Properties of a Polyharmonic Operator with Limit Periodic Potential in Dimension Two.

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In memory of our colleague and friend
Robert M. Kauffman.

ABSTRACT This is an announcement of the following results. We consider a polyharmonic operator $H=(-\Delta)^l+V(x)$ in dimension two with $l\geq 6$ and $V(x)$ being a limit-periodic potential. We prove that the spectrum of H contains a semiaxis and there is a family of generalized eigenfunctions at every point of this semiaxis with the following properties. First, the eigenfunctions are close to plane waves at the high energy region. Second, the isoenergetic curves in the space of momenta corresponding to these eigenfunctions have a form of a slightly distorted circles with holes (Cantor type structure). Third, the spectrum corresponding to the eigenfunctions (the semiaxis) is absolutely continuous.

1 Main Results.

We study the operator

$$H=(-\Delta)^l+V(x) \tag{1}$$

in two dimensions, $V(x)$ being a limit-periodic potential:

$$V(x)=\sum_{r=1}^{\infty}V_r(x), \tag{2}$$

where $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing L_∞-norms, namely, V_r has orthogonal periods $2^{r-1}\vec{b_1},\ 2^{r-1}\vec{b_2}$ and $\|V_r\|_\infty<exp(-2^{\eta r})$ for some $\eta>0$.

The one-dimensional analog of (1), (2) with $l=1$ has been already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1u=-u''+Vu$ is a Cantor type set with a positive Lebesgue measure [1, 6]. The spectrum is absolutely continuous [1, 2], [5]-[9]. Generalized eigenfunctions can be represented in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]-[14]. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operator are studied in [15]. As to the continuum multidimensional case, it is known that the integrated density of states for (1) is the limit of densities of states for periodic operators [14]. We concentrate here on properties of the spectrum and eigenfunctions of (1), (2) in the high energy region. We prove the following results for the case $d=2$, $l\geq 6$.

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1. The spectrum of the operator (1), (2) contains a semiaxis. We know that a proof of analogous result ($8l > d + 1$, $d \neq 1(\mathrm{mod} 4)$) by different means is to appear in the forthcoming paper [16].

2. There are generalized eigenfunctions $\Psi_\infty(\vec{k}, \vec{x})$, corresponding to the semiaxis, which are close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_\infty$ of $\mathbb{R}^2$, there is a solution $\Psi_\infty(\vec{k}, \vec{x})$ of the equation $H\Psi_\infty = \lambda_\infty\Psi_\infty$ which can be described by the formula:

$$\Psi_\infty(\vec{k}, \vec{x}) = e^{i\langle\vec{k}, \vec{x}\rangle}\left(1 + u_\infty(\vec{k}, \vec{x})\right), \tag{3}$$

$$\|u_\infty\| =_{|\vec{k}|\to\infty} O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1 > 0, \tag{4}$$

where $u_\infty(\vec{k}, \vec{x})$ is a limit-periodic function:

$$u_\infty(\vec{k}, \vec{x}) = \sum_{r=1}^{\infty} u_r(\vec{k}, \vec{x}), \tag{5}$$

$u_r(\vec{k}, \vec{x})$ being periodic with periods $2^{r-1}\vec{b_1}$, $2^{r-1}\vec{b_2}$. The eigenvalue $\lambda_\infty(\vec{k})$ corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_\infty(\vec{k}) =_{|\vec{k}|\to\infty} |\vec{k}|^{2l} + O(|\vec{k}|^{-\gamma_2}), \quad \gamma_2 > 0. \tag{6}$$

The "non-resonant" set $\mathcal{G}_\infty$ of the vectors $\vec{k}$, for which (3) – (6) hold, is an extensive Cantor type set: $\mathcal{G}_\infty = \cap_{n=1}^{\infty}\mathcal{G}_n$, where $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets with larger and larger number of holes in each bounded region, holes added on each step being of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$\lim_{R\to\infty} \frac{|(\mathcal{G}_\infty \cap \mathbf{B_R})|}{|\mathbf{B_R}|} = 1, \tag{7}$$

where $\mathbf{B_R}$ is the disk of radius R centered at the origin, $|\cdot|$ is the Lebesgue measure in $\mathbb{R}^2$.

3. The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \left\{\vec{k} \in \mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda\right\},$$

is proven to be a slightly distorted circle with infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \{\vec{k} : \vec{k} = \left(\varkappa_\infty(\lambda, \vec{\nu})\right)\vec{\nu},\ \vec{\nu} \in \mathcal{B}_\infty(\lambda)\}, \tag{8}$$

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_\infty(\lambda)$ can be interpreted as the set of possible propagation directions for the almost plane waves (3). The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on S_1 as $\lambda \to \infty$:

$$|\mathcal{B}_\infty(\lambda)| =_{\lambda\to\infty} 2\pi + O\left(\lambda^{-\gamma_3}\right), \quad \gamma_3 > 0. \tag{9}$$

The value $\varkappa_\infty(\lambda, \vec{\nu}) - \lambda^{1/2l}$ in (8) gives the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small

$$\varkappa_\infty(\lambda, \vec{\nu}) = \lambda^{1/2l} + O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4 > 0. \tag{10}$$

4. Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k},\vec{x})$ is proven.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method. This paper is inspired by [17, 18, 19], where the method is used for periodic problems. In [17] KAM method is applied to classical Hamiltonian systems. In [18, 19] the technique developed in [17] is applied to semiclassical approximation for multidimensional periodic Schrödinger operators at high energies.

We consider a sequence of operators

$$H_0=(-\Delta)^l, \qquad H^{(n)}=H_0+\sum_{r=1}^{M_n}V_r, \quad n\geq 1,\ M_n\to\infty \text{ as } n\to\infty.$$

Obviously, $\|H-H^{(n)}\|\to 0$ as $n\to\infty$ and $H^{(n)}=H^{(n-1)}+W_n$ where $W_n=\sum_{r=M_{n-1}+1}^{M_n}V_r$. We consider each operator $H^{(n)}$, $n\geq 1$, as a perturbation of the previous operator $H^{(n-1)}$. Every operator $H^{(n)}$ is periodic, however the periods go to infinity as $n\to\infty$. We show that there is a λ_0, $\lambda_0=\lambda_0(V)$, such that the semiaxis $[\lambda_0,\infty)$ is contained in the spectra of all operators $H^{(n)}$. For every operator $H^{(n)}$ there is a set of eigenfunctions (corresponding to the semiaxis) being close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k},\vec{x})$ of the differential equation $H^{(n)}\Psi_n=\lambda_n\Psi_n$, which can be described by the formula:

$$\Psi_n(\vec{k},\vec{x})=e^{i\langle\vec{k},\vec{x}\rangle}\left(1+\tilde{u}_n(\vec{k},\vec{x})\right), \quad \|\tilde{u}_n\|=_{|\vec{k}|\to\infty}O(|\vec{k}|^{-\gamma_1}), \quad \gamma_1>0, \tag{11}$$

where $\tilde{u}_n(\vec{k},\vec{x})$ is a periodic function. The corresponding eigenvalue $\lambda_n(\vec{k})$ is close to $|\vec{k}|^{2l}$:

$$\lambda_n(\vec{k})=_{|\vec{k}|\to\infty}|\vec{k}|^{2l}+O\left(|\vec{k}|^{-\gamma_2}\right), \quad \gamma_2>0.$$

The non-resonant set $\mathcal{G}_n$ is proven to be extensive in $\mathbb{R}^2$:

$$\lim_{R\to\infty}\frac{|\mathcal{G}_n\cap\mathbf{B_R}|}{|\mathbf{B_R}|}=1. \tag{12}$$

The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for non-resonant eigenvalue $\lambda_n(\vec{k})$:

$$\mathcal{D}_n(\lambda)=\left\{\vec{k}\in\mathcal{G}_n:\lambda_n(\vec{k})=\lambda\right\}.$$

This set is proven to be a slightly distorted circle with a finite number of holes, see figures 1 and 2.

It can be described by the formula:

$$\mathcal{D}_n(\lambda)=\{\vec{k}:\vec{k}=\varkappa_n(\lambda,\vec{\nu})\vec{\nu},\ \vec{\nu}\in\mathcal{B}_n(\lambda)\}, \tag{13}$$

where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle S_1. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves (11). It has an asymptotically full measure on S_1 as $\lambda\to\infty$:

$$L\left(\mathcal{B}_n(\lambda)\right)=_{\lambda\to\infty}2\pi+O\left(\lambda^{-\gamma_3}\right), \quad \gamma_3>0, \tag{14}$$

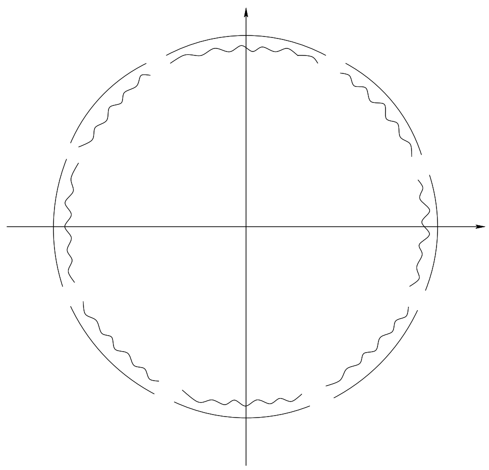

Figure 1: Distorted circle with holes, $\mathcal{D}_1(\lambda)$

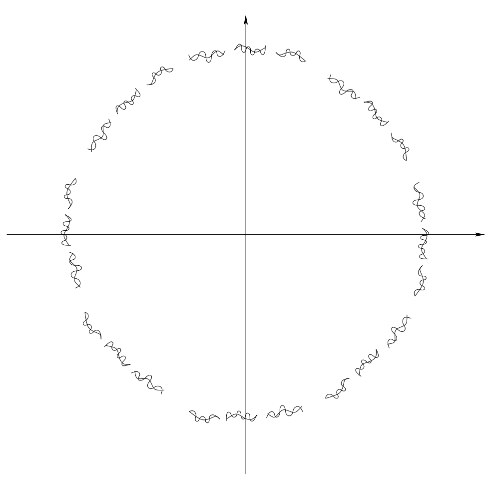

Figure 2: Distorted circle with holes, $\mathcal{D}_2(\lambda)$

here and below $L(\cdot)$ is the length of a curve. The set $\mathcal{B}_n$ has only a finite number of holes, however their number is growing with n. More and more holes of a smaller and smaller size are added at each step. The value $\varkappa_n(\lambda,\vec{\nu})-\lambda^{1/2l}$ gives the deviation of $\mathcal{D}_n(\lambda)$ from the perfect circle of the radius $\lambda^{1/2l}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small:

$$\varkappa_n(\lambda,\vec{\nu})=\lambda^{1/2l}+O\left(\lambda^{-\gamma_4}\right), \qquad \frac{\partial \varkappa_n(\lambda,\vec{\nu})}{\partial \varphi}=O\left(\lambda^{-\gamma_5}\right) \quad \gamma_4,\gamma_5>0, \tag{15}$$

φ being an angle variable $\vec{\nu}=(\cos\varphi,\sin\varphi)$. It is shown that $\mathcal{D}_1(\lambda)$ is strictly inside the perfect circle, here and below we assume without loss of generality that $\int V_r =$, for all r, integral being taken over the elementary cell of periods.

On each step more and more points are excluded from the non-resonant sets $\mathcal{G}_n$, thus $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $\mathcal{G}_\infty$ is defined as the limit set: $\mathcal{G}_\infty = \cap_{n=1}^{\infty}\mathcal{G}_n$. It has an infinite number of holes, but nevertheless satisfies the relation (7). For every $\vec{k}\in\mathcal{G}_\infty$ and every n, there is a generalized eigenfunction of $H^{(n)}$ of the type (11). It is proven that the sequence $\Psi_n(\vec{k},\vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ when $\vec{k}\in\mathcal{G}_\infty$. The function $\Psi_\infty(\vec{k},\vec{x})=\lim_{n\to\infty}\Psi_n(\vec{k},\vec{x})$ is a generalized eigenfunction of H. It can be written in the form (3) – (5), the functions u_n being related to $\tilde{u}_n$ in (11) by the formula $u_n=\tilde{u}_n-\tilde{u}_{n-1}$. Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda_n(\vec{k})$ as $n\to\infty$.

Obviously, $\{\mathcal{B}_n(\lambda)\}_{n=1}^{\infty}$ is a decreasing sequence of sets, since on each step more and more directions are excluded. We consider the limit $\mathcal{B}_\infty(\lambda)$ of $\mathcal{B}_n(\lambda)$:

$$\mathcal{B}_\infty(\lambda)=\bigcap_{n=1}^{\infty}\mathcal{B}_n(\lambda).$$

This set has a Cantor type structure on the unit circle. It is proven that $\mathcal{B}_\infty(\lambda)$ has an asymptotically full measure on the unit circle (see (9)). We prove that the sequence $\varkappa_n(\lambda,\vec{\nu})$, $n=1,2,...,$, describing the isoenergetic curves $\mathcal{D}_n$, quickly converges as $n\to\infty$. Hence, $\mathcal{D}_\infty(\lambda)$ can be described as the limit of $\mathcal{D}_n(\lambda)$ in the sense (8), where $\varkappa_\infty(\lambda,\vec{\nu})=\lim_{n\to\infty}\varkappa_n(\lambda,\vec{\nu})$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda,\vec{\nu})$ (with respect to the angle variable on the unit circle) have a limit as $n\to\infty$ for every $\vec{\nu}\in\mathcal{B}_\infty(\lambda)$. We denote this limit by $\frac{\partial\varkappa_\infty(\lambda,\vec{\nu})}{\partial\varphi}$. It follows from (15) that

$$\frac{\partial\varkappa_\infty(\lambda,\vec{\nu})}{\partial\varphi}=O\left(\lambda^{-\gamma_5}\right).$$

Thus, the limit curve $\mathcal{D}_\infty(\lambda)$ has a tangent vector in spite its Cantor type structure, the tangent vector being the limit of corresponding tangent vectors for $\mathcal{D}_n(\lambda)$ as $n\to\infty$. The curve $\mathcal{D}_\infty(\lambda)$ looks as a distorted circle with infinite number of holes.

Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to the functions $\Psi_\infty(\vec{k},\vec{x})$, $\vec{k}\in\mathcal{G}_\infty$ follows from the convergence of the spectral projections corresponding to $\Psi_n(\vec{k},\vec{x})$, $\vec{k}\in\mathcal{G}_\infty$, to spectral projections of H (in a strong sense uniformly in λ) and properties of the level curves $\mathcal{D}_\infty(\lambda)$, $\lambda>\lambda_0$.

The main technical difficulty overcome is construction of non-resonance sets $\mathcal{B}_n(\lambda)$ for every fixed sufficiently large λ. The set $\mathcal{B}_n(\lambda)$ is obtained by deleting a "resonant" part from $\mathcal{B}_{n-1}(\lambda)$.

Definition of $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ includes Bloch eigenvalues of $H^{(n-1)}$. To describe $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ one needs not only non-resonant eigenvalues of the type (6), but also resonant eigenvalues, for which no suitable formulae are known. Absence of formulae cause difficulties in estimating the size of $\mathcal{B}_n \setminus \mathcal{B}_{n-1}$. To deal with this problem we start with introducing an angle variable φ, $\varphi \in [0, 2\pi)$, $(\cos\varphi, \sin\varphi) \in S_1$ and consider sets $\mathcal{B}_n(\lambda)$ in terms of this variable. Next, we show that the resonant set $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ can be described as the set of zeros of determinants of the type $Det(I + S_n)$, $S_n = S_n(\varphi)$ being a trace type operator,

$$I + S_n(\varphi) = \Big(H^{(n-1)}\big(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\big) - \lambda - \epsilon\Big)\Big(H_0\big(\vec{\varkappa}_{n-1}(\varphi) + \vec{b}\big) + \lambda\Big)^{-1},$$

where $\vec{\varkappa}_{n-1}(\varphi)$ is a vector-function describing $\mathcal{D}_{n-1}(\lambda)$. To obtain $\mathcal{B}_{n-1} \setminus \mathcal{B}_n$ we take all values of ϵ in a small interval and values of $\vec{b}$ in a finite set, $\vec{b} \neq 0$. Further, we extend our considerations to a complex neighborhood Φ_0 of $[0, 2\pi)$. We show the determinants to be analytic functions of φ in Φ_0, and, by this, reduce the problem of estimating the size of the resonance set to a problem in complex analysis. We use theorems for analytic functions to count the zeros of the determinants and to investigate how far the zeros move when ε changes. It enables us to estimate the size of the zero set of the determinants, and, hence, the size of the non-resonance set $\Phi_n \subset \Phi_0$, which is defined as a non-zero set for the determinants. Proving that the non-resonance set Φ_n is sufficiently large, we obtain estimates (12) for $\mathcal{G}_n$ and (14) for $\mathcal{B}_n$, the set $\mathcal{B}_n$ being the real part of Φ_n. To obtain Φ_n we delete from Φ_0 more and more holes of smaller and smaller radii at each step. Thus, the non-resonance set $\Phi_n \subset \Phi_0$ has a structure of Swiss Cheese (Fig. 7, 8). Deleting resonance set from Φ_0 at each step of the recurrent procedure we call a "Swiss Cheese Method". The essential difference of our method from those applied in similar situations before (see e.g. [17, 18, 19]) is that we construct a non-resonance set not only on the whole space of a parameter ($\vec{k} \in \mathbb{R}^2$ here), but also on all isoenergetic curves $\mathcal{D}_n(\lambda)$ in the space of parameter corresponding to sufficiently large energies. Estimates for the size of non-resonance sets in this case require more subtle technical considerations than those sufficient for description of a non-resonant set in the whole space of the parameter.

The restriction $l \geq 6$ is technical, it is needed only for the first two steps of the recurrent procedure. The requirement for super exponential decay of $\|V_n\|$ as $n \to \infty$ is more essential, since it is needed to ensure convergence of the recurrent procedure.

Proofs of the described results (except absolute continuity) are presented in [22]. A paper containing all the proofs is in preparation. Below we sketch main steps of the recurrent procedure and a "Swiss cheese method".

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2 Recurrent Procedure.

2.1 The First Approximation.

2.1.1 The Main Operator H and the First Operator $H^{(1)}$.

We introduce the first operator $H^{(1)}$, which corresponds to a partial sum in the series (2):

$$H^{(1)} = (-\Delta)^l + W_1, \qquad W_1 = \sum_{r=1}^{M_1} V_r, \tag{16}$$

where M_1 is chosen in such a way that $2^{M_1} \approx k^{s_1}$ [2] for a $k > 1$, $s_1 = (2l - 11)/32$. Obviously, the periods of W_1 are $(a_1, 0) = 2^{M_1 - 1}(b_1, 0)$ and $(0, a_2) = 2^{M_1 - 1}(0, b_2)$, and $a_1 \approx k^{s_1} b_1/2$, $a_2 \approx k^{s_1} b_2/2$. Note that

$$\|W_1\|_\infty \leq \sum_{n=1}^{M_1} \|V_n\|_\infty = O(1) \text{ as } k \to \infty.$$

It is well-known (see e.g. [20]) that spectral analysis of a periodic operator $H^{(1)}$ can be reduced to analysis of a family of operators $H^{(1)}(t)$, $t \in K_1$, where K_1 is the elementary cell of the dual lattice, $K_1 = [0, 2\pi a_1^{-1}) \times [0, 2\pi a_2^{-1})$. The vector t is called *quasimomentum*. The operator $H^{(1)}(t)$, $t \in K_1$, acts in $L_2(Q_1)$, Q_1 being the elementary cell of the periods of the potential, $Q_1 = [0, a_1] \times [0, a_2]$. Operator $H^{(1)}(t)$ is described by formula (16) and the quasiperiodic conditions for a function itself and its derivatives:

$$u(a_1, x_2) = \exp(it_1 a_1) u(0, x_2), \quad u(x_1, a_2) = \exp(it_2 a_2) u(x_1, 0), \tag{17}$$

$$u^{(j)}_{x_1}(a_1, x_2) = \exp(it_1 a_1) u^{(j)}_{x_1}(0, x_2), \quad u^{(j)}_{x_2}(x_1, a_2) = \exp(it_2 a_2) u^{(j)}_{x_2}(x_1, 0), \quad 0 < j < 2l.$$

Each operator $H^{(1)}(t)$, $t \in K_1$, has a discrete bounded below spectrum $\Lambda^{(1)}(t)$:

$$\Lambda^{(1)}(t) = \cup_{n=1}^\infty \lambda_n^{(1)}(t), \ \lambda_n^{(1)}(t) \to_{n \to \infty} \infty.$$

The spectrum $\Lambda^{(1)}$ of the operator $H^{(1)}$ is the union of the spectra of the operators $H^{(1)}(t)$: $\Lambda^{(1)} = \cup_{t \in K_1} \Lambda(t) = \cup_{n \in N, t \in K_1} \lambda_n^{(1)}(t)$. The functions $\lambda_n^{(1)}(t)$ are continuous in t, so $\Lambda^{(1)}$ has a band structure:

$$\Lambda^{(1)} = \cup_{n=1}^\infty [q_n^{(1)}, Q_n^{(1)}], \ q_n^{(1)} = \min_{t \in K_1} \lambda_n^{(1)}(t), \ Q_n^{(1)} = \max_{t \in K_1} \lambda_n^{(1)}(t). \tag{18}$$

Eigenfunctions of $H^{(1)}(t)$ and $H^{(1)}$ are simply related. Extending all the eigenfunctions of the operators $H^{(1)}(t)$ quasiperiodically (see (17)) to $\mathbb{R}^2$, we obtain a complete system of generalized eigenfunctions of $H^{(1)}$.

Let $H_0^{(1)}$ be the operator (1) corresponding to $V = 0$. We consider that it has periods a_1, a_2, operators $H_0^{(1)}(t)$, $t \in K_1$, being defined in $L_2(Q_1)$. Eigenfunctions of an operator $H_0^{(1)}(t)$, $t \in K_1$, are plane waves satisfying (17). They are naturally indexed by points of $\mathbb{Z}^2$: $\Psi_j^0(t, x) = |Q_1|^{-1/2} \exp i\langle \vec{p}_j(t), x\rangle$, $j \in \mathbb{Z}^2$, the eigenvalue corresponding to $\Psi_j^0(t, x)$ being equal to $p_j^{2l}(t)$, here and below $\vec{p}_j(t) = 2\pi j/a + t$, $2\pi j/a = (2\pi j_1/a_1, 2\pi j_2/a_2)$, $j \in \mathbb{Z}^2$, $|Q_1| = a_1 a_2$ and $p_j^{2l}(t) = |\vec{p}_j(t)|^{2l}$.

[2] We write $a(k) \approx b(k)$ when the inequalities $\frac{1}{2} b(k) < a(k) < 2b(k)$ hold.

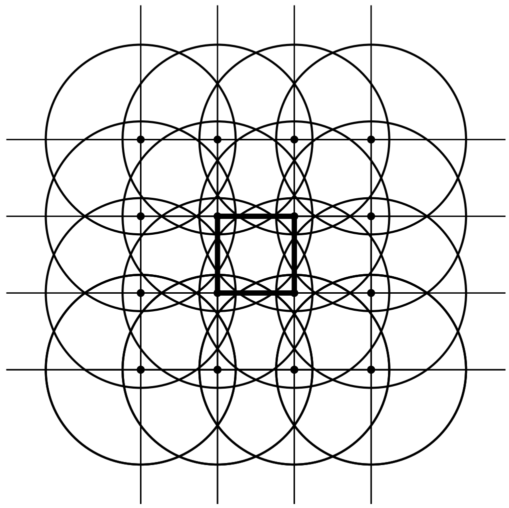

Figure 3: The isoenergetic surface $S_0(\lambda)$ of the free operator $H_0^{(1)}$

Let us introduce an isoenergetic surface[3] $S_0(\lambda)$ of the free operator $H_0^{(1)}$. A point t belongs to $S_0(\lambda)$ if and only if $H_0^{(1)}(t)$ has an eigenvalue equal to λ, i.e., if there exists a $j \in \mathbb{Z}^2$ such that $p_j^{2l}(t) = \lambda$. This surface can be obtained as follows: the circle of radius $k = \lambda^{\frac{1}{2l}}$ centered at the origin is divided into pieces by the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$, and then all pieces are translated in a parallel manner into the cell K_1 of the dual lattice. We also can get $S_0(\lambda)$ by drawing sufficiently many circles of radii k centered at the dual lattice $\{\vec{p}_q(0)\}_{q\in\mathbb{Z}^2}$ and by looking at the figure in the cell K_1. As a result of any of these two procedures we obtain a circle of radius k "packed into the bag K_1" as it is shown in the Fig. 3. Note that each piece of $S_0(\lambda)$ can be described by an equation $p_j^{2l}(t) = \lambda$ for a fixed j. If $t \in S_0(\lambda)$, then j can be uniquely defined from the last equation unless t is not the point of a self-intersection of the isoenergetic surface. A point t is a self-intersection of $S_0(\lambda)$ if and only if

$$p_q^{2l}(t) = p_j^{2l}(t) = k^{2l} \tag{19}$$

for at least on pair of indices q, j, $q \neq j$.

Note that any vector $\vec{\varkappa}$ in $\mathbb{R}^2$ can be uniquely represented in the form $\vec{\varkappa} = \vec{p}_j(t)$, where $j \in \mathbb{Z}^2$ and $t \in K_1$. Let $\mathcal{K}_1$ be the mapping: $\mathcal{K}_1 : \mathbb{R}^2 \to K_1$, $\mathcal{K}_1\big(\vec{p}_j(t)\big) = t$. Obviously, $\mathcal{K}_1 S_k = S_0(\lambda)$ and $L\big(S_0(\lambda)\big) = L\big(S_k\big) = 2\pi k$, $k = \lambda^{\frac{1}{2l}}$, S_k being the circle of radius k centered at the origin. Without the loss of generality we assume $\int_{Q_1} W_1(x)dx = 0$.

2.1.2 Perturbation Formulae.

In this section we consider operator $H^{(1)}(t)$ as a perturbation of $H_0^{(1)}(t)$. We show that for every sufficiently large λ there is a "non-resonant" subset $\chi_1(\lambda)$ of $S_0(\lambda)$, such that perturbation

[3] "surface" is a traditional term. In our case, it is a curve.

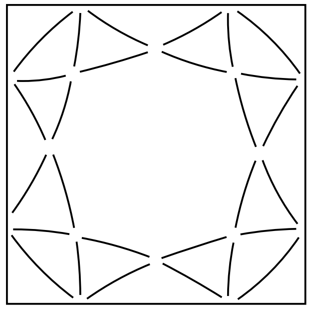

Figure 4: The first non-resonance set $\chi_1(\lambda)$

series for an eigenvalue and a spectral projection of $H^{(1)}(t)$ converge when $t \in \chi_1(\lambda)$. The set $\chi_1(\lambda)$ is obtained by deleting small neighborhoods of self-intersections of $S_0(\lambda)$, Fig. 4. The self-intersections are described by (19) and correspond to degenerated eigenvalues of $H_0^{(1)}(t)$. The size of the neighborhood is $k^{-1-4s_1-\delta}$, $k = \lambda^{\frac{1}{2l}}$, δ being a small positive number. The set $\chi_1(\lambda)$ is sufficiently large: its relative measure with respect to $S_0(\lambda)$ tends to 1 as $\lambda \to \infty$. The precise formulation of these results is given in the next Geometric Lemma. The lemma is proven by elementary geometric considerations. We follow [21] at this step.

Lemma 1 (Geometric Lemma)**.** *For an arbitrarily small positive δ, $2\delta < 2l - 2 - 4s_1$, and sufficiently large λ, $\lambda > \lambda_0(\delta)$, there exists a non-resonance set $\chi_1(\lambda, \delta) \subset S_0(\lambda)$, such that:*

1. *For any point $t \in \chi_1$ the following conditions hold:*
   - *(a) There exists a unique $j \in \mathbb{Z}^2$, such that $p_j(t) = k$, $k = \lambda^{\frac{1}{2l}}$.*
   - *(b) The inequality holds:*
   $$\min_{i \neq j} |p_j^2(t) - p_i^2(t)| > 2k^{-4s_1-\delta}. \tag{20}$$
2. *For any t in the $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, there exists a unique $j \in \mathbb{Z}^2$ such that*
   $$|p_j^2(t) - k^2| < 5k^{-4s_1-2\delta} \tag{21}$$
   and that the inequality (20) holds.
3. *The non-resonance set, χ_1, has an asymptotically full measure on $S_0(\lambda)$ in the following sense:*
   $$\frac{L(S_0(\lambda) \setminus \chi_1(\lambda, \delta))}{L(S_0(\lambda))} =_{\lambda \to \infty} O(k^{-\delta/2}). \tag{22}$$

Corollary 2. *If t belongs to the $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, then, for any z lying on the circle $C_1 = \{z : |z - k^{2l}| = k^{2l-2-4s_1-\delta}\}$ and any i in $\mathbb{Z}^2$, the inequality $2|p_i^{2l}(t) - z| > k^{2l-2-4s_1-\delta}$ holds.*

Let E_j be the spectral projection of the free operator, corresponding to the eigenvalue p_j^{2l} : $(E_j)_{rm} = \delta_{jr}\delta_{jm}$. In the $(2k^{-1-4s_1-2\delta})$-neighborhood of χ_1, we define functions $g_r^{(1)}(k,t)$ and operator-valued functions $G_r^{(1)}(k,t)$, $r = 1, 2, \cdots$ as follows:

$$g_r^{(1)}(k,t) = \frac{(-1)^r}{2\pi i r} \mathrm{Tr} \oint_{C_1} ((H_0^{(1)}(t) - z)^{-1} W_1)^r dz, \tag{23}$$

$$G_r^{(1)}(k,t) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_1} ((H_0^{(1)}(t) - z)^{-1} W_1)^r (H_0^{(1)}(t) - z)^{-1} dz. \tag{24}$$

They are well-defined, since Corollary 2 holds. To find $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$, one has to compute residues of rational functions of simple structure, whose numerator does not depend on z, while the denominator is a product of factors of the type $(p_i^{2l}(t) - z)$. For all t in the non-resonance set the integrand has a single pole within C_1 at the point $z = k^{2l} = p_j^{2l}(t)$. By computing the residue at this point, we obtain explicit expressions for $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$. For example, $g_1^{(1)}(k,t) = 0$, since $\int_{Q_1} W_1(x)dx = 0$;

$$g_2^{(1)}(k,t) = \sum_{q \in \mathbb{Z}^2, q \neq 0} |w_q|^2 (p_j^{2l}(t) - p_{j+q}^{2l}(t))^{-1}; \tag{25}$$

$$G_1^{(1)}(k,t)_{rm} = w_{j-m}(p_j^{2l}(t) - p_m^{2l}(t))^{-1}\delta_{rj} + w_{r-j}(p_j^{2l}(t) - p_r^{2l}(t))^{-1}\delta_{mj}, \quad G_1^{(1)}(k,t)_{jj} = 0, \tag{26}$$

w_q being Fourier coefficients of W_1. It can be easily shown that $g_2^{(1)} > 0$ when $W_1 \neq 0$. For technical reasons it is convenient to introduce parameter α in front of the potential W_1. Namely, we consider $H_\alpha^{(1)} = (-\Delta)^l + \alpha W_1$, $0 \leq \alpha \leq 1$, in special, $H_1^{(1)} = H^{(1)}$.

Theorem 3. *Let t belong to the $(2k^{-1-4s_1-2\delta})$-neighborhood in K_1 of the non-resonance set $\chi_1(\lambda, \delta)$, $0 < 2\delta < 2l - 2 - 4s_1$. Then for sufficiently large λ, $\lambda > \lambda_0(\|V\|, b_1, b_2, \delta)$, and for all α, $-1 \leq \alpha \leq 1$, there exists a unique eigenvalue of the operator $H_\alpha^{(1)}(t)$ in the interval $\varepsilon_1(k,\delta) := (k^{2l} - k^{2l-2-4s_1-\delta}, k^{2l} + k^{2l-2-4s_1-\delta})$. It is given by the series:*

$$\lambda_j^{(1)}(\alpha, t) = p_j^{2l}(t) + \sum_{r=2}^{\infty} \alpha^r g_r^{(1)}(k,t), \tag{27}$$

converging absolutely in the disk $|\alpha| \leq 1$, where the index j is determined according to Parts 1(a) and 2 of Geometric Lemma. The spectral projection, corresponding to $\lambda_j^{(1)}(\alpha,t)$ is given by the series:

$$E_j^{(1)}(\alpha, t) = E_j + \sum_{r=1}^{\infty} \alpha^r G_r^{(1)}(k,t), \tag{28}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha| \leq 1$.

For coefficients $g_r^{(1)}(k,t)$, $G_r^{(1)}(k,t)$ the following estimates hold:

$$|g_r^{(1)}(k,t)| < k^{2l-2-4s_1-\gamma_0 r-\delta}, \tag{29}$$

$$\|G_r^{(1)}(k,t)\|_1 < k^{-\gamma_0 r}, \tag{30}$$

where $\gamma_0 = 2l - 2 - 4s_1 - 2\delta$.

Corollary 4. *For the perturbed eigenvalue and its spectral projection the following estimates hold:*

$$\left|\lambda_j^{(1)}(\alpha,t)-p_j^{2l}(t)\right| \le 2\alpha^2 k^{2l-2-4s_1-2\gamma_0-\delta}, \tag{31}$$

$$\left\|E_j^{(1)}(\alpha,t)-E_j\right\|_1 \le 2|\alpha| k^{-\gamma_0}. \tag{32}$$

Let us introduce the notations:

$$T(m):=\frac{\partial^{|m|}}{\partial t_1^{m_1}\partial t_2^{m_2}},\quad m=(m_1,m_2),\quad |m|:=m_1+m_2,\quad m!:=m_1!m_2!,\quad T(0)f:=f$$

We show [21] that the coefficients $g_r^{(1)}(k,t)$, and $G_r^{(1)}(k,t)$ can be continued as holomorphic functions of two variables from the real $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set χ_1 to its complex neighborhood of the same size and the following estimates hold in the complex neighborhood:

$$|T(m)g_r^{(1)}(k,t)|<m!k^{2l-2-4s_1-\delta-\gamma_0 r+|m|(1+4s_1+2\delta)}, \|T(m)G_r^{(1)}(k,t)\|_1<m!k^{-\gamma_0 r+|m|(1+4s_1+2\delta)}.$$

From this the following theorem easily follows.

Theorem 5. *The series (27), (28) can be continued as holomorphic functions of two variables to the complex $(2k^{-1-4s_1-2\delta})$-neighborhood of the non-resonance set χ_1 from its neighborhood in K_1, and the following estimates hold in the complex neighborhood:*

$$|T(m)(\lambda_j^{(1)}(\alpha,t)-p_j^{2l}(t))|<2m!\alpha^2k^{2l-2-4s_1-2\gamma_0-\delta+|m|(1+4s_1+2\delta)}, \tag{33}$$

$$\|T(m)(E_j^{(1)}(\alpha,t)-E_j)\|_1<2m!\alpha k^{-\gamma_0+|m|(1+4s_1+2\delta)}. \tag{34}$$

2.1.3 Nonresonant Part of Isoenergetic Set of $H^{(1)}$.

Let $S_1(\lambda)$[4] be an isoenergetic set of the operator $H_\alpha^{(1)}$:

$$S_1(\lambda)=\{t\in K_1:\exists n\in\mathbb{N}:\ \lambda_n^{(1)}(\alpha,t)=\lambda\}, \tag{35}$$

where $\{\lambda_n^{(1)}(\alpha,t)\}_{n=1}^{\infty}$ is the complete set of eigenvalues of $H_\alpha^{(1)}(t)$. Next, we construct a "non-resonance" subset $\chi_1^*(\lambda)$ of $S_1(\lambda)$, which corresponds to non-resonance eigenvalues $\lambda_j^{(1)}(\alpha,t)$ given by the perturbation series (27). Let us note, that for every t belonging to the non-resonant set $\chi_1(\lambda,\delta)$, there is a single $j\in\mathbb{Z}^2$ such that $p_j(t)=k$, $k=\lambda^{\frac{1}{2l}}$. This means that the function $t\to\vec{p}_j(t)$ maps $\chi_1(\lambda,\delta)$ into the circle S_k of radius k. We denote the image of $\chi_1(\lambda,\delta)$ in S_k by $\mathcal{D}_0(\lambda)_{nonres}$. Obviously,

$$\chi_1(\lambda,\delta)=\mathcal{K}_1\mathcal{D}_0(\lambda)_{nonres}, \tag{36}$$

where $\mathcal{K}_1$ establishes one-to one relation between two sets. Let $\mathcal{B}_1(\lambda)$ be the set of unit vectors corresponding to $\mathcal{D}_0(\lambda)_{nonres}$:

$$\mathcal{B}_1(\lambda)=\{\vec{\nu}\in S_1:k\vec{\nu}\in\mathcal{D}_0(\lambda)_{nonres}\}.$$

[4] $S_1(\lambda)$ definitely depends on αW_1, however we omit this to keep the notation simple.

It is easy to see that $\mathcal{B}_1(\lambda)$ is a unit circle with holes centered at the origin. We denote by Θ_1 the set of angles φ in polar coordinates, corresponding to $\mathcal{B}_1(\lambda)$:

$$\Theta_1(\lambda) = \{\varphi \in [0, 2\pi) : \ (\cos\varphi, \sin\varphi) \in \mathcal{B}_1(\lambda)\}.$$

Let $\vec{\varkappa} \in \mathcal{D}_0(k)_{nonres}$. Then, by (36), there is $j \in \mathbb{Z}^2$, $t \in \chi_1(\lambda, \delta)$, such that $\vec{\varkappa} = \vec{p}_j(t)$, [5] where $t = \mathcal{K}_1\vec{\varkappa}$. According to Theorem 3, for sufficiently large k, there exists an eigenvalue of the operator $H_\alpha^{(1)}(t)$, $t = \mathcal{K}_1\vec{\varkappa}$, $0 \le \alpha \le 1$, given by (27). It is convenient here to denote $\lambda_j^{(1)}(\alpha, t)$ by $\lambda^{(1)}(\alpha, \vec{\varkappa})$, we can do this since there is one-to-one correspondence between $\vec{\varkappa}$ and the pair (t, j). We rewrite (27) in the form:

$$\lambda^{(1)}(\alpha, \vec{\varkappa}) = \varkappa^{2l} + f_1(\alpha, \vec{\varkappa}), \qquad \varkappa = |\vec{\varkappa}| \tag{37}$$

$$|T(m) f_1(\alpha, \vec{\varkappa})| \le 2\alpha^2 k^{2l-2-4s_1-2\gamma_0-\delta+|m|(1+4s_1+2\delta)}. \tag{38}$$

where $f_1(\alpha, \vec{\varkappa}) = \sum_{r=2}^{\infty} \alpha^r g_r^{(1)}(\vec{\varkappa})$, $g_r^{(1)}(\vec{\varkappa})$ being defined by (23) with j and t such that $\vec{p}_j(t) = \vec{\varkappa}$. By Theorem 3, the formulae (37), (38) hold in $(k^{-1-4s_1-2\delta})$-neighborhood of $\mathcal{D}_0(\lambda)_{nonres}$, i.e., they hold for any $\varkappa\vec{\nu}$ such that $\vec{\nu} \in \mathcal{B}_1(\lambda)$, $|\varkappa - k| < k^{-1-4s_1-2\delta}$. We define $\mathcal{D}_1(\lambda)$ as the level set of the function $\lambda^{(1)}(\alpha, \vec{\varkappa})$ in this neighborhood:

$$\mathcal{D}_1(\lambda) := \{\vec{\varkappa}_1 = \varkappa_1\vec{\nu} : \vec{\nu} \in \mathcal{B}_1(\lambda), \ |\varkappa_1 - k| < k^{-1-4s_1-2\delta}, \ \lambda^{(1)}(\alpha, \vec{\varkappa}_1) = \lambda\}. \tag{39}$$

Lemma 6. *1. The set $\mathcal{D}_1(\lambda)$ is a distorted circle with holes (Fig. 1): it can be described by the formula:*

$$\mathcal{D}_1(\lambda) = \left\{\vec{\varkappa}_1 \in R^2 : \vec{\varkappa}_1 = \varkappa_1(\lambda, \vec{\nu})\vec{\nu}, \quad \vec{\nu} \in \mathcal{B}_1(\lambda)\right\}, \tag{40}$$

where $\varkappa_1(\lambda, \vec{\nu}) = k + h_1(\lambda, \vec{\nu})$ and $h_1(\lambda, \vec{\nu})$ satisfies the estimates

$$|h_1| < ck^{-1-4s_1-2\gamma_0-\delta}, \tag{41}$$

$$\left|\frac{\partial h_1}{\partial \varphi}\right| \le ck^{-2\gamma_0+1}, \tag{42}$$

given $\vec{\nu} = (\cos\varphi, \sin\varphi)$.

2. Function $h_1(\lambda, \vec{\nu})$ can be extended as a holomorphic function of φ to the complex $(k^{-2-4s_1-2\delta})$-neighborhood of each connected component of Θ_1 and estimates (41), (42) hold.

3. The curve $\mathcal{D}_1(k)$ has a length which is asymptotically close to that of the whole circle in the following sense:

$$L\big(\mathcal{D}_1(\lambda)\big) \underset{\lambda\to\infty}{=} 2\pi k\Big(1 + O(k^{-\delta/2})\Big). \tag{43}$$

It can be easily shown that $\lambda_1(\vec{\varkappa}) > |\vec{\varkappa}|^{2l}$ (since $g_2^{(1)}(\vec{\varkappa}) > 0$). Therefore, $\mathcal{D}_1(\lambda)$ is strictly inside the perfect circle.

Next, define a non-resonance subset $\chi_1^*(\lambda)$ of isoenergetic set $S_1(\lambda)$ as the a parallel shift of $\mathcal{D}_1(\lambda)$ into K_1 (Fig. 5):

$$\chi_1^*(\lambda) := \mathcal{K}_1\mathcal{D}_1(\lambda). \tag{44}$$

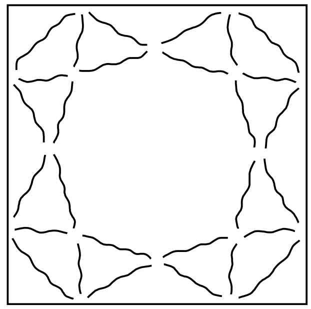

Figure 5: The set $\chi_1^*(\lambda)$

Lemma 7. *The set $\chi_1^*(\lambda)$ belongs to the $(ck^{-1-4s_1-2\gamma_0-\delta})$-neighborhood of $\chi_1(\lambda)$ in K_1. If $t \in \chi_1^*(\lambda)$, then the operator $H_\alpha^{(1)}(t)$ has a simple eigenvalue $\lambda_n^{(1)}(\alpha, t)$, $n \in \mathbb{N}$, equal to λ, no other eigenvalues being in the interval $\varepsilon_1(k,\delta)$, $\varepsilon_1(k,\delta) := (k^{2l} - k^{2l-2-4s_1-\delta}, k^{2l} + k^{2l-2-4s_1-\delta})$. The eigenvalue is given by the perturbation series (27), j being uniquely defined by t from the relation $p_j^{2l}(t) \in \varepsilon_1(k,\delta)$.*

Lemma 8. *Formula (44) establishes one-to-one correspondence between $\chi_1^*(\lambda)$ and $\mathcal{D}_1(\lambda)$.*

From geometric point of view, the lemma means that $\chi_1^*(\lambda)$ does not have self-intersections.

2.2 The Second Step of Approximation.

2.2.1 The Operator $H_\alpha^{(2)}$.

Choosing $s_2 = 2s_1$, we define the second operator $H_\alpha^{(2)}$ by the formula:

$$H_\alpha^{(2)} = H^{(1)} + \alpha W_2, \quad (0 \le \alpha \le 1), \qquad W_2 = \sum_{r=M_1+1}^{M_2} V_r, \tag{45}$$

where $H^{(1)}$ is defined by (16), M_2 is chosen in such a way that $2^{M_2} \approx k^{s_2}$. Obviously, the periods of W_2 are $2^{M_2-1}(b_1, 0)$ and $2^{M_2-1}(0, b_2)$. We will write them in the form: $N_1(a_1, 0)$ and $N_2(0, a_2)$, where a_1, a_2 are the periods of W_1 and $N_1 = 2^{M_2-M_1} \approx k^{s_2-s_1}$. Note that

$$\|W_2\|_\infty \le \sum_{n=M_1+1}^{M_2} \|V_n\|_\infty \le \sum_{n=M_1+1}^{M_2} \exp(-2^{\eta n}) < \exp(-k^{\eta s_1}). \tag{46}$$

2.2.2 Multiple Periods of $W_1(x)$.

The operator $H^{(1)} = H_0 + W_1(x)$ has the periods a_1, a_2. The corresponding family of operators, $\{H^{(1)}(t)\}_{t \in K_1}$, acts in $L_2(Q_1)$, where $Q_1 = [0, a_1] \times [0, a_2]$ and $K_1 = [0, 2\pi/a_1) \times [0, 2\pi/a_2)$.

[5] Usually the vector $\vec{p}_j(t)$ is denoted by $\vec{k}$, the corresponding plane wave being $e^{\langle \vec{k}, \vec{x} \rangle}$. We use less common notation $\vec{\varkappa}$, since we have already other k's in the text.

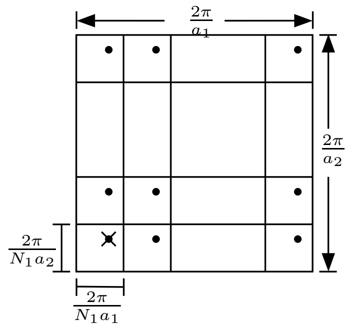

Figure 6: Relation between τ and t_p

Eigenvalues of $H^{(1)}(t)$ are denoted by $\lambda_n^{(1)}(t)$, $n \in \mathbb{N}$, and its spectrum by $\Lambda^{(1)}(t)$. Now let us consider the same $W_1(x)$ as a periodic function with the periods N_1a_1, N_1a_2. Obviously, the definition of the operator $H^{(1)}$ does not depend on the way how we define the periods of W_1. However, the family of operators $\{H^{(1)}(t)\}_{t\in K_1}$ does change, when we replace the periods a_1, a_2 by N_1a_1, N_1a_2. The family of operators $\{H^{(1)}(t)\}_{t\in K_1}$ has to be replaced by a family of operators $\{\tilde{H}_1(\tau)\}_{\tau\in K_2}$ acting in $L_2(Q_2)$, where $Q_2 = [0, N_1a_1] \times [0, N_1a_2]$ and $K_2 = [0, 2\pi/N_1a_1) \times [0, 2\pi/N_1a_2)$. We denote eigenvalues of $\tilde{H}_1(\tau)$ by $\tilde{\lambda}_n^{(1)}(\tau)$, $n \in \mathbb{N}$ and its spectrum by $\tilde{\Lambda}^{(1)}(\tau)$. The next lemma establishes a connection between spectra of operators $H^{(1)}(t)$ and $\tilde{H}_1(\tau)$. It easily follows from Bloch theory (see e.g. [20]).

Lemma 9. *For any* $\tau \in K_2$,

$$\tilde{\Lambda}^{(1)}(\tau) = \bigcup_{p\in P} \Lambda^{(1)}(t_p), \tag{47}$$

where

$$P = \{p = (p_1, p_2) \in \mathbb{Z}^2 : 0 \le p_1 \le N_1 - 1,\ 0 \le p_2 \le N_1 - 1\} \tag{48}$$

and $t_p = (t_{p,1}, t_{p,2}) = (\tau_1 + 2\pi p_1/N_1a_1, \tau_2 + 2\pi p_2/N_1a_2) \in K_1$, *as shown in the Figure 6.*

We defined isoenergetic set $S_1(\lambda) \subset K_1$ of $H^{(1)}$ by formula (35). Obviously, this definition is directly associated with the family of operators $H^{(1)}(t)$ and, therefore, with periods a_1, a_2, which we assigned to $W_1(x)$. Now, assuming that the periods are equal to N_1a_1, N_1a_2, we give an analogous definition of the isoenergetic set $\tilde{S}_1(\lambda)$ in K_2:

$$\tilde{S}_1(\lambda) := \{\tau \in K_2 : \exists n \in \mathbb{N} :\ \ \tilde{\lambda}_n^{(1)}(\tau) = \lambda\}.$$

By Lemma 9, $\tilde{S}_1(\lambda)$ can be expressed as follows:

$$\tilde{S}_1(\lambda) = \Big\{\tau \in K_2 : \exists n \in \mathbb{N},\ p \in P :\ \lambda_n^{(1)}\big(\tau + 2\pi p/N_1a\big) = \lambda\Big\}, \quad 2\pi p/N_1a = \left(\frac{2\pi p_1}{N_1a_1}, \frac{2\pi p_2}{N_1a_2}\right).$$

The relation between S_1 and $\tilde{S}_1$ can be easily understood from the geometric point of view as $\tilde{S}_1 = \mathcal{K}_2 S_1$, where $\mathcal{K}_2$ is the parallel shift into K_2:

$$\mathcal{K}_2 : \mathbb{R}^2 \to K_2,\ \mathcal{K}_2(\tau + 2\pi m/N_1a) = \tau,\ m \in \mathbb{Z}^2,\ \tau \in K_2.$$

Thus, $\tilde{S}_1(\lambda)$ is obtained from S_1 by cutting S_1 into pieces of the size K_2 and shifting them together in K_2.

Definition 10. *We say that τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $n, \hat{n} \in N$, $n \neq \hat{n}$ such that $\tilde{\lambda}_n^{(1)}(\tau) = \tilde{\lambda}_{\hat{n}}^{(1)}(\tau) = \lambda$.*

Remark 1. *By Lemma 9, τ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $p, \hat{p} \in P$ and a pair $n, \hat{n} \in N$ such that $|p - \hat{p}| + |n - \hat{n}| \neq 0$ and $\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1 a) = \lambda$.*

Now let us recall that the isoenergetic set $S_1(\lambda)$ consists of two parts: $\chi_1^*(\lambda)$ and $S_1(\lambda) \setminus \chi_1^*(\lambda)$, where $\chi_1^*(\lambda)$ is the first non-resonance set given by (44). Obviously $\mathcal{K}_2\chi_1^*(\lambda) \subset \mathcal{K}_2 S_1(\lambda) = \tilde{S}_1(\lambda)$ and can be described by the formula:

$$\mathcal{K}_2\chi_1^*(\lambda) = \{\tau \in K_2 : \ \exists p \in P : \tau + 2\pi p/N_1 a \in \chi_1^*(\lambda)\}. \tag{49}$$

Furthermore, we consider only those self-intersections of $\tilde{S}_1$ which belong to $\mathcal{K}_2\chi_1^*(\lambda)$, i.e., we consider the points of intersection of $\mathcal{K}_2\chi_1^*(\lambda)$ both with itself and with $\tilde{S}_1(\lambda) \setminus \mathcal{K}_2\chi_1^*(\lambda)$.

To obtain a new non-resonance set $\chi_2(\lambda)$ we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ neighborhoods of its self-intersections with $\tilde{S}_1(\lambda)$. Strictly speaking, we remove from $\mathcal{K}_2\chi_1^*(\lambda)$ the following set:

$$\Omega_1(\lambda) = \Big\{\tau \in \mathcal{K}_2\chi_1^*(\lambda) : \exists n, \hat{n} \in \mathbb{N}, \ p, \hat{p} \in P, \ p \neq \hat{p} : \ \lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda,$$
$$\tau + 2\pi p/N_1 a \in \chi_1^*(\lambda), \ \left|\lambda_n^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1 a)\right| < \epsilon_1\Big\}, \quad \epsilon_1 = e^{-\frac{1}{4}k^{\eta s_1}}.$$

We define $\chi_2(\lambda)$ by the formula:

$$\chi_2(\lambda) = \mathcal{K}_2\chi_1^*(\lambda) \setminus \Omega_1(\lambda). \tag{50}$$

2.2.3 Perturbation Formulae.

Lemma 11 (Geometric Lemma)**.** *For an arbitrarily small positive δ, $7\delta < 2l - 11 - 16s_1$ and sufficiently large λ, $\lambda > \lambda_0(V, \delta)$, there exists a non-resonance set $\chi_2(\lambda, \delta) \subset \mathcal{K}_2\chi_1^*$ such that:*

1. *For any $\tau \in \chi_2$, the following conditions hold:*
   - *(a) There exist a unique $p \in P$ such that $\tau + 2\pi p/N_1 a \in \chi_1^*$.* [6]
   - *(b) The following relation holds:*

     $$\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) = k^{2l},$$

     where $\lambda_j^{(1)}(\tau + 2\pi p/N_1 a)$ is given by the perturbation series (27) with $\alpha = 1$, j being uniquely defined by $t = \tau + 2\pi p/N_1 a$ as it is described in Part 2 of Geometric Lemma for the previous step.

[6] From geometric point of view this means that χ_2 does not have self-intersections.

(c) *The eigenvalue* $\lambda_j^{(1)}(\tau+2\pi p/N_1a)$ *is a simple eigenvalue of* $\tilde H^{(1)}(\tau)$ *and the distance between it and all other eigenvalues* $\lambda_{\hat n}^{(1)}(\tau+2\pi\hat p/N_1a)$, $\hat n\in\mathbb{N}$ *of* $\tilde H_1(\tau)$ *is greater than* $\epsilon_1=e^{-\frac14 k^{\eta s_1}}$:

$$|\lambda_j^{(1)}(\tau+2\pi p/N_1a)-\lambda_{\hat n}^{(1)}(\tau+2\pi\hat p/N_1a)|>\epsilon_1. \tag{51}$$

2. *For any* τ *in the* $(\epsilon_1k^{-2l+1-\delta})$*-neighborhood in* $\mathbb{C}^2$ *of* χ_2, *there exist a unique* $p\in P$ *such that* $\tau+2\pi p/N_1a$ *is in the* $(\epsilon_1k^{-2l+1-\delta})$*-neighborhood in* $\mathbb{C}^2$ *of* χ_1^* *and*

$$|\lambda_j^{(1)}(\tau+2\pi p/N_1a)-k^{2l}|<\epsilon_1k^{-\delta}. \tag{52}$$

j being uniquely defined by $\tau+2\pi p/N_1a$ as it is described in Part 2 of Geometric Lemma for the previous step.

3. *The second non-resonance set* χ_2 *has an asymptotically full measure in* χ_1^* *in the following sense:*

$$\frac{L(\mathcal{K}_2\chi_1^*\setminus\chi_2))}{L(\chi_1^*)}<k^{-2-2s_1}. \tag{53}$$

Remark 2. *Note that every point* $2\pi m/N_1a$ ($m\in\mathbb{Z}^2$) *of a dual lattice corresponding to the larger periods* N_1a_1, N_1a_2 *can be uniquely represented in the form* $2\pi m/N_1a=2\pi j/a+2\pi p/N_1a$, *where* $m=N_1j+p$ *and* $2\pi j/a$ *is a point of a dual lattice for periods* a_1, a_2, *while* $p\in P$ *is responsible for refining the lattice.*

Let us consider a normalized eigenfunction $\psi_n(t,x)$ *of* $H^{(1)}(t)$ *in* $L_2(Q_1)$. *We extended it quasiperiodically to* Q_2, *renormalize in* $L_2(Q_2)$ *and denote the new function by* $\tilde\psi_n(\tau,x)$, $\tau=\mathcal{K}_2t$. *The Fourier representations of* $\psi_n(t,x)$ *in* $L_2(Q_1)$ *and* $\tilde\psi_n(\tau,x)$ *in* $L_2(Q_2)$ *are simply related. If we denote Fourier coefficients of* $\psi_n(t,x)$ *with respect to the basis of exponents* $|Q_1|^{-1/2}e^{i\langle\vec p_j(t),x\rangle}$, $j\in\mathbb{Z}^2$, *in* $L_2(Q_1)$ *by* C_{nj}, *then, the Fourier coefficients* $\tilde C_{nm}$ *of* $\tilde\psi_n(\tau,x)$ *with respect to the basis of exponents* $|Q_2|^{-1/2}e^{i(2\pi m/N_1a+\tau,x)}$, $m\in\mathbb{Z}^2$, *in* $L_2(Q_2)$ *are given by the formula:*

$$\tilde C_{nm}=\begin{cases}C_{nj}, & \text{if } m=jN_1+p;\\ 0, & \text{otherwise,}\end{cases}$$

p being defined from the relation $t=\tau+2\pi p/N_1a$, $p\in P$. Hence, matrices of the projections on $\psi_n(t,x)$ and $\tilde\psi_n(\tau,x)$ with respect to the above bases are simply related:

$$(\tilde E_n)_{j\hat j}=\begin{cases}(E_n)_{m\hat m}, & \text{if } m=jN_1+p,\ \hat m=\hat jN_1+p;\\ 0, & \text{otherwise,}\end{cases}$$

$\tilde E_n$ and E_n being projections in $L_2(Q_2)$ and $L_2(Q_1)$, respectively.

Let us denote by $\tilde E_j^{(1)}\big(\tau+2\pi p/N_1a\big)$ *the spectral projection* $E_j^{(1)}(\alpha,t)$ *(see (28)) with* $\alpha=1$ *and* $t=\tau+2\pi p/N_1a$, *"extended" from* $L_2(Q_1)$ *to* $L_2(Q_2)$.

By analogy with (23), (24), we define functions $g_r^{(2)}(k,\tau)$ and operator-valued functions $G_r^{(2)}(k,\tau)$, $r=1,2,\cdots$, as follows:

$$g_r^{(2)}(k,\tau)=\frac{(-1)^r}{2\pi ir}\mathrm{Tr}\oint_{C_2}\Big(\big(\tilde{H}_1(\tau)-z\big)^{-1}W_2\Big)^r dz, \tag{54}$$

$$G_r^{(2)}(k,\tau)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_2}\Big(\big(\tilde{H}_1(\tau)-z\big)^{-1}W_2\Big)^r\big(\tilde{H}_1(\tau)-z\big)^{-1}dz. \tag{55}$$

We consider the operators $H_\alpha^{(2)}=H^{(1)}+\alpha W_2$ and the family $H_\alpha^{(2)}(\tau)$, $\tau\in K_2$, acting in $L_2(Q_2)$.

Theorem 12. *Suppose τ belongs to the $(\epsilon_1 k^{-2l+1-\delta})$-neighborhood in K_2 of the second non-resonance set $\chi_2(\lambda,\delta)$, $0<7\delta<2l-11-16s_1$, $\epsilon_1=e^{-\frac{1}{4}k^{\eta s_1}}$. Then, for sufficiently large λ, $\lambda>\lambda_0(\|V\|,\delta,\eta)$ and for all α, $0\le\alpha\le 1$, there exists a unique eigenvalue of the operator $H_\alpha^{(2)}(\tau)$ in the interval $\varepsilon_2(k,\delta):=(k^{2l}-\epsilon_1/2,k^{2l}+\epsilon_1/2)$. It is given by the series:*

$$\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)=\lambda_j^{(1)}\big(\tau+2\pi p/N_1a\big)+\sum_{r=1}^{\infty}\alpha^r g_r^{(2)}(k,\tau),\qquad \tilde{j}=j+p/N_1, \tag{56}$$

converging absolutely in the disk $|\alpha|\le 1$, where $p\in P$ and $j\in Z^2$ are described as in Geometric Lemma 11. The spectral projection, corresponding to $\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)$ is given by the series:

$$E_{\tilde{j}}^{(2)}(\alpha,\tau)=\tilde{E}_j^{(1)}\big(\tau+2\pi p/N_1a\big)+\sum_{r=1}^{\infty}\alpha^r G_r^{(2)}(k,\tau), \tag{57}$$

which converges in the trace class $\mathbf{S_1}$ uniformly with respect to α in the disk $|\alpha|\le 1$.

The following estimates hold for coefficients $g_r^{(2)}(k,\tau)$, $G_r^{(2)}(k,\tau)$, $r\ge 1$:

$$\big|g_r^{(2)}(k,\tau)\big|<\frac{3\epsilon_1}{2}(4\epsilon_1^3)^r, \tag{58}$$

$$\big\|G_r^{(2)}(k,\tau)\big\|_1<6r(4\epsilon_1^3)^r. \tag{59}$$

Corollary 13. *The following estimates hold for the perturbed eigenvalue and its spectral projection:*

$$\Big|\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)-\lambda_j^{(1)}\big(\tau+2\pi p/N_1a\big)\Big|\le 12\alpha\epsilon_1^4, \tag{60}$$

$$\Big\|E_{\tilde{j}}^{(2)}(\alpha,\tau)-\tilde{E}_j^{(1)}\big(\tau+2\pi p/N_1a\big)\Big\|_1\le 48\alpha\epsilon_1^3. \tag{61}$$

The proof of the theorem is analogous to that of Theorem 3 and it is based on expanding the resolvent $(H_\alpha^{(2)}(\tau)-z)^{-1}$ in a perturbation series for $z\in C_2$, C_2 being the contour around the unperturbed eigenvalue k^{2l}: $C_2=\{z:|z-k^{2l}|=\frac{\epsilon_1}{2}\}$. Integrating the resolvent yields the formulae for an eigenvalue of $H_\alpha^{(2)}$ and its spectral projection.

Theorem 14. *Under the conditions of Theorem 12 the series (56), (57) can be continued as holomorphic functions of two variables from the real* $(\epsilon_1 k^{-2l+1-\delta})$*-neighborhood of the non-resonance set* χ_2 *to its complex* $(\epsilon_1 k^{-2l+1-\delta})$*-neighborhood and the following estimates hold in the complex neighborhood:*

$$\left|T(m)\left(\lambda_{\tilde{j}}^{(2)}(\alpha,\tau)-\lambda_j^{(1)}(\tau+2\pi p/N_1a)\right)\right| < m!\cdot\alpha\cdot 3\cdot 2^{2+|m|}\epsilon_1^{4-|m|}k^{|m|(2l-1+\delta)}, \quad (62)$$

$$\left\|T(m)\left(E_{\tilde{j}}^{(2)}(\alpha,\tau)-\tilde{E}_j^{(1)}(\tau+2\pi p/N_1a)\right)\right\|_1 < m!\cdot\alpha\cdot 3\cdot 2^{4+|m|}\epsilon_1^{3-|m|}k^{|m|(2l-1+\delta)}. \quad (63)$$

2.2.4 Proof of the Geometric Lemma.

Parts 1 and 2 of Geometric Lemma easily follow from the definition of the non-resonance set. The main problem is to prove that the non-resonance set exists and is rather extensive, i.e., Part 3. We outline a proof of Part 3 below.

Determinants. Intersections and Quasi-intersections. Description of the set Ω_1 **in terms of determinants.** We consider self-intersections of $\tilde{S}_1(\lambda)$ belonging to $\mathcal{K}_2\chi_1^*$. We describe self-intersections as zeros of determinants of operators of the type $I+A$, $A\in\mathbf{S_1}$ (see e.g. [20]). In fact, let us represent the operator $(H^{(1)}(t)-\lambda)(H_0(t)+\lambda)^{-1}$ in the form $I+A_1$:

$$(H^{(1)}(t)-\lambda)(H_0(t)+\lambda)^{-1}=I+A_1(t),\quad A_1(t)=(W_1-2\lambda)(H_0(t)+\lambda)^{-1}. \quad (64)$$

Obviously, $A_1(t)$ is in the trace class $\mathbf{S_1}$. From properties of determinants and the definition of $S_1(\lambda)$ it easily follows that the isoenergetic set is the zero set of $\det\Big(I+A_1(t)\Big)$ in K_1.

Next, we recall that the set $\mathcal{D}_1(\lambda)$ can be described in terms of vectors $\vec{\varkappa}_1(\varphi)=\varkappa_1(\lambda,\vec{\nu})\vec{\nu}$, $\vec{\nu}=(\cos\varphi,\sin\varphi)$, $\varphi\in\Theta_1(\lambda)$, see Lemma 6. By definition, $\chi_1^*(\lambda)=\mathcal{K}_1\mathcal{D}_1(\lambda)$. By Lemma 8, the curve $\chi_1^*(\lambda)$ does not have self-intersections, i.e., for every $t\in\chi_1^*(\lambda)$, there is a single $\vec{\varkappa}_1(\varphi)\in\mathcal{D}_1(\lambda)$, such that $t=\mathcal{K}_1\vec{\varkappa}_1(\varphi)$. If $\tau\in\mathcal{K}_2\chi_1^*(\lambda)$, there is $p\in P$ such that $\tau+2\pi p/N_1a\in\chi_1^*(\lambda)$. Note that p is not uniquely defined by τ, since $\mathcal{K}_2\chi_1^*(\lambda)$ may have self-intersections. Hence, every $\tau\in\mathcal{K}_2\chi_1^*(\lambda)$ can be represented as $\tau=\mathcal{K}_2\vec{\varkappa}_1(\varphi)$, where $\vec{\varkappa}_1(\varphi)$ is not necessary uniquely defined. The next lemma describes self-intersections of $\mathcal{K}_2\chi_1^*(\lambda)$ as zeros of a group of determinants.

Lemma 15. *If* τ *is a point of self-intersection of* $\tilde{S}_1$ *(Definition 10), belonging to* $\mathcal{K}_2\chi_1^*(\lambda)$, *then* $\tau=\mathcal{K}_2\vec{\varkappa}_1(\varphi)$, *where* $\varphi\in\Theta_1(\lambda)$ *and satisfies the equation*

$$\det\Big(I+A_1\left(\vec{y}(\varphi)\right)\Big)=0,\quad \vec{y}(\varphi)=\vec{\varkappa}_1(\varphi)+\vec{b},\quad \vec{b}=2\pi p/N_1a, \quad (65)$$

for some $p\in P\setminus\{0\}$. *Conversely, if (65) is satisfied for some* $p\in P\setminus\{0\}$, *then* $\tau=\mathcal{K}_2\vec{\varkappa}_1(\varphi)$ *is a point of self-intersection.*

Definition 16. *Let* Φ_1 *be the complex* $\left(k^{-2-4s_1-2\delta}\right)$*-neighborhood of* $\Theta_1(\lambda)$.

By Lemma 6, $\vec{\varkappa}_1(\varphi)$ is an analytic function in Φ_1, and, hence, $\det\Big(I+A\big(\vec{y}(\varphi)\big)\Big)$, $\vec{y}(\varphi)=\vec{\varkappa}_1(\varphi)+\vec{b}$, $\vec{b}\in K_1$, is analytic too.

Definition 17. *We say that $\varphi \in \Phi_1$ is a quasi-intersection of $\mathcal{K}_2\chi_1^*$ with $\tilde{S}_1(\lambda)$ if the relation (65) holds for some $p \in P \setminus \{0\}$.*

Thus, real intersections correspond to real zeros of the determinant, while quasi-intersections may have a small imaginary part (quasi-intersections include intersections).

Next we describe the non-resonance set in terms of determinants.

Lemma 18. *If $\tau \in \Omega_1$, then $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$ where $\varphi \in \Theta_1(\lambda)$ and satisfies the equation*

$$\det\left(\frac{H^{(1)}\big(\vec{y}(\varphi)\big) - k^{2l} - \epsilon}{H_0\big(\vec{y}(\varphi)\big) + k^{2l}}\right) = 0, \quad \vec{y}(\varphi) = \vec{\varkappa}_1(\lambda, \vec{\nu}) + \vec{b}, \quad \vec{b} = 2\pi p/N_1 a, \tag{66}$$

for some $p \in P \setminus \{0\}$ and $|\epsilon| < \epsilon_1$. Conversely, if (66) is satisfied for some $p \in P \setminus \{0\}$, and $|\epsilon| < \varepsilon_1$, then $\tau = \mathcal{K}_2\vec{\varkappa}_1(\varphi)$ belongs to Ω_1.

We denote the set of $\varphi \in \Theta_1(\lambda)$ corresponding to Ω_1 by ω_1, i.e., $\omega_1 = \{\varphi \in \Theta_1(\lambda) : \mathcal{K}_2\vec{\varkappa}_1(\varphi) \in \Omega_1\}$.

Complex resonant set. Further we consider a complex resonance set $\omega_1^*(\lambda)$, which is the set of zeros of the determinants (66) in Φ_1 ($p \in P \setminus \{0\}$, $|\epsilon| < \epsilon_1$). Obviously, $\omega_1 = \omega_1^* \cap \Theta_1(\lambda)$. We prefer to consider quasi-intersections, instead of intersections, and the complex resonance set, instead of just the real one, for the following reason: the determinants (65) and (66), involved in the definitions of quasi-intersections and the complex resonance set ω_1^* are holomorphic functions of φ in Φ_1. We can apply complex analysis theorem to these determinants. Rouché's theorem is particularly important here since it states stability of zeros of a holomorphic function with respect to small perturbations of the function. We take the determinant (65) as a holomorphic function, its zeros being quasi-intersections, the perturbation of the determinant being obtained by "switching on" a potential W_1. Since there is no analog of Rouché's theorem for real functions on the real axis, introduction of the region Φ_1 and analytic extension of the determinants into this region is in the core of our considerations. We will use also a well-known inequality for the determinants (e.g. see [20]):

$$\Big|\det(I + A) - \det(I + B)\Big| \leq \|A - B\|_1 \exp\Big(\|A\|_1 + \|B\|_1 + 1\Big), \ A, B \in \mathbf{S}_1. \tag{67}$$

Note that $\omega_1^* = \cup_{p\in P\setminus\{0\}}\omega_{1,p}^*$, where $\omega_{1,p}^*$ corresponds to a fixed p in (66); and similarly, $\omega_1 = \cup_{p\in P\setminus\{0\}}\omega_{1,p}$. We fix $p \in P$ and study $\omega_{1,p}^*$. We start with the case $W_1 = 0$. The corresponding determinant (65) is

$$\det\Big(I + A_0\big(\vec{y}_0(\varphi)\big)\Big), \ I + A_0\big(\vec{y}_0(\varphi)\big) = \Big(H_0\big(\vec{y}_0(\varphi)\big) - \lambda\Big)\Big(H_0\big(\vec{y}_0(\varphi)\big) + \lambda\Big)^{-1}, \ \vec{y}_0(\varphi) = k(\cos\varphi, \sin\varphi) + \vec{b}. \tag{68}$$

This determinant can be investigated by elementary means. We easily check that the number of zeros of the determinant in Φ_1 does not exceed $c_0 k^{2+2s_1}$. The resolvent $\Big(H_0\big(\vec{y}_0(\varphi)\big) - \lambda\Big)^{-1}$ has poles at zeros of the determinant. The resolvent norm at $\varphi \in \Phi_1$ can be easily estimated by the distance which φ has to the nearest zero of the determinant. Next, we introduce a set $\mathcal{O}(\vec{b})$, which is the union of all disks of the radius $r = k^{-4-6s_1-3\delta}$ surrounding zeros of the

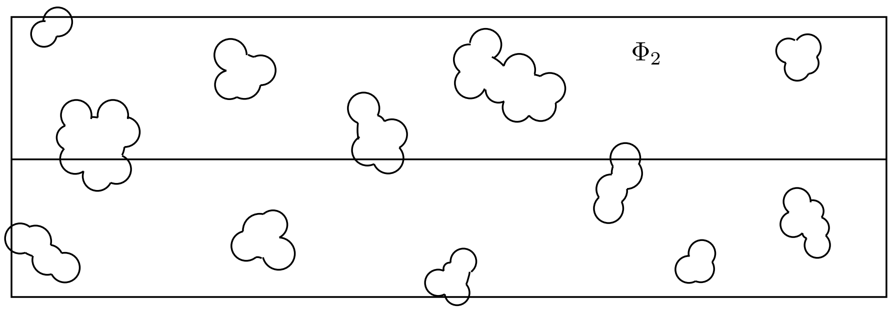

Figure 7: The set Φ_2.

determinant (68) in Φ_1. Obviously, any $\varphi \in \Phi_1 \setminus \mathcal{O}(\vec{b})$ is separated from zeros of the determinant (68) by the distance no less than r. This estimate on the distance yields an estimate for the norm of the resolvent $\Big(H_0\big(\vec{y}_0(\varphi)\big) - \lambda\Big)^{-1}$, when $\varphi \in \Phi_1 \setminus \mathcal{O}(\vec{b})$. Furthermore, we introduce a potential W_1 and ϵ. Our goal in this section is to prove that the number of zeros of each determinant (66) is preserved in a connected component $\Gamma(\vec{b})$ of $\mathcal{O}(\vec{b})$, when we switch from the case $W_1 = 0, A_1 = A_0$ to the case of nonzero W_1 and from $\epsilon = 0$ to $|\epsilon| < \epsilon_1$. We also show that estimates for the resolvent are stable under such change, if $\varphi \in \Phi_1 \setminus \mathcal{O}(\vec{b})$. We "switch on" potential W_1 and ϵ in two steps. First, we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$, i.e., we consider $\det\Big(I + A_0\big(\vec{y}(\varphi)\big)\Big)$ and $\Big(H_0\big(\vec{y}(\varphi)\big) - k^{2l}\Big)^{-1}$ in Φ_1. We take into account that $\vec{y}(\varphi) - \vec{y}_0(\varphi)$ is small and holomorphic in Φ_1 (Lemma 6), use (67) on the boundary of Γ, and apply Rouché's theorem. This enables us to conclude that the number of zeros of the determinant in $\Gamma(\vec{b})$ is preserved when we replace $\vec{y}_0(\varphi)$ by $\vec{y}(\varphi)$. Applying Hilbert relation for resolvents, we show that the estimates for the resolvent in $\Phi_1 \setminus \mathcal{O}(\vec{b})$ are also stable under such change. In the second step we replace $H_0\big(\vec{y}(\varphi)\big)$ by $H^{(1)}\big(\vec{y}(\varphi)\big) + \epsilon I$ and prove similar results. From this lemma we see that $\omega^*_{1,p} \subset \mathcal{O}(\vec{b})$, $\vec{b} = 2\pi p/N_1 a$ and

$$\omega_1^* \subset \mathcal{O}_* = \cup_{p \in P \setminus \{0\}} \mathcal{O}\big(2\pi p/N_1 a\big).$$

Considering that that $\mathcal{O}(\vec{b})$ is formed by no more than $c_0 k^{2+2s_1}$ disks of the radius $r = k^{-4-6s_1-3\delta}$ and the set P contains no more than $4k^{2s_2-2s_1}$ elements, $s_2 = 2s_1$, we easily obtain that $\mathcal{O}_*$ contains no more than $4c_0k^{2+4s_1}$ disks. Taking the real parts of the sets, we conclude: $\omega_1 \subset \mathcal{O}_* \cap \Theta_1(\lambda)$. Considering that that $\mathcal{O}_*$ is formed by disks of the radius $r = k^{-4-6s_1-3\delta}$ and using the estimate for the number of disks, we easily obtain, that the total length of ω_1 does not exceed $k^{-2-2s_1-3\delta}$ and hence the length of Ω_1 does not exceed $k^{-1-2s_1-3\delta}$.

Let us introduce new notation:

$$\Phi_2 = \Phi_1 \setminus \mathcal{O}_*, \tag{69}$$

Φ_1, being given by Definition 16. Obviously, to obtain Φ_2 we produced round holes in each connected component of Φ_1. The set Φ_2 has a structure of Swiss cheese (Fig. 7); we will add more holes of a smaller size at each step of approximation.

Basing on perturbation formulae (56), (57), we construct $\mathcal{B}_2(\lambda)$, $\mathcal{D}_2(\lambda)$ (Fig. 2) and $\chi_2^*(\lambda)$ in the way analogous to the step 1.

2.3 Next Steps of Approximation.

On the n-th step, $n \geq 3$, we choose $s_n = 2s_{n-1}$ define the operator $H_\alpha^{(n)}$ by the formula:

$$H_\alpha^{(n)} = H^{(n-1)} + \alpha W_n, \quad (0 \leq \alpha \leq 1), \qquad W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r,$$

where M_n is chosen in such a way that $2^{M_n} \approx k^{s_n}$. Obviously, the periods of W_n are $2^{M_n-1}(b_1, 0)$ and $2^{M_n-1}(0, b_2)$. We will write the periods in the form: $N_{n-1} \cdot ... \cdot N_1(a_1, 0)$ and $N_{n-1} \cdot ... \cdot N_1(0, a_2)$, where N_{n-1} is of order of $k^{s_n - s_{n-1}}$, namely, $N_{n-1} = 2^{M_n - M_{n-1}}$. Note that $\|W_n\|_\infty \leq \sum_{r=M_{n-1}+1}^{M_n} \|V_r\|_\infty \leq \exp(-k^{\eta s_{n-1}})$.

The geometric lemma for n-th step is the same as that for Step 2 up to a shift of indices. Note only that we need an inductive procedure to define the set χ_{n-1}^*: it is defined by (44) for $n = 1$ and in the analogous way for for $n \geq 2$. The estimate (53) for n-th step takes the form:

$$\frac{L\left(\mathcal{K}_n \chi_{n-1}^* \setminus \chi_n\right)}{L\left(\chi_{n-1}^*\right)} < k^{-\mathcal{S}_n}, \qquad \mathcal{S}_n = 2\sum_{i=1}^{n-1}(1 + s_i). \tag{70}$$

It is easy to see that $\mathcal{S}_n = 2(n-1) + (2^n - 2)\, s_1$ and $\mathcal{S}_n \approx 2^{n-1} s_1 = s_n$. The formulation of the main results (perturbation formulae) for n-th step is the same as for the third step up to the shift of indices. The formula for the resonance set Ω_{n-1} and non-resonance set χ_n are analogous to those for Ω_1, χ_2 (see (50)). Proof of the first and second statement of Geometric Lemma follows from the definition of the non-resonance set. Now we describe shortly a proof of the third statement.

In the second step we defined the set $\mathcal{O}\left(\vec{b}\right) \equiv \mathcal{O}^{(1)}\left(\vec{b}^{(1)}\right)$. Now we define $\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$, $\vec{b}^{(n-1)} \in K_{n-1}$, $n \geq 3$, by the analogous formula:

$$\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right) = \cup_{p^{(n-2)} \in P^{(n-2)}} \mathcal{O}_s^{(n-2)}\left(\vec{b}^{(n-1)} + 2\pi p^{(n-2)}/\hat{N}_{n-2} a\right), \quad \hat{N}_{n-2} \equiv N_{n-2} \cdot .... \cdot N_1, \tag{71}$$

here and below $P^{(m)} = \left\{p^{(m)} = \left(p_1^{(m)}, p_2^{(m)}\right),\ 0 \leq p_1^{(m)} < N_m - 1, 0 \leq p_2^{(m)} < N_m - 1\right\}$; the set $\mathcal{O}_s^{(m)}(\vec{b}^{(m)})$, $m \geq 1$, is a collection of disks of the radius $r^{(m+1)} = r^{(m)} k^{-2-4s_{m+1}-\delta}$, $r^{(1)} = r = k^{-4-6s_1-3\delta}$, around zeros of the determinant $\det\left(I + A_m\left(\vec{y}^{(m)}(\varphi)\right)\right)$ in $\mathit{\Phi}_m$, $\vec{y}^{(m)} = \vec{\varkappa}^{(m)}(\varphi) + \vec{b}^{(m)}$, the set $\mathit{\Phi}_m$ being defined above for $m = 1, 2$ (Definition 16 and formula (69)) and below for $m \geq 3$. Next, let

$$\mathcal{O}_*^{(n-1)} = \cup_{p^{(n-1)} \in P^{(n-1)} \setminus \{0\}} \mathcal{O}^{(n-1)}\left(2\pi p^{(n-1)}/\hat{N}_{n-1} a\right), \qquad \mathit{\Phi}_n = \mathit{\Phi}_{n-1} \setminus \mathcal{O}_*^{(n-1)}. \tag{72}$$

If $n = 2$ then (71) and (72) give $\mathcal{O}\left(\vec{b}\right)$ and $\mathcal{O}_*$ respectively. Note that the complex non-resonance set $\mathit{\Phi}_n$ is defined by the recurrent formula analogous to (69).

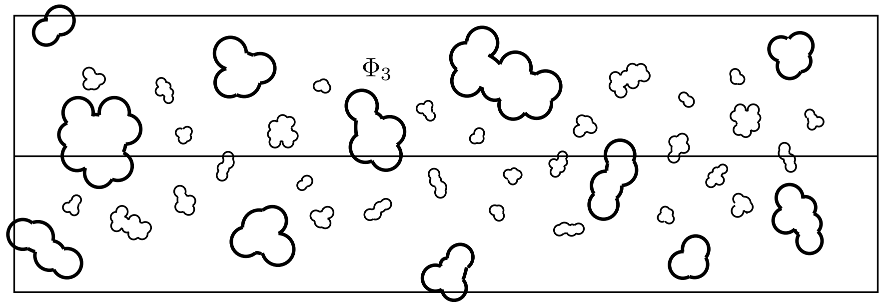

Figure 8: The set Φ_3.

Lemma 19. *The set* $\mathcal{O}_s^{(m)}\left(\vec{b}^{(m)}\right)$, $\vec{b}^{(m)} \in K_m$ *contains less than* $4^{m-1}c_o k^{2+2s_m}$ *disks.*

Corollary 20. *The set* $\mathcal{O}^{(n-1)}\left(\vec{b}^{(n-1)}\right)$ *contains less than* $4^{n-2}c_o k^{2+2s_{n-1}}$ *disks.*

Corollary 21. *The set* $\mathcal{O}_*^{(n-1)}$ *contains less than* $4^{n-1}c_o k^{2+2s_n}$ *disks.*

The lemma is proved by induction procedure. Corollaries 20 and 21 are based on the fact that $P^{(n-2)}$ contains less that $4k^{2(s_{n-1}-s_{n-2})}$ elements and a similar estimate holds for $P^{(n-1)}$.

Obviously, Φ_n has a structure of Swiss cheese, more and more holes of smaller and smaller radii appear at each step of approximation (Fig. 8). Note that the disks (holes) are more and more precisely "targeted" at each step of approximation. At the n-th step the disks of $\mathcal{O}_*^{(n-1)}$ are centered around the zeros of the determinants

$$\det\Big(I + A_{\mathbf{n-2}}\left(\vec{\varkappa}_{n-2}(\varphi) + 2\pi p^{(n-2)}/\hat{N}_{n-2}a + 2\pi p^{(n-1)}/\hat{N}_{n-1}a\right)\Big),$$

$$p^{(n-2)} \in P^{(n-2)}, \quad p^{(n-1)} \in P^{(n-1)},$$

where $\vec{\varkappa}_{n-2}(\varphi) \in \mathcal{D}_{n-2}$, the corresponding to $A_{\mathbf{n-2}}$ operator $H^{(n-2)}$ being closer and closer to the operator H. Here, $\lambda^{(n-2)}(\vec{\varkappa}_{n-2}(\varphi)) = \lambda$. If $W_1 = W_2 = ... = W_{n-2} = 0$, then $\mathcal{O}_*^{(n-1)}$ is just the union of disks centered at quasi-intersections of the "unperturbed" circle $\vec{k} = k(\cos\varphi, \sin\varphi)$, $k = \lambda^{\frac{1}{2l}}$, $\varphi \in [0, 2\pi)$ with circles of the same radius centered at points $2\pi j/a + 2\pi p^{(1)}/N_1 a + ... + 2\pi p^{(n-1)}/\hat{N}_{n-1}a$, these points being nodes of the dual lattice corresponding to the periods $\hat{N}_{n-1}a_1, \hat{N}_{n-1}a_2$. After constructing $\chi_n(\lambda)$ as the real part of Φ_n, we define a non-resonance subset $\chi_n^*(\lambda)$ of the isoenergetic set $S_n(\lambda)$ of $H_\alpha^{(n)}$, $S_n(\lambda) \subset K_n$. It corresponds to non-resonance eigenvalues given by perturbation series. The sets $\chi_1^*(\lambda)$, $\chi_2^*(\lambda)$, are defined in the previous steps, as well as the non-resonance sets $\chi_1(\lambda)$, $\chi_2(\lambda)$. Let us recall that we started with the definition of $\chi_1(\lambda)$ (Fig. 4) and then use it to define $\mathcal{D}_1(\lambda)$ (Fig. 1) and $\chi_1^*(\lambda)$, $\chi_1^* = \mathcal{K}_1\mathcal{D}_1$ (Fig. 5). In the second step we constructed $\chi_2(\lambda)$, using $\chi_1^*(\lambda)$. Next, we defined $\mathcal{D}_2(\lambda)$ (Fig. 2) and $\chi_2^*(\lambda)$, $\chi_2^* = \mathcal{K}_2\mathcal{D}_2$. Thus, the process looks like $\chi_1 \to \mathcal{D}_1 \to \chi_1^* \to \chi_2 \to \mathcal{D}_2 \to \chi_2^* \to \chi_3 \to \mathcal{D}_3 \to \chi_3^* \to ....$ Every next χ_n is defined on χ_{n-1}^* by a formula analogous to (50).

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