That is, we assume that $\mathcal{G}_\delta = \{I_1, \ldots, I_{m(\delta)}\}$, where $a_{\nu+1} > b_\nu$ for every $1 \leq \nu < m(\delta)$.

We define a positive function $\eta(\delta)$ to be the maximal distance between neighboring elements of $\mathcal{G}_\delta$, that is,

$$\eta(\delta) = \max\{a_{\nu+1} - b_\nu \mid I_\nu, I_{\nu+1} \in \mathcal{G}_\delta\}\,.$$

Clearly, $\eta(\delta)$ is monotonely decreasing. Moreover, since $\sigma(H_0)$ is nowhere dense, one easily sees that $\eta(\delta) \to 0$ as $\delta \to 0$. We can thus pick a monotonely decreasing sequence $\{\delta_j\}_{j=1}^{\infty} \subset (0,1)$, such that $\sum_{j=1}^{\infty}(\eta(\delta_j) + \delta_j) < \infty$. We also define, for each j,

$$m_j = [2\eta(\delta_j)/\delta_j] + 3\,,$$

where $[\,\cdot\,]$ denotes integer part.

We will now show that we can construct recursively a sequence $\{L_j\}_{j=1}^\infty \subset \mathbb{N}$, numbers $\{n_{j,1}, \ldots, n_{j,m_j}\}$, so that $n_{j,k+1} > n_{j,k}$, $n_{j,1} = L_j$, $n_{j,m_j} = L_{j+1}$, and a bounded monotonely increasing step potential V, so that H_0+V has no absolutely continuous spectrum. The resulting V will not be decaying, but converge to some constant $\lim_{n\to\infty} V(n)$. Since we can simply subtract this constant from V to obtain a decaying potential, Theorem 1.3 would follow.

To accomplish the recursive construction, let $L_1 = 1$, $V(1) = 0$, and assume that L_j and $V(n)$ for $1 < n \le L_j$ are determined. Define $V(n)$, for $L_j < n \le L_{j+1}$, by

$$V(n) = V(L_j) + k\delta_j/2$$

for $n_{j,k} < n \le n_{j,k+1}$, where $n_{j,2}, \ldots, n_{j,m_j}$ will be determined later. Note that, by the construction, $V(L_{j+1}) - V(L_j) = m_j\delta_j/2 \le \eta(\delta_j) + 3\delta_j/2$, and thus V is positive and $\sup_n V(n) = \lim_{n\to\infty} V(n) \le \sum_{j=1}^\infty (\eta(\delta_j) + 3\delta_j/2)$.

Consider now some fixed $k \in \{1, \ldots, m_j\}$. Suppose that $n_{j,2}, \ldots, n_{j,k}$ are already determined and that $E \in \mathbb{R}$ obeys

$$(E - \delta_j/4, E + \delta_j/4) \cap \sigma(H_0 + V(L_j) + k\delta_j/2) = \emptyset\,. \tag{3.1}$$

Since $V(n)$ coincides with the constant $V(L_j) + k\delta_j/2$ for $n_{j,k} < n \le n_{j,k+1}$, it follows from Proposition 2.2 that for $4 < \ell \le n_{j,k+1} - n_{j,k}$,

$$\|\Phi_{n_{j,k}+1, n_{j,k}+\ell}(E)\| \ge \frac{1}{2}(\delta_j/4)^2(1 + (\delta_j/4)^2)^{\frac{\ell-4}{2}}\,, \tag{3.2}$$

where we use the notations of Section 2 for $\tilde V = V_0 + V$. If $\|\Phi_{n_{j,k}}(E)\| > (jn_{j,k})^{1/2}\log n_{j,k}$, then we clearly have

$$\frac{1}{N(\log N)^2}\sum_{n=1}^{N} \|\Phi_n(E)\|^2 > j \tag{3.3}$$

for $N = n_{j,k}$. Otherwise, since

$$\|\Phi_{n_{j,k}+1, n_{j,k}+\ell}(E)\| = \|\Phi_{n_{j,k}}(E)^{-1}\Phi_{n_{j,k}+\ell}(E)\| \le \|\Phi_{n_{j,k}}(E)^{-1}\|\|\Phi_{n_{j,k}+\ell}(E)\|$$

and $\det(\Phi_{n_{j,k}}(E)) = 1$ implies $\|\Phi_{n_{j,k}}(E)^{-1}\| = \|\Phi_{n_{j,k}}(E)\|$, (3.2) implies

$$\|\Phi_{n_{j,k}+\ell}(E)\| \ge \frac{\|\Phi_{n_{j,k}+1, n_{j,k}+\ell}(E)\|}{\|\Phi_{n_{j,k}}(E)\|} \ge \frac{(\delta_j/4)^2}{2(jn_{j,k})^{1/2}\log n_{j,k}}(1 + (\delta_j/4)^2)^{\frac{\ell-4}{2}}\,. \tag{3.4}$$

This implies that, by choosing $n_{j,k+1}$ to be sufficiently large, we can ensure that for any E obeying (3.1), (3.3) will hold either for $N = n_{j,k}$ or for $N = n_{j,k+1}$.

Now given any fixed $E \in \mathbb{R}$, the definitions of $\eta(\delta)$ and m_j ensure that there will be some $k \in \{1, \ldots, m_j\}$ for which (3.1) holds. We thus see that we can choose $n_{j,2}, \ldots, n_{j,m_j}$ so that for every $E \in \mathbb{R}$, there will be some $L_j \leq N \leq L_{j+1}$ for which (3.3) holds. Thus, by choosing $n_{j,2}, \ldots, n_{j,m_j}$ appropriately for each j, we ensure that

$$\limsup_{N \to \infty} \frac{1}{N(\log N)^2} \sum_{n=1}^{N} \|\Phi_n(E)\|^2 = \infty \,, \tag{3.5}$$

for every $E \in \mathbb{R}$. By Proposition 2.1, this implies $\Sigma_{\mathrm{ac}}(H_V) = \emptyset$, which completes the proof. $\square$

While we formulated and proved Theorem 1.3 for discrete Schrödinger operators, the theorem also applies to continuous Schrödinger operators on $L^2([0,\infty))$, as well as to cases where H_0 is a more general tridiagonal operator on $\ell^2(\mathbb{N})$, which may have unbounded absolutely continuous spectrum. The proof for these cases is very similar to the above. The main difference is that we cannot assume $\sigma(H_0)$ to be bounded. This technical difference can be easily accommodated by considering in the jth stage of the construction only $E \in (-j,j)$. More explicitly, instead of considering $\mathcal{G}_\delta$ and $\eta(\delta)$ as above, we consider $\mathcal{G}_{j,\delta} = \{I_\nu \in \mathcal{G} \,|\, |I_\nu| > \delta \,, I_\nu \cap (-j,j) \neq \emptyset\}$ and $\eta_j(\delta)$, which is defined to be the maximal distance between neighboring elements of $\mathcal{G}_{j,\delta}$. Since, for a fixed j, $\lim_{\delta \to 0} \eta_j(\delta) = 0$, we can find $\{\delta_j\}_{j=1}^\infty \subset (0,1)$ so that $\sum_{j=1}^\infty (\eta_j(\delta_j) + \delta_j) < \infty$. By proceeding as above with $\eta_j(\delta_j)$ instead of $\eta(\delta_j)$, we then obtain that for every $E \in (-j,j)$ (instead of every $E \in \mathbb{R}$), (3.3) holds for some $L_j \leq N \leq L_{j+1}$. It then follows that (3.5) holds for every $E \in \mathbb{R}$, and thus $\Sigma_{\mathrm{ac}}(H_V) = \emptyset$.

4 Proof of Theorem 1.4

The potential V_0 will be constructed in conjunction with three sequences, $\{q_j\}_{j=1}^\infty, \{\ell_j\}_{j=1}^\infty, \{L_j\}_{j=1}^\infty \subset \mathbb{N}$, connected by $L_1 = 0$, $L_{j+1} = L_j + \ell_j q_j$, and a sequence of coupling constants $\{\lambda_j\}_{j=1}^\infty \subset (0,1)$. We let

$$V_0(n) = \lambda_j \cos(2\pi(n - L_j)/q_j) \tag{4.1}$$

for $L_j < n \leq L_{j+1}$. For the sequence $\{q_j\}_{j=1}^\infty$, we could take, in principle, any strictly increasing sequence of positive integers, but it is more convenient to take q_j's that are odd and obey $\sum_{j=1}^\infty q_j^{-1} < \infty$. We thus fix, once and for all, $q_j = (2j+1)^2$, and so $\{\ell_j\}_{j=1}^\infty$ uniquely determines $\{L_j\}_{j=1}^\infty$. We have the following:

Theorem 4.1 *For V_0 as above, there exists a sequence $\{\lambda_j\}_{j=1}^\infty \subset (0,1)$, obeying $\lambda_j \to 0$ as $j \to \infty$, such that for any choice of the sequence $\{\ell_j\}_{j=1}^\infty$, $\Sigma_{\rm ac}(\Delta + V_0) = (-2,2)$.*

Remark. The proof below actually establishes a more general theorem in the following sense: $\{q_j\}_{j=1}^\infty$ may be any arbitrary sequence of positive integers and $\cos(2\pi(n-L_j)/q_j)$, for $L_j < n \leq L_{j+1}$, can be replaced by any real-valued periodic sequence of period q_j and norm one.

Proof. Given $\lambda \in (0,1)$ and $q \in \mathbb{N}$, we denote by $V_{\lambda,q}$ the periodic potential given by $V_{\lambda,q}(n) = \lambda \cos(2\pi n/q)$, $n \in \mathbb{Z}$, and let $H_{\lambda,q} = \Delta + V_{\lambda,q}$ on $\ell^2(\mathbb{Z})$. We also denote, for $\ell = 0, \ldots, q$, $E_{q,\ell} = 2\cos(\pi\ell/q)$ and

$$S(\lambda,q) = [-2,2] \setminus \bigcup_{\ell=0}^{q} [E_{q,\ell} - \lambda,\ E_{q,\ell} + \lambda]\,.$$

The periodic spectrum $\sigma(H_{\lambda,q})$ is well known to consist of q bands (closed intervals) separated by $q-1$ gaps. The edges of these bands are well known to coincide with eigenvalues of certain $q \times q$ matrices (see, e.g., [15]), from which it is easy to see (and also well known) that all of them are contained in the set

$$\bigcup_{\ell=0}^{q} [E_{q,\ell} - \lambda,\ E_{q,\ell} + \lambda]\,.$$

Thus, denoting by $\tilde{\sigma}_{\lambda,q}$ the union of the interiors of the bands that make up $\sigma(H_{\lambda,q})$, we have $S(\lambda,q) \subset \tilde{\sigma}_{\lambda,q} \subset \sigma(H_{\lambda,q})$. Moreover, if $\lambda \in (0,\lambda_0)$, then $S(\lambda_0,q) \subset \tilde{\sigma}_{\lambda,q}$. For any $\ell \in \mathbb{N}$, let $\Phi_\ell(q,\lambda,E)$ be the transfer matrix for $H_{\lambda,q}$ from 1 to ℓ, namely,

$$\Phi_\ell(q,\lambda,E) = \begin{pmatrix} E - V_{\lambda,q}(1) & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - V_{\lambda,q}(\ell) & -1 \\ 1 & 0 \end{pmatrix}.$$

$\sigma(H_{\lambda,q})$ is well known to coincide with the set $\{E \in \mathbb{R} \,|\, \mathrm{Tr}\,\Phi_q(q,\lambda,E) \leq 2\}$ and $\tilde{\sigma}_{\lambda,q} = \{E \in \mathbb{R} \,|\, \mathrm{Tr}\,\Phi_q(q,\lambda,E) < 2\}$. Thus, for $E \in \tilde{\sigma}_{\lambda,q}$, $\Phi_q(q,\lambda,E)$ has two complex eigenvalues of the form $e^{\pm iq\theta}$. Let $(\varphi_q(\lambda,E),1)^T$ be an appropriately normalized eigenvector of $\Phi_q(q,\lambda,E)$, corresponding to the eigenvalue $e^{iq\theta}$, then $(\varphi_q^*(\lambda,E),1)^T$, where $\cdot^*$ denotes complex conjugation, is an eigenvector corresponding to the eigenvalue $e^{-iq\theta}$. It is thus easy to see that for $E\in\tilde\sigma_{\lambda,q}$, we have the equality

$$\Phi_q(q,\lambda,E)=U_q(\lambda,E)\,R(q\theta)\,U_q(\lambda,E)^{-1}\,,$$

where

$$R(q\theta)=\begin{pmatrix} e^{iq\theta} & 0\\ 0 & e^{-iq\theta}\end{pmatrix}\,,\qquad U_q(\lambda,E)=\begin{pmatrix}\varphi_q(\lambda,E) & \varphi_q^*(\lambda,E)\\ 1 & 1\end{pmatrix}\,.$$

Consider now some $j\in\mathbb{N}$, $E\in S(j^{-2}q_j^{-1},q_j)$, and $\lambda\in(0,j^{-2}q_j^{-1})$. As $\lambda\to 0$ inside $(0,j^{-2}q_j^{-1})$, $E\in\tilde\sigma_{\lambda,q_j}$ and thus $\varphi_{q_j}(\lambda,E)$ is well defined and converges (continuously) to $\varphi_{q_j}(0,E)=e^{i\theta(E)}$, where $\theta(E)\in(0,\pi)$ is determined by $E=2\cos\theta(E)$. Similarly, $\max_{1\le\ell\le q_j}\|\Phi_\ell(q_j,\lambda,E)\|$ converges to the corresponding value for $\lambda=0$, which is bounded from above by $1/\sin\theta(E)$. By the Egoroff theorem, there is a measurable subset $S_j\subset S(j^{-2}q_j^{-1},q_j)$, with $|S_j|>|S(j^{-2}q_j^{-1},q_j)|-j^{-2}\ge 4-3j^{-2}$, on which these convergences are uniform. Thus, we can pick $\lambda_j\in(0,j^{-2}q_j^{-1})$, so that, for every $E\in S_j$, $|\varphi_{q_j}(\lambda_j,E)-e^{i\theta(E)}|<j^{-2}$ and $\max_{1\le\ell\le q_j}\|\Phi_\ell(q_j,\lambda_j,E)\|<1+(\sin\theta(E))^{-1}$.

Now let $j_0,j_1\in\mathbb{N}$, $2<j_0<j_1$, and let $E\in\bigcap_{j=j_0}^\infty S_j$ be such that $\sin\theta(E)>j_0^{-2}$. The transfer matrix for $\Delta+V_0$ from $L_{j_0}+1$ to L_{j_1} is given by

$$\Phi_{L_{j_0}+1,L_{j_1}}(E)=\prod_{j=j_0}^{j_1-1}(\Phi_{q_j}(q_j,\lambda_j,E))^{\ell_j}\,.$$

Since

$$\Phi_{q_j}(q_j,\lambda_j,E)=U_{q_j}(\lambda_j,E)\,R(q_j\theta_j)\,U_{q_j}(\lambda_j,E)^{-1}\,,\tag{4.2}$$

$$(\Phi_{q_j}(q_j,\lambda_j,E))^{\ell_j}=U_{q_j}(\lambda_j,E)\,(R(q_j\theta_j))^{\ell_j}\,U_{q_j}(\lambda_j,E)^{-1}\,,$$

and thus, since $\|R(q_j\theta_j)\|=1$, we see that $\|\Phi_{L_{j_0}+1,L_{j_1}}(E)\|$ is bounded from above by

$$\|U_{q_{j_0}}(\lambda_{j_0},E)\|\|U_{q_{j_1-1}}(\lambda_{j_1-1},E)^{-1}\|\prod_{j=j_0}^{j_1-2}\|U_{q_j}(\lambda_j,E)^{-1}\,U_{q_{j+1}}(\lambda_{j+1},E)\|\,.$$

Denote $\varphi_j = \varphi_{q_j}(\lambda_j, E)$, then we see by a simple calculation that

$$U_{q_j}(\lambda_j, E)^{-1}\, U_{q_{j+1}}(\lambda_{j+1}, E) = \frac{1}{\varphi_j - \varphi_j^*} \begin{pmatrix} \varphi_{j+1} - \varphi_j^* & \varphi_{j+1}^* - \varphi_j^* \\ \varphi_{j+1} - \varphi_j & \varphi_j - \varphi_{j+1}^* \end{pmatrix} ,$$

and thus, since $|\varphi_j - e^{i\theta(E)}| < j^{-2}$, we obtain

$$\|U_{q_j}(\lambda_j, E)^{-1}\, U_{q_{j+1}}(\lambda_{j+1}, E)\| \leq 1 + \frac{2j^{-2}}{\sin\theta(E) - j^{-2}} .$$

Similarly, it's easy to see that $\|U_{q_{j_0}}(\lambda_{j_0}, E)\| < 3$ and

$$\|U_{q_{j_1-1}}(\lambda_{j_1-1}, E)^{-1}\| < \frac{3}{2(\sin\theta(E) - (j_1-1)^{-2})} ,$$

and so we can conclude that

$$\|\Phi_{L_{j_0}+1, L_{j_1}}(E)\| \leq \frac{9}{2(\sin\theta(E) - j_0^{-2})} \prod_{j=j_0}^{j_1-2} \left(1 + \frac{2j^{-2}}{\sin\theta(E) - j^{-2}}\right) .$$

Since the product in the last expression clearly converges as $j_1 \to \infty$, we conclude that $\limsup_{j_1\to\infty} \|\Phi_{L_{j_0}+1, L_{j_1}}(E)\| < \infty$.

Consider now any $L_{j_1} < n < L_{j_1+1}$. By using (4.2), one easily sees that

$$\|\Phi_{L_{j_1}+1,n}(E)\| \leq \|U_{q_{j_1}}(\lambda_{j_1}, E)\| \|U_{q_{j_1}}(\lambda_{j_1}, E)^{-1}\| \max_{1\leq \ell \leq q_{j_1}} \|\Phi_\ell(q_{j_1}, \lambda_{j_1}, E)\|$$

and thus, since $\max_{1\leq\ell\leq q_{j_1}} \|\Phi_\ell(q_{j_1}, \lambda_{j_1}, E)\| < 1 + (\sin\theta(E))^{-1}$,

$$\|\Phi_{L_{j_1}+1,n}(E)\| \leq \frac{9(1 + (\sin\theta(E))^{-1})}{2(\sin\theta(E) - j_1^{-2})} .$$

Since

$$\|\Phi_{L_{j_0}+1,n}(E)\| \leq \|\Phi_{L_{j_0}+1, L_{j_1}}(E)\| \|\Phi_{L_{j_1}+1,n}(E)\| ,$$

we conclude that $\limsup_{n\to\infty} \|\Phi_{L_{j_0}+1,n}(E)\| < \infty$.

Now given any $E \in \liminf_{j\to\infty} S_j \subset (-2,2)$, there will be some j_0 for which $E \in \bigcap_{j=j_0}^\infty S_j$ and $\sin\theta(E) > j_0^{-2}$. Thus, we conclude that for any $E \in \liminf_{j\to\infty} S_j$, $\limsup_{n\to\infty} \|\Phi_{1,n}(E)\| < \infty$. By well known results (see, e.g., [17, 19]), this implies $\liminf_{j\to\infty} S_j \subset \Sigma_{\rm ac}(\Delta + V_0)$. Since $|(-2,2)\setminus S_j| \leq 3j^{-2}$, it follows from the Borel-Cantelli lemma that $|(-2,2) \setminus \liminf_{j\to\infty} S_j| = 0$ and thus $\Sigma_{\rm ac}(\Delta + V_0) = (-2,2)$ as required. □

Proof of Theorem 1.4. We let V_0 be as above and fix $\{\lambda_j\}_{j=1}^\infty \subset (0,1)$ to be some sequence of the type whose existence is ensured by Theorem 4.1, namely, $\lambda_j \to 0$ as $j \to \infty$ and $\Sigma_{\rm ac}(\Delta+V_0) = (-2,2)$ for any choice of $\{\ell_j\}_{j=1}^\infty$. Theorem 1.4 would now follow by showing that we can simultaneously pick the sequence $\{\ell_j\}_{j=1}^\infty$ and construct a bounded monotonely increasing potential V, such that $\Sigma_{\rm ac}(\Delta + V_0 + V) = \emptyset$. This construction is very similar to (and will rely upon) the one done in the proof of Theorem 1.3.

Consider $j \in \mathbb{N}$ and suppose that $\ell_1, \dots, \ell_{j-1}$ and thus L_j are determined and that $V(n)$ is determined for $1 \le n \le L_j$. By a result of Choi-Elliott-Yui [3], $\sigma(H_{\lambda_j,q_j})$ is known to have $q_j - 1$ open gaps and the length of each of these gaps is larger than $\delta_j = (\lambda_j/16)^{q_j}$. By a general result of Deift-Simon [6, Corollary 1.5], the distance between every two neighboring gaps is bounded from above by $\eta_j = 4\pi/q_j$. Denote

$$m_j = [2\eta_j/\delta_j] + 3\,.$$

We will construct the potential $V(n)$, for $L_j < n \le L_{j+1}$, in conjunction with numbers $\{n_{j,1}, \dots, n_{j,m_j}\}$, so that $n_{j,k+1} > n_{j,k}$, $(n_{j,k+1} - n_{j,k})/q_j$ is an integer, $n_{j,1} = L_j$ and $n_{j,m_j} = L_{j+1}$. Note that ℓ_j will be determined by this construction as $\ell_j = (n_{j,m_j} - n_{j,1})/q_j$. Similarly to the proof of Theorem 1.3, we define

$$V(n) = V(L_j) + k\delta_j/2$$

for $n_{j,k} < n \le n_{j,k+1}$, and determine $n_{j,2}, \dots, n_{j,m_j}$ recursively. Note that $V(L_{j+1}) - V(L_j) = m_j\delta_j/2 \le \eta_j + 3\delta_j/2$, and thus, since $\sum_{j=1}^\infty(\eta_j + \delta_j) < \infty$, V will be bounded.

Assuming that $n_{j,2}, \dots, n_{j,k}$ are determined, we can repeat the considerations of (3.1)–(3.4), except that we replace (3.1) by

$$(E - \delta_j/4, E + \delta_j/4) \cap \sigma(H_{\lambda_j,q_j} + V(L_j) + k\delta_j/2) = \emptyset\,, \qquad (4.3)$$

and use the fact that $V_0(n)$ coincides with $V_{\lambda_j,q_j}(n)$ for $L_j < n \le L_{j+1}$, so that $\Phi_{n_{j,k}+1,n_{j,k}+\ell}(E)$ doesn't change if we change the potential from V_0 to V_{λ_j,q_j}. We thus show that we can choose $n_{j,k+1}$ to be sufficiently large, so that for any E obeying (4.3), (3.3) will hold either for $N = n_{j,k}$ or for $N = n_{j,k+1}$. Another difference from the proof of Theorem 1.3 is that we now also need to ensure that $(n_{j,k+1} - n_{j,k})/q_j$ is an integer, but this is clearly not a problem.

Now given any fixed $E \in \mathbb{R}$, the definitions of δ_j, η_j, and m_j ensure that there will be some $k \in \{1, \dots, m_j\}$ for which (4.3) holds. We thus see that we can choose $n_{j,2}, \dots, n_{j,m_j}$ so that for every $E \in \mathbb{R}$, there will be some $L_j \leq N \leq L_{j+1}$ for which (3.3) holds. Thus, by choosing $n_{j,2}, \dots, n_{j,m_j}$ appropriately for each j, we ensure that (3.5) holds for every $E \in \mathbb{R}$. By Proposition 2.1, this implies $\Sigma_{\mathrm{ac}}(H_V) = \emptyset$. □

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