TOPOLOGICAL METHODS IN THE INSTABILITY PROBLEM OF HAMILTONIAN SYSTEMS

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Abstract. We use topological methods to investigate some recently proposed mechanisms of instability (Arnol'd diffusion) in Hamiltonian systems.

In these mechanisms, chains of heteroclinic connections between whiskered tori are constructed, based on the existence of a normally hyperbolic manifold Λ, so that: (a) the manifold Λ is covered rather densely by transitive tori (possibly of different topology), (b) the manifolds W^{s}_{Λ}, W^{u}_{Λ} intersect transversally, (c) the systems satisfies some explicit non-degeneracy assumptions, which hold generically.

In this paper we use the method of correctly aligned windows to show that, under the assumptions (a), (b) (c), there are orbits that move a significant amount.

As a matter of fact, the method presented here does not require that the tori are exactly invariant, only that they are approximately invariant. Hence, compared with the previous papers, we do not need to use KAM theory. This lowers the assumptions on differentiability.

Also, the method presented here allows to produce concrete estimates on the time to move, which were not considered in the previous papers.

1. Introduction

The paper [Arn64] described a mechanism of global instability in Hamiltonian systems which are arbitrarily closed to integrable. The paper showed that in the remarkable two parameter family

$$\begin{aligned} H_{\varepsilon,\mu}(A_1, A_2, \varphi_1, \varphi_2, t) &= H_0 + \varepsilon H_\varepsilon + \mu H_\mu \\ &= \frac{1}{2}A_1^2 + \frac{1}{2}A_2^2 + \varepsilon(\cos\varphi_1 - 1) \\ &\quad + \mu(\cos\varphi_1 - 1)\,(\sin\varphi_2 + \cos t) \end{aligned} \tag{1.1}$$

for all $0 < |\mu| \ll |\varepsilon| \ll 1$, the system (1.1) admits orbits for which the action A_2 changes by 1 over time.

The mechanism of [Arn64] is based on the existence of whiskered tori. The perturbation is chosen in such a way that it does not affect the tori but causes the stable and unstable manifolds to intersect. Therefore, there is a chain of whiskered tori with heteroclinc intersections. The change of action moving along the chain is independent of μ, ε. By an obstruction argument, there is one orbit that follows the chain and provides with the desired unstable orbit.

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An important ingredient in the remarkable example in [Arn64] is that the the perturbation is chosen carefully so that it vanishes on a smooth manifold. Since this feature does not seem realistic in systems appearing in practice, there has been a great deal of interest in finding more robust mechanisms that hold for systems in which the perturbation is similar to those one encounters in real systems. It is to be remarked that, numerical experiments [Chi79, Ten82, LT83] suggest that generic perturbations that indeed destroy the foliation of whiskered tori, indeed generate more diffusion. We recall that it is expected that a perturbation of size ε will create gaps of size $\varepsilon^{1/2}$ in the foliation of whiskered tori, but will move the invariant manifolds only an amount ε. This is the so-called large gap problems.

In the last five years, there have appeared a number of papers developing the mathematical theory of these problems, and a variety of methods have been suggested. (See [DdlLS03a, DdlLS04] for a discussion of the recent literature). Of particular interest for the present paper are [DdlLS00, DdlLS03b, DdlLS03a, DdlLS04]. The above papers developed an argument to establish diffusion in dynamical systems and verify it in several models, including models presenting the large gap problem. The argument in the above papers is based on identifying a normally hyperbolic manifold Λ with transverse homoclinic intersections, and then studying the interplay between the dynamics restricted to Λ and the dynamics given by the homoclinic excursions.

In this paper, we extend and simplify some of the proofs of the existence of diffusing orbits in Hamiltonian systems the above mentioned papers by incorporating the use of the topological method of *"correctly aligned windows"* introduced in [Eas81, Eas75, Eas78] and extended in [ZG04, GZ04, GR03, GR04]. In particular, we obtain estimates for the time required for the orbits to diffuse and we lower the requirements of differentiability in the models considered.

The strategy of [DdlLS00, DdlLS03b, DdlLS03a] was to use the theory of persistence of normally hyperbolic manifolds to study the manifold Λ and its homoclinic intersections, then apply averaging methods and the KAM theorem to establish the existence of invariant tori in Λ. These invariant tori in Λ are whiskered tori in the full system. A more detailed study of the intersections of the stable and unstable manifolds of Λ – obtained through the use of the *scattering map* – shows that, under appropriate nondegeneracy conditions on the potential, there are transition chains of whiskered tori. Then, an obstruction argument similar to that sketched in [Arn64, AA67], but incorporating technical refinements, establishes that there are orbits of the system which follow the transition chain. The main difference between the models in [DdlLS00] and the models in [DdlLS03a] is that the transition chains in [DdlLS03a] incorporate, besides the regular KAM tori, secondary tori or lower dimensional tori. These extra tori fit the gaps and overcome the large gap problem. Another technical difference is that in the model in [DdlLS03a], the stable and unstable manifolds of Λ did not intersect for $\varepsilon = 0$. Hence, the existence of intersections of the stable and unstable manifolds has to be established using perturbations and a non-degeneracy assumption. The paper [DdlLS04] generalizes [DdlLS00] to higher dimensions and to quasi-periodic perturbations.

In Section 2, we will review the main steps of the constructions in the above papers. As mentioned in the papers above, the argument is a sequence of well defined steps and milestones so that different arguments could be used for different steps.

The goal of this paper to accomplish several of the steps in the above strategy through topological methods. More specifically, in the scheme of [DdlLS03b], the topological argument presented here takes over once we have implemented the persistence of the normally hyperbolic manifold, transversality of the stable and unstable manifolds, and averaging in the normally hyperbolic manifold. Then, the argument presented in this paper takes the place of the KAM and the obstruction argument in the papers above.

In Section 3 we present a set up that, as shown in the corresponding papers, applies to the models described in Section 2 and state our main result, Theorem 3.1. This result shows that, under the assumptions of the set up, there are orbits whose action experiences changes of order 1.

The main tool in our approach is the method of correctly aligned windows. We discuss in detail the method in Section 4 but we anticipate that the windowing method is a refinement of index theory techniques. The windowing argument shows that, if there exist sequences of windows such that the image of each window *"goes through"* the next window in the sequence, then there is an orbit that shadows the windows. In our problem, we will construct the sequence of correctly aligned windows using some of the geometric structures identified in [DdlLS00] and in [DdlLS03b].

One advantage of the method presented here is that, since the correct alignment of windows is very robust and the argument is topological, it requires significantly less detail than the more analytic arguments used in [DdlLS03b]. We do not need to use invariant objects but only approximately invariant ones. In particular, we do not need to invoke the KAM theorem as it was done in the previous papers. The application of the averaging theory provides us with approximately invariant objects which is all that the windowing method requires. The elimination of the KAM theorem from the proof lowers the number of derivatives that one needs to assume in the problem. We also note that, since the constructions are more explicit, we can produce estimates – albeit not optimal – of the diffusion time in the mechanism.

We think the introduction of the widowing mechanism opens new possibilities for the extension of the methods of the above papers. Two such possibilities that appear rather immediate are: 1) eliminating the KAM step in [DdlLS04] — this would not only lower the number of derivatives required but it would also eliminate assumption that the external perturbation has a Diophantine frequency, 2) the existence of correctly aligned windows can be verified numerically by finite precision calculations even in systems far from integrable. We hope to come to these problems in future work.

Notation. We will denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$, the sets of all reals, integers, and positive integers, respectively. We will say that a quantity is of order ε^p, where $\varepsilon > 0$ and $p \in \mathbb{N}$, if it is between $C_1\varepsilon^p$ and $C_2\varepsilon^p$, for some $0 < C_1 < C_2$ independent of ε. To simplify notation, we will often omit the constant C when we will refer to quantities of the type $C\varepsilon^p$.

2. Description of the mechanisms in [DdlLS00, DdlLS03b, DdlLS03a]

In this section, we describe informally the problems considered in [DdlLS00, DdlLS03b, DdlLS03a]. This will serve as motivation for the set up described precisely in the subsequent Section 3. The present section is only motivational and

it is not used in subsequent sections, which only use the more formal set up in Section 3.

2.1. **Orbits of unbounded energy in periodic perturbations of geodesic flows.** The paper [DdlLS00] considers geodesic flows in $\mathbb{T}^2$ perturbed by a periodic potential. This is generalized in [DdlLS04] to geodesic flows in a any manifold perturbed by a quasi-periodic potential. We will follow the presentation of the later paper since it is more general. We refer to Section 1.1 of [DdlLS04] for more details on the model.

Let M be a d-dimensional manifold with a generic metric (Riemannian, Finsler, or Lorentz). We can describe the geodesic flow as a Hamiltonian flow on the $(2d)$-dimensional exact symplectic manifold T^*M. The Hamiltonian is just $H = \frac{1}{2}|p|^2$, where the norm $|\cdot|$ refers to the metric. The $(2d-1)$-dimensional energy surfaces $\Sigma_E = \{x \in T^*M \,|\, H(x) = E\}$ are invariant under the geodesic flow.

The two assumptions of [DdlLS03a] about the geodesic flow (later there will be another assumption about the potential) are:

A1: There exists a periodic orbit $\hat{\Lambda}$ which is hyperbolic (in the sense of dynamical systems) for the geodesic flow on the unit energy surface Σ_1.

A2: There exists an orbit

$$\hat{\Gamma} = \{\gamma(t) \,|\, t \in \mathbb{R}\} \subset \Sigma_1,$$

which is a transverse homoclinic orbit to $\hat{\Lambda}$.

If we denote by $W^{\mathrm{u}}_{\hat{\Lambda}}$, $W^{\mathrm{s}}_{\hat{\Lambda}}$ the usual stable and unstable manifolds in the sense of dynamical systems, this condition means that

$$\begin{aligned} &T_{\gamma(t)} W^{\mathrm{s}}_{\hat{\Lambda}} + T_{\gamma(t)} W^{\mathrm{u}}_{\hat{\Lambda}} = T_{\gamma(t)} \Sigma_1, \quad t \in \mathbb{R}, \\ &T_{\gamma(t)} W^{\mathrm{s}}_{\hat{\Lambda}} \cap T_{\gamma(t)} W^{\mathrm{u}}_{\hat{\Lambda}} = T_{\gamma(t)} \hat{\Gamma}, \quad t \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Note that the transversality conditions above imply that $\hat{\Gamma}$ is a locally unique intersection between the stable and unstable manifolds of $\hat{\Lambda}$.

The geodesic flow satisfies some scaling properties, so that the flow in different energy surfaces is just a rescaling (of the time and the momenta) of the flow on the unit energy surface. In particular, in any energy surface Σ_E with $E > 0$ we can find periodic orbits $\hat{\Lambda}_E$ and homoclinic orbits $\hat{\Gamma}_E$ which are rescalings of the orbits in the unit energy surface.

We fix E_0 large enough and consider

$$\begin{aligned} \Lambda &= \bigcup_{E \geq E_0} \hat{\Lambda}_E, \\ \Gamma &= \bigcup_{E \geq E_0} \hat{\Gamma}_E. \end{aligned} \tag{2.2}$$

Both Λ and Γ are invariant under the geodesic flow.

Furthermore, we have that Λ is normally hyperbolic in the sense of [Fen72, HPS77]. (There are a few technicalities that arise because Λ is not compact and because the flow is not bounded on it. They are dealt with in Appendix A of [DdlLS04].) We can associate to Λ its stable and unstable manifolds. Moreover, Λ and Γ satisfy relationships analogous to (2.1).

Since we consider periodic perturbations of the system, it is convenient to introduce an extra angle variable t, which moves at a constant rate of 1, and so we

extend the phase space to $T^*M \times \mathbb{T}^1$. We denote by $\tilde{\Lambda} = \Lambda \times \mathbb{T}^1$ and $\tilde{\Gamma} = \Gamma \times \mathbb{T}^1$ the corresponding objects in the extended phase space, on which we define the Hamiltonian function

$$H(p,q,t) = \frac{1}{2}|p|^2 + V(q,t), \tag{2.3}$$

where $V : M \times \mathbb{T}^1 \to \mathbb{R}$ is a potential function satisfying $V(q,t+1) = V(q,t)$. (The extended phase space is not symplectic — it has odd dimension — and so the energy function defined above is not a Hamiltonian function. In order to be able to use the Hamiltonian formalism, one considers the symplectic extended phase space $T^*M \times \mathbb{R} \times \mathbb{T}^1$, where a is a real variable symplectically conjugate to t, and defines the Hamiltonian $H(p,q,a,t) = a + \frac{1}{2}|p|^2 + V(q,t)$. The new variable a has no dynamical role, and the restriction of the Hamiltonian flow to each of the manifolds $a = \text{const.}$ is identical to the flow of $\frac{1}{2}|p|^2 + V(q,t)$ in the extended phase space.)

The paper [DdlLS04] considers the more general case that the potential V depends quasi-periodically on time. In this paper, we will only consider periodic perturbations. We postpone the application of the windowing method to quasi-periodic perturbations to future work. Of course, arguments of [DdlLS04] apply also to the case of periodic perturbation and, incorporate several technical improvements. So, we will often refer to [DdlLS04] with preference to [DdlLS00].

Applying a scaling transformation that transforms high energy into energies of order 1 (see [DdlLS04, Sec. 4.2]), we can write the problem for high energies as

$$H_\varepsilon(p,q,t) = \frac{1}{2}|p|^2 + \varepsilon^2 V(q,\varepsilon t), \tag{2.4}$$

which makes it clear that, for high energies, the problem with an external potential is a small and slow perturbation of the geodesic flow.

If we consider F_ε, the time $1/\varepsilon$ map for the flow corresponding to (2.4), the theory of normally hyperbolic manifolds shows that the manifold $\tilde{\Lambda}$ persists for ε sufficiently small; that is, there exists $\tilde{\Lambda}_\varepsilon$ normally hyperbolic for F_ε, for all ε sufficiently small. Also, the stable and unstable manifolds persist and the manifold $\tilde{\Gamma}$ defined as the locally unique intersection also persists; that is, there exists a transverse homoclinic orbit $\tilde{\Gamma}_\varepsilon$ to $\tilde{\Lambda}_\varepsilon$ for F_ε, for all ε sufficiently small. (See Appendix A of [DdlLS04] for a very detailed proof that takes into account technicalities such as that the manifold $\tilde{\Lambda}$ is not compact.)

The map $f_\varepsilon = F_\varepsilon|_{\tilde{\Lambda}_\varepsilon}$ is an exact symplectic map which satisfies a twist condition. The map f_0 is just a standard twist map. Using the fact that the perturbation is slow, in [DdlLS00] it is shown that one can use an averaging method and establish that, in some coordinates, the system can be reduced to a standard twist plus an error of order ε^3. Applying the KAM theorem to this averaged system, [DdlLS03a, DdlLS00] show that there are KAM tori with gaps of order not larger than $\varepsilon^{3/2}$. These KAM tori are very close to level sets of the averaged energy. (One of the goals of this paper is to eliminate the KAM step).

Remark 2.1. Note that, in this model, the external perturbation is slow and, therefore, there are no small divisors in the averaging method. The averaging method could be repeated to any order needed. So that, in this model the gaps are very

small. Hence, it is very similar to the model in [Arn64] and does not present the large gap model.

To the intersection $\tilde{\Gamma}_\varepsilon$ we can associate a scattering map S_ε from $\tilde{\Lambda}_\varepsilon$ to $\tilde{\Lambda}_\varepsilon$ defined by

$$S_\varepsilon(x_-) = x_+ \iff \exists z \in \Gamma, \quad \operatorname{dist}(F_\varepsilon^n(x_\pm), F_\varepsilon^n(z)) \leq C\lambda^n \quad \text{as} \quad n \to \pm\infty. \tag{2.5}$$

That is, $S_\varepsilon(x_-) = x_+ \iff W^s_{x_+} \cap W^u_{x_-} \cap \Gamma_\varepsilon \neq \emptyset$.

The scattering map can be computed perturbatively. In [DdlLS00, Sec. 4.4] the perturbation theory of the scattering map is compared with the averaged energy. The conclusion of the calculations there is that, under explicit conditions on V — which hold for generic potentials V — and given a level set of the averaged energy $\mathcal{L}_h$, with h sufficiently large, we have

$$S_\varepsilon(\mathcal{L}_h) \pitchfork_{\tilde{\Lambda}_\varepsilon} \mathcal{L}_{h'}$$

for all $h' \in [h - A\varepsilon, h + A\varepsilon]$ with $A > 0$.

The conclusion drawn in [DdlLS04, Sec. 4.4] is that there is an abundance of KAM tori $\mathcal{T}_i$ — close to the energy surfaces with good rotation numbers — such that $S_\varepsilon(\mathcal{T}_i) \pitchfork_{\tilde{\Lambda}_\varepsilon} \mathcal{L}_{\mathcal{T}_{i+1}}$. From this, in [DdlLS04, Sec. 4.5], it is shown that there are transition chains and orbits following them. Actually, the results of [DdlLS04, Sec. 4.5] are formulated in terms of *transition paths*, which is a somewhat more precise description of the orbits that the customary transition chains. The difference is that the transition paths specify the tori visited and the connecting orbits followed, whereas the usual transition chains just specify the orbits visited.

In this paper, for periodic perturbations, we eliminate the use of the KAM theorem and the study in [DdlLS04, Sec. 4.4, 4.5]. The description of the constructed orbits includes also a specification of the transition path. Some of the arguments in [DdlLS04, Sec. 4.5] were non-constructive point set topology arguments and this caused that there were no estimates on the time of the orbits. In this paper we produce some explicit estimates of the diffusion time for the orbits we construct.

2.2. **Overcoming the large gap problem.** One of the problems in the perturbative verification of the Arnol'd instability is that the main ingredient in the proof is the existence of a chain of whiskered tori. In the example in [Arn64], all the whiskered tori persist because the perturbation is cleverly chosen. In a generic perturbation, one expects that there are gaps of size $O(\varepsilon^{1/2})$ between the whiskered tori. On the other hand, the oscillation of the manifolds that can be controlled by perturbative methods is only $O(\varepsilon)$. A very lucid discussion of these problems appears in [Moe96].

The large gap problem has been overcome in [DdlLS03b] by geometric methods and in [CY02] by variational methods for convex systems (we think that the diffusing orbits of these two papers are presumably related). We also note that in the models considered in those papers, there are other mechanisms of instability [dlL04, Tre04].

The methods here are designed to take the place of some of the arguments in [DdlLS03b, DdlLS03a] and to produce estimates of the time used for the orbits produced. The diffusion time is presumably very different from the orbits constructed using other mechanisms.

For a more detailed overview of the mechanism we discuss here, we refer to [DdlLS03b],[DdlLS03a, Chapter 2].

The model can be considered as an approximation on the behavior near a simple resonance. It was already considered in [HM82] – which however did not consider the problem of crossing the gaps of size $O(\varepsilon^{1/2})$ in the foliation of tori – is:

$$\begin{aligned} H_\varepsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \\ &= P_\pm(p,q) + \frac{1}{2}I^2 + \varepsilon h(p,q,I,\varphi,t;\varepsilon) \end{aligned} \tag{2.6}$$

where

$$P_\pm(p,q) = \pm(\frac{1}{2}p^2 + V(q)) \tag{2.7}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the *pendulum*.

We make the following assumptions:

B1: The potential V has a non-degenerate maximum, which we will set, for convenience, at 0. That is, we will assume that $V'(0) = 0$, $V''(0) < 0$. We denote by $(p_0(t), q_0(t))$ an orbit of the pendulum $\pm(\frac{1}{2}p^2 + V(q))$ homoclinic to $(0,0)$.

B2: The perturbation h is a trigonometric polynomial in the angle variables q, φ, t. This ensures that there exists a finite number of resonances.

This assumption was made in [DdlLS03b, DdlLS03a] for convenience, however this is not essential and can be substituted by an assumption that the function h is differentiable enough [Hug].

B3: The Melnikov potential associated to h satisfies certain non-degeneracy conditions.

These conditions imply that the stable and unstable manifolds of the set $p = 0$, $q = 0$, have a transverse intersection for $0 < |\varepsilon| < \varepsilon^* \ll 1$ and that the transverse intersection is a smooth family for $|\varepsilon| < \varepsilon^*$ (see [DdlLS03b]).

More precisely, we consider the Poincaré function, also called Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0) mentioned in **B1**):

$$\begin{aligned} \mathcal{L}(I,\phi,s) = -\int_{-\infty}^{+\infty} \Big(& h\left(p_0(\sigma), q_0(\sigma), I, \phi + I\sigma, s+\sigma; 0\right) \\ & - h(0,0,I,\phi + I\sigma, s+\sigma; 0)\Big)\, d\sigma. \end{aligned}$$

The non-degeneracy assumption is that Assume that for all (I,ϕ,s) in some open set, the map

$$\tau \in \mathbb{R} \mapsto \mathcal{L}(I, \phi - I\tau, s - \tau)$$

has a non-degenerate critical point τ which is locally given, by the implicit function theorem, in the form $\tau = \tau^*(I,\phi,s)$ with τ^* a smooth function. Assume moreover that the function

$$\frac{\partial \mathcal{L}}{\partial \phi}(I, \phi - I\tau^*, a - \tau^*)$$

is non-constant and negative (respectively, positive).

Note that the non-degeneracy conditions assumed here are true for generic perturbations h.

Remark 2.2. We note that the intersections corresponding to critical sets of the Poicaré function are the only intersections that can be continued smoothly across

$\varepsilon = 0$. It is, however known [BW95] that there are other intersections for $0 < |\varepsilon| \ll 1$. These *"secondary intersections"* could well be used to construct diffusing orbits (see [GR03]), provided that we can verify the rest of the assumptions. Even if a verification in concrete models seems difficult, it is very plausible that they hold for generic systems. As mentioned before, the only reason we concentrate in the orbits which extend smoothly across $\varepsilon = 0$ is to obtain estimates on the diffusion time.

B4: The perturbation terms h and $\partial h/\partial \varepsilon$ satisfy some non-degeneracy conditions (see [DdlLS03b, DdlLS03a]).

These assumptions amount to the fact that the behavior of the averaged system at resonances indeed has hyperbolic orbits. These assumptions are also true for generic systems of the form (2.6).

We note that for $\varepsilon = 0$ the system is just the product of a pendulum and a rotator. The energy of the pendulum and the action of the rotator are conserved.

The goal is to show that for $0 < |\varepsilon| \ll 1$ there are orbits for which I changes by a quantity which is independent of ε. For simplicity, we will just take F_ε to be the time-one map of the periodic Hamiltonian (2.6).

The geometric mechanism is based on the observation that for $\varepsilon = 0$, the manifold

$$\Lambda_0 = \{\tilde{x} \in (\mathbb{R} \times \mathbb{T})^2 \,|\, p = q = 0, I \in [I_-, I_+]\} \tag{2.8}$$

is a 2-dimensional invariant manifold which is normally hyperbolic for F_0. The stable and unstable directions correspond to the stable and unstable directions of the fixed point of the pendulum. The stable and unstable manifolds of Λ_0 agree.

For $|\varepsilon| \ll 1$, because of normal hyperbolicity, the manifold Λ_0 persists, giving rise to another manifold Λ_ε. Moreover, a first order calculation shows that, under non-degeneracy conditions for the perturbation h, when $0 < |\varepsilon| \ll 1$ the stable and unstable manifolds of Λ_ε intersect along a manifold Γ_ε that satisfies analogues of (2.1). (As a matter of fact, the calculation produces several of those manifolds, but we choose just one). This allows us to locally define the scattering map associated to Γ_ε, which is also computed in first order perturbation theory. See [DdlLS03b, Ch. 9]. It is interesting to note that, due to the symplectic structure, both the generation of intersections and the scattering map can be expressed as partial derivatives of the same Melnikov potential. This justifies the hypothesis **B3** and **B4** from above.

The motion on the invariant manifold Λ_ε is rather different than in the case considered in the previous subsection. Averaging theory shows that the system restricted to Λ_ε can be transformed up to small errors into a time independent system. There are resonances present in this system. Far away from the resonances, standard KAM theorem shows that there are closely spaced KAM tori. Near the resonances, the system can be accurately described by systems similar to a pendulum. Switching to singular action-angle variables near the resonances show that, under appropriate non-degeneracy conditions, there exist primary KAM tori and secondary KAM tori, close to the separatrices of the pendulum. These tori are closer than $O(\varepsilon^{3/2})$, and they possess stable and unstable manifolds. The hypothesis **B3** ensures that the scattering map associated to Γ_ε maps pieces of these tori transversally in Λ_ε to other tori at a distance $O(\varepsilon)$. The transversality of the scattering map on these tori reflects the transversally of the stable and unstable manifolds of the corresponding tori.

Therefore, all these objects form a transition chain, which shows the existence of diffusing pseudo-orbits. Then an obstruction argument show that there are orbits that shadow the diffusing pseudo-orbits.

3. Set up and main result

Let M be a C^k-smooth $(2n+2)$-dimensional manifold, and Λ_0 be an embedded copy of $\mathbb{R}\times\mathbb{T}$ in M, where $\mathbb{T}$ denotes the 1-dimensional torus. We describe the points in Λ_0 through a system of 'action-angle' coordinates $(I,\phi)\in\mathbb{R}\times\mathbb{T}$.

On M we consider a family of C^k-diffeomorphisms $F_\varepsilon : M\to M$, for $\varepsilon\in(-a_0,a_0)$, for some $a_0>0$. We will think of F_ε as a perturbation of the diffeomorphism $F_0 : M\to M$. The unperturbed mapping F_0 is assumed to preserve some 'energy' functional. More precisely, there exists a C^k-differentiable function $H_0 : M\to\mathbb{R}$ such that $H_0\mid_{\Lambda_0}$ depends only on the action variable I, and $H_0(F_0\mid_{\Lambda_0})=H_0$. This implies that the restriction of F_0 to Λ_0 preserves the action coordinate I. The problem of diffusion is to show that, for all sufficiently small $\varepsilon>0$, there exist orbits for which the the changes of energy are of order 1. We assume that Λ_0 is a normally hyperbolic manifold in M relative to F_0. This means that Λ_0 is F_0-invariant, the tangent space to M at each point $x\in\Lambda_0$ splits into

$$T_xM = T_x\Lambda_0\oplus E^s_x(0)\oplus E^u_x(0),$$

with $E^s_x(0)$ and $E^u_x(0)$ n-dimensional vector subspaces of T_xM, and there exist

$$0<\lambda_-<\lambda_+<\lambda_1<1<\mu_1<\mu_-<\mu_+$$

such that the following growth rate conditions are satisfied

$$\begin{aligned}\lambda_- <&\|DF_{0|E^s_x(0)}\|<\lambda_+,\\ \mu_- <&\|DF_{0|E^u_x(0)}\|<\mu_+,\\ \lambda_1 <&\|DF_{0|E^c_x}\|<\mu_1.\end{aligned}\tag{3.1}$$

The (strong) stable and unstable manifolds $W^{\mathrm{s}}_x(0)$ and $W^{\mathrm{u}}_x(0)$, respectively, are defined at every point $x\in\Lambda_0$ and they are C^k-smooth. In general, the stable and unstable spaces $E^s_x(0), E^u_x(0)$, and the stable and unstable manifolds $W^{\mathrm{s}}_x(0)$ and $W^{\mathrm{u}}_x(0)$ vary only Hölder continuously with respect to $x\in\Lambda_0$. The stable and unstable manifolds vary C^l-smoothly with x provided the contraction in the stable direction is stronger than the contraction within the center. More precisely, if there exist $\alpha,\beta>1$ such that $\|DF_0^{-1}{}_{|E^u(0)\oplus E^c(0)}\|<\alpha$, $\|DF_{0|E^s(0)\oplus E^c(0)}\|<\beta$, and $\lambda_+\alpha^l<1$, $\mu_-^{-1}\beta^l<1$, then the stable and unstable manifolds vary C^l-smoothly with x.

Note that the conditions in (3.1) depend on the metric chosen, but that the conclusions of regularity do not. Also, one can consider iterates of the map and the hypothesis in (3.1) can be significantly better. In our situation, since the restriction of F to Λ_0 preserves the action coordinate I, one can chose a metric (See e.g. [FdlL92] for explicit formulas) in the invariant manifold Λ in such a way that

$$\|DF_{\varepsilon|T\Lambda_\varepsilon}\|<1+O(\varepsilon^{1/2}).$$

Therefore, the stable and unstable manifolds are as smooth as desired, provided that ε is sufficiently small.

The standard theory of normally hyperbolic systems guarantees that there exist a normally hyperbolic manifold Λ_ε in M relative to F_ε, for all ε sufficiently small.

We describe each point of Λ_ε by a pair of coordinates $(I_\varepsilon, \phi_\varepsilon) \in \mathbb{R} \times \mathbb{T}$. We assume that a_0 is small enough such that Λ_ε is normally hyperbolic for all $\varepsilon \in (-a_0, a_0)$. It means that for each ε, the set Λ_ε is F_ε-invariant, and

$$T_x M = T_x \Lambda_\varepsilon \oplus E^s_x(\varepsilon) \oplus E^u_x(\varepsilon),$$

at each $x \in \Lambda_\varepsilon$, for some n-dimensional vector subspaces $E^s_x(\varepsilon)$ and $E^u_x(\varepsilon)$ of TM. We assume that a_0 is small enough such that, for all $\varepsilon \in (-a_0, a_0)$, we have $\lambda_- < \|DF_{\varepsilon|E^s_x(\varepsilon)}\| < \lambda_+$, $\mu_- < \|DF_{\varepsilon|E^u_x}\| < \mu_+$, and $\lambda_1 < \|DF_{\varepsilon|E^c_x}\| < \mu_1$. A standard reference for normal hyperbolicity theory is [HPS77].

For all $\varepsilon \in (-a_0, a_0)$, there exist stable and unstable manifolds of Λ_ε

$$W^{\mathrm{s}}_{\Lambda_\varepsilon} = \bigcup_{x \in \Lambda_\varepsilon} W^{\mathrm{s}}_x(\varepsilon), \quad W^{\mathrm{u}}_{\Lambda_\varepsilon} = \bigcup_{x \in \Lambda_\varepsilon} W^{\mathrm{u}}_x(\varepsilon).$$

These manifolds are of dimension $n+2$ and they are C^l-differentiable, for some $l \geq 1$. As it is standard in nornal hyperbolicity theory, the points in W^{s}_x characterized by the fact that their orbits approach the orbit of x at an exponential rate with the rate bounded from below. Similarly for the other manifolds.

We assume that $W^{\mathrm{s}}_{\Lambda_\varepsilon}$ and $W^{\mathrm{u}}_{\Lambda_\varepsilon}$ have a transverse intersection along a 2-dimensional manifold Γ_ε, for all $\varepsilon > 0$ in $(-a_0, a_0)$. For each $z \in \Gamma_\varepsilon$, we have

$$T_z \Gamma_\varepsilon = T_z(W^{\mathrm{s}}_{\Lambda_\varepsilon}) \cap T_z(W^{\mathrm{u}}_{\Lambda_\varepsilon}), \quad \text{and} \tag{3.2}$$

$$T_z M = T_z \Gamma_\varepsilon \oplus T_z(W^{\mathrm{s}}_x(\varepsilon)) \oplus T_z(W^{\mathrm{u}}_y(\varepsilon)), \tag{3.3}$$

for some $x = x(z) \in \Lambda_\varepsilon$ and some $y = y(z) \in \Lambda_\varepsilon$, where x and y are locally uniquely defined.

Under the assumption above of existence of a transversal intersection, there are two naturally defined dynamics on Λ_ε: an *inner dynamics*, defined by the restriction of F_ε to Λ_ε — and which we will denote by f_ε — and an *outer dynamics*, defined by the homoclinic excursions to Λ_ε. We will refer to f_ε as the *inner map*. The dynamics of the homoclinic excursions is described by a *scattering map (outer map)* S_ε associated to the family of homoclinic orbits Γ_ε, and defined as follows:

- Let z be the locally unique intersection of $W^{\mathrm{u}}_x(\varepsilon)$ with Γ_ε;
- There exists a unique point $y(x) \in \Lambda_\varepsilon$ such that $z \in W^{\mathrm{s}}_\varepsilon(\Lambda_\varepsilon)$;
- Set $S_\varepsilon(x) = y$.

That is, we say that $S_\varepsilon(x_-) = x_+$ when $W^u_{x_-} \pitchfork W^s_{x_+}$ and the – locally unique – intersection ranges over Γ_ε. The fact that the scattering map is locally well defined is a consequence of the implicit function theorem and the smoothness of the foliation. Note that, of course, the intersections are not unique, so that it is quite important that the local uniqueness is taken into account. Since the scattering map is defined only through local considerations, it is important to remark that there is some global uniqueness. By the properties of the foliation $W^{\mathrm{s}}_{\Lambda_\varepsilon}(\varepsilon)$ by stable submanifolds $W^{\mathrm{s}}_y(\varepsilon)$ with $y \in \Lambda_\varepsilon$, we have $z \in W^{\mathrm{s}}_y(\varepsilon) \cap W^{\mathrm{s}}_{y'}(\varepsilon)$ implies $y = y'$, hence S_ε is unique. On the other hand, we note that, as pointed out in [DdlLS00], the point z, even if locally unique, it may present some nontrivial monodromy. This happens e.g. in the Geodesic Flow Model.

Notice that, since the scattering map is defined through the implicit function theorem, if the homoclinic intersection depends differentiably on parameters, the scattering map will depend differentiably on parameters too. A particularly interesting case – which appears in the Large Gap Model considered in [DdlLS03b, DdlLS03a]

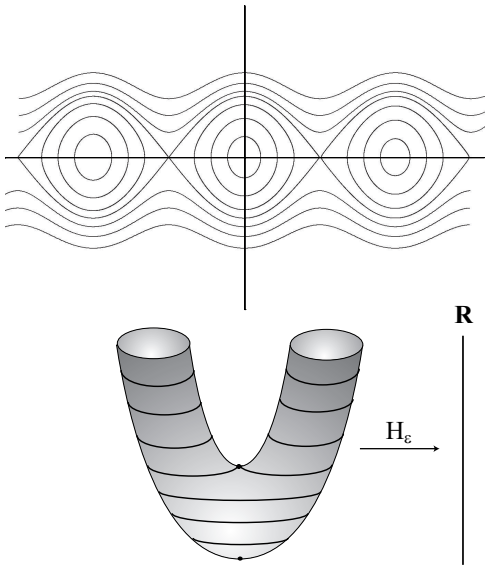

FIGURE 1. Foliation with regular and singular leaves.

– is when the intersection depends smoothly on the parameter ε and is transversal for $0 < |\varepsilon|$, but it is not transversal for $\varepsilon = 0$. In such a case, the scattering map has a well defined limit as $\varepsilon \to 0$ and can be extended to a smooth family of maps.

Alternatively, we can describe the scattering map as a composition of two homoclinic maps. Consider the maps Φ^u_ε from a neighborhood of x in Λ_ε to a neighborhood of z in Γ_ε along the unstable manifold, and Φ^u_ε from a neighborhood of x in Λ_ε to z in Γ_ε along the stable manifold. Then,

$$S_\varepsilon = (\Phi^{\mathrm{s}}_\varepsilon)^{-1} \circ \Phi^{\mathrm{u}}_\varepsilon.$$

The scattering map was introduced in [DdlLS00] and [Gar00].

In the sequel, we will make a series of assumptions on the inner and the outer dynamics.

Notice that some of the assumptions involve just the behavior for a fixed ε and some assumptions involve the behavior of a family of intersections. To establish the existence of diffusing orbits for a fixed ε, we will only use the assumptions for a fixed ε. The assumptions on the behavior of the family enter only when we want to obtain bounds for the diffusion time as a function of ε in Section 8.

C1: We will assume that there is a homoclinic intersection for Λ_ε depending differentiably on the parameter for $|\varepsilon| < \varepsilon^*$. The homoclinic intersection is transversal for $0 < |\varepsilon| < \varepsilon^*$.

When the homoclinic intersection is transverse, we can define a scattering map, depending differentiably on ε for $|\varepsilon| < \varepsilon^*$.

We assume that the scattering map $S_\varepsilon : U_\varepsilon \to \Lambda_\varepsilon$ is defined on some open set $U_\varepsilon \subseteq \Lambda_\varepsilon$.

If we take a parameterization $\mathcal{F}_\varepsilon \Lambda_0 = \Lambda_\varepsilon$ as in Appendix A of [DdlLS00], we obtain that $U_\varepsilon = \mathcal{F}_\varepsilon(U_0)$. In particular, the size of the domain of S_ε is bounded from below.

In order to estimate the times of diffusion, we will need to make some quantitative assumptions on the values of the angle of the intersections.

C1a (For the "Geodesic Flow Model", Section 2.1) As ε goes to zero, the angle between $T_{p_\varepsilon}W^s_{\Lambda_\varepsilon}$ and $T_{p_\varepsilon}W^u_{\Lambda_\varepsilon}$ remains bounded from below.

C1b (For the "Large Gap Model", Section 2.2)
Denoting by $C(\varepsilon)$ the angle between $T_{p_\varepsilon}W^s_{\Lambda_\varepsilon}$ and $T_{p_\varepsilon}W^u_{\Lambda_\varepsilon}$ we have:

$$K^{-1}\varepsilon < C(\varepsilon) < K\varepsilon$$

for some $K > 0$.

C2: We assume that each f_ε satisfies a *uniform twist condition* on Λ_ε, that is, there exists $T > 0$ such that

$$\frac{\partial(\Pi_\phi f_\varepsilon)}{\partial I_\varepsilon} > T \tag{3.4}$$

for all $\varepsilon > 0$ in $(-a_0, a_0)$, where $\Pi_\phi : \Lambda_\varepsilon \to \mathbb{R}$ represents the projection into the ϕ-coordinate.

Later on, we will make another twist assumption. (See **C5**.)

C3: We assume that for each $\varepsilon > 0$ in $(-a_0, a_0)$, there exists a C^k-differentiable Morse function $H_\varepsilon : M \to \mathbb{R}$ such that

$$||H_\varepsilon \mid_{\Lambda_\varepsilon} (f_\varepsilon) - H_\varepsilon \mid_{\Lambda_\varepsilon} ||_{C^k} \leq C\varepsilon^2.$$

In the sequel, with an abuse of notation, we will denote $H_\varepsilon \mid_{\Lambda_\varepsilon}$ also by H_ε. Each function F_ε determines a foliation of Λ_ε by leaves of the form $\mathcal{L}^h_\varepsilon = H_\varepsilon^{-1}(\mathrm{h})$.

We will make some extra hypothesis on the behavior of this Morse function which capture the behavior of the models.

C3a: (For the "Geodesic Flow Model", Section 2.1)
We assume that $H_\varepsilon(I_\varepsilon, \phi_\varepsilon) = \frac{1}{2}I^2 + \varepsilon\widehat{H}(I_\varepsilon, \phi_\varepsilon)$ where $\widehat{H}(I_\varepsilon, \phi_\varepsilon)$ is a uniformly bounded function. Hence, the leaves are homeomorphic to circles which are small perturbations of the circles $I =$ constant.

C3b: (For the "Large Gap Model". See Section 2.2.)
We assume that H_ε has a finite number of critical points that are degenerate of the first order and a finite number of critical points that are degenerate of the second order (a critical point c_ε of H_ε is said to be of order j if $D^2H_\varepsilon(c_\varepsilon)$ has one eigenvalue of order 1 and another eigenvalue which is of order ε^j). The regular values h of H_ε correspond to regular leaves Λ^h_ε in Λ_ε, while the critical values h of H_ε correspond to regular leaves Λ^h_ε in Λ_ε. See Figure 1. Moreover, we will assume that the leaves that contain critical points of an order do not contain critical points of another order. A leaf may contain several critical points.

C4: We assume that the *scattering map is transverse to leaves along leaves*, that is, there is an interval $[h^-, h^+]$ with $\mathcal{L}^h_\varepsilon \cap U_\varepsilon \neq \emptyset$, and there is a constant $c > 0$ such that, for each $\varepsilon > 0$, and for each regular value h of H_ε in $[h^-, h^+]$, there exists an interval $[k^-, k^+]$ depending on ε and h, with

$$c^{-1}\varepsilon < k^- < k^+ < c\varepsilon, \tag{3.5}$$

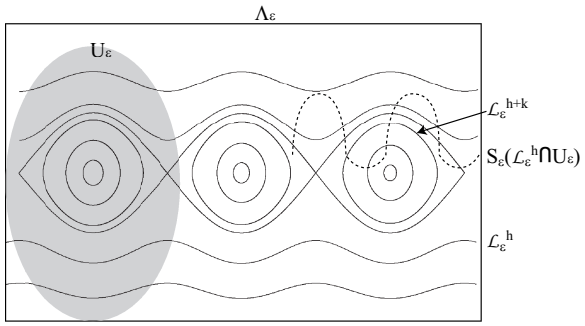

FIGURE 2. The effect of the scattering map on the leaves of the foliation.

such that, for each $k \in [k^-, k^+]$ we have that $h + k$ is a regular value for H_ε, and

$$S_\varepsilon(\mathcal{L}_\varepsilon^h \cap U_\varepsilon) \cap \mathcal{L}_\varepsilon^{h+k} \neq \emptyset \quad \text{and} \quad S_\varepsilon(\mathcal{L}_\varepsilon^h \cap U_\varepsilon) \pitchfork \mathcal{L}_\varepsilon^{h+k}. \tag{3.6}$$

Condition **C4** says that the scattering map makes jumps of order ε in the action direction, and it maps leaves transversally across leaves. See Figure 2.

C5: The approximately conserved quantity obtained in **C3** admits level sets which are circles. Some of them are homotopically non-trivial and some of them – in the "Large Gap Model" – are homotopically trivial.

In the neighborhood of these circles, we can introduce action angle coordinates in such a way that these circles correspond to constant action.

We will denote these coordinates around the level sets of the conserved quantity as r – for radius – or a – for angle. Note that there are two types of non-critical level sets. Some of them are contractible to points and others are not.

The non-contractible level sets correspond to the KAM tori while the contractible ones correspond to the secondary KAM tori.

We will assume that except for the critical sets, it is possible to choose these coordinates in such a way that it satisfies a twist condition similar to **C2**.

We note that in the "Geodesic Flow Model", the twist condition follows because the conserved quantity is the averaged energy. Hence, r, a are very similar to the action angle variables. In the "Large Gap Model", we see that the twist condition for the circles non-homotopic to a point can be verified in the action-angle variables. For the circles that are homotopic to identity, the twist condition is actually very large. The explicit computations are done in detail in [DdlLS03a, Chapter 8.5].

The main result of this paper is the following:

Theorem 3.1. *In the conditions* **C1-C5** *above.*
Let $\varepsilon \in (-a_0, a_0)$ for some $a_0 > 0$ sufficiently small. Let

$$h_0 = h^-, h_1, \ldots, h_n = h^+$$

be a finite sequence of regular values of H_ε, such that

$$\mathcal{L}^{h_i} \cap U_\varepsilon \neq \emptyset, \text{ for all } i = 0, \ldots, n.$$

For each $\eta > 0$, there exists an orbit $z_0, z_1, \ldots, z_n$ of the system, with

$$z_{i+1} = F_\varepsilon^{n_i}(z_i),$$

for some $n_i > 0$, and

$$d(z_i, \mathcal{L}_\varepsilon^{h_i}) < \eta,$$

as $i \in \{0, 1, \ldots, n\}$.

Moreover, for every loop $h_0, h_1, \ldots, h_n$ of regular values of H_ε in $[h^-, h^+]$, with $h_n = h_0$, there exists a closed orbit $z_0, z_1, \ldots, z_n$ of the system, with

$$z_N = z_0 \text{ and } z_{i+1} = F_\varepsilon^{n_i}(z_i)$$

for some $n_i > 0$, and

$$d(z_i, \mathcal{L}_\varepsilon^{h_i}) < \eta,$$

as $i \in \{0, 1, \ldots, n\}$.

This result applies straightforwardly to the models discussed in Subsection 2.1 and Subsection 2.2. We will provide an estimate for the diffusion time and discuss the assumptions on differentiability for these models in Section 8.

Remark 3.2. Using the same argument as in the proof of Theorem 1.2 in [GR03, GR04], we can prove the existence of symbolic dynamics over finitely many symbols in a small neighborhood of any finite collection of leaves. However, providing symbolic dynamics over infinitely many symbols in a small neighborhood of an infinite collection of leaves (as it is done in [DdlLS00, DdlLS03b]) appears to be challenging for our approach.

Remark 3.3. Note that the set up included here does not use at all that the stable and unstable manifolds of Λ_ε are one-dimensional.

The paper [DdlLS04] considers the geodesic flow model in any number of dimensions. The paper [DdlLS03a] uses the 1-dimensionality of the stable manifolds only in some of the most explicit calculations of the Melnikov function.

On the other hand, the fact that the manifold Λ_ε is 2-dimensional is used in several places. Since diffusion is easier to achieve the larger the dimension is, it seems that this restriction does not belong and should be removed.

4. The windowing method

The method of correctly aligned windows, as presented in [ZG04, GZ04, GR03, GR04], is an extension of the method introduced in [Eas81].

In this section, we collect some definitions and prove the results we will need.

4.1. **Correctly aligned windows.**

A window is a homeomorphic copy of a multi-dimensional rectangle, with a distinguished C^0-coordinate system.

Definition 4.1. An (n_1, n_2)-window in an n-dimensional manifold M is a compact subset W of M together with a parametrization given by a homeomorphism $\chi^W : [0,1]^{n_1} \times [0,1]^{n_2} \to W$, where $n_1 + n_2 = n$. The set $W^- = \chi^W(\partial[0,1]^u \times [0,1]^s)$ is called the 'exit set' and the set $W^+ = \chi^W([0,1]^u \times \partial[0,1]^s)$ is called the 'entry set' of W.

Here ∂ denotes the topological boundary of a set. Since χ^W is merely a homeomorphism, in the above definition one can always replace the rectangle $[0,1]^{n_1} \times [0,1]^{n_2}$

by a a homeomorphic copy of it. In the sequel, we shall adopt the following notation convention:

$$\begin{aligned} W_\chi &= (\chi^W)^{-1}(W), \\ (W^-)_\chi &= (\chi^W)^{-1}(W^-), \\ (W^+)_\chi &= (\chi^W)^{-1}(W^+). \end{aligned}$$

Two windows are correctly aligned under some map, provided that the image of the first window under the map can be stretched out and flattened down to a disk crossing the second window all the way through its exist set, so that the induced map on that disk has non-zero degree.

Definition 4.2. Let W_1, W_2 be two (n_1, n_2)-windows in M, and f be a continuous map on M with $f(\mathrm{im}(\chi^{W_1})) \subseteq \mathrm{im}(h^{W_2})$. Denote $f_\chi = (\chi^{W_2})^{-1} \circ f \circ \chi^{W_1}$. We say that W_1 is correctly aligned with W_2 under f provided that the following conditions are satisfied:

(i)

$$\begin{aligned} f_\chi((W_1^-)_\chi) \cap (W_2)_\chi = \emptyset, \\ f_\chi((W_1)_\chi) \cap ((W_2)_\chi^+ = \emptyset. \end{aligned}$$

(ii) there exists a point $y_0 \in [0,1]^{n_2}$ such that
  (ii.a) $f_\chi([0,1]^{n_1} \times \{y_0\}) \subseteq \mathrm{int}\,([0,1]^{n_1} \times [0,1]^{n_2} \cup (\mathbb{R}^{n_1} \setminus (0,1)^{n_1}) \times \mathbb{R}^{n_2})$,
  (ii.b) If $n_1 = 0$, then $f_\chi((W_1)_\chi) \subseteq \mathrm{int}((W_2)_\chi)$. If $n_1 > 0$, then the map $A_{y_0} : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ defined by $A_{y_0}(x) = \pi_{n_1}(f_\chi(x, y_0))$ satisfies

$$\begin{aligned} A_{y_0}(\partial[0,1]^{n_1}) \subseteq \mathbb{R}^{n_1} \setminus [0,1]^{n_1}, \\ \deg(A_{y_0}, 0) \neq 0. \end{aligned}$$

Here π_x denotes the projection $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to x \in \mathbb{R}^{n_1}$. Note that this definition is equivalent to the definition for correctly aligned windows provided in [ZG04, GZ04]. See Figure 3. Here $\deg(A, 0)$ denotes the Brouwer degree of the mapping A at the point 0. The number $\deg(A, 0)$ will be referred as the 'degree of alignment'. The definition and properties of the Brouwer degree can be found in [Hir94]. Below we state a property of the Brouwer degree which will be needed later.

Proposition 4.3. *Let $U_i \subset \mathbb{R}^{k_i}$ be open sets, and $f_i : U_i \to \mathbb{R}^{k_i}$ be continuous mappings, for $i = 1, 2$. Let $z_i \in \mathbb{R}^{k_i}$, for $i = 1, 2$. The map $(f_1, f_2) : \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ is given by $(f_1, f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. We have*

$$\deg((f_1, f_2), U_1 \times U_2, (z_1, z_2)) = \deg(f_1, U_1, z_1) \cdot \deg(f_2, U_2, z_2), \tag{4.1}$$

whenever the right hand side is defined.

Proposition 4.4, Theorem 4.5, Theorem 4.6, and Corollary 4.7, stated below, are proved in [ZG04, GZ04].

The following result is a simple criterion for correct alignment of windows.

Proposition 4.4. *Let W_1 and W_2 be (n_1, n_2)-windows, and f be a continuous map on M with $f(im(\chi^{W_1})) \subseteq im(\chi^{W_2})$. Assume that the following conditions are satisfied:*

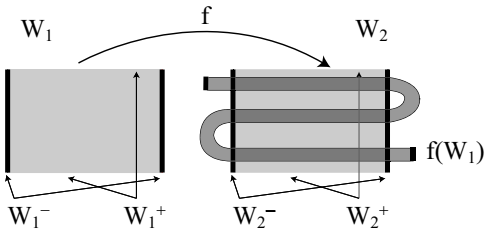

FIGURE 3. A pair of correctly aligned windows in the plane.

(i) *There exists a continuous homotopy* $h : [0,1] \times W_\chi \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, *such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_\chi, \\ h([0,1], (W_1^-)_\chi) \cap (W_2)_\chi &= \emptyset, \\ h([0,1], (W_1)_\chi) \cap (W_2^+)_\chi &= \emptyset, \end{aligned}$$

(ii) *there exists a linear map* $A : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$, *such that*
(ii.a) $h_1(x,y) = (Ax, 0)$ *for* $x \in [0,1]^{n_1}$ *and* $y \in [0,1]^{n_2}$,
(ii.b) $A(\partial[0,1]^u) \subset \mathbb{R}^{n_1} \setminus [0,1]^{n_1}$.

Then W_1 *is correctly aligned with* W_2 *under* f.

In the sequel, if W_1 and W_2 are as in the above proposition, we will say that W_1 is *linearly correctly aligned* with W_2 under f.

The following results says that correct alignment of windows is robust.

Theorem 4.5. *Suppose that the* (n_1, n_2)*-window* W_1 *correctly aligns with the* (n_1, n_2)*-window* W_2 *under a continuous map* f *on* M. *There exists* $\varepsilon > 0$ *such that for continuous map* g *on* M *for which* g_χ *is* ε*-close to* f_χ *in the compact-open topology,* W_1 *correctly aligns with* W_2 *under* g.

The following result is sometimes referred in the mathematical folklore as: 'one can see through a sequence of correctly aligned windows'.

Theorem 4.6. *Let* W_i *be a collection of* (n_1, n_2)*-windows in* M, *where* $i \in \mathbb{Z}$ *or* $i \in \{0, \ldots, d-1\}$, *with* $d > 0$ *(in the latter case, for convenience, we let* $W_i = W_{(i \bmod d)}$ *for all* $i \in \mathbb{Z}$*). Let* f_i *be a collection of continuous maps on* M. *If* N_i *is correctly aligned with* W_{i+1}, *for all* i, *then there exists a point* $p \in N_0$ *such that*

$$f_i \circ \ldots \circ f_0(p) \in W_{i+1},$$

Moreover, if $W_{i+k} = W_i$ *for some* $k > 0$ *and all* i, *then the point* p *can be chosen so that*

$$f_{k-1} \circ \ldots \circ f_0(p) = p.$$

Corollary 4.7. *Let* $W_0, \ldots, W_{d-1}$ *be a collection of mutually disjoint* (n_1, n_2)*-windows and* f *a continuous map on* M. *Assume that for every* $i, j \in \{0, \ldots, d-1\}$, *the window* W_i *is correctly aligned with the window* W_j *under* f. *There exist a maximal* f*-invariant set* S *in* $\bigcup_{i=0,\ldots,d-1} int(W_i)$, *and a continuous surjective map* $\rho : S \to \Sigma_d$ *such that* $\rho \circ f = \sigma \circ \rho$, *and the inverse image of every periodic orbit of* σ *contains a periodic orbit of* f.

The difficulty of the application of this technique is to explicitly construct windows satisfying correct alignment relations in concrete problems.

Remark 4.8. We now describe a situation in which it is very natural to construct correctly aligned windows. Consider a pair of submanifolds with boundary, of complementary dimensions, such that one submanifold is mapped across the other in some non-trivial way. One can carefully fatten up these submanifolds to full-dimensional windows in the phase space. A sufficient condition that ensures the non-trivial crossing of the windows is that the submanifolds of complementary dimensions are transverse or topologically crossing. Two immersed curves W^1 and W^2 in $\mathbb{R}^2$ have a topological crossing provided that there exist an open set $U \subseteq \mathbb{R}^2$, two embedded submanifolds with boundary $V^1 \subseteq W^1$ and $V^2 \subseteq W^2$ such that $\mathrm{bd}(V^1) \cap V^2 = \mathrm{bd}(V^2) \cap V^1 = \emptyset$, and $V^1 \cap V^2 \subseteq U$, and a coordinate system (x, y) on U such that $\mathrm{int}(V^1) = \{y = 0\}$ and V^2 contains a pair of points (x_1, y_1) and (x_2, y_2) such that $y_1 y_2 < 0$. For a definition of topological crossing in higher dimensions, see [BW95]. For an application of the correctly aligned window method in the context of topologically crossing invariant manifolds, see [GR04].

4.2. **Products of windows.** In order to construct correctly aligned windows one can use 'products' of correctly aligned windows in lower dimensions. The product of windows is not simply the Cartesian product of the images of the underlying homeomorphisms, but is the product of the multi-rectangles corresponding through the local coordinates.

Suppose that $n_1, n_2, m_1, m_2 \geq 0$ and $n_1 + n_2 + m_1 + m_2 = k$. Let $n_1 + m_1 = k_1$ and $n_2 + m_2 = k_2$. We describe points in $\mathbb{R}^k$ by their coordinates (x_1, x_2, y_1, y_2) where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $y_1 \in \mathbb{R}^{m_1}$, and $y_2 \in \mathbb{R}^{m_2}$. We denote by $\pi_{(x_1,x_2)}$, and $\pi_{(y_1,y_2)}$, the projection mapping into the (x_1, x_2), and (y_1, y_2)-coordinate subspaces, respectively.

Let W_1 and W_2 be two (k_1, k_2)-windows in M and χ^{W_1}, χ^{W_2} be their corresponding parametrizations. For each $(y_1^0, y_2^0) \in [0,1]^{m_1} \times [0,1]^{m_2}$, the mapping $\chi^{W_1^n}_{(y_1^0,y_2^0)}$ defined by

$$(x_1, x_2) \in [0,1]^{n_1} \times [0,1]^{n_2} \to \chi^{W_1^n}_{(y_1^0,y_2^0)}(x_1, x_2) := \chi^{W_1}(x_1, x_2, y_1^0, y_2^0)$$

is a homeomorphism, defining a (n_1, n_2)-window which we denote $(W_1^n)_{(y_1^0,y_2^0)}$. The exit set of this window is defined as

$$(W_1^n)^-_{(y_1^0,y_2^0)} = \chi^{W_1^n}_{(y_1^0,y_2^0)}(\partial[0,1]^{n_1} \times [0,1]^{n_2}).$$

Similarly, for each $(x_1^0, x_2^0) \in [0,1]^{n_1} \times [0,1]^{n_2}$, the mapping $\chi^{W_1^m}_{(x_1^0,x_2^0)}$ defined by

$$(y_1, y_2) \in [0,1]^{m_1} \times [0,1]^{m_2} \to \chi^{W_1^m}_{(x_1^0,x_2^0)}(y_1, y_2) := \chi^{W_1}(x_1^0, x_2^0, y_1, y_2)$$

is a homeomorphism, defining a (m_1, m_2)-window which we denote $(W_1^m)_{(x_1^0,x_2^0)}$. The exit set of this window is defined as

$$(W_1^m)^-_{(x_1^0,x_2^0)} = \chi^{W_1^m}_{(x_1^0,x_2^0)}(\partial[0,1]^{m_1} \times [0,1]^{m_2}).$$

When we write these windows in local coordinates, we obtain

$$\begin{aligned}
((W_1^n)_{(y_1^0,y_2^0)})_\chi : &= (\chi^{W_1^n}_{(y_1^0,y_2^0)})^{-1}((W_1^n)_{(y_1^0,y_2^0)}) = [0,1]^{n_1}\times[0,1]^{n_2},\\
((W_1^n)^-_{(y_1^0,y_2^0)})_\chi : &= (\chi^{W_1^n}_{(y_1^0,y_2^0)})^{-1}((W_1^n)^-_{(y_1^0,y_2^0)}) = \partial[0,1]^{n_1}\times[0,1]^{n_2},\\
((W_1^m)_{(x_1^0,x_2^0)})_\chi : &= (\chi^{W_1^m}_{(x_1^0,x_2^0)})^{-1}((W_1^m)_{(x_1^0,x_2^0)}) = [0,1]^{m_1}\times[0,1]^{m_2},\\
((W_1^m)^-_{(x_1^0,x_2^0)})_\chi : &= (\chi^{W_1^m}_{(x_1^0,x_2^0)})^{-1}((W_1^m)^-_{(x_1^0,x_2^0)}) = \partial[0,1]^{m_1}\times[0,1]^{m_2}.
\end{aligned}$$

Thus, we can write

$$(W_1)_\chi = ((W_1^n)_{(y_1^0,y_2^0)})_\chi \times ((W_1^m)_{(x_1^0,x_2^0)})_\chi, \tag{4.2}$$

$$\begin{aligned}
(W_1^-)_\chi &= \left[((W_1^n)^-_{(y_1^0,y_2^0)})_\chi \times ((W_1^m)_{(x_1^0,x_2^0)})_\chi\right]\\
&\cup \left[((W_1^n)_{(y_1^0,y_2^0)})_\chi \times ((W_1^m)^-_{(x_1^0,x_2^0)})_\chi\right],
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
(W_1^+)_\chi &= \left[((W_1^n)^+_{(y_1^0,y_2^0)})_\chi \times ((W_1^m)_{(x_1^0,x_2^0)})_\chi\right]\\
&\cup \left[((W_1^n)_{(y_1^0,y_2^0)})_\chi \times ((W_1^m)^+_{(x_1^0,x_2^0)})_\chi\right],
\end{aligned} \tag{4.4}$$

for any $(y_1^0,y_2^0)\in[0,1]^{m_1}\times[0,1]^{m_2}$ and $(x_1^0,x_2^0)\in[0,1]^{n_1}\times[0,1]^{n_2}$.

From this point of view, we can say that the window W_1 is, in local coordinates, the product of its components $(W_1^n)_{(y_1^0,y_2^0)}$ and $(W_1^m)_{(x_1^0,x_2^0)}$. Note that the exit set of W_1 is not, in local coordinates, the product of the exit sets of $(W_1^n)_{(y_1^0,y_2^0)}$ and $(W_1^m)_{(x_1^0,x_2^0)}$.

Below, we will consider the (n_1,n_2)-window $(W_2^n)_{(0,0)}$, corresponding to the choice $(y_1^0,y_2^0)=(0,0)$, and the (m_1,m_2)-window $(W_2^m)_{(0,0)}$, corresponding to the choice $(x_1^0,x_2^0)=(0,0)$.

We now describe a situation in which the correct alignment of the windows $(W_1^n)_{(y_1^0,y_2^0)}$ and $(W_2^n)_{(0,0)}$, and of the windows $(W_1^m)_{(x_1^0,x_2^0)}$ and $(W_2^m)_{(0,0)}$, implies the correct alignment of the windows W_1 and W_2.

Proposition 4.9. *Let W_1 and W_2 be (k_1,k_2)-windows as above. Assume*

(i) *For each $(y_1^0,y_2^0)\in[0,1]^{m_1}\times[0,1]^{m_2}$, the (n_1,n_2)-window $((W_1^n)_{(y_1^0,y_2^0)})_\chi$ is linearly correctly aligned with the (n_1,n_2)-window $((W_2^n)_{(0,0)})_\chi$ under $\pi_{(x_1,x_2)}\circ f_\chi$,*

(ii) *For each $(x_1^0,x_2^0)\in[0,1]^{n_1}\times[0,1]^{n_2}$, the (m_1,m_2)-window $((W_1^m)_{(x_1^0,x_2^0)})_\chi$ is linearly correctly aligned with the (m_1,m_2)-window $((W_2^m)_{(0,0)})_c$ under $\pi_{(y_1,y_2)}\circ f_\chi$.*

Then W_1 is correctly aligned with W_2 under f.

See Figure 4.

Proof. We want to verify the conditions from Definition 4.2. We only consider the general case $n_1,m_1>0$, the other possibilities are left as an exercise to the reader.

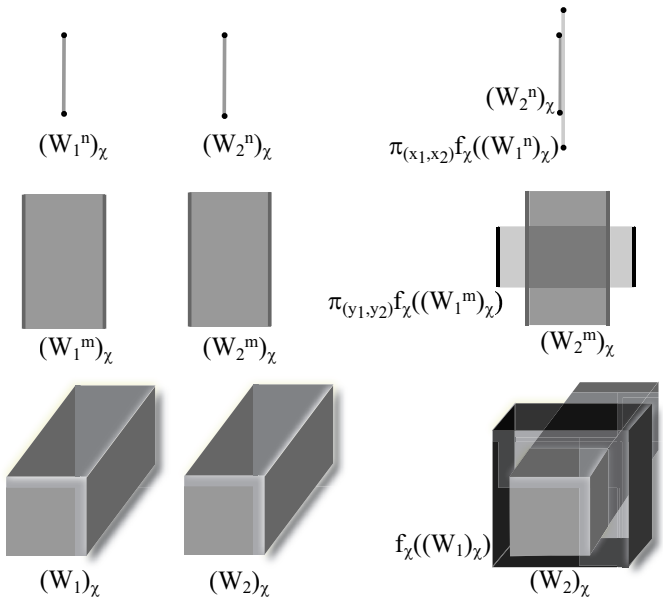

FIGURE 4. Products of correctly aligned windows. The exit set of each window is shown in darker color.

First, note that by (4.3) we have

$$(W_1^-)_\chi = \left(\bigcup_{(y_1^0,y_2^0)\in[0,1]^{m_2}\times[0,1]^{m_2}} ((W_1^n)^-_{(y_1^0,y_2^0)})_\chi \right) \bigcup \left(\bigcup_{(x_1^0,x_2^0)\in[0,1]^{n_1}\times[0,1]^{n_2}} ((W_1^m)^-_{(x_1^0,x_2^0)})_\chi \right).$$

The correct alignment of $((W_1^n)_{(y_1^0,y_2^0)})_\chi$ with $((W_2^n)_{(0,0)})_\chi$ under $\pi_{(x_1,x_2)}\circ f_\chi$, and the correct alignment of $((W_1^m)_{(x_1^0,x_2^0)})_\chi$ with $((W_2^m)_{(0,0)})_c$ under $\pi_{(y_1,y_2)}\circ f_\chi$, ensure that $f_\chi((W_1^-)_\chi)\cap(W_2)_\chi = \emptyset$. A similar argument shows that $f_\chi((W_1)_\chi)\cap(W_2^+)_\chi = \emptyset$. □

Remark 4.10. The situation described in the above proposition is quite special. It is however easier to verify provided that f_χ preserves, for example, the 'vertical fibers', that is, for any $(y_1^0, y_2^0) \in [0,1]^{m_1}\times[0,1]^{m_2}$,

$$f_\chi(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\{y_1^0\}\times\{y_2^0\}) = \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\{\hat{y}_1^0\}\times\{\hat{y}_2^0\}$$

for some $(\hat{y}_1^0, \hat{y}_2^0) \in [0,1]^{m_1}\times[0,1]^{m_2}$. In Section 6 we will apply Proposition 4.9 for a mapping that satisfies this extra condition.

5. Behavior in a neighborhood of a normally hyperbolic manifold

In this section, we introduce a system of coordinates which allows us to analyze comfortably the behavior of a system in a neighborhood of a normally hyperbolic invariant manifold Λ_ε. The ε-subscript will be omitted to simplify the notation.

We will find a C^1 system of coordinates in which the motion is a skew product of the center directions and the hyperbolic directions. Such a system will simplify significantly the construction of correctly aligned windows. Indeed, the original construction of windows in [Eas81] was done for product systems. We will also show how this system can be extended by propagating it by the map. Given a homoclinic point, since the forward orbit and the backward orbit arrive to a neighborhood of Λ, there are two ways of propagating the system of coordinates and we will need to compare them.

Systems of coordinates linearizing the behavior around a normally hyperbolic manifold have been considered in [Rob71, PS70, KP90].

5.1. **Preliminaries and notation.** Let Λ be a normally hyperbolic invariant manifold for a map F. We will not assume it is compact, we will rather assume that several functions that appear in the proof are uniformly continuous and that the derivatives are uniformly bounded and uniformly continuous. We will say that such maps are uniformly C^1 or in C^1_u. Indeed, in many standard presentations of the theory of normally hyperbolic manifolds, given one compact manifold, one considers extensions to the whole space enjoying the above uniformity properties.

By the implicit function theorem, we can identify a neighborhood of Λ with a neighborhood of the zero section of the bundle $(E^s \oplus E^u)_\Lambda$. For example, we can write all points x in a neighborhood of Λ as $x = \exp_c(v)$ where $c \in \Lambda$, $v \in E^s_c \oplus E^u_c$. For typographical simplicity, we write the coordinates as (c, s, u) where $s, u \in E^{s,u}_c$, and denote (s, u) by v.

We also write $A_s(c) = DF(c)|_{E^s_c}$, $A_u(c) = DF(c)|_{E^u_c}$ and denote $f = F|_\Lambda$.

We will use the notation c to denote a coordinate that takes values on Λ. This coordinate c will be part of the description of a neighborhood of Λ. Even if the use of the coordinate c ranging over a manifold Λ is natural, it will happen that we have to use action or angle coordinates to, in turn, describe Λ. For example, we have formulated assumption **C2** in terms of the original action-angle coordinates and assumption **C5** in terms of some other action-angle variables. Some statements (e.g. those related to the homoclinic excursions) are easier to state in the coordinates c but others are easier to state in the action-angle coordinates.

In these coordinates, recalling the invariance of the stable and unstable subspaces, the map F can be written as

$$F(c, s, u) = (f(c) + N_c(c, s, u), A_s(c)s + N_s(c, s, u), A_u(c)u + N_u(c, s, u)), \tag{5.1}$$

where $N_\sigma(c, 0, 0) = 0$, $D_s N_\sigma(c, 0, 0) = 0$, $D_u N_\sigma(c, 0, 0) = 0$ for $\sigma = c, s, u$.

Note that F is really a map on a neighborhood U of the zero section of the $E^s \oplus E^u$ bundle.

The next step is quite standard in Hartman-Grobman theorem. It is the observation that, since we are interested only on a neighborhood of the zero section, we can assume that the map is defined in the whole bundle and that N is globally uniformly C^1 and the C^1 norm is small. We just choose a bump function Ψ (a C^∞ function identically 1 on $[-1, 1]$ and identically 0 on $\mathbb{R} \setminus (-2, 2)$) and we denote $\Psi_a(t) = \Psi(at)$.

Given a map F as in (5.1), for large enough a we consider the function given in coordinates by

$$\begin{aligned}\tilde{F}(c,s,u) = \big(&f(c) + N_c(c,s,u)\Psi_a(s^2+u^2),\\ &A_s(c)s + N_s(c,s,u)\Psi_a(s^2+u^2),\\ &A_u(c)u + N_u(c,s,u)\Psi_a(s^2+u^2)\big).\end{aligned} \tag{5.2}$$

The map $\tilde{F}$ in (5.2) is, strictly speaking, only defined in the domain of the map N. On the other hand, since it is clear that $N(c,s,u)\Psi_a(s^2+u^2) = 0$ for all s, u outside of a ball, there is no problem in extending the map N by 0 outside of its domain. In the sequel, we will suppress the $\tilde{}$ from $\tilde{F}$ and use only F.

It is standard to show that, by choosing a large enough, we arrange that

$$\sup_{c,s,u} |N(c,s,u)\Psi_a(s^2+u^2)|, \quad \sup_{c,s,u} |DN(c,s,u)\Psi_a(s^2+u^2)|$$

are as small as we wish.

For this extension, we can apply the unstable foliation theorem [Fen74, HPS77, Pes04] and obtain that there there is a manifold $W^u_\Lambda = \bigcup_{c\in\Lambda} W^u_c$, where W^u_Λ consists the of points y such that $\mathrm{dist}(F^{-n}(y),\Lambda) \leq C\mu_-^{-n}$ for $n > 0$. The manifolds W^u_c consist of the points y such that $\mathrm{dist}(F^{-n}(y), F^{-n}(c)) \leq C\mu_-^{-n}$ for $n > 0$. Notice that, in the theory of normally hyperbolic systems, it is customary to denote by W^s_c not just the points whose orbits approach the orbit of c but only those which approach faster than an exponential rate.

The manifolds W^u_c are as smooth as the map F. The manifold W^u, however, may be less differentiable, depending on the rates in (3.1). However, the conditions of normal hyperbolicity imply that it is C^1 and, in our case, because the map restricted to the stable manifold is close to integrable, the manifold will be in any C^n, $n \in \mathbb{N}$, for sufficiently small ε.

Furthermore, we note that the manifolds W^u_c constitute a normally hyperbolic lamination in the sense of [HPS77]. Therefore, we can associate stable manifolds to them. We have that $W^s_{W^u_\Lambda}$ is the whole space. If we consider the leaves $\mathcal{W}_c \equiv W^s_{W^u_c}$ it constitutes a uniformly C^1 foliation. Note that we have $F(\mathcal{W}_c) = \mathcal{W}_{f(c)}$. Moreover, $\mathcal{W}_c$ are uniformly C^1-close to $E^s_c \oplus E^u_c$.

We can take coordinates s, u on each of the manifolds $\mathcal{W}_c$, and we obtain a C^1_u system of coordinates – which we denote by $c, \tilde{s}, \tilde{u}$ – in which

$$F(c,\tilde{s},\tilde{u}) = (f(c), A_s(c)\tilde{s} + N_s(c,\tilde{s},\tilde{u}), A_u(c)\tilde{u} + N_u(c,\tilde{s},\tilde{u}))$$

with N_s, N_u also globally small.

Using the exponential mapping, we can consider $c, \tilde{s}, \tilde{u}$ as a coordinate system from a neighborhood of Λ to a neighborhood of the zero section. In this coordinate system, the map has the same expression as above.

Note that even if the stable and unstable manifolds are unique once we consider the extension to the whole bundle, the construction of the extension involves several choices – including notably the cutoff function. In general, the constructed foliations depend on the cutoffs considered. Examples when the manifold Λ is a fixed point are well known. See, for example [dlLW95]. Hence, the system of coordinates thus constructed is non-unique.

5.2. **Extending the decoupling coordinate system across a homoclinic intersection.** If we have a coordinate system that decouples the center behavior in a neighborhood U of Λ, it is possible to extend it to a neighborhood of W^u_Λ and to a neighborhood of W^s_Λ, by propagating the system of coordinates either forward in time or backwards in time.

When $p \in W^u_\Lambda \cap W^s_\Lambda$, in a neighborhood of p there are several possible extensions. These extensions may not agree since the propagation of the system of coordinates in U is done through different paths. For the purpose of constructing correctly aligned windows, it will be crucial for us to obtain estimates on the change of variables between the two extensions considered.

We will see that, when the intersection of W^u_Λ and W^s_Λ is transverse at p, the two systems of coordinates are related by a C^1 change of variables. The C^1 properties of this change of variables will depend only on a few parameters related to the geometry of the intersection. For the purposes of controlling our constructions as ε tends to zero, we note that the C^1 properties of the change of variables depend on the angle of the intersection. In particular, in the geodesic flow model, they will remain bounded as ε tends to zero. In the Large gap model, the angle of intersection between $T_p W^s_\Lambda$ and $T_p W^u_\Lambda$ tends to zero linearly with ε.

We now make precise the construction. Given $p \in W^u_\Lambda \cap W^s_\Lambda$ we can find numbers $N_+, N_- \in \mathbb{N}$ such that

$$F^{N_+}(p) \in U, \qquad F^{-N_-}(p) \in U. \tag{5.3}$$

We will assume that $N_\pm$ are chosen once and for all. We will assume without loss of generality that they are the smallest numbers satisfying (5.3). Notice that, by assumption **C1**, the numbers $N_\pm$ are constant for a range of ε.

If the intersection is transverse at p, the intersection is a locally unique manifold. There is a neighborhood V of p in $W^u_\Lambda \cap W^s_\Lambda$ for which $F^{N_+}(V) \subset U$, $F^{-N_-}(V) \subset U$.

We have $p \in W^u_{c_-} \cap W^s_{c_+}$ for some unique c_-, c_+. Indeed, given any other point $q \in V$, we can find unique points $c_-(q), c_+(q)$. The mapping that assigns to the point $c_-(q)$ the point $c_+(q)$ is called the scattering map associated to V.

We only need to construct our system of coordinates in U and in neighborhoods of $\{F^i(q) | i = -N_-, \dots N_+,\ q \in V\}$ that are sufficiently small in the stable and unstable directions. All the orbits that we will consider lie there and the windows will be constructed in these neighborhoods. We will construct the coordinates and make sure that several of the quantities that measure the change of coordinates are uniform in $N_\pm$.

We consider a neighborhood V_{N_-} of $F^{-N_-}(p)$ which is sufficiently small in the stable and unstable directions, such that $V^{-i}_{N_-} \equiv F^i(V_{N_-})$ are mutually disjoint for $i = 0, \dots, N_-$ (later on, we will impose some other conditions).

We can define a system of coordinates on $V^{-i}_{N_-}$ by setting $\mathcal{P}^-_i = M_i \circ \mathcal{P} \circ F^{-i}$ where $M_i(c, s, u) = (c, \lambda^i_* s, \mu^i_* u)$, and $\lambda_* = (\lambda_+ \cdot \lambda_-)^{1/2}$ and $\mu_* = (\mu_+ \cdot \mu_-)^{1/2}$.

Clearly, in these coordinates, we see that the contraction rate along the s directions is λ_* which satisfies the bounds $\lambda_- \le \lambda_* \le \lambda_+$. Similarly the expansion rate in the u variables is just μ_* which satisfies $\mu_- \le \mu_* \le \mu_+$. We also note that in this system of coordinates, the action of F is just M_i. That is, $\mathcal{P}^-_{i+1} \circ F \circ \mathcal{P}^-_i = M_i$.

We therefore obtain a system of coordinates $\mathcal{P}^-_{N_-}$ defined on a neighborhood V_0 of the homoclinic intersection.

Proceeding similarly, we can produce another coordinate system $\mathcal{P}_i^+$ defined on neighborhoods of $F^i(p)$, $i = 0, \ldots, N_+$.

The only place where these two systems of coordinates are defined simultaneously is a neighborhood of p.

We need to compare these two systems of coordinates and study the properties of the change of variables between them. We use the notation $c^\pm, s^\pm, u^\pm$ to denote the coordinates in the systems $\mathcal{P}^\pm$ obtained as before.

We first observe that, since the c coordinates are propagated without change through the steps, the c coordinates in $\mathcal{P}^-$ and the c coordinates in $\mathcal{P}^+$ are such that the scattering map expressed in these new coordinates $\mathcal{P}^+ \circ S \circ (\mathcal{P}^-)^{-1}$ is close to the identity. The distance to the identity is small if the neighborhood U is small.

We also observe that in a sufficiently small neighborhood of p, the sets that correspond to keeping the c, s coordinates fixed in the $\mathcal{P}^-$ coordinate systems and letting the u coordinates vary are C^1-close to a segment of $W^u_{c^-}$. Similarly, the sets that correspond to keeping the c, u constant are C^1-close to a segment of $W^s_{c_+}$.

The transversality of the intersection around p tells us that in a neighborhood of p, we can use as coordinates c^+, s^+, u^- or c^-, s^+, u^- besides, of course, using the coordinate systems c^+, u^+, s^+ and c^-, u^-, s^-.

For the purpose of estimating later the time of diffusion, we note that in the "Large Gap Model" – see Section 2.2 – the angle between $T_pW^s_\Lambda$ and $T_pW^u_\Lambda$ is $O(\varepsilon)$ (condition **C1b**). On the other hand in the "Geodesic Flow Model", this angle remains bounded independently of ε (condition **C1a**). In both the "Large Gap Model" and in the "Geodesic Flow Model" the properties of the coordinate systems c^+, u^+, s^+ and c^-, u^-, s^- remain uniform as ε approaches zero. Note that in both cases, the properties of these coordinate systems depend only on the properties of the change of variables in U – which has a limit as ε tends to zero – and on the finite number of iterates $N_\pm$.

Similarly, we note that the changes of variables in the c directions remain bounded since the scattering maps have a limit as ε goes to zero in both cases. In the "Geodesic Flow Model" the scattering map tends to a twist map. In the "Large Gap Model" the scattering map tends to the identity.

As we will see in Section 5, the fact that we have to change coordinates systems will lead to the need of introducing some factors when comparing the lengths of the coordinate intervals used to construct correctly aligned windows. In the case of the "Large Gap Model" these factors will become important since they will blow up as ε tends to zero. In the "Geodesic Flow Model" these factors will remain bounded.

We also note that in some of the constructions, we will not use the coordinates c ranging over the manifold Λ but we will need in turn to describe Λ using action and angle coordinates. So that in this case, the expression of coordinate system will involve the composition with the coordinate maps that give us the manifold Λ in terms of the action-angle variables. In our applications we have two types of action-angle variables. One is the standard type of action-angle variables, and the other is the action angle variables that appear in the secondary tori in the large gap model. We note that, outside the singularities of the average energy – which only happen at the resonances – the two variables differ by $O(\varepsilon)$. Hence, we can find intervals of size $O(1)$ in the angle variables where the secondary coordinates are close to the original ones.

6. Construction of windows along a transition chain

In this section, $\varepsilon > 0$ is fixed. Most of the ε-subscripts will be omitted to simplify the notation.

We describe a construction of correctly aligned windows near the normal hyperbolic manifold, by following a chain of heteroclinic connections to almost invariant leaves forward in time. (A similar construction of correctly aligned windows can be done by following a chain of of heteroclinic connections to almost invariant leaves backwards in time; the details of such a construction will be omitted.)

We first describe a procedure for construction of correctly aligned windows along a short chain consisting of three almost invariant leaves and their stable and unstable manifolds, for which the first two leaves are linked by a heteroclinic connection and the last two leaves are linked by a different heteroclinic connection. Then we describe how the windows constructed along two adjacent short chains as above can be put together into a longer sequence of correctly aligned windows. The conclusion of these two procedures will be that, using this algorithm repeatedly, a sequence of correctly aligned windows associated to any given chain of heteroclinic connections to almost invariant leaves can be produced.

We give the outline this construction. Assume that $\mathcal{L}^{h_{i-1}}$, $\mathcal{L}^{h_i}$, and $\mathcal{L}^{h_{i+1}}$ are almost invariant leaves in Λ, and $W^u(\mathcal{L}^{h_{i-1}}) \pitchfork W^s(\mathcal{L}^{h_i})$ and $W^u(\mathcal{L}^{h_i}) \pitchfork W^s(\mathcal{L}^{h_{i+1}})$. Let $q_i \in \Gamma$ be an intersection point of $W^u(\mathcal{L}^{h_{i-1}})$ with $W^s(\mathcal{L}^{h_i})$. There exist $p_{i-1}^- \in \mathcal{L}^{h_{i-1}}$ and $p_i^+ \in \mathcal{L}^{h_i}$ such that $q_i \in W^u(p_{i-1}^-) \cap W^s(p_i^+)$. Suppose that we want to obtain a pair of windows W_{i-1}' and W_i near q_i, such that W_{i-1}' is correctly aligned with W_i under the identity mapping. There exist stable, unstable, angle, and radius local coordinates near $\mathcal{L}^{h_{i-1}}$ that can be propagated along $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$. There also exist stable, unstable, angle, and action local coordinates near $\mathcal{L}^{h_i}$ that can be propagated along $W^{\mathrm{s}}(\mathcal{L}^{h_i})$. The unstable manifold $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$ corresponds to keeping the stable and radius coordinates fixed, while varying the unstable and angle coordinates. The stable manifold $W^{\mathrm{s}}(\mathcal{L}^{h_i})$ corresponds to keeping the corresponding unstable and radius coordinates fixed, while varying the stable and angle coordinates. The transversality of the intersection of $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$ with $W^{\mathrm{s}}(\mathcal{L}^{h_i})$ at q_i means that the unstable and angle direction corresponding to $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$ are transverse to the stable and angle directions corresponding to $W^{\mathrm{s}}(\mathcal{L}^{h_i})$. We will construct W_{i-1}' by fattening up $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$ near q_i, and W_i by fattening up $W^{\mathrm{s}}(\mathcal{L}^{h_i})$. The exit sets are naturally defined by the dynamics: the exit set W_{i-1}' is in the unstable and angle direction corresponding to $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$, while the exit set W_i is in the stable and radius direction corresponding to $W^{\mathrm{u}}(\mathcal{L}^{h_{i-1}})$. Since the two local coordinate systems may not agree, we will need to make this construction more carefully, as it will be detailed in Subsection 6.2. See Figure 5. Then we will push W_i forward in time along $W^{\mathrm{s}}(\mathcal{L}^{h_i})$ and align its image with a window $\tilde{W}_i$ constructed about $\mathcal{L}^{h_i}$. The exit set of $\tilde{W}_i$ will still be in the unstable and radius directions. In order to get in a place from where it can escape along $W^{\mathrm{s}}(\mathcal{L}^{h_i})$, we need to move $\tilde{W}_i$ along $\mathcal{L}^{h_i}$. Due to the twist condition on the inner dynamics, $\tilde{W}_i$ will shear in the angle direction. We construct a new window $\hat{W}_i$ about $\mathcal{L}^{h_i}$, such that the image of $\tilde{W}_i$ stretches across $\hat{W}_i$. Therefore, we need to choose the exit set of $\hat{W}_i$ in the unstable and angle directions. Then we will push $\hat{W}_i$ forward in time along $W^{\mathrm{u}}(\mathcal{L}^{h_i})$, and construct a new window W_i' near q_{i+1}, such that $\hat{W}_i$ is correctly aligned with W_i'. The exit set of W_i' in the unstable and angle directions.

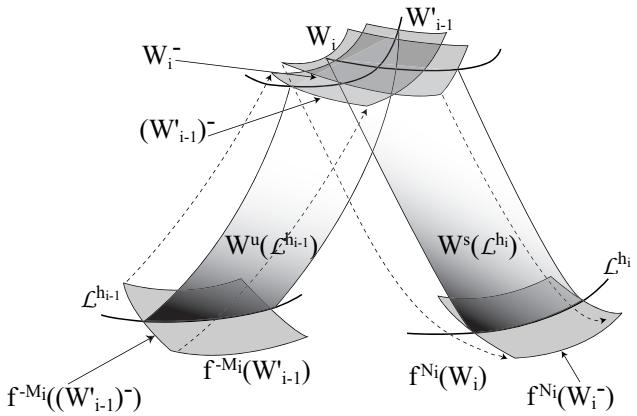

FIGURE 5. The transversality-torsion phenomenon.

The details of this construction are given in Subsection 6.1. Note that the directions of the exit set of W_i' agree with those of W_{i-1}'. Therefore, we will be able to repeat this construction and extend the sequence of correctly aligned windows along any finite transition chain.

Remark 6.1. In the above construction, the reversal of roles of the radius and angle directions in terms of which one acts as an exit direction and which one acts as an entry direction is a topological version of the transversality-torsion phenomenon (compare with [Cre03]).

Consider a local coordinate system (s, u, a, r) near a leaf $\mathcal{L}^{h_i}$. When we propagate this local coordinate along the unstable manifold and stable manifolds of $\mathcal{L}^{h_i}$, we obtain the local coordinate systems (s^-, u^-, a^-, r^-) (s^+, u^+, a^+, r^+), respectively. The windows involved in the construction outlined above are defined in terms in terms of these local coordinates. We introduce some notation conventions. Assume that we want to construct a window relative to the (s, u, a, r) coordinate system. The window will be regarded as a product of two windows, one corresponding to the hyperbolic directions, and the other corresponding to the center directions. Each of these two windows will be a product of two rectangles. Formally, each window will be described as a product of the form

$$W = [S \times U] \times [A \times R],$$

with $S = [l_1, r_1]^n$ being an n-dimensional rectangle in the s-coordinate, $U = [l_2, r_2]^n$ being an n-dimensional rectangle in the u-coordinate, $A = [l_3, r_3]$ being a segment in the a-coordinate, and $R = [l_4, r_4]$ being a segment in the r-coordinates. In general, by the 'size' of a rectangle component $[l_i, r_i]^{p_i}$, where $i = 1, ..., 4$ and $p_i \in \{n, 1\}$, we will mean the length of the segment $[l_i, r_i]$. By the 'center' of a window W we will mean the point of (s, u, a, r)-coordinates equal to

$$\left(\frac{(l_1 + r_1)}{2}, \ldots, \frac{(l_1 + r_1)}{2}, \frac{(l_2 + r_2)}{2}, \ldots, \frac{(l_2 + r_2)}{2}, \frac{(l_3 + r_3)}{2}, \frac{(l_4 + r_4)}{2} \right).$$

Since E^u and E^s are invariant under F, and since the coordinate system was chosen so that s and u are mutually orthogonal and also orthogonal to the a and r-coordinates, in order to verify correctly alignment of windows, we only need to verify the correct alignment of the components and apply Proposition 4.9 (and also Remark 4.10). We will do so without further explicit mention.

Similar conventions will hold in the case of the $(s^{\pm}, u^{\pm}, a^{\pm}, r^{\pm})$ local coordinates, or even in the case when we will mix coordinate systems, such as in Subsection 6.2.

6.1. **Construction of correctly aligned windows along a short chain.** We construct correctly aligned windows

- W_i about q_i,
- $\widetilde{W}_i$ about $\mathcal{L}^{h_i}$,
- $\widehat{W}_i$ about $\mathcal{L}^{h_i}$,
- W_i' about q_{i+1},

such that

- W_i is correctly aligned with $\widetilde{W}_i$ under F^{N_i},
- $\widetilde{W}_i$ is correctly aligned with $\widehat{W}_i$ under F^{K_i},
- $\widehat{W}_i$ is correctly aligned with W_i' under F^{M_i}.

Step 1. Consider a coordinate system (s, u, a, r) near $\mathcal{L}^{h_i}$. Consider the coordinate system (s^-, u^-, a^-, r^-) near q_i, obtained by propagating (s, u, a, r) along the stable manifold of $\mathcal{L}^{h_i}$. Consider the coordinate system (s^+, u^+, a^+, r^+) near q_{i+1}, obtained by propagating (s, u, a, r) along the unstable manifold of $\mathcal{L}^{h_i}$.

We define a window W_i near q_i in terms of (a^-, r^-, s^-, u^-), by

$$W_i \quad = \quad [S_i \times U_i] \times [A_i \times R_i]\,,$$

where S_i, U_i, A_i, R_i denote closed rectangles in coordinates s^-, u^-, a^-, r^-, respectively. We choose the exit set of the hyperbolic rectangle $[S_i \times U_i]$ in the unstable direction, and the exit set of the rectangle $[A_i \times R_i]$ parallel to the center in the radius direction. In other words, we let

$$\begin{aligned} [A_i \times R_i]^- &= A_i \times \partial R_i, \\ [S_i \times U_i]^- &= S_i \times \partial U_i. \end{aligned}$$

The exit set of W_i is given by the definition of the product of windows, as defined in Section 4. Assume that the sizes of the rectangles S_i, U_i, A_i, R_i are α_i, β_i, γ_i and δ_i respectively. There are no extra conditions on W_i provided that this is the first window in the long sequence of correctly aligned windows that we construct. If this is not the first window, in other words if q_i is the arrival point of a short chain consisting of three invariant leaves and their heteroclinic connections, then on W_i we impose conditions analogue to those specified in Subsection 6.2.

Let a_i be the distance between the homoclinic point q_i and p_i^+ measured along the stable manifold of $\mathcal{L}^{h_i}$.

We take a forward iterate $F^{N_i}(W_i)$ of W_i. Note that the coordinate system (s^-, u^-, a^-, r^-) are transformed by F^{N_i} into the coordinate system (s, u, a, r) near $\mathcal{L}_i$. The distance between the center $F^{N_i}(q_i)$ of $F^{N_i}(W_i)$ and $f^{N_i}(p_i^+)$, measured along the stable manifold, is in between $a_i\lambda_-^{N_i}$ and $a_i\lambda_+^{N_i}$. The size of the component of $F^{N_i}(W_i)$ in the s-direction is in between $\alpha_i\lambda_-^{N_i}$ and $\alpha_i\lambda_+^{N_i}$. The size of the component of $F^{N_i}(W_i)$ in the u-direction is in between $\beta_i\mu_-^{N_i}$ and $\beta_i\mu_+^{N_i}$. Under this iteration, the twist condition **C2** determines a shearing in the positive a-direction of

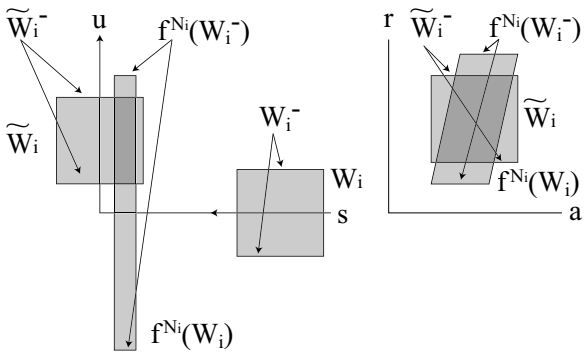

FIGURE 6. Step 1.

order $T\delta_i N_i$. Condition **C3** says that the leaves of the foliation are almost invariant under F within a margin of order ε^2. This implies that the sizes of the components of $F^{N_i}(W_i)$ in the a and r-directions are of order $\gamma_i \pm \varepsilon^2 N_i$ and $T\delta_i N_i \pm \varepsilon^2 N_i$, respectively.

We define a window $\widetilde{W_i}$ centered at $f^{N_i}(p_i^+)$ relative to the coordinates (s, u, a, r) near $\mathcal{L}_i$ also as a product of four rectangles, the first two corresponding to the hyperbolic directions, and the others corresponding to the center directions:

$$\widetilde{W_i} = \left[\widetilde{S}_i \times \widetilde{U}_i\right] \times \left[\widetilde{A}_i \times \widetilde{R}_i\right].$$

The exit sets of the component rectangles are defined by

$$\begin{aligned} \left[\widetilde{S}_i \times \widetilde{U}_i\right]^- &= \widetilde{S}_i \times \partial\widetilde{U}_i, \\ \left[\widetilde{A}_i \times \widetilde{R}_i\right]^- &= \widetilde{A}_i \times \partial\widetilde{R}_i, \end{aligned}$$

while the exit set of $\widetilde{W_i}$ is given by the definition of the product of windows from Section 4. We choose the rectangles such that W_i is correctly aligned with $\widetilde{W_i}$ under F^{N_i}. Assume that the sizes of the rectangles $\widetilde{S}_i$, $\widetilde{U}_i$, $\widetilde{A}_i$, $\widetilde{R}_i$ are $\widetilde{\alpha}_i$, $\widetilde{\beta}_i$, $\widetilde{\gamma}_i$ and $\widetilde{\delta}_i$, respectively. To satisfy the correct alignment condition, we require the following conditions:

$$\widetilde{\alpha}_i > a_i\lambda_+^{N_i} + \alpha_i\lambda_+^{N_i}, \tag{6.1}$$

$$\widetilde{\beta}_i < \beta_i\mu_-^{N_i}, \tag{6.2}$$

$$\widetilde{\gamma}_i > \gamma_i + \varepsilon^2 N_i + T\delta_i N_i, \tag{6.3}$$

$$\widetilde{\delta}_i < \delta_i - \varepsilon^2 N_i. \tag{6.4}$$

See Figure 6.

The outcome of Step 1 is the pair of windows W_i near q_i and $\widetilde{W_i}$ near $f^{N_i}(p_i^+)$, such that W_i is correctly aligned with $\widetilde{W_i}$ under F^{N_i}.

Step 2. We take a forward iterate $F^{K_i}(\widetilde{W_i})$ of $\widetilde{W_i}$. The center $f^{N_i}(p_i^+)$ of the window $\widetilde{W_i}$ is displaced to $f^{N_i+K_i}(p_i^+)$. The component of $F^{K_i}(\widetilde{W_i})$ in the s-direction is of a size in between $\widetilde{\alpha}_i\lambda_-^{K_i}$ and $\widetilde{\alpha}_i\lambda_+^{K_i}$. The component of $F^{K_i}(\widetilde{W_i})$ in the u-direction is of a size in between $\widetilde{\beta}_i\mu_-^{K_i}$ and $\widetilde{\beta}_i\mu_+^{K_i}$. During this iteration, the

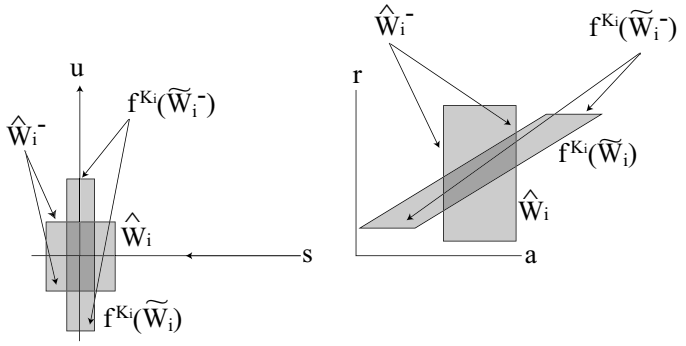

FIGURE 7. Step 2.

twist condition **C2** determines a shearing in the positive a-direction of order $T\widetilde{\delta}K_i$. In addition to the shearing, due to condition **C3**, the a and r coordinates of point at each iteration change by an amount between $-\varepsilon^2$ and ε^2. Thus, the components of $F^{K_i}(\widetilde{W}_i)$ in the a and r-directions are of sizes of order $\tilde{\gamma}_i + T\delta_i K_i \pm \varepsilon^2 K_i$ and $\widetilde{\delta}_i \pm \varepsilon^2 K_i$, and , respectively.

We construct a window $\widehat{W}_i$ centered at $f^{N_i+K_i}(p_i^+)$, such that $\widetilde{W}_i$ is correctly aligned with $\widehat{W}_i$ under f^{K_i}. The window $\widehat{W}_i$ is defined in terms of the coordinate system (s, u, a, r) near $\mathcal{L}_i$ as a product of two windows, one corresponding to the hyperbolic directions, and the other corresponding to the center directions:

$$\widehat{W}_i = \left[\widehat{S}_i \times \widehat{U}_i\right] \times \left[\widehat{A}_i \times \widehat{R}_i\right],$$

with

$$\begin{aligned} \left[\widehat{S}_i \times \widehat{U}_i\right]^- &= \widehat{S}_i \times \partial\widehat{U}_i, \\ \left[\widehat{A}_i \times \widehat{R}_i\right]^- &= \partial\widehat{A}_i \times \widehat{R}_i, \end{aligned}$$

and $\widehat{W}_i^-$ resulting from the definition of the product of windows. Note that the exit set of the center part of $\widehat{W}_i$ is in the angle direction, due to the shearing of $F^{K_i}(\widetilde{W}_i)$ across $\widehat{W}_i$. The sizes of the rectangles $\widehat{S}_i$, $\widehat{U}_i$, $\widehat{A}_i$ and $\widehat{R}_i$ are $\widehat{\alpha}_i$, $\widehat{\beta}_i$, $\widehat{\gamma}_i$ and $\widehat{\delta}_i$, respectively. In order to ensure correct alignment, we choose these numbers so that they satisfy the following inequalities:

$$\widehat{\alpha}_i > \widetilde{\alpha}_i \lambda_+^{K_i}, \tag{6.5}$$

$$\widehat{\beta}_i < \widetilde{\beta}_i \mu_-^{K_i}, \tag{6.6}$$

$$\widehat{\gamma}_i < T\widetilde{\delta}_i K_i - \widetilde{\gamma}_i - \varepsilon^2 K_i \tag{6.7}$$

$$\widehat{\delta}_i > \widetilde{\delta}_i + \varepsilon^2 K_i. \tag{6.8}$$

See Figure 7.

Step 3. Let $q_{i+1} \in \Gamma$ be an intersection point of $W^u(\mathcal{L}^{h_i})$ with $W^s(\mathcal{L}^{h_{i+1}})$, $p_i^- \in \mathcal{L}^{h_i}$ and $p_{i+1}^+ \in \mathcal{L}^{h_i}$ such that $q_{i+1} \in W^u(p_i^-) \cap W^s(p_{i+1}^+)$, and let a_i' be the distance between q_{i+1} and p_i^- measured along the unstable manifold. We define a

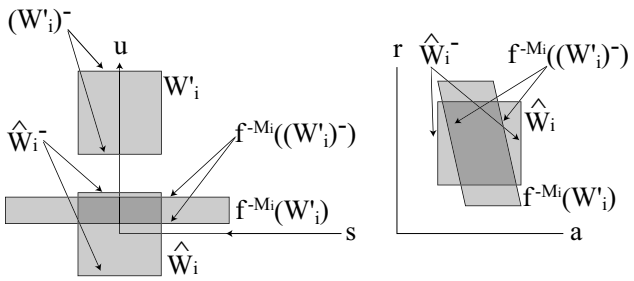

FIGURE 8. Step 3.

windows W_i' near q_{i+1}, relative to the coordinate system (s^+, u^+, a^+, r^+) by

$$\begin{aligned} W_i' &= [S_i' \times U_i'] \times [A_i' \times R_i'], \\ [S_i' \times U_i']^- &= S_i' \times \partial U_i', \\ [A_i' \times R_i']^- &= \partial A_i' \times R_i', \end{aligned}$$

and the sizes of the rectangles S_i', U_i', A_i', R_i' are α_i', β_i', γ_i' and δ_i', respectively.

We take a negative iterate $F^{-M_i}(W_i')$ of W_i'. Note that the coordinate system (s^+, u^+, a^+, r^+) are transformed by F^{-M_i} into the coordinate system (s, u, a, r) near $\mathcal{L}_i$. The distance between the center $F^{-M_i}(q_{i+1})$ of $F^{-M_i}(W_i')$ and $f^{-M_i}(p_i^-)$, measured along the unstable manifold, is in between $a_i'\mu_+^{-M_i}$ and $a_i'\mu_-^{-M_i}$. The size of the component of $F^{-M_i}(W_i')$ in the s-direction is of size in between $\alpha_i'\lambda_+^{M_i}$ and $\alpha_i'\lambda_-^{-M_i}$. The size of the component of $F^{-M_i}(W_i')$ in the u-direction is of size in between $\beta_i'\mu_+^{-M_i}$ and $\beta_i'\mu_-^{-M_i}$. The size of the components of $F^{-M_i}(W_i')$ in the a and r-directions are of order $\gamma_i' \pm \varepsilon^2 M_i$ and $\delta_i' \pm \varepsilon^2 M_i$, respectively. During this iteration, the twist condition **C2** determines a shearing in the negative a-direction of order $T\delta_i' M_i$. In addition to the shearing, due to condition **C3**, the a and r coordinates of point change by an amount between $-\varepsilon^2 M_i$ and $\varepsilon^2 M_i$. Thus, the components of $F^{K_i}(\widetilde{W}_i)$ in the a and r-directions are of sizes of order $\gamma_i' + T\delta_i' M_i \pm \varepsilon^2 M_i$ and $\delta_i' \pm \varepsilon^2 M_i$, respectively. We would like to ensure the correct alignment of $\widehat{W}_i$ with W_i' under the identity mapping. Hence we require

$$\alpha_i'\lambda_+^{-M_i} > \widehat{\alpha}_i, \tag{6.9}$$

$$a_i'\mu_-^{-M_i} + \beta_i'\mu_-^{-M_i} < \widehat{\beta}_i, \tag{6.10}$$

$$T\delta_i' M_i + \gamma_i' + \varepsilon^2 M_i < \widehat{\gamma}_i, \tag{6.11}$$

$$\delta_i' - \varepsilon^2 M_i > \widehat{\delta}_i. \tag{6.12}$$

See Figure 8.

This ends the description of the three-step procedure.

6.2. **Gluing short sequences of correctly aligned windows near the homoclinic points.** Suppose that we have constructed a short sequence of correctly aligned windows $W_i, \widetilde{W}_i, \widehat{W}_i$, and W_i' as in the previous subsection. The last window W_i' is centered about the homoclinic point $q_{i+1} \in W^{\mathrm{u}}(\mathcal{L}^{h_{i+1}}) \cap W^{\mathrm{s}}(\mathcal{L}^{h_{i+2}})$. We know how to continue this short sequence of correctly aligned windows by another sequence $W_{i+1}, \widetilde{W}_{i+1}, \widehat{W}_{i+1}$, and W_{i+1}' provided that we correctly align W_i' to W_{i+1}.

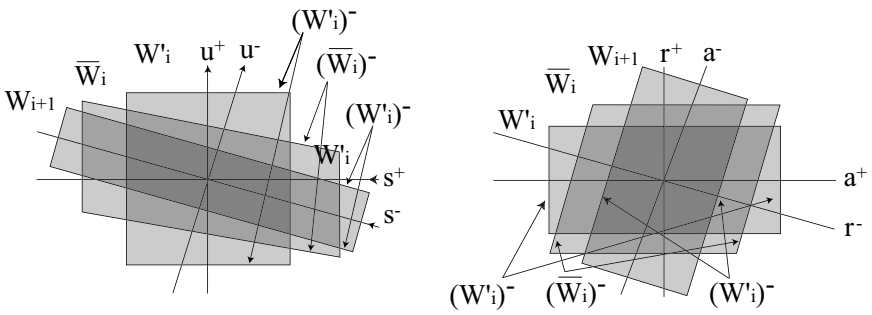

FIGURE 9. Construction of the window $\bar{W}_i$.

The problem we have is that the local coordinate system (s^+, u^+, a^+, r^+), obtained by propagating the local coordinates (s, u, a, r) near $\mathcal{L}^{h_i}$ forward in time along $W^{\mathrm{u}}(\mathcal{L}^{h_i})$, may not agree near q_{i+1} with the coordinate system (s^-, u^-, a^-, r^-), obtained by propagating the local coordinates (s, u, a, r) near $\mathcal{L}^{h_{i+1}}$ backwards in time along $W^{\mathrm{s}}(\mathcal{L}^{h_{i+1}})$. Also note that the local coordinates near $\mathcal{L}^{h_i}$ and the local coordinates near $\mathcal{L}^{h_{i+1}}$ may be different (although we have used the same notation for them).

We define a 'mixed coordinate system' (s^-, u^+, a^+, a^-) near q_{i+1}. In these coordinates $W^{\mathrm{u}}(\mathcal{L}^{h_i})$ is given by keeping s^- and a^- equal to 0, and varying u^+ and a^+, and $W^{\mathrm{s}}(\mathcal{L}^{h_{i+1}})$ is given by keeping u^+ and a^+ equal to 0, and varying s^- and a^-. As described in Section 5, the variables a^+ and a^- differ by $O(\varepsilon)$.

We define a new window $\bar{W}_i$ near q_{i+1} such that W_i is correctly aligned with $\bar{W}_i$ under the identity mapping, and $\bar{W}_i$ is correctly aligned with W'_{i+1} under the identity mapping. The window $\bar{W}_i$ is defined as a product of four rectangles. We let

$$\bar{W}_i \;=\; \left[\bar{S}_i \times \bar{U}_i\right] \times \left[\bar{A}_i \times \bar{A}'_i\right],$$

where $\bar{S}_i$, $\bar{U}_i$, $\bar{A}_i$, $\bar{A}'_i$ denote closed rectangles in coordinates s^-, u^+, a^+, a^-, respectively. We let

$$\begin{aligned} \left[\bar{S}_i \times \bar{U}_i\right]^- &= \bar{S}_i \times \partial\bar{U}_i, \\ \left[\bar{A}_i \times \bar{A}'_i\right]^- &= \partial\bar{A}_i \times \bar{A}'_i. \end{aligned}$$

The exit set of $\bar{W}_i$ is given by the definition of the product of windows, as defined in Section 4. Assume that the sizes of the rectangles $\bar{S}_i$, $\bar{U}_i$, $\bar{A}_i$, $\bar{A}'_i$ are $\bar{\alpha}_i$, $\bar{\beta}_i$, $\bar{\gamma}_i$ and $\bar{\delta}_i$ respectively. To ensure correct alignment, we need to compare windows that are defined in different coordinate systems. As indicated in Subsection 5.2, the lengths of the coordinate intervals used in the construction of the windows change by some factor $C(\varepsilon)$ when we change the coordinates one way, and by some factor of $C(\varepsilon)^{-1}$ when we change the coordinates the other way.

Therefore, we require

$$\bar{\alpha}_i \;>\; \alpha'_i, \tag{6.13}$$

$$\bar{\beta}_i \;<\; C(\varepsilon)\beta'_i, \tag{6.14}$$

$$\bar{\gamma}_i \;<\; C(\varepsilon)\gamma'_i, \tag{6.15}$$

$$\bar{\delta}_i \;>\; \delta'_i. \tag{6.16}$$

See Figure 9.

We define a new window W_{i+1} near q_{i+1}, relative to the (s^-, u^-, a^-, r^-)-coordinates, such that $\bar{W}_i$ is correctly aligned with W_{i+1} under the identity mapping. We let

$$\begin{aligned} W_{i+1} &= [S_{i+1} \times U_{i+1}] \times [A_{i+1} \times R_{i+1}], \\ [S_{i+1} \times U_{i+1}]^- &= S_{i+1} \times \partial U_{i+1}, \\ [A_{i+1} \times R_{i+1}]^- &= A_{i+1} \times \partial R_{i+1}. \end{aligned}$$

The exit set of W_{i+1} is given by the definition of the product of windows, as defined in Section 4. Assume that the sizes of the rectangles S_{i+1}, U_{i+1}, A_{i+1}, R_{i+1} are α_{i+1}, β_{i+1}, γ_{i+1} and δ_{i+1} respectively. As before, the windows $\bar{W}_i$ and W_{i+1} are defined in different coordinate systems. Therefore, we require

$$\alpha_{i+1} > C(\varepsilon)^{-1}\bar{\alpha}_i, \tag{6.17}$$

$$\beta_{i+1} < \bar{\beta}_i, \tag{6.18}$$

$$\gamma_{i+1} > C(\varepsilon)^{-1}\bar{\delta}_i, \tag{6.19}$$

$$\delta_{i+1} < \bar{\gamma}_i. \tag{6.20}$$

See Figure 9.

This construction ensures the correct alignment of W_i' with $\bar{W}_i$ under the identity mapping, and of $\bar{W}_i$ with W_{i+1} under the identity mapping. Combining (6.17), (6.14), (6.15), (6.16) with (6.13), (6.18), (6.19), (6.20) we obtain the conditions

$$\alpha_{i+1} > C(\varepsilon)^{-1}\alpha_i', \tag{6.21}$$

$$\beta_{i+1} < C(\varepsilon)\beta_i', \tag{6.22}$$

$$\gamma_{i+1} > C(\varepsilon)^{-1}\delta_i', \tag{6.23}$$

$$\delta_{i+1} < C(\varepsilon)\gamma_i'. \tag{6.24}$$

In this way, we can continue the short sequence of correctly aligned windows $W_i, \widetilde{W}_i, \widehat{W}_i, W_i'$ with the short sequence of correctly aligned windows $W_{i+1}, \widetilde{W}_{i+1}, \widehat{W}_{i+1}, W_{i+1}'$, via the intermediary sequence of correctly aligned windows $W_i', \bar{W}_i, W_{i+1}$.

6.3. **Construction of long sequences of correctly aligned windows.** Now we will impose more conditions on the sizes of the windows and discuss the aspect ratio of the windows. We choose $\widetilde{\alpha}_i$, $\widehat{\alpha}_i$, α_i', $\widetilde{\beta}_i$, $\widehat{\beta}_i$, β_i' of order 1 (independent of ε). Let $\eta > 0$ be as in Theorem 3.1. Since we want to produce trajectories that visit η-neighborhoods of leaves in some prescribed fashion, we choose $\widetilde{\alpha}_i, \widehat{\alpha}_i < \eta$ and $\widetilde{\beta}_i, \widehat{\beta}_i < \eta$. By (6.21) and (6.22), α_i is of order $1/\varepsilon$ and β_i is of order ε.

We choose γ_i, $\widetilde{\gamma}_i$, $\widehat{\gamma}_i$, γ_i' of order 1, and δ_i, $\widetilde{\delta}_i$, $\widehat{\delta}_i$, δ_i' of order ε. Due to (6.23) and (6.24), γ_i can be chosen of order 1 provided δ_{i-1}' is of order ε, and δ_i can be chosen of order ε provided that γ_{i-1}' is of order 1. We also require $\delta_i, \delta_{i+1} < \eta$. The condition on δ_i, δ_{i+1} to be of order ε is imposed since we want the radius components of the windows $\widetilde{W}_i$ and $\widehat{W}_i$ to be of size of order ε, in order to ensure that $\widetilde{W}_i \cap \mathcal{L}^{h_{i\pm1}} = \emptyset$ and $\widehat{W}_i \cap \mathcal{L}^{h_{i\pm1}} = \emptyset$ (recall also (3.5)).

We want to show that these choices can be made consistently provided N_i, K_i and M_i are chosen sufficiently large.

By (6.1) and (6.2), we should require that N_i is sufficiently large, of order $\ln(1/\varepsilon)$, so that $(a_i + \alpha_i)\lambda_+^{N_i}$ and $\beta_i \mu_-^{N_i}$ are of order 1. Due to (6.4) and (6.3), $\widetilde{\gamma}_i$ can be

chosen of order 1, provided γ_i is of order 1, and $\widetilde{\delta}_i$ can be chosen of order ε, provided δ_i is of order ε.

Since $\widetilde{\gamma}_i$ and $\widehat{\gamma}_i$ are of order 1, the ergodization time necessary to obtain the center of the window $\widetilde{W}_i$ sufficiently close to the center of the window $\widehat{W}_i$ is negligible. Note that in order to satisfy (6.7), in order to be able to choose both $\widetilde{\gamma}_i$ and $\widehat{\gamma}_i$ of order 1, we need to choose K_i sufficiently large and of order $1/\varepsilon$. Then, by (6.8) we can choose $\widetilde{\delta}_i$ and $\widehat{\delta}_i$ of order ε. Here note that the exponent 2 of ε in (6.8) (and so in condition **C3**) is optimal, in the sense that any lower exponent will make impossible to choose $\widehat{\delta}_i$ of order ε. By (6.5) and (6.6), $\widehat{\alpha}_i$ and $\widehat{\alpha}_i$ can be chosen of order 1.

By (6.9) and (6.10), we should require that M_i is sufficiently large, but of order 1, so that $(a_i' + \beta_i')\mu_-^{-M_i}$ and $\alpha_i'\mu_-^{-M_i}$ are of order 1. Note that the inequalities (6.11), and (6.12) allow us to choose γ_i' of order 1 provided $\widehat{\gamma}_i$ is of order 1, and δ_i' of order ε provided $\widehat{\delta}_i$ is of order ε.

In conclusion, $N_i = O(\ln(1/\varepsilon))$, $K_i = O(1/\varepsilon)$ and $M_i = O(1)$. Thus, in order to travel the distance of order ε in the action direction between $\mathcal{L}^{h_i}$ and $\mathcal{L}^{h_{i+1}}$, an orbit shadowing the above windows would spend a time $N_i + K_i + M_i = O(1/\varepsilon)$.

7. Conclusion of the proof of Theorem 3.1

Proof of Theorem 3.1. Fix $\varepsilon \in (-\varepsilon^*, \varepsilon^*)$ with $\varepsilon \neq 0$, and

$$h_0 = h^-, h_1, \ldots, h_{n-1}, h_n = h_+.$$

By (3.5), for each h_i there exist $k_i^-, k_i^+ > 0$ satisfying (3.6). One can assume that $h_i + k_i^- < h_{i+1} < h_i + k_i^+$ for all $i = 0, \ldots, n-1$. Otherwise, one can always partition all intervals $[h_i, h_{i+1}]$ into subintervals of length at least $c^{-1}\varepsilon$, whose endpoints are regular values, and apply Theorem 3.1 for the resulting sequence of regular values.

We start with the short heteroclinic chain consisting of $W^{\mathrm{u}}(\mathcal{L}^{h_0}) \pitchfork W^{\mathrm{s}}(\mathcal{L}^{h_1})$ and $W^{\mathrm{u}}(\mathcal{L}^{h_1}) \pitchfork W^{\mathrm{s}}(\mathcal{L}^{h_2})$. Using the algorithm described above, we construct the windows $\widehat{W}_0$ about $\mathcal{L}^{h_0}$, W_0', $\bar{W}_0$ and W_1 about q_1 in $W^{\mathrm{u}}(\mathcal{L}^{h_0}) \pitchfork W^{\mathrm{s}}(\mathcal{L}^{h_1})$, $\widetilde{W}_1$ about $\mathcal{L}^{h_1}$, $\widehat{W}^1$ about $\mathcal{L}^{h_1}$, W_1', $\bar{W}_1$ and W_2 about q_2 in $W^{\mathrm{u}}(\mathcal{L}^{h_2}) \pitchfork W^{\mathrm{s}}(\mathcal{L}^{h_i})$, such that such that $\widehat{W}_0$ is correctly aligned with W_0' under F^{m_0}, W_0' is correctly aligned with $\bar{W}_0$ under the identity mapping, $\bar{W}_0$ is correctly aligned with W_1 under the identity map, W_1 is correctly aligned with $\widetilde{W}_1$ under F^{N_1}, $\widetilde{W}_1$ is correctly aligned with $\widehat{W}_1$ under F^{K_1}, $\widehat{W}_1$ is correctly aligned with W_1' under F^{M_1}, W_1' is correctly aligned with $\bar{W}_1$ under the identity mapping, and $\bar{W}_1$ is correctly aligned with W_2 under the identity mapping. We continue this construction until we reach the leaf $\mathcal{L}^{h_n}$.

By Theorem 4.6, there exists an orbit z_i, $i = 0, \ldots, n$ with $z_i \in \widehat{W}_i$, such that $F^{n_i}(z_i) = z_{i+1}$ for all $i = 0, \ldots, n-1$, where $n_i = M_i + N_{i+1} + K_{i+1}$. By choosing the sizes $\widehat{\alpha}_i$, $\widehat{\beta}_i$, $\widehat{\gamma}_i$ and $\widehat{\delta}_i$ sufficiently small, we can ensure that the diameter of $\widehat{W}_i$ is less than the prescribed $\eta > 0$. Hence $d(z_i, \mathcal{L}^{h_i}) < \eta$ for all i. The same theorem says that there exist closed orbits that visit η-neighborhoods of the prescribed sequence of almost invariant leaves. $\square$

8. Remarks on differentiability conditions and on diffusion time

The windowing argument requires only C^0-differentiability. The normal hyperbolicity conditions require the diffeomorphisms F_ε to be C^1-smooth. We also need the foliation given by the level sets of H_ε to be C^1-smooth.

For the model discussed in Subsection 2.1, it is sufficient that the metric and the potential to be C^r smooth with $r \geq 9$. This order of differentiability is assumed in order to perform averaging theory in order to reduce the perturbation (see Theorem 4.6 and Lemma 4.8 in [DdlLS00]). By comparison, the geometric perturbation theory argument used in [DdlLS00] to prove the existence of orbits whose energy grow unbounded, which makes use of the KAM theorem, requires the metric and the potential V to be differentiable of class C^r, with $r \geq 15$.

For the model discussed in Subsection 2.2, in order to prove the existence of diffusing orbits through the above windowing method, it is sufficient that the potential V and the perturbation h are differentiable of class C^r, with $r \geq 6$ (see Proposition 28 and Theorem 35 in [DdlLS03b]). This order of differentiability is assumed in order to be able to produce almost invariant tori through the averaging method (see Theorem in [DdlLS03b]). By comparison, the obstruction argument used in [DdlLS03b] to prove the existence of diffusion orbits, which makes use of the KAM theorem, requires the potential V and the perturbation h to be differentiable of class C^r, with $r \geq 44$.

Now we would like to estimate the diffusion time. As noted in Section 6, the time required for an orbit to travel from a η-neighborhood of $\mathcal{L}^{h_i}$ to a η-neighborhood of $\mathcal{L}^{h_{i+1}}$, is of the order $1/\varepsilon$. Since $\mathcal{L}^{h_i}$ and $\mathcal{L}^{h_{i+1}}$ are order ε apart in the action direction, it results that the time required for an orbit to travel a distance of order 1 in the action direction, is of order $1/\varepsilon^2$. This estimate may not be sharp. There are several non-optimal choices we made in the construction of the correctly aligned windows, in to avoid dealing with estimates on the ergodization time. The optimization of the diffusion time will make the subject of future investigations.

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References

[AA67] V.I. Arnold and A. Avez. *Ergodic problems of classical mechanics*. Benjamin, New York, 1967.

[Arn64] V.I. Arnold. Instability of dynamical systems with several degrees of freedom. *Sov. Math. Doklady*, 5:581–585, 1964.

[BW95] Keith Burns and Howard Weiss. A geometric criterion for positive topological entropy. *Comm. Math. Phys.*, 172(1):95–118, 1995.

[Chi79] B.V. Chirikov. A universal instability of many-dimensional oscillator systems. *Phys. Rep.*, 52(5):264–379, 1979.

[Cre03] Jacky Cresson. Symbolic dynamics and Arnold diffusion. *J. Differential Equations*, 187(2):269–292, 2003.

[CY02] Chong-Quin Cheng and Jun Yan. Existence of diffusion orbits in a priori unstable Hamiltonian systems. Preprint 03-360, `http://www.ma.utexas.edu/mp_arc/`, 2002.

[DdlLS00] A. Delshams, R. de la Llave, and T.M. Seara. A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbb{T}^2$. *Comm. Math. Phys.*, 209(2):353–392, 2000.

[DdlLS03a] A. Delshams, R. de la Llave, and T.M. Seara. A geometric mechanism for diffusion in hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model, 2003. To appear in Memoirs of the AMS, Preprint 03-182 `http://www.ma.utexas.edu/mp_arc/`.

[DdlLS03b] Amadeu Delshams, Rafael de la Llave, and Tere M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: announcement of results. *Electron. Res. Announc. Amer. Math. Soc.*, 9:125–134, 2003.

[DdlLS04] A. Delshams, R. de la Llave, and T.M. Seara. Orbits of unbounded energy in generic quasiperiodic perturbations of geodesic flows of certain manifolds. To appear Adv. in Mathematics, Preprint 04-280 `http://www.ma.utexas.edu/mp_arc/`, 2004.

[dlL04] R. de la Llave. Orbits of unbounded energy in perturbations of geodesic flows by periodic potentials. A simple construction. Preprint, 2004.

[dlLW95] R. de la Llave and C. E. Wayne. On Irwin's proof of the pseudostable manifold theorem. *Math. Z.*, 219(2):301–321, 1995.

[Eas75] Robert W. Easton. Isolating blocks and symbolic dynamics. *J. Differential Equations*, 17:96–118, 1975.

[Eas78] Robert W. Easton. Homoclinic phenomena in Hamiltonian systems with several degrees of freedom. *J. Differential Equations*, 29(2):241–252, 1978.

[Eas81] Robert W. Easton. Orbit structure near trajectories biasymptotic to invariant tori. In *Classical mechanics and dynamical systems (Medford, Mass., 1979)*, volume 70 of *Lecture Notes in Pure and Appl. Math.*, pages 55–67. Dekker, New York, 1981.

[FdlL92] C. Falcolini and R. de la Llave. A rigorous partial justification of Greene's criterion. *J. Statist. Phys.*, 67(3-4):609–643, 1992.

[Fen72] Neil Fenichel. Persistence and smoothness of invariant manifolds for flows. *Indiana Univ. Math. J.*, 21:193–226, 1971/1972.

[Fen74] N. Fenichel. Asymptotic stability with rate conditions. *Indiana Univ. Math. J.*, 23:1109–1137, 1973/74.

[Gar00] Antonio García. Transition tori near an elliptic fixed point. *Discrete Contin. Dynam. Systems*, 6(2):381–392, 2000.

[GR03] Marian Gidea and Clark Robinson. Topologically crossing heteroclinic connections to invariant tori. *J. Differential Equations*, 193(1):49–74, 2003.

[GR04] Marian Gidea and Clark Robinson. Symbolic dynamics for transition tori-II. In *New advances in celestial mechanics and Hamiltonian systems*, pages 95–108. Kluwer/Plenum, New York, 2004.

[GZ04] Marian Gidea and Piotr Zgliczyński. Covering relations for multidimensional dynamical systems. II. *J. Differential Equations*, 202(1):59–80, 2004.

[Hir94] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[HM82] P.J. Holmes and J.E. Marsden. Melnikov's method and Arnol'd diffusion for perturbations of integrable Hamiltonian systems. *J. Math. Phys.*, 23(4):669–675, 1982.

[HPS77] M.W. Hirsch, C.C. Pugh, and M. Shub. *Invariant manifolds*, volume 583 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1977.

[Hug] G. Huget. Work in progress.

[KP90] U. Kirchgraber and K. J. Palmer. *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization*, volume 233 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1990.

[LT83] M. A. Lieberman and Jeffrey L. Tennyson. Chaotic motion along resonance layers in near-integrable Hamiltonian systems with three or more degrees of freedom. In Claude Wendell Horton, Jr. and Linda E. Reichl, editors, *Long-time prediction in dynamics (Lakeway, Tex., 1981)*, pages 179–211. Wiley, New York, 1983.

[Moe96] Richard Moeckel. Transition tori in the five-body problem. *J. Differential Equations*, 129(2):290–314, 1996.

[Pes04] Yakov B. Pesin. *Lectures on partial hyperbolicity and stable ergodicity*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2004.

[PS70] Charles Pugh and Michael Shub. Linearization of normally hyperbolic diffeomorphisms and flows. *Invent. Math.*, 10:187–198, 1970.

[Rob71] Clark Robinson. Differentiable conjugacy near compact invariant manifolds. *Bol. Soc. Brasil. Mat.*, 2(1):33–44, 1971.

[Ten82] Jeffrey Tennyson. Resonance transport in near-integrable systems with many degrees of freedom. *Phys. D*, 5(1):123–135, 1982.

[Tre04] D. Treschev. Evolution of slow variables in a priori unstable hamiltonian systems. *Nonlinearity*, 17(5):1803–1841, 2004.

[ZG04] Piotr Zgliczyński and Marian Gidea. Covering relations for multidimensional dynamical systems. *J. Differential Equations*, 202(1):32–58, 2004.

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