KAM FOR THE NON-LINEAR SCHRÖDINGER EQUATION – A SHORT PRESENTATION

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ABSTRACT. We consider the d-dimensional nonlinear Schrödinger equation under periodic boundary conditions:

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x),\ x \in \mathbb{T}^d$$

where $V(x) = \sum \hat{V}(a) e^{i\langle a, x\rangle}$ is an analytic function with $\hat{V}$ real and F is a real analytic function in $\Re u$, $\Im u$ and x. (This equation is a popular model for the 'real' NLS equation, where instead of the convolution term $V * u$ we have the potential term Vu.) For $\varepsilon = 0$ the equation is linear and has time–quasi-periodic solutions u,

$$u(t, x) = \sum_{s \in \mathcal{A}} \hat{u}_0(a) e^{i(|a|^2 + \hat{V}(a))t} e^{i\langle a, x\rangle}, \quad 0 < |\hat{u}_0(a)| \le 1,$$

where $\mathcal{A}$ is any finite subset of $\mathbb{Z}^d$. We shall treat $\omega_a = |a|^2 + \hat{V}(a)$, $a \in \mathcal{A}$, as free parameters in some domain $U \subset \mathbb{R}^{\mathcal{A}}$.

This is a Hamiltonian system in infinite degrees of freedom, degenerate but with external parameters, and we shall describe a KAM-theory which, in particular, will have the following consequence:

If $|\varepsilon|$ is sufficiently small, then there is a large subset U' of U such that for all $\omega \in U'$ the solution u persists as a time–quasi-periodic solution which has all Lyapounov exponents equal to zero and whose linearized equation is reducible to constant coefficients.

This is a short presentation of the basic ideas. A detailed proof is given in [EK06].

1. INTRODUCTION

We consider the d-dimensional nonlinear Schrödinger equation

$$-i\dot{u} = \Delta u + V(x) * u + \varepsilon \frac{\partial F}{\partial \bar{u}}(x, u, \bar{u}), \quad u = u(t, x)$$

under the periodic boundary condition $x \in \mathbb{T}^d$. The convolution potential $V : \mathbb{T}^d \to \mathbb{C}$ have real Fourier coefficients $\hat{V}(a)$, $a \in \mathbb{Z}^d$, and we shall suppose it is analytic. F is an analytic function in $\Re u, \Im u$ and x.

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1.1. **An ∞-dimensional Hamiltonian system.** If we write

$$\begin{cases} u(x) = \sum_{a\in\mathbb{Z}^d} u_a e^{i<a,x>} \\ \overline{u(x)} = \sum_{a\in\mathbb{Z}^d} v_a e^{i<a,x>}, \end{cases}$$

and let

$$\zeta_a = \begin{pmatrix} \xi_a \\ \eta_a \end{pmatrix} = C \begin{pmatrix} u_a \\ v_a \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}.$$

then, in the symplectic space

$$\{(\xi_a, \eta_a) : a \in \mathbb{Z}^d\} = \mathbb{C}^{\mathbb{Z}^d} \times \mathbb{C}^{\mathbb{Z}^d}, \quad \sum_{a\in\mathbb{Z}^d} d\xi_a \wedge d\eta_a,$$

the equation becomes a real Hamiltonian system with an integrable part

$$\frac{1}{2}\sum_{a\in\mathbb{Z}^d}(|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2)$$

plus a perturbation.

Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^d$ and fix

$$0 < p_a(0), \quad a \in \mathcal{A}.$$

The $(\#\mathcal{A})$-dimensional torus

$$\begin{array}{ll} \frac{1}{2}(\xi_a^2 + \eta_a^2) = p_a(0) & a \in \mathcal{A} \\ \xi_a = \eta_a = 0 & a \in \mathcal{L} = \mathbb{Z}^d \setminus \mathcal{A}, \end{array}$$

is invariant for the Hamiltonian flow when $\varepsilon = 0$. In a neighborhood of this torus we introduce action-angle variables (φ_a, r_a)

$$\begin{array}{l} \xi_a = \sqrt{2(r_a(0) + r_a)}\cos(\varphi_a) \\ \eta_a = \sqrt{2(r_a(0) + r_a)}\sin(\varphi_a). \end{array}$$

The integrable Hamiltonian now becomes

$$h = \sum_{a\in\mathcal{A}} \omega_a r_a + \frac{1}{2}\sum_{a\in\mathcal{L}}(|a|^2 + \hat{V}(a))(\xi_a^2 + \eta_a^2),$$

where

$$\omega_a = |a|^2 + \hat{V}(a), \quad a \in \mathcal{A},$$

are the *basic frequencies*, and

$$|a|^2 + \hat{V}(a), \quad a \in \mathcal{L},$$

are the *normal frequencies* (of the invariant torus). The perturbation $\varepsilon f(\varphi, r, \xi, \eta)$ will be a function of all variables.

1.2. **The topology.** We define the complex domain

$$\mathcal{O}^\gamma(\sigma,\rho,\mu)=\left\{\begin{array}{l}\|\zeta\|_\gamma=\sqrt{\sum_{a\in\mathcal{L}}(|\xi_a|^2+|\eta_a|^2)\langle a\rangle^{2m_*}e^{2\gamma|a|}}<\sigma\\ |\Im\varphi|<\rho\\ |r|<\mu,\end{array}\right.$$

$\langle a\rangle=\max(|a|,1)$ and $m_*>\frac{d}{2}$. In this space the Hamitonian equations

$$\left\{\begin{array}{l}\dot\zeta=J(\frac{\partial h}{\partial\zeta}+\varepsilon\frac{\partial f}{\partial\zeta}),\quad J=\begin{pmatrix}0&I\\-I&0\end{pmatrix}\\ \dot\varphi=\frac{\partial h}{\partial r}+\varepsilon\frac{\partial f}{\partial r}\\ \dot r=-\varepsilon\frac{\partial f}{\partial\varphi}\end{array}\right.$$

have a well-defined local flow.

1.3. **Statemant of the result.** The Hamiltonian $h+\varepsilon f$ is a standard form for the perturbation theory of lower-dimensional (isotropic) tori with one exception: it is strongly degenerate. We therefore need external parameters to control the basic frequencies and the simplest choice is to let the basic frequencies (i.e. the potential itself) be our free parameters. The parameters will belong to a set

$$U\subset\{\omega\in\mathbb{R}^{\mathcal{A}}:|\omega|\le C_1\}\,. \tag{1}$$

The potential V will be analytic and

$$|\hat V(a)|\le C_2e^{-C_3|a|},\ C_3>0,\ a\in\mathbb{Z}^d. \tag{2}$$

The normal frequencies will be assumed to verify

$$\begin{array}{ll}||a|^2+\hat V(a)|\ge C_4 & \forall\,a,b\in\mathcal{L}\,,\\ ||a|^2+\hat V(a)+|b|^2+\hat V(b)|\ge C_4 & \forall\,a,b\in\mathcal{L}\,,\\ ||a|^2+\hat V(a)-|b|^2-\hat V(b)|\ge C_4 & \forall\,a,b\in\mathcal{L},|a|\ne|b|.\end{array} \tag{3}$$

This is fulfilled, for example, if V is small and $\mathcal{A}\ni 0$ or if V is arbitrary and $\mathcal{A}$ is sufficiently large.

Theorem A. *Under the above assumptions, for ε sufficiently small there exist a subset $U'\subset U$, which is large in the sense that*

$$\mathrm{Leb}\,(U\setminus U')\le \mathrm{cte}.\varepsilon^{exp_1}\,,$$

and for each $\omega\in U'$, a real analytic symplectic diffeomorphism Φ

$$\mathcal{O}^0(\frac{\sigma}{2},\frac{\rho}{2},\frac{\mu}{2})\to\mathcal{O}^0(\frac{\sigma}{2}+\varepsilon^{1/2},\frac{\rho}{2}+\varepsilon^{1/2},\frac{\mu}{2}+\varepsilon^{1/2})$$

and a vector $\omega'=\omega'(\omega)$ such that $(h_{\omega'}+\varepsilon f)\circ\Phi$ equals

$$c+<\omega,r>+\frac{1}{2}<\zeta,A(\omega)\zeta>+\varepsilon f'\,,$$

where

$$f' \in \mathcal{O}(|r|^2, |r| \|\zeta\|_0, \|\zeta\|_0^3)$$

and

$$A(\omega) = \begin{pmatrix} \Omega_1(\omega) & \Omega_2(\omega) \\ {}^t\Omega_2(\omega) & \Omega_1(\omega) \end{pmatrix}$$

is block-diagonal matrix with finite-dimensional blocks and $\Omega(\omega) = \Omega_1(\omega) + i\Omega_2(\omega)$ *is Hermitian.*

This theorem, as well as a more generalized version, is proven in [EK06].

1.4. **Notations.** The dimension d will be fixed and m_* will be a fixed constant $> \frac{d}{2}$. $\lesssim$ means $\leq$ modulo a multiplicative constant that only, unless otherwise specified, depends on d, m_* and $\#\mathcal{A}$.

The points in the lattice $\mathbb{Z}^d$ will be denoted $a, b, c, \ldots$.

A matrix on $\mathcal{L}$ is just a mapping $A : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ or $gl(2, \mathbb{C})$. Its components will be denoted A_a^b.

$<,>$ is the standard scalar product in $\mathbb{R}^d$. $\| \ \|$ is an operator or l^2-norm. $| \ |$ will in general denote a supremum norm, with a notable exception: for a lattice vector $a \in \mathbb{Z}^d$ we use $|a|$ for the l^2-norm.

For any two compact subsets X, Y of $\mathbb{R}^n$, $\mathrm{dist}(X, Y)$ is the Hausdorff distance and

$$X - Y = \{x - y : x \in X, \ y \in Y\}.$$

2. KAM-tori

A *KAM-torus* is a tripple object consisting of

- (i) an invariant torus;
- (ii) a flow on the torus which is conjugate to a linear flow $\varphi \mapsto \varphi + t\omega$;
- (iii) reducibility of the linearized equations on the torus to a constant coefficient system

$$\begin{cases} \dot{\zeta} = JA\zeta \\ \dot{\varphi} = ar \\ \dot{r} = 0. \end{cases}$$

The imaginary part of the eigenvalues of JA are the normal frequencies of the KAM-torus.

In general a KAM-torus is a much stronger property than just being an invariant torus or just being an invariant torus with a linear flow. For finite-dimensional Hamiltonian systems there are two notable exceptions: if the torus is one-dimensional it is just a periodic solution and (ii) is automatic and (iii) is a general fact called Floquet theory; if the torus is Lagrangian then (iii) follows from (i)+(ii) [dlL01].

2.1. **Normal form Hamiltonians.** This is a Hamiltonian of the form

$$h = c + <\omega, r> + \frac{1}{2}<\zeta, A(\omega)\zeta>,$$

where

$$A = \begin{pmatrix} \Omega_1 & \Omega_2 \\ {}^t\Omega_2 & \Omega_1 \end{pmatrix}$$

is block-diagonal matrix with finite-dimensional blocks and $\Omega = \Omega_1 + i\Omega_2$ is Hermitian. We shall say more about these blocks in Section 4.

Clearly $\{\zeta = r = 0\}$ is a KAM-torus for h. Moreover, since Ω is Hermitian and block diagonal the eigenvalues of JA are

$$\pm i\Omega_a, \quad a \in \mathcal{L}$$

where $\{\Omega_a : a \in \mathcal{L}\}$ are the eigenvalues of Ω. Therefore the linearized equation has only quasi-periodic solutions and, hence, the torus is linearly stable.

2.2. **Consequences of Theorem A.** The consequences of the theorem is a KAM-torus for $h_{\omega'} + \varepsilon f$. The dynamics of the Hamiltonian vector field of $h_{\omega'} + \varepsilon f$ on the image of Φ is the same as that of

$$<\omega, r> + \frac{1}{2}<\zeta, A(\omega)\zeta> .$$

The torus $\{\zeta = r = 0\}$ is invariant, since the Hamiltonian vector field on it is

$$\begin{cases} \dot{\zeta} = 0 \\ \dot{\varphi} = \omega \\ \dot{r} = 0, \end{cases}$$

and the flow on the torus is linear

$$t \mapsto \varphi + t\omega.$$

Moreover, the linearized equations on this torus becomes

$$\begin{cases} \dot{\zeta} = JA(\omega)\zeta + J\varepsilon\frac{\partial^2}{\partial r\partial\zeta}f'(0, \varphi + t\omega, 0)r \\ \dot{\varphi} = <J\varepsilon\frac{\partial^2}{\partial r\partial\zeta}f'(0, \varphi + t\omega, 0), \zeta> + J\varepsilon\frac{\partial^2}{\partial r^2}f'(0, \varphi + t\omega, 0)r \\ \dot{r} = 0. \end{cases}$$

Since $\Omega(\omega)$ is Hermitian and block diagonal the eigenvalues of $JA(\omega)$ are purely imaginary $\pm i\Omega_a(\omega), \quad a \in \mathcal{L}$. The linearized equation is reducible to constant coefficients if all $\Omega_a(\omega)$ are non-resonant with respect to ω, something which can be assumed if we restrict the set U' arbitrarily little. Then the ζ-component (and of course also the

r-component) will have only quasi-periodic (in particular bounded) solutions. The φ-component may have a linear growth in t, the growth factor (the "twist") being linear in r. It follows that the torus is linearly stable.

Reducibility is not only an important outcome of KAM-theory it is also an essential ingredient in the proof – it simplifies the iteration since it makes possible to reduce all approximate linear equations to constant coefficients. But it does not come for free as we shall see below.

3. The homological equations

Let Tf be the Taylor polynomial

$$f(0,\varphi,0)+ <\frac{\partial f}{\partial r}(0,\varphi,0), r> + <\frac{\partial f}{\partial \zeta}(0,\varphi,0),\zeta> +\frac{1}{2}<\zeta, \frac{\partial^2 f}{\partial \zeta^2}(0,\varphi,0)\zeta>$$

of f – it also depends on ω.

If $\varepsilon Tf = 0$ then $\{\zeta = r = 0\}$ is a KAM-torus for $h + \varepsilon f$. Now

$$\varepsilon Tf \in \mathcal{O}(\varepsilon).$$

Suppose we have a Taylor polynomial s, i.e. $s = Ts$, and a normal form Hamiltonian εk

$$c+ <\omega'(\omega) - \omega, r> +\frac{1}{2}<\zeta, (A' - A)(\omega)\zeta> \ .$$

verifying the homological equation

$$\{h, s\} = -Tf + k. \tag{4}$$

If Φ^t is the flow of

$$\begin{cases} \dot\zeta = \varepsilon J\frac{\partial s}{\partial \zeta}(\zeta,\varphi,r) \\ \dot\varphi = \varepsilon\frac{\partial s}{\partial r}(\zeta,\varphi,r) \\ \dot r = -\varepsilon\frac{\partial s}{\partial \varphi}(\zeta,\varphi,r), \end{cases}$$

then

$$\begin{aligned} (h+\varepsilon f)\circ\Phi^1 \quad &= h + \varepsilon k + \int_0^1 \tfrac{d}{dt}(h + t\varepsilon f + (1-t)\varepsilon k)\circ\Phi^t dt \\ &= h + \varepsilon k + \int_0^1 (\{h + t\varepsilon f + (1-t)\varepsilon k, \varepsilon s\} + \varepsilon f - \varepsilon k)\circ\Phi^t dt \\ &= h + \varepsilon k + \int_0^1 (\varepsilon^2\{tf + (1-t)k, s\} + \varepsilon f - \varepsilon Tf)\circ\Phi^t dt \\ &= h + \varepsilon k + \varepsilon f'. \end{aligned}$$

So Φ^1 transforms $h_\omega + \varepsilon f$ to a new normal form $h_\omega + \varepsilon k$ plus a new perturbation $\varepsilon f'$. It is easy to verify that

$$\varepsilon Tf' \in \mathcal{O}(\varepsilon^2).$$

3.1. **The homological equations.** In order to solve (4) we write s as

$$S_{01}(\varphi)+ <S_{02}(\varphi), r> + <S_1(\varphi), \zeta> + \frac{1}{2} <\zeta, S_2(\varphi)\zeta> .$$

Then the equation (4) decomposes into three homological equations corresponding to the three KAM-objects:

$$\left\{ \begin{array}{l} \partial_\omega S_1(\varphi) + JAS_1(\varphi) = \frac{\partial f}{\partial \zeta}(0, \varphi, 0) \\ \partial_\omega S_{01}(\varphi) = \frac{\partial f}{\partial \varphi}(0, \varphi, 0); \end{array} \right. \tag{5}$$

$$\partial_\omega S_{02}(\varphi) = \frac{\partial f}{\partial r}(0, \varphi, 0) - \frac{1}{\varepsilon}(\omega' - \omega); \tag{6}$$

$$\begin{array}{c} \partial_\omega S_2(\varphi) + AJS_2(\varphi) - S_2(\varphi)JA \\ = \frac{\partial^2 f}{\partial \zeta^2}(0, \varphi, 0) - \frac{1}{\varepsilon}(A' - A), \end{array} \tag{7}$$

where ∂_ω is the directional derivative in the direction ω.

The most delicate of these equations is the third one. This is a matrix equation since

$$A \quad \text{and} \quad F(\varphi) = \frac{\partial^2 f}{\partial \zeta^2}(0, \varphi, 0)$$

are ∞-dimensional matrices $\mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{R})$. For such matrices X let us define $\tilde{X} = {}^tCXC : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$ through

$$(\tilde{X})_a^b = {}^tCX_a^bC.$$

Then the equation (7) becomes

$$\partial_\omega \tilde{S}_2 + \tilde{A}J\tilde{S}_2 - \tilde{S}_2J\tilde{A} = \tilde{F} - \frac{1}{\varepsilon}(\tilde{A}' - \tilde{A}).$$

Since $\tilde{A}$ has the form

$$\begin{pmatrix} 0 & \Omega \\ {}^t\Omega & 0 \end{pmatrix}$$

this equation decouples into four equations for (scalar-valued) matrices $\mathcal{L} \times \mathcal{L} \to \mathbb{C}$

$$\partial_\omega R(\varphi) \pm i(\Omega R(\varphi) + R(\varphi){}^t\Omega) = G(\varphi) \tag{8}$$

$$\partial_\omega R(\varphi) \pm i(\Omega R(\varphi) - \Omega R(\varphi)) = G(\varphi) - \frac{1}{\varepsilon}(\Omega' - \Omega). \tag{9}$$

These equations can be solved (formally) in Fourier series and to get a solution we must prove the convergence of these Fourier series and estimate the solution. It is the equations (9) which give rise to problem. We define

$$(\Omega' - \Omega)_a^b = \begin{cases} \varepsilon(\hat{G}(0))_a^b & \text{if } |a| = |b| \\ 0 & \text{if not} \end{cases}$$

and

$$\hat{R}(0)_a^b = 0 \quad \text{if} \quad |a| = |b|.$$

The remaining part of R is determined by

$$i <k,\omega> \hat{R}(k) \pm i(\Omega\hat{R}(k) - \hat{R}(k)\Omega) = \hat{G}(k), \quad k \in \mathbb{Z}^{\mathcal{A}}. \tag{10}$$

3.2. **Small Divisors.** In order to get a solution with estimates we need a lower bound on the small divisors

$$|<k,\omega> + \Omega_a(\omega) - \Omega_b(\omega)|, \quad k \in \mathbb{Z}^{\mathcal{A}},\ a, b \in \mathcal{L},\ |a| \neq |b|. \tag{11}$$

The basic frequencies ω will be keep fixed during the iteration – that's what the parameters are there for – but the normal frequencies will vary. Indeed, $\Omega_a(\omega)$ and $\Omega_b(\omega)$ are perturbations of $|a|^2 + \hat{V}(a)$ and $|b|^2 + \hat{V}(b)$ which are not known a priori but are determined by the approximation process. [1]

This is a lot of conditions for a few parameters ω. Due to the exponential decay of space modes and Fourier modes we can truncate G to $G_{\Delta'}$

$$(\hat{G}_{\Delta'}(k))_a^b = \begin{cases} \hat{G}(k)_a^b & \text{if } |a-b| \leq \Delta' \text{ and } |k| \leq \Delta' \\ 0 & \text{if not} \end{cases}$$

for some sufficiently large (scale-dependent) $\Delta' = \Delta'_\varepsilon$. To solve the truncated equation it is enough to control the small divisors for

$$|k|,\ |a-b| \leq \Delta' \tag{12}$$

which improves the situation a bit. Indeed, in one space-dimension ($d = 1$) it improves a lot, and $(11+12)$ reduces to only finitely many cases. Not so however when $d \geq 2$, in which case the number of cases remains infinite.

How to control (11+12) is the main difficulty in the proof. But before we turn to this question (in Section 5) we shall discuss the normal form and how it changes during the iteration.

[1]A lower bound on (11), often known as the second Melnikov condition, is strictly speaking not necessary at all for reducibility. It is necessary, however, for reducibility with a reducing transformation close to the identity.

4. Block decomposition and Normal forms

4.1. **Blocks.** For a non-negative integer Δ we define an *equivalence relation* on $\mathcal{L}$ generated by the pre-equivalence relation

$$a \sim b \iff \begin{cases} |a|^2 = |b|^2 \\ |a-b| \le \Delta. \end{cases}$$

Let $[a]_\Delta$ denote the equivalence class (*block*) of a, and let $\mathcal{E}_\Delta$ be the set of equivalence classes.

It is trivial that each block $[a]$ is finite with cardinality

$$\lesssim |a|^{d-1}$$

that depends on a. But there is also a uniform Δ-dependent bound.

Lemma 4.1. *Let*

$$d_\Delta = \sup_a(\#[a]_\Delta).$$

Then

$$d_\Delta \lesssim \ \Delta^{\frac{(d+1)!}{2}}.$$

The blocks $[a]_\Delta$ have a rigid structure when $|a|$ is large. For a vector $c \in \mathbb{Z}^d \setminus 0$ let

$$a_c \in (a+\mathbb{R}c) \cap \mathbb{Z}^d$$

be the lattice point b on the line $a+\mathbb{R}c$ with smallest norm – if there are two such b's we choose the one with $<b, c> \ge 0$.

Lemma 4.2. *Given a and $c \neq 0$ in $\mathbb{Z}^d$. For all $t \ge 0$, such that*

$$|a+tc| \ge d_\Delta^2(|a_c| + |c|)\,|c|\,,$$

the set $[a+tc]_\Delta - (a+tc)$ is independent of t and $\perp$ to c.

Description of blocks when $d = 2, 3$. For $d = 2$, we have outside $\{|a| :\le d_\Delta \approx \Delta^3\}$

- $\star$ $\operatorname{rank}[a]_\Delta = 1$ if, and only if, $a \in \frac{b}{2} + b^\perp$ for some $0 < |b| \le \Delta$ – then $[a]_\Delta = \{a, a-b\}$;
- $\star$ $\operatorname{rank}[a]_\Delta = 0$ otherwise – then $[a]_\Delta = \{a\}$.

For $d = 3$, we have outside $\{|a| :\le d_\Delta \approx \Delta^{12}\}$

- $\star$ $\operatorname{rank}[a]_\Delta = 2$ if, and only if, $a \in \frac{b}{2} + b^\perp \cap \frac{c}{2} + c^\perp$ for some $0 < |b|\,, |c| \le \Delta_2$ linearly independent – then $[a]_\Delta \supset \{a, a-b, a-c\}$;
- $\star$ $\operatorname{rank}[a]_\Delta = 1$ if, and only if, $a \in \frac{b}{2} + b^\perp$ for a unique(!) $0 < |b|\,, \le \Delta$ – then $[a]_\Delta = \{a, a-b\}$;
- $\star$ $\operatorname{rank}[a]_\Delta = 0$ otherwise – then $[a]_\Delta = \{a\}$.

4.2. **Normal form matrices and Hamiltonians.** We say that a (scalar-valued) matrix $X : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ is on normal form – denoted $\mathcal{NF}_\Delta$ – if

(i) X is Hermitian;
(ii) X is *block-diagonal* over over $\mathcal{E}_\Delta$, i.e.

$$X_a^b = 0 \quad \text{if} \quad [a]_\Delta \neq [b]_\Delta.$$

We say that our normal form Hamiltonians

$$h = c + <\omega, r> + \frac{1}{2}<\zeta, A(\omega)\zeta>,$$

$$A = \begin{pmatrix} \Omega_1 & \Omega_2 \\ {}^t\Omega_2 & \Omega_1 \end{pmatrix},$$

is $\mathcal{NF}_\Delta$ if $\Omega = \Omega_1 + i\Omega_2$ is $\mathcal{NF}_\Delta$. Clearly if h is $\mathcal{NF}_\Delta$ for some $\Delta \leq \Delta'$ then k and $h + \varepsilon k$ are $\mathcal{NF}_{\Delta'}$, where k is determined by the homological equation (4) under the truncation (12).

5. Töplitz-Lipschitz matrices

5.1. **Töplitz at ∞.** We say that a matrix

$$X : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$$

has a Töplitz-limit at ∞ in the direction c if, for all a, b

$$\lim_{t\to\infty} X_{a+tc}^{b+tc} \ \exists \ = X_a^b(c).$$

$X(c)$ is a new matrix which is Töplitz in the direction c, i.e.

$$X_{a+c}^{b+c}(c) = X_a^b(c).$$

We say that X is 1-Töplitz if all Töplitz-limits $X(c)$ exist, and we define, inductively, that X is n-Töplitz if all Töplitz-limits $X(c)$ are (n-1)-Töplitz. We say that X is *Töplitz* if it is (d-1)-Töplitz.

Example. Consider the equation (10) and assume for simplicity that

$$\Omega = \text{diag}(|a|^2 + \hat{V}(a)).$$

Then

$$\hat{R}(k)_a^b = \frac{1}{i} \frac{\hat{G}(k)_a^b}{<k, \omega> + |a|^2 - |b|^2 + \hat{V}(a) - \hat{V}(b)}$$

and if the small divisors are all $\neq 0$ then $\hat{R}(k)$ is a well-defined matrix $\mathcal{L} \times \mathcal{L} \to \mathbb{C}$. Replacing a, b by $a + tc, b + tc$ and letting $t \to \infty$ we see two different cases. If $<a - b, c> \neq 0$ then the limit exist and is $= 0$ as long as $|\hat{G}(k)_{a+tc}^{b+tc}|$ is bounded. If $<a-b,c>=0$ then the limit exist as long as $|\hat{G}(k)_{a+tc}^{b+tc}|$ has a limit:

$$\hat{R}(k)_a^b(c) = \frac{1}{i}\frac{\hat{G}(k)_a^b(c)}{<k,\omega> + |a|^2 - |b|^2}.$$

Hence the matrix $\hat{R}(k)$ is Töplitz at ∞ if $\hat{G}(k)$ is Töplitz at ∞.

5.2. **Lipschitz domains.** For a non-negative constant Λ and for any $c \in \mathbb{Z}^d \setminus 0$, let the Lipschitz domain

$$D_\Lambda(c) \subset \mathcal{L} \times \mathcal{L}$$

be the set of all (a,b) such that there exist $a',\ b' \in \mathbb{Z}^d$ and $t \geq 0$ such that

$$\left\{ \begin{array}{lcl} |a = a' + tc| & \geq & \Lambda(|a'| + |c|)\,|c| \\ |b = b' + tc| & \geq & \Lambda(|b'| + |c|)\,|c| \end{array} \right.$$

and

$$\frac{|a|}{|c|},\quad \frac{|b|}{|c|} \ \geq\ 2\Lambda^2.$$

The Lipschitz domains are not so easy to grasp, but it is easy to verify

Lemma 5.1. *For $\Lambda \geq 3$*

$$\frac{|a|}{|c|} \approx \frac{|b|}{|c|} \approx \frac{<a,c>}{|c|} \approx \frac{<b,c>}{|c|} \approx t \gtrsim \Lambda|c|$$

and

$$|a'| \leq \frac{t}{\Lambda - 1}.$$

The most important property is that finitely many Lipschitz domains cover a "neighborhood of ∞" in the following sense.

Lemma 5.2. *For any N, the subset*

$$\{|a| + |b| \gtrsim \Lambda^{2d-1}\} \cap \{|a-b| \leq N\} \subset \mathbb{Z}^d \times \mathbb{Z}^d$$

is contained in

$$\bigcup_{|c| \lesssim \Lambda^{d-1}} D_\Omega(c)$$

for any

$$\Omega \leq \frac{\Lambda}{N+1} - 1.$$

5.3. **Töplitz-Lipschitz matrices.** We define the supremum-norm

$$|X|_\gamma = \sup_{a,b\in\mathcal{L}} |X|_a^b e^{\gamma|a-b|},$$

the Lipschitz-constant

$$\mathrm{Lip}_{\Lambda,\gamma} X = \sup_{c\in\mathbb{Z}^d\setminus 0} \sup_{(a,b)\in D_\Lambda(c)} |X_a^b - X_a^b(c)| \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) e^{\gamma|a-b|}$$

and the Lipschitz-norms

$${}^1\!\!<X>_{\Lambda,\gamma} = \mathrm{Lip}_{\Lambda,\gamma} X + |X|_\gamma\,,$$

and, inductively,

$${}^n\!\!<X>_{\Lambda,\gamma} = \sup_{c\in\mathbb{Z}^d} {}^{n-1}\!\!<X(c)>_{\Lambda,\gamma}$$

– this norm is defined if X is n-Töplitz.

We define

$$<X>_{\Lambda,\gamma} = {}^{d-1}\!\!<X>_{\Lambda,\gamma}$$

and we say that the matrix X is *Töplitz-Lipschitz* if $<X>_{\Lambda,\gamma}<\infty$ for some Λ,γ.

Example. Consider $\hat{R}(k)$ from the example above. If

$$(a,b) = (a'+tc, b'+tc) \in D_\Lambda(c), \quad \Lambda \geq 3,$$

then

$$\frac{|a|}{|c|} \approx \frac{|b|}{|c|} \approx t \geq \Lambda.$$

If $<a-b,c>\neq 0$ then

$$\left|\hat{R}(k)_a^b - 0\right| \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) e^{\gamma|a-b|}$$

$$\approx \left| \frac{\hat{G}(k)_a^b}{<a-b,c> + \frac{1}{t}(<k,\omega> + |a'|^2 - |b'|^2 + \hat{V}(a) - \hat{V}(b))} \right| e^{\gamma|a-b|}$$

which is

$$\approx \left| \frac{\hat{G}(k)_a^b}{<a-b,c>} \right| e^{\gamma|a-b|} \lesssim |G|_\gamma$$

if Λ is sufficiently large.

If $<a-b,c>=0$ then

$$\left|\hat{R}(k)_a^b - \hat{R}(k)(c)_a^b\right| \max(\frac{|a|}{|c|}, \frac{|b|}{|c|}) e^{\gamma|a-b|}$$

$$\lesssim \left|\frac{1}{<k,\omega> + |a'|^2 - |b'|^2}\right| \mathrm{Lip}_{\Lambda,\gamma}(\hat{G}(k)) + \left|\frac{1}{<k,\omega> + |a'|^2 - |b'|^2}\right|^2 \left|\hat{G}(k)\right|_\gamma$$

if Λ is sufficiently large.

In particular, the matrix $\hat{R}(k)$ is Töplitz-Lipschitz if $\hat{G}(k)$ is Töplitz-Lipschitz.

5.4. **How do we use this property.** Let us discuss the case $d=2$. Assume that

$$\Omega(\omega)=\mathrm{diag}(|a|^2+\hat{V}(a))+H(\omega)$$

where $H(\omega)$ and $\frac{\partial H}{\partial\omega}(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ for all $\omega\in U$ and verify

$$\left\|\frac{\partial H}{\partial\omega}(\omega)\right\|\le\frac{1}{4},\quad \omega\in U. \tag{13}$$

(Here $\|\cdot\|$ is the operator norm.)

Let

$$\begin{cases}(a,b)=(a'+t_0c,b'+t_0c)\in D_\Lambda(c),\ \Lambda\ge d_{\Delta^2}\\ |a-b|\le\Delta.\end{cases}$$

For $t\ge t_0$, by Lemma 4.2,

$$\sigma(\Omega_{[a'+tc]_\Delta}^{[a'+tc]_\Delta}(\omega))-\sigma(\Omega_{[b'+tc]_\Delta}^{[b'+tc]_\Delta}(\omega))=\sigma(\Omega_{[a]_\Delta+tc}^{[a]_\Delta+tc}(\omega))-\sigma(\Omega_{[b]_\Delta+tc}^{[b]_\Delta+tc}(\omega))$$

[2] and

$$\mathrm{dist}(\sigma(\Omega_{[a]_\Delta+tc}^{[a]_\Delta+tc}(\omega))-\sigma(\Omega_{[b]_\Delta+tc}^{[b]_\Delta+tc}(\omega)),0)$$

is equal to

$$|t<a'-b',c>+|a'|^2-|b'|^2|$$

with an error of size at most

$$C_2+\|H(\omega)\|.$$

By Lemma 5.1

$$|t<a'-b',c>+|a'|^2-|b'|^2|\ge\Lambda(|<a'-b',c>|-2\Delta\frac{1}{\Lambda-1}).$$

If $<a-b,c>=<a'-b',c>\neq 0$ then the small divisors (11) are large for all

$$a\in[a]_\Delta,\ b\in[b]_\Delta,\ |k|\le\Delta$$

if Λ is sufficiently large.

If $<a-b,c>=<a'-b',c>=0$ then

$$\sigma(\Omega_{[a]_\Delta+tc}^{[a]_\Delta+tc}(\omega))-\sigma(\Omega_{[b]_\Delta+tc}^{[b]_\Delta+tc}(\omega))\to$$
$$\sigma(\mathrm{diag}(|a|^2)_{a\in[a]_\Delta}+H_{[a]_\Delta}^{[a]_\Delta}(c,\omega))-\sigma(\mathrm{diag}(|b|^2)_{b\in[b]_\Delta}+H_{[b]_\Delta}^{[b]_\Delta}(c,\omega))$$

as $t\to\infty$. Notice that the limit does not change if we replace (a,b) by $(a+c,b+c)$.

[2] $\sigma(\Omega_X^Y)$ is the spectrum of the matrix Ω_X^Y.

We denote the limit-set as

$$\{\Omega_a(c,\omega)-\Omega_b(c,\omega):(a,b)\in[a]_\Delta\times[b]_\Delta\}$$

and we notice that he small divisors at "∞c", i.e.

$$|<k,\omega>+\Omega_a(c,\omega)-\Omega_b(c,\omega)|\,, \tag{14}$$

are only finitely many under the restriction

$$|k|\,,\ |a-b|\le\Delta' \quad\text{and}\quad <a-b,c>=0 \tag{15}$$

due to invariance under c-translations. Therefore we can bound (14+15) for ω in an appropriate subset U' of U – here we need (13) – and using the Lipschitz-property we can propagate this bound into the domain $D_\Lambda(c)$ if Λ is sufficiently large – the size of Λ depends in particular on

$$<H>_{\left\{\substack{\Lambda\\U}\right\}}=\sup_{\omega\in U}(<H(\omega)>_{\Lambda,0},<\frac{\partial H}{\partial_\omega}(\omega)>_{\Lambda,0}).$$

By Lemma 5.2 the set

$$\mathcal{L}\times\mathcal{L}\cap\{|a-b|\le\Delta\}$$

is covered by finitely many Lipschitz-domains and a finite set. For each Lipschitz-domain the small divisor condition holds, as above, for ω in some subset of U. For (a,b) in the finite set it also holds for ω in some subset of U.

Carrying out the estimates and making an induction of d we prove

Proposition 5.3. *Let $\Delta'>0$ and $\kappa>0$. Assume that U verifies (1), that $\hat V$ is real and verifies (2) and that $H(\omega)$ and $\frac{\partial H}{\partial\omega}(\omega)$ are Töplitz at ∞ and $\mathcal{NF}_\Delta$ and verify (13) for all $\omega\in U$.*

Then there exists a subset $U'\subset U$,

$$\begin{aligned}Leb(U\setminus U') &\le\\ &\text{cte.}\max(\Delta',d_\Delta^2,\Lambda)^{exp+\#\mathcal{A}\text{-}1}(1+<H>_{\left\{\substack{\Lambda\\U}\right\}})^d\kappa^{\frac{1}{d}}C_1^{d-1},\end{aligned}$$

such that, for all $\omega\in U'$, $0<|k|\le\Delta'$ and all

$$|a-b|\le\Delta'$$

$$|<k,\omega>+\alpha(\omega)-\beta(\omega)|\ge\kappa\quad\forall\begin{cases}\alpha(\omega)\in\sigma(\Omega(\omega)_{[a]_\Delta})\\ \beta(\omega)\in\sigma(\Omega(\omega)_{[b]_\Delta}).\end{cases}$$

Moreover the κ-neighborhood of $U\setminus U'$ satisfies the same estimate.

The exponent exp depends only on d. The constant cte. *depends on the dimensions d and $\#\mathcal{A}$ and on C_2,C_3.*

This proposition permits to control the small divisors and, hence, estimate the solution of the homological equation if $\Omega(\omega)$ satisfies the assumptions of the proposition and if we can bound

$$<H>_{\{\substack{\Lambda \\ U}\}} .$$

In order to iterate this construction and, hence, prove the theorem, we must grant that the modified normal form

$$h+\varepsilon k = c + <\omega'(\omega), r> + \frac{1}{2}<\zeta, A'(\omega)\zeta> ,$$

also verifies the assumptions and control

$$<H+\varepsilon H'>_{\{\substack{\Lambda' \\ U'}\}}$$

for some $\Lambda' \geq \Lambda$. The essential points in doing this is discussed in the next section.

6. Function with Töplitz-Lipshitz property

6.1. **Töplitz structure of $\frac{\partial^2 f}{\partial \zeta^2}$.** The quadratic differential

$$<\zeta, \frac{\partial^2}{\partial \zeta^2} f(0,\varphi,r)\zeta>$$

has the form

$$<\zeta, A\zeta> = \sum_{a,b\in\mathcal{L}} <\zeta_a, A_a^b \zeta_b>,$$

where $A : \mathcal{L}\times\mathcal{L} \to gl(2,\mathbb{R})$ is a $gl(2,\mathbb{R})$-valued matrix. It is uniquely determined by the symmetry condition

$${}^t\!A_a^b = A_b^a.$$

Its properties are best seen in the complex variables

$$({}^t\!CAC)_a^b = \begin{pmatrix} P_a^b & Q_a^b \\ \bar{Q}_b & \bar{P}_a^b \end{pmatrix}.$$

Consider for example the Schrödinger equation with a cubic potential, i.e.

$$F(x,u,\bar{u}) = u^2\bar{u}^2.$$

Then

$$P_{a_1}^{a_2} = \sum_{\substack{b_1,b_2\in\mathcal{A} \\ b_1+b_2=a_1+a_2}} 2\sqrt{r_{b_1} r_{b_2}}\, e^{-i(\varphi_{b_1}+\varphi_{b_2})}$$

and

$$Q_{a_2}^{b_2} = \sum_{\substack{a_1,b_1\in\mathcal{A} \\ a_1-b_1=a_2-b_2}} 8\sqrt{r_{a_1} r_{b_1}}\, e^{i(\varphi_{a_1}-\varphi_{b_1})}.$$

In particular

$$\begin{cases} P \text{ is symmetric} \\ Q \text{ is Hermitian.} \end{cases}$$

Moreover Q is Töplitz,

$$Q_{a+c}^{b+c} = Q_a^b \quad \forall a, b, c,$$

and (since $\mathcal{A}$ is finite) its elements are zero at finite distance from the diagonal. In particular, this matrix is Töplitz-Lipschitz and has exponential decay off the diagonal $a = b$. P is also Töplitz-Lipschitz with exponential decay but *in a different sense*:

$$P_{a+c}^{b-c} = P_a^b \quad \forall a, b, c,$$

and has exponential decay off the "anti-diagonal" $\{a = -b\}$.

6.2. **Töplitz-Lipschitz matrices $\mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{R})$.** We consider the space $gl(2, \mathbb{C})$ of all complex 2×2-matrices provided with the scalar product

$$Tr({}^t\bar{A}B).$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

and consider the orthogonal projection π of $gl(2, \mathbb{C})$ onto the subspace

$$M = \mathbb{C}I + \mathbb{C}J.$$

For a matrix

$$A : \mathcal{L} \times \mathcal{L} \to gl(2, \mathbb{C})$$

we define πA through

$$(\pi A)_a^b = \pi A_a^b, \quad \forall a, b.$$

We define the supremum-norms

$$|A|_\gamma^\pm = \sup_{(a,b) \in \mathcal{L} \times \mathcal{L}} |A_a^b| e^{\gamma |a \mp b|}$$

and

$$|A|_\gamma = \max(|\pi A|_\gamma^+, |A - \pi A|_\gamma^-).$$

A is said to have a Töplitz-limit at ∞ in the direction c if, for all a, b the two limits

$$\lim_{t \to +\infty} A_{a+tc}^{b \pm tc} \ \exists \ = A_a^b(\pm, c).$$

$A(\pm, c)$ are new matrices which are Töplitz/"anti-Töplitz" in the direction c, i.e.

$$A_{a+c}^{b+c}(+, c) = A_a^b(+, c) \quad \text{and} \quad A_{a+c}^{b-c}(-, c) = A_a^b(-, c).$$

If $|A|_\gamma < \infty$, $\gamma > 0$, then

$$\pi A(-,c) = (A - \pi A)(+,c) = 0.$$

We say that A is 1-Töplitz if all Töplitz-limits $A(\pm,c)$ exist, and we define, inductively, that X is n-Töplitz if all Töplitz-limits $A(\pm,c)$ are (n-1)-Töplitz. We say that A is *Töplitz* if it is (d-1)-Töplitz.

We define the Lipschitz-constants

$$\mathrm{Lip}^{\pm}_{\Lambda,\gamma}A = \sup_{c\neq 0}\sup_{(a,b)\in D_\Lambda(c)} |(A - A(\pm,c))_a^{\pm b}| \max(\frac{|a|}{|c|},\frac{|b|}{|c|})e^{\gamma|a\mp b|}$$

and the Lipschitz-norms

$$^1\!< A >_{\Lambda,\gamma} = \max(\mathrm{Lip}^+_{\Lambda,\gamma}\pi A + |\pi A|^+_\gamma \,, \mathrm{Lip}^-_{\Lambda,\gamma}(I-\pi)A + |(I-\pi)A|^-_\gamma)$$

and, inductively,

$$^n\!< A >_{\Lambda,\gamma} = \sup_c(^{n-1}\!< A(+,c) >_{\Lambda,\gamma}, {}^{n-1}\!< A(-,c) >_{\Lambda,\gamma})$$

– it is defined if A is n-Töplitz.

We define

$$< A >_{\Lambda,\gamma} = {}^{d-1}d\!< A >_{\Lambda,\gamma}$$

and we say that A *Töplitz-Lipschitz* if $< A >_{\Lambda,\gamma} < \infty$ for some Λ, γ. (For a more general formulation see [EK05].) The most important property is a product formula.

Lemma 6.1.

$$\begin{gathered} < A_1 \cdots A_n >_{\Lambda+6,\gamma'} \leq \\ (\mathrm{cte.})^n \Lambda^2 (\tfrac{1}{\gamma-\gamma'})^{(n-1)d+1} [\textstyle\sum_{1\leq k\leq n} \prod_{\substack{1\leq j\leq n \\ j\neq k}} |A_j|_{\gamma_j} < A_k >_{\Lambda,\gamma_k}], \end{gathered}$$

where all $\gamma_1, \ldots, \gamma_n$ are $= \gamma$ except one which is $= \gamma'$.

6.3. **Functions with Töplitz-Lipschitz property.** Let $\mathcal{O}^\gamma(\sigma)$ be the set of vectors in the complex space $l^2_\gamma(\mathcal{L},\mathbb{C})$ of norm less than σ, i.e.

$$\mathcal{O}^\gamma(\sigma) = \{\zeta \in \mathbb{C}^{\mathcal{L}} \times \mathbb{C}^{\mathcal{L}} : \|\zeta\|_\gamma < \sigma\}.$$

Our functions $f : \mathcal{O}^0(\sigma) \to \mathbb{C}$ will be defined and real analytic on the domain $\mathcal{O}^0(\sigma)$. [3] Its first differential

$$l^2_0(\mathcal{L},\mathbb{C}) \ni \hat\zeta \mapsto <\hat\zeta, \frac{\partial f}{\partial \zeta}(\zeta)>$$

[3]The space $l^2_\gamma(\mathcal{L},\mathbb{C})$ is the complexification of the space $l^2_\gamma(\mathcal{L},\mathbb{R})$ of real sequences. “real analytic” means that it is a holomorphic function which is real on $\mathcal{O}^0(\sigma) \cap l^2_\gamma(\mathcal{L},\mathbb{R})$.

defines a unique vector $\frac{\partial f}{\partial \zeta}(\zeta)$ in $l_0^2(\mathcal{L},\mathbb{C})$, and its second differential

$$l_0^2(\mathcal{L},\mathbb{C}) \ni \hat{\zeta} \mapsto <\hat{\zeta}, \frac{\partial^2 f}{\partial \zeta^2}(\zeta)\hat{\zeta}>$$

defines a unique matrix $\frac{\partial^2 f}{\partial \zeta^2}(\zeta)$ $\mathcal{L}\times\mathcal{L} \to gl(2,\mathbb{C})$ which is symmetric, i.e.

$$^t A_a^b = A_b^a.$$

We say that f is *Töplitz at* ∞ if the matrix $\frac{\partial^2 f}{\partial \zeta^2}(\zeta)$ is Töplitz at ∞ for all $\zeta \in \mathcal{O}^0(\sigma)$. We define the norm

$$[f]_{\Lambda,\gamma,\sigma}$$

to be the smallest C such that

$$\begin{cases} |f(\zeta)| \le C & \forall \zeta \in \mathcal{O}^0(\sigma) \\ \|\partial_\zeta f(\zeta)\|_{\gamma'} \le \frac{1}{\sigma} C & \forall \zeta \in \mathcal{O}^{\gamma'}(\sigma),\ \forall \gamma' \le \gamma, \\ <\partial_\zeta^2 f(\zeta)>_{\Lambda,\gamma'} \le \frac{1}{\sigma^2} C & \forall \zeta \in \mathcal{O}^{\gamma'}(\sigma),\ \forall \gamma' \le \gamma. \end{cases}$$

We study the behavior of this norm under truncations, Poisson brackets, flows and compositions in order to control it during the KAM-step.

7. Some References

For finite dimensional Hamiltonian systems the first proof of persistence of stable (i.e. vanishing of all Lyapunov exponents) lower dimensional invariant tori was obtained in [Eli85, Eli88] and there are now many works on this subjects. There are also many works on reducibility (see for example [Kri99, Eli01]) and the situation in finite dimension is now pretty well understood. Not so, however, in infinite dimension.

If $d = 1$ and the space-variable x belongs to a finite segment supplemented by Dirichlet or Neumann boundary conditions, this result was obtained in [Kuk88] (also see [Kuk93, Pös96]). The case of periodic boundary conditions was treated in [Bou96], using another multi–scale scheme, suggested by Fröhlich–Spencer in their work on the Anderson localization [FS83]. This approach, often referred to as the Craig-Wayne scheme, is different from KAM. It avoids the, sometimes, cumbersome bounds on the small divisors (11) but to a high cost: the approximate linear equations are not of constant coefficients. Moreover, it gives persistence of the invariant tori but no reducibility and no information on the linear stability. A KAM-theorem for periodic boundary conditions has recently been proved in [GY05] (with a perturbation F independent of x) and the perturbation theory for quasi-periodic solutions of one-dimensional Hamiltonian PDE is now sufficiently well developed (see for example [Kuk93, Cra00, Kuk00]).

The study of the corresponding problems for $d \geq 2$ is at its early stage. Developing further the scheme, suggested by Fröhlich–Spencer, Bourgain proved persistence for the case $d = 2$ [Bou98]. More recently, the new techniques developped by him and collaborators in their work on the linear problem has allowed him to prove persistence in any dimension d[Bou04]. (In this work he also treats the wave equation.)

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