Exotic Spectra: A Review of Barry Simon's Central Contributions

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Dedicated to Barry Simon on the occasion of his 60th birthday

ABSTRACT. We review some of Barry Simon's central contributions concerning what is often called exotic spectral properties. These include phenomena such as Cantor spectrum, thick point spectrum and singular continuous spectrum.

CONTENTS

1. Introduction

> "General wisdom used to say that Schrödinger operators should have absolutely continuous spectrum plus some discrete point spectrum, while singular continuous spectrum is a pathology that should not occur in examples with V bounded."

The above quote starts Section 10.4 of the 1987 book "*Schrödinger Operators*" by Cycon, Froese, Kirsch and Simon [**15**]. A few lines below, it further states: "Another correction to the 'general picture' is that point spectrum may be dense in some region of the spectrum rather than being a discrete set." These statements

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are made as background to the introduction of yet a third exotic spectral phenomenon, which is the occurrence of *Cantor spectrum*, and to the characterization of absolutely continuous spectrum associated with Cantor spectrum as *recurrent*.

The book [**15**] is mostly a summary of a summer school course that Simon gave in 1982. Section 10.4 mainly discusses ideas that were introduced in the seminal 1981 Avron–Simon paper "*Transient and recurrent spectrum*" [**3**] along with some relevant results of Avron–Simon [**5**], Bellissard–Simon [**11**], and Chulaevsky [**13**]. In particular, the above quotations echo similar statements previously made in [**3**].

For insight regarding the "general wisdom" of the time, it may be illuminating to look at the 1978 fourth (and last) volume of "*Methods of Modern Mathematical Physics*" by Reed and Simon [**49**]. This volume is called "*Analysis of Operators*" and its Section XIII.6 starts with the following statement: "Spectral analysis of an operator A concentrates on identifying the five sets $\sigma_{\rm ess}(A)$, $\sigma_{\rm disc}(A)$, $\sigma_{\rm ac}(A)$, $\sigma_{\rm sing}(A)$, $\sigma_{\rm pp}(A)$." This section is called "*The absence of singular continuous spectrum* I" and it is followed by similarly titled sections up to "*The absence of singular continuous spectrum* IV." These "absence of singular continuous spectrum" sections occupy roughly 16% of the volume and point at the main role played by singular continuous spectrum in pre-1978 spectral theory: It was a non-occurring phenomenon which complicated life by requiring some effort to prove it did not occur.

The Avron–Simon paper [**3**] posed a significant challenge to the "general wisdom" of its time. First, it extended the above "five sets" by defining four new spectral types (*recurrent absolutely continuous*, *transient absolutely continuous*, *thick pure point* and *thin pure point*). Second, it made the prediction that what they called "extraordinary" spectra (and we call here *exotic spectra*) "will become more and more commonly encountered than one might have thought!" Indeed, Avron–Simon point out this prediction as a potential objection to their choice of the term "extraordinary." The fact that their prediction materialized so fully is why we decided to reject their proposed terminology and adopt "exotic spectra" instead.

The historical background concerning exotic spectra which preceded [**3**] consisted of several fairly isolated results. In particular, one should mention the 1977 Goldsheid–Molchanov–Pastur [**25**] proof of Anderson localization [**1**] (namely, the occurrence of pure point spectrum with eigenvalues dense in an interval) in a random Schrödinger operator. They considered a continuous one-dimensional operator of the form $-\Delta + V$ on $L^2(\mathbb{R})$ with a certain type of random potential V.

Even earlier, the Ishii–Pastur theorem (see [**15**]) indicated that some random one-dimensional Schrödinger operators have no absolutely continuous spectrum in spite of their spectrum being an interval (which says they must have either thick point spectrum or singular continuous spectrum or both). Anderson localization for discrete Schrödinger operators with i.i.d.r.v. potentials was proven in 1980 by Kunz–Souillard [**43**]. Another notable result of the era is Pearson's seminal 1978 paper [**47**], which gave an explicit construction of a one-dimensional Schrödinger operator having purely singular continuous spectrum.

[**3**] was essentially part of a series of papers concerned mostly with almost periodic Schrödinger operators that Simon wrote with several coauthors in the early 1980's [**4, 3, 5, 6, 7, 11, 14, 16, 55, 56**]. These works were predated by the 1975 paper of Dinaburg–Sinai [**20**], which established absolutely continuous spectrum for some almost periodic Schrödinger operators, and by several papers of

Shubin (see [**53**]). They were paralleled by a number of works by other authors (e.g., [**9, 10, 12, 13, 21, 22, 37, 38, 46, 52**]).

The phenomenon of sudden broad interest in almost periodic Schrödinger operators has been named the "almost periodic flu" by Simon and is very nicely presented in his review paper [**54**], which is itself often called "the almost periodic flu paper." [**3**] seems to have been primarily motivated by the Avron–Simon discovery [**5**] of Cantor absolutely continuous spectrum for certain almost periodic Schrödinger operators, but, while other papers in the series were focused purely on almost periodic, or very similar, potentials, [**3**] took a more general perspective: It looked at the potential implications of the new spectral phenomenon to quantum mechanics, identified natural mathematical structures which arise in this context and, most notably, identified the connection between the newly discovered spectral phenomenon for almost periodic operators and the prior findings of Goldsheid–Molchanov–Pastur [**25**] and Pearson [**47**]. We thus believe that it is [**3**], more than any other single work, that marks the beginning of a new era in spectral analysis: the era of exotic spectra.

We note that by classical inverse spectral theory, one should actually expect the full spectral richness allowed by measure theory to find its way into Schrödinger operators. In particular, we have

THEOREM 1.1 (Gel'fand–Levitan [**24**]). *Given any finite Borel measure ν on $[a,b] \subset \mathbb{R}$, there exists a continuous half-line Schrödinger operator for which the spectral measure coincides with ν on $[a,b]$.*

Thus, one could argue that the "general wisdom" expecting only "absolutely continuous spectrum plus some discrete point spectrum" was never really justified. Indeed, in looking back, it seems to be very much the consequence of mathematical physicists concentrating their attention on operators arising from atomic and molecular physics, mathematical problems associated with scattering experiments, periodic problems, etc., namely, on problems which happen to have such spectra.

The exotic spectra era can thus be at least partially attributed to mathematical physicists starting to look into some rich problems of modern condensed matter physics which yield rich spectral phenomena. Indeed, several developments in condensed matter physics in the early 1980's contributed to the growing interest in almost periodic Schrödinger operators and exotic spectra. The 1980 discovery of the integer quantum Hall effect by von Klitzing [**40**] (for which he got the Nobel Prize in 1985), led to a beautiful theory by Thouless, Kohmoto, Nightingale and den Nijs [**66**], which explains the quantization of charge transport in this effect as connected with certain topological invariants. Central to their theory is the use of the almost Mathieu operator as a model for Bloch electrons in a magnetic field (in which case the frequency α is proportional to the magnetic flux; see below).

Another strong source of interest in almost periodic problems came from the 1984 discovery of quasicrystals by Shechtman et al. [**48**], as almost periodic Schrödinger operators provide elementary models for electronic properties in such media. Yet another motivating development occurred in the context of quantum chaos theory, notably in works of Fishman, Grempel, and Prange [**23, 30, 31**], as discrete one-dimensional Schrödinger operators with rich potentials (and, in particular, almost periodic ones) appeared in studies of dynamics of some elementary quantum models and, in particular, in studies aimed towards distinguishing quantum from classical dynamics in chaotic systems.

Interestingly, while the above developments in physics certainly contributed to broader interest in problems connected with exotic spectra, the timeline doesn't point to a simple scenario of mathematical physicists following the footsteps of physicists. Anderson localization was discovered in 1958 [**1**] and has been an active field of research by condensed matter physicists throughout the 1960's and 1970's, yet the Goldsheid–Molchanov–Pastur paper [**25**] came only in 1977. On the other hand, the "almost periodic flu" was in full motion *before* the above developments in physics that made it all the more interesting.

It is also interesting to note that Pearson's example [**47**] of a one-dimensional Schrödinger operator having purely singular continuous spectrum isn't directly connected with anything that has been of much interest to physics. The same is true for quite a few other models that have been considered in the context of exotic spectra. Thus, it appears that at least some of the exotic spectra era is associated with the willingness of spectral analysts, largely headed by Simon, to look at problems that are interesting from a mathematical perspective and to loosen some of the ties with physics. The fact that the "almost periodic flu" was very soon followed by discoveries that made it more interesting from a physics perspective than what one might have initially thought is an intriguing historical sidelight.

Another important phase in the field of exotic spectra came with Simon's "singular continuous spectrum revolution" which started around 1994. The core of this revolution has been the realization that singular continuous spectrum is a much more common phenomenon than what was previously thought and that it is, in fact, a generic phenomenon for broad classes of operators. In particular, the revolution supplied numerous new examples for operators with singular continuous spectrum. The revolution started with the three papers [**19, 36, 61**] by Simon and coauthors, along with Gordon's work [**27, 28**], which independently obtained roughly the same results as [**19**] a little earlier (also see the announcement paper [**17**], which summarizes the central findings of [**19, 36, 61**]). These papers soon inspired many more works by Simon and coauthors, as well as by others.

Alongside the central theme of identifying more and more operators with singular continuous spectrum, some of the focus (e.g., in [**18, 44**]) has also been on improving the understanding of singular continuous spectrum itself, namely, on issues such as the implications of such spectra to dynamical properties of quantum systems, natural ways of distinguishing different types of singular continuous spectrum, etc. While the main spike of the revolution can be timed, roughly, to the 1994–1996 period, related work continued throughout the 1990's and continues to this day.

Simon's work in the area of exotic spectra occupies roughly fifty-five research papers written in the 1981–1999 period. They make up a vast collection of results, and this article makes no attempt to achieve any sort of systematic coverage of all this work. Instead, we focus on a relatively small subset of these results, which tend to emphasize certain aspects of Simon's work and/or to fit in certain historical contexts. Many of Simon's important contributions to the subject have been left out.

The rest of this paper is organized as follows. In Section 2 we discuss the almost periodic flu, in Section 3 we review some of Simon's central contributions to thick point spectrum and in Section 4 we discuss the singular continuous spectrum revolution.

2. The Almost Periodic Flu

Simon's 1982 review paper [**54**] starts with the following:

> "In many years, flu sweeps the world. The actual strain varies from year to year; some years it has been Hong Kong flu, some years swine flu. In 1981, it was the almost periodic flu!"

The paper then moves to counting specific contributions by Avron and Simon [**4, 3, 5, 6, 7**], Bellissard and Testard [**12**], Bellissard et al. [**9**], Chulaevsky [**13**], Johnson [**37**], Moser [**46**], Johnson and Moser [**38**], and Sarnak [**52**].

This colorful opening led to the phenomenon of sudden broad interest in almost periodic Schrödinger operators becoming known as the "almost periodic flu" and to [**54**] becoming known as the "almost periodic flu paper." The flu paper came in the midst of the actual flu season and was closely followed by additional important developments such as Kotani theory [**41, 55**] and the paper by Deift–Simon [**16**] that built upon it.

The first of Simon's flu season papers was the Avron–Simon paper [**3**], which has already been mentioned above. The central theme of this paper has been to extend the scope of spectral analysis by making the following definitions of new spectral types. Given a self-adjoint operator A on a separable Hilbert space $\mathcal{H}$ and $\psi \in \mathcal{H}$, we denote by μ_ψ the spectral measure for A and ψ. It is the unique Borel measure on $\mathbb{R}$ obeying

$$\langle \psi, f(A)\psi \rangle = \int f(x)\, d\mu_\psi(x)$$

for any bounded Borel function f. We further denote by $\hat{\mu}_\psi$ the Fourier transform of μ_ψ, namely,

$$\hat{\mu}_\psi(t) \equiv \int e^{-itx}\, d\mu_\psi(x) \; .$$

Avron-Simon [**3**] defined the *transient subspace*, $\mathcal{H}_{\mathrm{tac}}$, by

$$\mathcal{H}_{\mathrm{tac}} = \overline{\{\psi \mid \hat{\mu}_\psi \in L^1\}} \; ,$$

where $\overline{\cdot}$ denotes closure. This should be be compared with the well-known fact that the absolutely continuous subspace, $\mathcal{H}_{\mathrm{ac}}$, obeys $\mathcal{H}_{\mathrm{ac}} = \overline{\{\psi \mid \hat{\mu}_\psi \in L^2\}}$. Since $|\hat{\mu}_\psi(t)|^2$ coincides with the quantum mechanical survival probability of ψ, $\mathcal{H}_{\mathrm{tac}}$ is a closed subspace of $\mathcal{H}_{\mathrm{ac}}$ which is made of vectors that have the fastest escape rate from their original position under the Schrödinger time evolution. The *recurrent subspace*, $\mathcal{H}_{\mathrm{r}}$, is then defined by $\mathcal{H}_{\mathrm{r}} = \mathcal{H}_{\mathrm{tac}}^{\perp}$, and the *recurrent absolutely continuous subspace*, $\mathcal{H}_{\mathrm{rac}}$, by $\mathcal{H}_{\mathrm{rac}} = \mathcal{H}_{\mathrm{tac}}^{\perp} \cap \mathcal{H}_{\mathrm{ac}}$. Corresponding spectra are defined by $\sigma_{\mathrm{tac}}(A) = \sigma(A \restriction \mathcal{H}_{\mathrm{tac}})$ and $\sigma_{\mathrm{rac}}(A) = \sigma(A \restriction \mathcal{H}_{\mathrm{rac}})$. σ_{tac} is called *transient absolutely continuous spectrum* and σ_{rac} is called *recurrent absolutely continuous spectrum.*

Avron–Simon showed that $\mathcal{H}_{\mathrm{tac}} = P_{\mathrm{ei}}\mathcal{H}_{\mathrm{ac}}$, where P_{ei} is the spectral projection on the essential interior of the essential support of the absolutely continuous part of the spectral measure class of A. This implies that the absolutely continuous spectrum of the free Laplacian (on $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$) and of similar problems such as periodic Schrödinger operators and atomic Hamiltonians is purely transient. If, however, A has nowhere dense spectrum, then $\sigma_{\mathrm{tac}} = \emptyset$, and so any absolutely continuous spectrum of operators having Cantor spectrum is purely recurrent.

[**3**] also provided the following distinction between thin and thick point spectrum (σ_{pp} denotes the closure of the set of eigenvalues): $\lambda \in \sigma_{\mathrm{pp}}$ is said to be in

the *thin point spectrum* if $(\lambda-\epsilon,\lambda+\epsilon)\cap\sigma_{\rm pp}$ is countable for some $\epsilon>0$. λ is said to be in the *thick point spectrum* if $(\lambda-\epsilon,\lambda+\epsilon)\cap\sigma_{\rm pp}$ is uncountable for every $\epsilon>0$. Avron–Simon showed that the thin point spectrum is countable and the thick point spectrum is a perfect set (namely, a closed set with no isolated points) which is empty if and only if $\sigma_{\rm pp}$ is countable.

Additional central flu season results by Simon and coauthors included the following:

The Avron–Simon paper [**5**] studied one-dimensional Schrödinger operators with limit periodic potentials, namely, potentials which are norm-limits of periodic potentials. A typical example (for the continuous Schrödinger operator case) is $V(x)=\sum_{j=1}^{\infty}a_j\cos(x/2^j)$ with $\sum_{j=1}^{\infty}|a_j|<\infty$. They considered the space of all such potentials with the ℓ^∞ (namely, operator norm) topology and proved that for a dense G_δ set of such potentials the spectrum must be a Cantor set. Moreover, for a dense set, the spectrum must be both a Cantor set and purely absolutely continuous and it is thus purely recurrent absolutely continuous. Similar results also hold for some subsets of limit periodic potentials. We note that roughly the same results were independently obtained around the same time by Chulaevsky [**13**] and that a partial result (Cantor spectrum for a dense set) was also independently obtained by Moser [**46**].

The Avron–Simon paper [**6**] studied the almost Mathieu operator, which is the discrete Schrödinger operator $H_{\lambda,\alpha,\theta}$ on $\ell^2(\mathbb{Z})$ given by

$$(H_{\lambda,\alpha,\theta}\psi)(n)=\psi(n+1)+\psi(n-1)+\lambda\cos(2\pi\alpha n+\theta)\psi(n)\ .$$

They showed, by utilizing a result of Gordon [**26**], that if $|\lambda|>2$ and α is a Liouville number (namely, an irrational for which there is a sequence of rationals obeying $|\alpha-p_n/q_n|<n^{-q_n}$) then $H_{\lambda,\alpha,\theta}$ has purely singular continuous spectrum for Lebesgue a.e. θ. This provided a second concrete example (after Pearson's [**47**]) for a Schrödinger operator with purely singular continuous spectrum.

The Avron–Simon paper [**7**] and Craig–Simon paper [**14**] focused on the general theory of almost periodic Schrödinger operators and particularly on the spectrum, Lyapunov exponent and density of states. These papers prove many fundamental results (some of which were also independently obtained by others around the same time). Among the results are a rigorous proof of the Thouless formula $\gamma(E)=\int\ln|E-E'|\,dk(E')$, which connects the Lyapunov exponent $\gamma(E)$ with the density of states $dk(E)$, subharmonicity of the Lyapunov exponent and log-Hölder continuity of the density of states. [**7**] also had some results for the almost Mathieu operator, including a rigorous version of the Aubry duality [**2**] saying that for irrational α, $\sigma(\alpha,\lambda)=(\lambda/2)\sigma(\alpha,4/\lambda)$, where $\sigma(\alpha,\lambda)$ is the spectrum of $H_{\lambda,\alpha,\theta}$ (it is independent of θ).

The Bellissard–Simon paper [**11**] provided the first rigorous result concerning Cantor spectrum for the almost Mathieu operator. They showed that $\sigma(\alpha,\lambda)$ is a Cantor set for a dense G_δ set of pairs (α,λ). (It is known by now that the spectrum of $H_{\lambda,\alpha,\theta}$ is a Cantor set whenever α is irrational, as first conjectured in 1964 by Azbel [**8**].)

The Deift–Simon paper [**16**] built on Kotani theory [**41**], a version of which for discrete Schrödinger operators has been worked out by Simon in [**55**]. The Deift–Simon paper obtained several fundamental results for ergodic one-dimensional Schrödinger operators, most of which concern the absolutely continuous spectrum and its essential support, which, by Kotani theory, coincides with the set

$\{E \,|\, \gamma(E) = 0\}$. One of their results is the remarkable inequality (we give the versions of the results for discrete Schrödinger operators)

$$\frac{dk}{dE}\bigg|_{\{E \,|\, \gamma(E)=0\}} \geq \frac{1}{2\pi \sin(\pi k)}$$

which says that the restriction of $\frac{dk}{dE}$ to the set $\{E \,|\, \gamma(E) = 0\}$ is bounded from below by the value of $\frac{dk}{dE}$ for the free Laplacian at corresponding values of the integrated density of states. It also implies

$$|\{E \,|\, \gamma(E) = 0\}| \leq 4\;,$$

where $|\cdot|$ denotes Lebesgue measure. Another result is the averaged boundedness of absolutely continuous spectrum eigenfunctions, more explicitly, the existence, for Lebesgue a.e. $E \in \{E \,|\, \gamma(E) = 0\}$, of two linearly independent solutions $u_\pm$ of the corresponding Schrödinger equation, each of which is the complex conjugate of the other, normalized to have Wronskian $-2i$ and obeying

$$\lim_{L\to\infty} \frac{1}{L} \sum_{n=1}^{L} |u_\pm(n)|^2 \leq 2\pi \frac{dk}{dE}\;.$$

Yet another result of [**16**] is the mutual singularity of the singular parts of spectral measures for different realizations of an ergodic potential.

The paper which can be naturally considered as Simon's last flu season paper is [**56**], which studies the Maryland model. This is the discrete Schrödinger operator $\tilde{H}_{\lambda,\alpha,\theta}$ on $\ell^2(\mathbb{Z})$ given by

$$(\tilde{H}_{\lambda,\alpha,\theta}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda \tan(\pi\alpha n + \theta)\psi(n)\;.$$

The same name is also used for the multidimensional analog on $\ell^2(\mathbb{Z}^d)$ of this operator. $\tilde{H}_{\lambda,\alpha,\theta}$ was discovered by Fishman–Grempel–Prange [**23, 30**] (in the University of Maryland, from which it got its name) and has the remarkable property of being roughly precisely solvable. It exhibits singular continuous spectrum if α is a Liouville number and thick pure point spectrum (the spectrum as a set equals $\mathbb{R}$ for any irrational α) with precisely computable eigenvalues if α is an irrational with typical Diophantine properties. The point spectrum result extends also to the multidimensional case. [**56**] extended and made rigorous some of the original results of the Maryland team. Similar work was done independently around the same time by Figotin–Pastur [**21**].

Aside from Simon being the leading worker on almost periodic spectral theory around the flu season (in terms of having the largest volume of results), he also had great impact on the field in terms of drawing the map for future progress. The flu paper [**54**] had a list of thirteen open problems and conjectures, of which five were devoted to the almost Mathieu operator and the rest were more general. Some of these were also repeated in Simon's 1984 "*Fifteen problems in mathematical physics*" paper [**57**]. These problems helped to inspire a considerable amount of ongoing work in the 24 years that passed since [**54**]. The community of contributors to the field has been quite diverse, ranging from well-established world-class mathematicians to young entrants who were in various stages between starting kindergarten to finishing high school when [**54**] was written.

While probably no period since the spike of the flu season in 1981–1982 matched the level of almost periodic activity of that time, the overall progress made since then is quite vast. In particular, most of the original open problems of [**54**] are solved by now (although certainly not all of them). Some of the conjectures turned out to be false! Considerable progress in the field continues as we write.

3. Thick Point Spectrum

Thick point spectrum tends to largely coincide with the phenomenon of Anderson localization [**1**]. The most central family of operators for which this phenomenon has been studied consists of various random multidimensional Schrödinger operators (with or without magnetic fields). Anderson's paper [**1**] gave some freedom regarding what may be rightfully called "the Anderson model." For simplicity of exposition, we use the name *Anderson model* for the (semi-concrete) random operator H_ω on $\ell^2(\mathbb{Z}^d)$ given by

$$H_\omega = \Delta + \sum_{n\in\mathbb{Z}^d} \lambda_n(\omega)\langle\delta_n,\,\cdot\,\rangle\delta_n \ ,$$

where the $\lambda_n(\omega)$'s are independent, identically distributed random variables (i.i.d.r.v.) with uniform distribution in an interval and $\{\delta_n\}_{n\in\mathbb{Z}^d}$ is the natural basis of $\ell^2(\mathbb{Z}^d)$. Various variants and analogs have also been considered, including variants with different distributions, analogous continuous Schrödinger operators on $L^2(\mathbb{R}^d)$, discrete operators with more than just next-near neighbor interactions, etc. While all of the results discussed below for "the Anderson model" are valid for more general families of random operators, the precise range of validity tends to be different for different results. We will thus keep things simple by discussing results almost exclusively for the above semi-concrete H_ω. We say that an Anderson model operator H_ω is "strongly coupled" if the interval over which the $\lambda_n(\omega)$'s are distributed is "large."

Anderson localization has been the first exotic spectral phenomenon to be discovered (both by physicists and in terms of proving rigorous mathematical results) and also the one which has drawn the most attention (the largest number of workers and papers) over the years. In a sense, one can say that the field of exotic spectra has been largely dominated by Anderson localization. Unlike almost periodic Schrödinger operators and the singular continuous spectrum revolution that we discuss in the next section, where the current state of those fields would be hard to imagine without Simon's crucial contributions, Simon's involvement in studies of Anderson localization was relatively minor and it is likely that the current state of the field would have been similar to what it is even without him. Nevertheless, Simon had quite a few papers and results involving point spectrum and Anderson localization and some of them are very important. The purpose of this section is to point out a small subset of these results.

Probably the most famous of Simon's contributions to Anderson localization is the 1986 Simon–Wolff criterion obtained by Simon–Wolff in [**65**]. It says that an Anderson model Hamiltonian H_ω has only point spectrum in an energy interval I, for a.e. ω, if and only if

$$\sum_{n\in\mathbb{Z}^d} |\langle\delta_n, (H_\omega - E - i0)^{-1}\delta_0\rangle|^2 < \infty$$

for a.e. ω and Lebesgue a.e. $E \in I$. For this to hold, the distribution of the $\lambda_n(\omega)$'s need not be uniform in an interval, but it does need to be absolutely continuous with respect to the Lebesgue measure. The Simon–Wolff criterion thus reduces the problem of establishing Anderson localization to proving appropriate square summability of resolvent matrix elements. This provides considerable simplification to many proofs of Anderson localization, and the Simon–Wolff criterion continues to play an important role in many settings of proving Anderson localization.

Another notable result is the absence of ballistic motion for point spectrum proven by Simon in [**58**]. It says that if H is a discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$, X^2 is the squared position operator on this space, namely, $(X^2\psi)(n) = n^2\psi(n)$, and H has only point spectrum, then for any vector ψ in the domain of $|X|$,

$$\frac{1}{t^2}\langle \psi, e^{iHt}X^2e^{-iHt}\psi\rangle \to 0$$

as $t \to \infty$. This shows that operators with purely point spectrum cannot induce ballistic propagation of initially localized wavepackets. (Note that ballistic motion is always an upper bound on the propagation rate of wavepackets for discrete Schrödinger operators, regardless of spectral properties.) [**58**] also obtained similar results for continuous Schrödinger operators on $L^2(\mathbb{R}^d)$, as long as the potential is sufficiently close to being bounded from below.

Yet another notable result was given in Simon's paper [**60**] on cyclic vectors in the Anderson model. This paper established that for the Anderson model H_ω, with probability one, each of the δ_n vectors is a cyclic vector for the restriction of H_ω to its pure point subspace. In particular, this says that the δ_n vectors are cyclic vectors for H_ω whenever it has only point spectrum and that all of the point spectrum of H_ω must be simple (namely, the probability of H_ω having any degenerate eigenvalues is zero).

The final set of results we discuss in this section is from the work of del Rio–Jitomirskaya–Last–Simon [**18**]. They have shown, by constructing an explicit example, that from the fact that a Schrödinger operator has only point spectrum with exponentially localized eigenvectors, one cannot conclude anything for the growth rate of $\langle \psi, e^{iHt}X^2e^{-iHt}\psi\rangle$ beyond the absence of strict ballistic motion as discussed above. That is, for such an operator one can still have

$$\limsup_{t\to\infty} \frac{1}{t^{2-\epsilon}}\langle \psi, e^{iHt}X^2e^{-iHt}\psi\rangle = \infty$$

for any $\epsilon > 0$. Since the Anderson model is known to exhibit dynamical localization (namely, $\langle \psi, e^{iH_\omega t}X^2e^{-iH_\omega t}\psi\rangle$ is bounded with probability one), this called for extending the understanding of its precise spectral characteristics. To achieve this goal, [**18**] introduced the following definitions: A self-adjoint operator A on $\ell^2(\mathbb{Z}^d)$ is said to have SULE (Semi-Uniformly Localized Eigenvectors) if it has only point spectrum and there exists a constant $\gamma > 0$ such that for any $b > 0$, there exists a constant $C(b) > 0$, such that for any eigenvector ψ_s of A one can find $n(s) \in \mathbb{Z}^d$, so that

$$|\langle \delta_k, \psi_s\rangle| \leq C(b)e^{b|n(s)|-\gamma|k-n(s)|}$$

for all $k \in \mathbb{Z}^d$. A is said to exhibit SUDL (Semi-Uniform Dynamical Localization) if there exists a constant $\tilde{\gamma} > 0$, such that for any $b > 0$, there exists a constant $\tilde{C}(b) > 0$, so that

$$\limsup_{t} |\langle \delta_n, e^{-itA}\delta_\ell\rangle| \leq \tilde{C}(b)e^{b|\ell|-\tilde{\gamma}|n-\ell|}.$$

[**18**] proved that SULE implies SUDL and that SUDL, along with simple spectrum, implies SULE. In particular, for operators on $\ell^2(\mathbb{Z}^d)$ with simple spectrum, SULE

$\Leftrightarrow$ SUDL. It also proved that a sufficiently strongly coupled Anderson model H_ω exhibits SUDL and thus it also has SULE. Some additional results of [**18**] are discussed in Section 4 below.

4. The Singular Continuous Spectrum Revolution

Simon's singular continuous spectrum revolution started around 1994, when he discovered that singular continuous spectrum is a much more common phenomenon than what was previously thought, and proved results establishing it as a topologically generic phenomenon for many classes of operators. The initial results inspired much more work by Simon and coauthors, as well as by others, and within a few years the landscape concerning operators with singular continuous spectrum was drastically changed.

The most notable paper of the revolution is probably Simon's paper [**61**], which is sometimes called "the Wonderland paper," following how Simon named one of its central theorems. This paper focused on general self-adjoint operators and its central results provide conditions for topological families of operators to exhibit singular continuous spectrum for dense G_δ sets. The most central theorem of [**61**] is probably the following:

THEOREM 4.1 (Simon's Wonderland theorem). *Let X be a complete metric space of self-adjoint operators in which convergence implies strong resolvent convergence. Suppose*
(a) $\{A \mid A$ has purely absolutely continuous spectrum$\}$ *is dense in X;*
(b) $\{A \mid A$ has purely point spectrum$\}$ *is dense in X.*
Then for a dense G_δ set of A's, A has only singular continuous spectrum.

An illuminating corollary of Theorem 4.1 is that in a natural topological sense, a generic strongly coupled Anderson model operator has only singular continuous spectrum (as opposed to having only point spectrum with probability one). More precisely, if the exact same set of potentials of the Anderson model is considered with the natural product topology rather than as a probability space, then for a dense G_δ set of potentials in this topology, the corresponding operator has only singular continuous spectrum.

Another notable paper is the del Rio–Makarov–Simon paper [**19**], which studied the genericity of singular continuous spectrum in the context of rank one perturbations (see [**59**]). Its central result, which was also independently obtained a little earlier by Gordon [**27, 28**], is the following:

THEOREM 4.2. *Let A be a self-adjoint operator with a cyclic vector φ. Suppose $[a,b] \subset \sigma(A)$ and $\sigma_{\mathrm{ac}}(A) \cap [a,b] = \emptyset$. Then for a dense G_δ set of λ's, $A + \lambda\langle\varphi, \cdot\rangle\varphi$ has purely singular continuous spectrum on (a,b).*

An important corollary of Theorem 4.2 (which also uses the cyclicity of δ_n vectors in the Anderson model discussed in Section 3) is that if one considers a typical realization of the strongly coupled Anderson model H_ω, which has only point spectrum, and continuously changes the value of the potential at a single point of $\mathbb{Z}^d$, then for a dense G_δ set of potential values at this point, the corresponding operator has only singular continuous spectrum. In particular, this says that the phenomenon of Anderson localization is extremely unstable, since the point spectrum can be converted into singular continuous spectrum by making arbitrarily small changes to the value of the potential at a single point.

The third paper of the trio which started the revolution is the Jitomirskaya–Simon paper [**36**], which studied genericity of singular continuous spectrum in the almost periodic context. They proved the following:

THEOREM 4.3. *For $|\lambda| > 2$ and any irrational α, there is a dense G_δ set of θ's for which the almost Mathieu operator on $\ell^2(\mathbb{Z})$, given by*

$$(H_{\lambda,\alpha,\theta}\psi)(n) = \psi(n+1) + \psi(n-1) + \lambda\cos(2\pi\alpha n + \theta)\psi(n) \ ,$$

has only singular continuous spectrum.

Theorem 4.3 can be considered as an almost periodic analog of Theorem 4.2, since there are many other similarities between making rank one perturbations and changing realizations within the hull of one-dimensional almost periodic potentials. We note that the actual central result of [**36**] ensured the absence of eigenvalues, for an appropriate G_δ set in the hull, for any even almost periodic potential. The application to the almost Mathieu operator with $|\lambda| > 2$ is done in order to ensure the absence of absolutely continuous spectrum.

Another major paper of the revolution was the del Rio–Jitomirskaya–Last–Simon paper [**18**]. Some of its results were already discussed in Section 3. A central role in [**18**] was played by the fact that singular continuous spectra can be naturally decomposed into many spectral sub-types by using Hausdorff measures and dimensions. The measure-theoretic foundations for such decompositions go back at least to the works of Rogers–Taylor [**50, 51**] and they were introduced into spectral theory by Last [**44**], who was impacted by the singular continuous spectrum revolution along with Guarneri's seminal papers on quantum dynamics [**32, 33**] and Avron–Simon [**3**].

Recall that for any subset S of $\mathbb{R}$ and $\alpha \in [0,1]$, the α-dimensional Hausdorff measure, h^α, is given by

$$h^\alpha(S) \equiv \lim_{\delta\to 0} \inf_{\delta\text{-covers}} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha \ ,$$

where a δ-cover is a cover of S by a countable collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ. h^α is an outer measure on $\mathbb{R}$ whose restriction to Borel sets is a Borel measure. h^1 coincides with the Lebesgue measure and h^0 is the counting measure, such that the family $\{h^\alpha \,|\, 0 \le \alpha \le 1\}$ can be viewed as a way of continuously interpolating between the counting measure and the Lebesgue measure. Given any nonempty set $S \subseteq \mathbb{R}$, there exists a unique $\alpha(S) \in [0, 1]$, called the Hausdorff dimension of S, such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$.

The following basic notions and facts stem from the Rogers–Taylor theory [**50, 51**]: Given α, a measure μ is called α-continuous (αc) if $\mu(S) = 0$ for every set S with $h^\alpha(S) = 0$. It is called α-singular (αs) if it is supported on some set S with $h^\alpha(S) = 0$. μ is said to be one-dimensional (od) if it is α-continuous for every $\alpha < 1$. It is said to be zero-dimensional (zd) if it is α-singular for every $\alpha > 0$. A measure μ is said to have exact dimension α if, for every $\epsilon > 0$, it is both $(\alpha-\epsilon)$-continuous and $(\alpha+\epsilon)$-singular. Given a (positive, finite) measure μ and $\alpha \in [0,1]$, one defines

$$D_\mu^\alpha(x) \equiv \limsup_{\epsilon\to 0} \frac{\mu((x-\epsilon,\, x+\epsilon))}{(2\epsilon)^\alpha}$$

and $T_\infty \equiv \{x \mid D_\mu^\alpha(x) = \infty\}$. The restriction $\mu(T_\infty \cap \cdot\,) \equiv \mu_{\alpha s}$ is α-singular, and $\mu((\mathbb{R} \setminus T_\infty) \cap \cdot\,) \equiv \mu_{\alpha c}$ is α-continuous. Thus, each measure decomposes uniquely into an α-continuous part and an α-singular part: $\mu = \mu_{\alpha c} + \mu_{\alpha s}$.

Consider now a separable Hilbert space $\mathcal{H}$ and a self-adjoint operator H. By letting $\mathcal{H}_{\alpha c} \equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-continuous}\}$ and $\mathcal{H}_{\alpha s} \equiv \{\psi \mid \mu_\psi \text{ is } \alpha\text{-singular}\}$, one obtains a decomposition $\mathcal{H} = \mathcal{H}_{\alpha c} \oplus \mathcal{H}_{\alpha s}$, of $\mathcal{H}$ into mutually orthogonal closed subspaces which are invariant under H. The α-continuous spectrum ($\sigma_{\alpha c}$) and α-singular spectrum ($\sigma_{\alpha s}$) are then naturally defined as the spectra of the restrictions of H to the corresponding subspaces. Thus, the standard spectral theoretical scheme which uses the Lebesgue decomposition of a Borel measure into absolutely-continuous, singular-continuous, and pure-point parts can be extended to include further decompositions with respect to Hausdorff measures. As described in [**44**], the full picture is richer, and for every dimension $\alpha \in (0,1)$, there is a natural unique decomposition (of finite Borel measures and thus also of $\mathcal{H}$) into five parts: one below the dimension α, one above it, and three within it—of which the middle one is absolutely-continuous with respect to h^α.

A major focus in [**18**] was rank one perturbations and attempting to understand the above discussed instability of Anderson localization under such small perturbations. While [**18**] showed that it is fully possible for a rank one perturbation to change the spectral type all the way from point spectrum to one-dimensional spectrum, it also established some spectral semi-stability for the Anderson model (in fact, for any self-adjoint operator on $\ell^2(\mathbb{Z}^d)$ that has SULE).

Explicitly, [**18**] showed that the singular continuous spectrum which is obtained in the Anderson model by changing the value of the potential at a single point must be purely zero-dimensional (namely, the spectral measures are supported on a set of zero Hausdorff dimension). This implies that if one focuses on the spectral dimension rather than on distinguishing point spectrum from continuous spectrum, the situation appears to be stable, since changing the value of the potential at a single point never changes the fact that the spectrum is zero-dimensional. This semi-stability result was given two proofs in [**18**]. One obtained it directly as a spectral result, while the other deduced it (using results of [**44**]) as a corollary of a dynamical result controlling the growth rate of $\langle \psi, e^{iHt} X^2 e^{-iHt} \psi \rangle$. That is, while $\langle \psi, e^{iHt} X^2 e^{-iHt} \psi \rangle$ cannot be bounded if H has continuous spectrum, [**18**] shows that if H is obtained from an operator with SULE by changing the value of the potential at a single point, then $\langle \psi, e^{iHt} X^2 e^{-iHt} \psi \rangle$ cannot grow with t faster than logarithmically.

Another interesting result in [**18**] is the fact that if an ergodic operator on $\ell^2(\mathbb{Z}^d)$ has what they called ULE (Uniformly Localized Eigenvectors), which means that all its eigenvectors can be fitted by shifting under a single exponential envelope, then this localization property is stable and there cannot be any potential in the support of the corresponding ergodic measure for which the spectrum is continuous. Thus, while the occurrence of singular continuous spectrum in the Anderson model is a zero-probability event, it is nevertheless saying something important about the probabilistic problem: With probability one, the Anderson model doesn't have ULE (and so its localization properties cannot be much stronger than SULE).

Another interesting paper of the revolution was Simon's paper [**63**], which studied, among other things, singular continuous spectrum of one-dimensional sparse barrier potentials. These include, in particular, the potential used by Pearson [**47**] to construct the first explicit example with purely singular continuous spectrum. [**63**] proves that whenever such potentials are sufficiently sparse, the corresponding spectrum must be purely one-dimensional (namely, the spectral measure gives no weight to sets of Hausdorff dimension less than one). This roughly says that the singular continuous spectrum in Pearson's example [**47**] is one-dimensional. This should be contrasted with results of Last [**44**] and Jitomirskaya–Last [**35**] saying that all of the singular continuous spectrum which occurs for the almost Mathieu operator with coupling $|\lambda| > 2$ is purely zero-dimensional (this is roughly an almost periodic analog of the above discussed semi-stability result of [**18**] for the Anderson model). Thus, while the first concrete example of a Schrödinger operator with purely singular continuous spectrum had purely one-dimensional spectrum, the second example (of Avron–Simon [**6**]) had zero-dimensional spectrum.

To briefly mention a few more of Simon's revolution era results: Hof–Knill–Simon [**34**] established purely singular continuous spectrum for generic subsets in several families of discrete one-dimensional Schrödinger operators with potentials taking finitely many values. Simon–Stolz [**64**] established singular continuous spectrum for sufficiently sparse one-dimensional barrier potentials with growing barriers. Simon [**62**] proved the generic occurrence of purely singular continuous spectrum for certain topological families of graph Laplacians and Laplace–Beltrami operators. Gordon–Jitomirskaya–Last–Simon [**29**] established a new version of the Aubry duality for the almost Mathieu operator and proved that at the critical coupling $|\lambda| = 2$, it has purely singular continuous spectrum for Lebesgue a.e. pair α, θ.

Last–Simon [**45**] obtained some fundamental results for spectral analysis of one-dimensional Schrödinger operators, which yielded new proofs for many known results as well as important new results. Their results include characterizations of the absolutely continuous spectrum in terms of the behavior of eigenfunctions and transfer matrices, from which they also deduce that the absolutely continuous spectrum must be contained in the intersection of absolutely continuous spectra of a natural family of limiting operators. Among the applications of these results are a proof that the absolutely continuous spectrum of almost periodic Schrödinger operators is constant everywhere on the hull (which was also independently obtained around the same time by Kotani [**42**]), and the absence of absolutely continuous spectrum for strongly coupled (namely, any $|\lambda| > 2$) discrete potentials of the form $V(n) = \lambda \cos(n^\beta)$ with any $\beta > 1$.

Kiselev–Last–Simon [**39**] used modified Prüfer and EFGP transforms to study one-dimensional Schrödinger operators with decaying potentials. Among their results is a variant of Pearson's result [**47**] yielding purely singular continuous spectrum for suitable decaying sparse barrier potentials. They also obtain singular continuous spectrum with precisely computable fractional spectral Hausdorff dimensions for certain random decaying potentials.

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