SPECTRAL AND LOCALIZATION PROPERTIES FOR THE ONE-DIMENSIONAL BERNOULLI DISCRETE DIRAC OPERATOR

CÉSAR R. DE OLIVEIRA AND ROBERTO A. PRADO

ABSTRACT. A 1D Dirac tight-binding model is considered and it is shown that its nonrelativistic limit is the 1D discrete Schrödinger model. For random Bernoulli potentials taking two values (without correlations), for typical realizations and for all values of the mass, it is shown that its spectrum is pure point, whereas the zero mass case presents dynamical delocalization for specific values of the energy. The massive case presents dynamical localization (excluding some particular values of the energy). Finally, for general potentials the dynamical moments for distinct masses are compared, especially the massless and massive Bernoulli cases.

1. INTRODUCTION

Besides the huge amount of mathematical works on spectral problems related to the one-dimensional Dirac model [4, 26], in physics it has also been used in comparative studies of relativistic and nonrelativistic electron-localization phenomena [2], in relativistic investigations of electrical conduction in disordered systems [21], in the construction of supertransparent models with supersymmetric structures [25] and in relativistic tunnelling problems [20].

In this paper a discrete version of the 1D Dirac model is discussed, which can be interpreted as a relativistic version of the well-known tight-binding Schrödinger Hamiltonian (with $\hbar = 1$)

$$(1) \quad (H\psi)_n = -\frac{1}{2m}(\Delta\psi)_n + V_n\psi_n = \frac{1}{2m}(-\psi_{n+1} - \psi_{n-1} + 2\psi_n) + V_n\psi_n.$$

The model was first reported in [11] and this work is its very expanded and mathematical detailed version. Consider a particle of mass $m \geq 0$ in the one-dimensional lattice $\mathbb{Z}$ under the real site potential $\tilde{V} = (V_n)$. The proposed 1D Dirac tight-binding operator is

$$(2) \qquad \mathbb{D}(m,c) = \mathbb{D}_0(m,c) + \tilde{V}\, Id_2 = c\mathcal{B} + mc^2\sigma_3 + \tilde{V}\, Id_2,$$

1991 *Mathematics Subject Classification.* 81Q10 (11B,47B99).

CRdeO was partially supported by CNPq (Brazil).
RAP was supported by CAPES (Brazil).

with $c > 0$ representing the speed of light,

$$\mathcal{B} = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix},$$

σ_3 the usual Pauli matrix, Id_2 the 2×2 identity matrix and d the finite difference operator (a discrete counterpart of the first derivative) defined by

$$(d\psi)_n = \psi_{n+1} - \psi_n.$$

$(d^*\psi)_n = \psi_{n-1} - \psi_n$ is the adjoint of d so that $\mathbb{D}_0(m,c) = c\mathcal{B} + mc^2\sigma_3$ is a bounded self-adjoint operator acting on $\ell^2(\mathbb{Z};\mathbb{C}^2)$ and its square is

$$\mathbb{D}_0(m,c)^2 = \begin{pmatrix} c^2 dd^* + m^2c^4 & 0 \\ 0 & c^2 dd^* + m^2c^4 \end{pmatrix}.$$

This equality is reminiscent of the relation between momentum $\vec{p}$ and energy E in relativistic quantum mechanics [4], given by $E^2 = c^2\vec{p}^2 + m^2c^4$. Denoting by $\sigma(A)$ the spectrum of a self-adjoint operator A, it is well known that $\sigma(-\Delta) = [0,4]$, and since $d^*d = dd^* = -\Delta$,

$$\sigma(\mathbb{D}_0(m,c)) = \left[-c\sqrt{4+m^2c^2}, -mc^2\right] \cup \left[mc^2, c\sqrt{4+m^2c^2}\right].$$

In case the potential $\tilde{V}$ is a bounded sequence, $\mathbb{D}(m,c)$ is also a bounded self-adjoint operator acting on $\ell^2(\mathbb{Z};\mathbb{C}^2)$.

It will be shown that the nonrelativistic limit of the resolvent of the discrete Dirac operator (2) is the resolvent of discrete Schrödinger operator (1) (when projected on a proper subspace; see Section 2). This is an important support for such Dirac model.

The study of quantum transport depends, of course, on the admitted definitions. In the physics literature terms like "extended states" and "zero Lyapunov exponents" have been used to crudely designate quantum transport. For instance, in [27] it was claimed that "extended states" were found in one-dimensional Schrödinger systems with off-diagonal randomness, but in [24] it was argued that although the localization length diverges the "transmission coefficient" vanishes as the system size goes to infinity. Up to recently, in the mathematical literature pure point spectrum (sometimes with exponentially decaying eigenfunctions) was considered synonymous of absence of transport. Currently the transport has been probed via the time behavior of the moments of the position operator, and in this work this idea will be followed. See ahead for precise definitions and related comments.

One of the goals of this paper is to study the phenomenon called dynamical localization (in the sense of time-boundedness of all moments of the position operator) for the Bernoulli-Dirac model, that is, the model (2) with the site potentials V_n, $n \in \mathbb{Z}$, being independent identically distributed Bernoulli random variables taking the values $\pm V$, $V > 0$. In this case it will be shown that almost surely the spectrum of $\mathbb{D}(m,c)$ is pure point for all values of the mass, the massive case has dynamical localization (excluding some particular values of the energy for which a more careful analysis is needed) and the zero mass case presents dynamical delocalization (that is, absence of localization) for specific values of the energy.

The problem of dynamical localization has been intensively studied during last years, especially in the case of random discrete and continuous Schrödinger operators (in particular for the Bernoulli-Anderson model, that is, the Schrödinger model with Bernoulli potentials); see [10, 15, 16] and references there in. What one usually proves is the so-called exponential localization [1, 6, 28], i.e., pure point spectrum and exponentially decaying eigenfunctions. On the other hand, it is also known that exponential localization does not imply dynamical localization [12]; it is usually needed a precise control of the decay of the eigenfunctions, called SULE [12, 16], that can be obtained through the method of multiscale analysis, a technique set out by Fröhlich and Spencer [14, 15].

One motivation for studying dynamical localization for the Bernoulli-Dirac operator comes from the random dimer model [13, 10], i.e., the Bernoulli-Anderson model with the site energies V_n assigned for pairs of lattices: $V_{2n} = V_{2n+1} = \pm V$ for all n. This model almost surely presents pure point spectrum for all values of $V \neq 0$ [10]. It was also numerically found in [13] and rigorously shown in [10, 18] the existence of *critical energies* (in the sense of [18]; see ahead) at which the Lyapunov exponent vanishes; dynamical localization was obtained in [10] only after projecting onto closed energy intervals not containing such critical energies. Despite the similarity between the transfer matrices of the two models, it is not immediate the adaptation of the localization (delocalization) results to the Bernoulli-Dirac model and each step needs to be verified; here, many points will not be detailed when they follow exactly the same lines of their Schrödinger counterpart.

With respect to nontrivial quantum transport, probed via dynamical delocalization (unbounded moments of the position operator), it was found in random polymer models [18] and in random palindrome models [7] (both including the important random dimer model), due to existence of critical energies [18]. Recently, for 1D discrete Schrödinger operators, Damanik, Sütő and Tcheremchantsev [9] have developed a general method which allows one to derive quantum dynamical lower bounds from upper bounds on the growth of norms of transfer matrices, and they applied this method to some substitution, Sturmian and prime models, among others. Damanik, Lenz and Stolz [8] presented an extension of this method to 1D continuous Schrödinger operators, with application to the continuous Bernoulli-Anderson model. Another method to obtain quantum transport from upper bounds on transfer matrix was lately developed by Germinet, Kiselev and Tcheremchantsev [17], with application to Schrödinger operators with random decaying potentials, providing new examples of Schrödinger operator with point spectrum and nontrivial quantum transport.

In the zero mass case, the one-dimensional Bernoulli-Dirac model presented here has pure point spectrum and nontrivial quantum transport for potentials with no correlations nor decaying properties (see Section 3). This phenomenon does not take place in the corresponding Schrödinger tight-binding model [16], and this also motivates the interest to better understanding the Dirac case. Presumably, this tight-binding model is the simplest one presenting such phenomenon.

Now the localization results for the Bernoulli-Dirac model will be briefly summarized. By using as the main tool a particular form of Furstenberg Theorem (Lemma 2 ahead), it is shown (see Theorems 2, 3 and 4) that the Lyapunov exponent $\gamma_m(E)$ is strictly positive for the energies $E \in \sigma(\mathbb{D}(m,c))$, except for: $E = \pm V,\ V \in (0,c],\ V \neq c/\sqrt{2}$, in the case $m = 0$; and if $m \geq 0$ for $\big(E_V = 0,\ V = c\sqrt{2+m^2c^2}\big)$ and the four energy-potential pairs $\big(E_V = \pm c\sqrt{2+m^2c^2} \pm c/\sqrt{2},\ V = c/\sqrt{2}\big)$.

For all energies E for which $\gamma_m(E) > 0$, a initial estimate for localization (Lemma 4) and the Wegner's estimate (Lemma 3) will be checked; by adapting the method multiscale analysis [28, 15, 16] to this model, it will be shown (see Theorems 2 and 3) that for typical realizations the spectrum of $\mathbb{D}(m,c)$ is pure point and the corresponding eigenfunctions are semi-uniformly exponentially localized (SULE) [12, 16]. This and the results of [16] (properly adapted to $\mathbb{D}(m,c)$) imply dynamical localization.

In the massless ($m = 0$) case, the values $E = \pm V$ with $V \in (0,c],\ V \neq c/\sqrt{2}$, are critical energies for the operator $\mathbb{D}(0,c)$ and this implies (almost surely) upper boundedness for the transfer matrices in the vicinity of these energies (see Lemmas 5 and 6). By adapting the ideas of [18] (see also [9]) to $\mathbb{D}(0,c)$ it will follow (see Theorem 5) that for an initial spinor Ψ well localized in space, there is $0 < C_q < \infty$ such that

$$\int_0^\infty \frac{1}{T}\, e^{-2t/T} M_\Psi^{(q)}(0,t)dt \geq C_q T^{q-1/2},$$

for almost all realization of the potential (or exponent $q-1$ instead of $q-1/2$ for every realization), where (X is the usual position operator)

$$M_\Psi^{(q)}(m,t) := \langle e^{-i\mathbb{D}(m,c)t}\Psi, |X|^q e^{-i\mathbb{D}(m,c)t}\Psi\rangle,$$

i.e., there is nontrivial quantum transport despite the absence of a continuous component in the spectrum of $\mathbb{D}(0,c)$.

In the case of the set of pairs $\big(E_V = \pm c\sqrt{2+m^2c^2} \pm c/\sqrt{2},\ V = c/\sqrt{2}\big)$ and $\big(E_V = 0,\ V = c\sqrt{2+m^2c^2}\big)$ it is shown (see Theorem 4) that the Lyapunov exponent γ_m vanishes, but it was not possible to give an answer about dynamical localization for them. Nevertheless, for these cases there is a general dynamical upper bound (in fact valid for all potentials $\tilde{V}$) established in Theorem 6.

For distinct masses $m, m' \geq 0$, but m close to m', it is expected that the moments $M_\Psi^{(q)}(m,t)$ follow closely the moments $M_\Psi^{(q)}(m',t)$ (both with the same potential), at least for a small period of time. The final result to be reported is an inequality confirming such expectative; by making using of Duhamel's formula, it will be shown (see Theorem 7) that, for the initial state Ψ with only one nonzero component, there exists $K_q > 0$ so that, for all $t > 0$,

$$\left| M_\Psi^{(q)}(m,t) - M_\Psi^{(q)}(m',t) \right| \le K_q |m - m'| c^2 \, t^{q+2}.$$

In particular, for the Bernoulli-Dirac model this relation with $m' = 0$ gives quantitatively an estimate of how, for small times, the dynamics of the localized regime follows the delocalized one (see also Corollary 1 in Section 6).

This paper is organized as follows: In Section 2 the nonrelativistic limit for the discrete Dirac model (2) is discussed. In Section 3 the results about spectral properties of such model, dynamical localization (delocalization) and a dynamical upper bound for moments are presented, whose proofs appear in Section 5. In Section 4 some tools used in those proofs are collected. Finally, in the Section 6 the dynamical moments with different masses are compared; in particular the dynamics of the massless and massive Dirac-Bernoulli cases.

2. Nonrelativistic Limit

In this section consider $\mathbb{D}(m,c)$ with $m > 0$ fixed and c as a parameter. For simplicity, $\mathbb{D}(c)$ will denote $\mathbb{D}(m,c)$, which is supposed to be self-adjoint with (real) potential $\tilde{V}$.

The nonrelativistic limit means c going to infinity, and since the rest energy mc^2 is a purely relativistic quantity, (as usual) it must be subtracted before taking this limit. The norm convergence of the resolvent operators $\left(\mathbb{D}(c) - mc^2 - z\right)^{-1}$, for $z \in \mathbb{C}\backslash\mathbb{R}$ will be considered. Λ below is the projector onto the subspace of "positive energies," and so ΛH_∞ corresponds to the Schrödinger operator (1). It is interesting to compare the approach presented here with the one in [11].

Theorem 1. *If $z \in \mathbb{C}\backslash\mathbb{R}$, then*

$$\lim_{c\to\infty} \left(\mathbb{D}(c) - mc^2 - z\right)^{-1} = \Lambda \left(H_\infty - z\right)^{-1},$$

where $\Lambda = \dfrac{1}{2}\left(Id_2 + \sigma_3\right)$ and $H_\infty = \dfrac{\mathcal{B}^2}{2m} + \tilde{V}\Lambda$, and the limit is in the norm of bounded operators.

Lemma 1. *If $z \in \mathbb{C}\backslash\mathbb{R}$, then*

$$\left(\mathbb{D}(c) - mc^2 - z\right)^{-1} = \left(\Lambda + \frac{c\mathcal{B} + z}{2mc^2}\right) S(c) \left(Id + \tilde{V}\frac{c\mathcal{B} + z}{2mc^2} S(c)\right)^{-1}, \tag{3}$$

where Id is the identity operator and

(4)

$$S(c) = \left(H_\infty - z - \frac{z^2}{2mc^2}\right)^{-1} = \left(Id - \frac{z^2}{2mc^2}\left(H_\infty - z\right)^{-1}\right)^{-1}\left(H_\infty - z\right)^{-1}.$$

Proof. Note that

$$\left(\mathbb{D}_0(c) + mc^2 + z\right)\left(\mathbb{D}_0(c) - mc^2 - z\right) = c^2\mathcal{B}^2 - 2mc^2 z - z^2.$$

Hence

$$(5)\quad \left(\mathbb{D}_0(c) - mc^2 - z\right)^{-1} = \frac{\mathbb{D}_0(c) + mc^2 + z}{2mc^2}\left(\frac{\mathcal{B}^2}{2m} - z - \frac{z^2}{2mc^2}\right)^{-1} = \left(\Lambda + \frac{c\mathcal{B} + z}{2mc^2}\right)S_0$$

with $S_0 = \left(\frac{\mathcal{B}^2}{2m} - z - \frac{z^2}{2mc^2}\right)^{-1}$. On the other hand, by using the operator relation

$$(A + B)^{-1} = (Id - A^{-1}B)^{-1}A^{-1}$$

with $A = \frac{\mathcal{B}^2}{2m} - z - \frac{z^2}{2m}$ and $B = \tilde{V}\Lambda$, one obtains

$$(6)\qquad S(c) = S_0\left(Id + \tilde{V}\Lambda S_0\right)^{-1}.$$

Therefore, by (5) and (6) it is found that

$$\begin{aligned}\left(\mathbb{D}(c) - mc^2 - z\right)^{-1} &= \left(\mathbb{D}_0(c) - mc^2 - z\right)^{-1}\left(Id + \tilde{V}\left(\mathbb{D}_0(c) - mc^2 - z\right)^{-1}\right)^{-1}\\ &= \left(\Lambda + \frac{c\mathcal{B} + z}{2mc^2}\right)S_0\left(Id + \tilde{V}\Lambda S_0 + \tilde{V}\frac{c\mathcal{B} + z}{2mc^2}S_0\right)^{-1}\\ &= \left(\Lambda + \frac{c\mathcal{B} + z}{2mc^2}\right)S(c)\left(Id + \tilde{V}\frac{c\mathcal{B} + z}{2mc^2}S(c)\right)^{-1}.\end{aligned}$$

□

Proof. **(Theorem 1)** Since $(H_\infty - z)^{-1}$ is bounded for $z \in \mathbb{C}\backslash\mathbb{R}$ and

$$\left\|\frac{z^2}{2mc^2}\left(H_\infty - z\right)^{-1}\right\| < 1$$

for c sufficiently large, one can expand

$$(7)\qquad S(c) = \sum_{n=0}^{\infty}\left(\frac{z^2}{2mc^2}\left(H_\infty - z\right)^{-1}\right)^n\left(H_\infty - z\right)^{-1},$$

where the sum is convergent in the operator norm.

For any fixed $z \in \mathbb{C}\backslash\mathbb{R}$ and c sufficiently large,

$$\left\| T(c) := \tilde{V}\frac{c\mathcal{B} + z}{2mc^2} S(c) \right\| < 1$$

and so

$$(Id + T(c))^{-1} = \sum_{n=0}^{\infty} (-T(c))^n \ . \tag{8}$$

Replacing (7) and (8) into (3) one obtains the expansion

$$\left(\mathbb{D}(c) - mc^2 - z\right)^{-1} = \sum_{n=0}^{\infty} \frac{R_n(z)}{c^n}$$

with

$$R_0(z) = \Lambda \left(H_\infty - z\right)^{-1},$$

$$R_1(z) = \Lambda \left(H_\infty - z\right)^{-1} \frac{\mathcal{B}}{2m} + \frac{\mathcal{B}}{2m} \left(H_\infty - z\right)^{-1} \Lambda,$$

and so on, and the sum is convergent in the operator norm. The result then follows. □

3. Localization Results

Consider the family of Dirac operators

$$\mathbb{D}_\omega(m, c) = \begin{pmatrix} mc^2 & cd^* \\ cd & -mc^2 \end{pmatrix} + V_\omega Id_2, \quad \omega \in \Omega = \{-V, V\}^{\mathbb{Z}}, \tag{9}$$

on $\ell^2(\mathbb{Z}; \mathbb{C}^2)$, where $V_\omega(n)$, $n \in \mathbb{Z}$, are i.i.d. Bernoulli random variables taking the values $\pm V$, $V > 0$, with common (nontrivial) probability measure μ and product measure $\mathbf{P} = \prod_{n\in\mathbb{Z}} \mu\left(V_\omega(n)\right)$. Let $P^\omega_{I,m}$ be the spectral projector of $\mathbb{D}_\omega(m, c)$ onto the interval $I \subset \mathbb{R}$.

Denote by $\delta_n^\pm$ the elements of the canonical position basis of $\ell^2(\mathbb{Z}; \mathbb{C}^2)$, for which all entries are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ except at the nth entry, which is given by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for the superscript indices $+$ and $-$, respectively. If $\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}$ is a solution of the eigenvalue equation

$$(\mathbb{D}_\omega(m, c) - E)\Psi = 0,$$

then it is simple to check that

$$\begin{pmatrix} \psi^+(n+1) \\ \psi^-(n) \end{pmatrix} = T_m^{V_\omega(n)}(E) \begin{pmatrix} \psi^+(n) \\ \psi^-(n-1) \end{pmatrix},$$

with

$$T_m^V(E) = \begin{pmatrix} 1 + \dfrac{m^2c^4 - (E-V)^2}{c^2} & \dfrac{mc^2 + E - V}{c} \\ \dfrac{mc^2 - E + V}{c} & 1 \end{pmatrix}.$$

The transfer matrix from site k to site n is introduced by

$$T_m^\omega(E; n, k) = T_m^{V_\omega(n-1)}(E) T_m^{V_\omega(n-2)}(E) \cdots T_m^{V_\omega(k)}(E) \ , \quad n > k$$

and $T_m^\omega(E; n, n) = Id_2$. For $q > 0$, let $|X|^q$ be the moment of order q of the position operator on $\ell^2(\mathbb{Z}; \mathbb{C}^2)$, i.e.,

$$|X|^q \Psi = \sum_n |n|^q \left(\langle \delta_n^+, \Psi \rangle \delta_n^+ + \langle \delta_n^-, \Psi \rangle \delta_n^- \right).$$

Definition 1. *The operator $\mathbb{D}_\omega(m,c)$ is dynamically localized on a spectral interval I if for all $q > 0$ and for all exponentially decaying initial state $\Psi \in \ell^2(\mathbb{Z}; \mathbb{C}^2)$*

$$\sup_t M_{\Psi,I,\omega}^{(q)}(m,t) := \sup_t \langle P_{I,m}^\omega e^{-i\mathbb{D}_\omega(m,c)t} \Psi, |X|^q P_{I,m}^\omega e^{-i\mathbb{D}_\omega(m,c)t} \Psi \rangle < \infty$$

***P** almost surely (**P**-a.s.). Otherwise $\mathbb{D}_\omega(m,c)$ is dynamically delocalized on I. If $I = \sigma(\mathbb{D}_\omega(m,c))$, then $M_{\Psi,I,\omega}^{(q)}(m,t)$ will be denoted by $M_{\Psi,\omega}^{(q)}(m,t)$.*

It is important to notice that although the Dirac operator acts on spinors, its eigenvalue equation, in the transfer matrix form, looks exactly like the equation for a one-dimensional Schrödinger operator acting on scalar valued functions, with the transfer matrix being in $SL(2, \mathbb{R})$. Hence the methods used in studies of the usual one-dimensional Anderson model, as Furstenberg's Theorem, can be applied for this Dirac model; see Sections 4 and 5.

The localization results are gathered in the following set of theorems.

Theorem 2. *Let $(\mathbb{D}_\omega(m,c))_{\omega\in\Omega}$ be as in (9) and $V \in (0,c], V \neq c/\sqrt{2}$. Then, **P** almost surely, the Lyapunov exponent*

$$\gamma_m(E) = \lim_{n\to\infty} \frac{1}{|n|} \ln \|T_m^\omega(E; n, 1)\|$$

exists, is independent of ω, and

(i)(i.1) $\gamma_m(E \neq \pm V) > 0$ *for* $m \geq 0$,

(i.2) $\gamma_m(E = \pm V) > 0$ *for* $m > 0$,

(i.3) $\gamma_0(E = \pm V) = 0$.

(ii) *Let $m \geq 0$; then **P**-a.s. $\sigma(\mathbb{D}_\omega(m,c))$ is pure point.*

(iii)(iii.1) *Let $m > 0$. Then **P**-a.s. the operator $\mathbb{D}_\omega(m,c)$ is dynamically localized on its spectrum.*

(iii.2) *For any closed interval* $I \subset \sigma(\mathbb{D}_\omega(0,c))$, *with* $\pm V \notin I$, *the operator* $\mathbb{D}_\omega(0,c)$ *is dynamically localized on* I.

Theorem 3. *Let* $(\mathbb{D}_\omega(m,c))_{\omega\in\Omega}$ *be as in (9),* $m \geq 0$ *and* $V > c$, $V \neq c\sqrt{2+m^2c^2}$. *Then,* $\boldsymbol{P}$*-a.s. the spectrum of* $\mathbb{D}_\omega(m,c)$ *is pure point and this operator is dynamically localized on its spectrum.*

Theorem 4. *Let* $(\mathbb{D}_\omega(m,c))_{\omega\in\Omega}$ *be as in (9),* $m \geq 0$ *and* $V = c/\sqrt{2}$ *(respectively* $V = c\sqrt{2+m^2c^2}$*). Then the same conclusions of Theorem 2 (resp. Theorem 3) hold except at the four possibilities of energies* $E_V = \pm c\sqrt{2+m^2c^2} \pm c/\sqrt{2}$ *(resp.* $E_V = 0$*). (The point is that* $E_V = \pm c\sqrt{2+m^2c^2} \pm c/\sqrt{2}$ *(resp.* $E_V = 0$*) are energies so that* $\gamma_m(E_V) = 0$*.)*

For the next result it is convenient to use the average dynamical moments

$$A^{(q)}_{\Psi,\omega}(m,T) := \int_0^\infty \frac{1}{T} e^{-2t/T} M^{(q)}_{\Psi,\omega}(m,t)\, dt, \tag{10}$$

defined for $m \geq 0$ and $T > 0$. The main reason for working with this kind of Laplace transform average is relation (14) ahead.

Theorem 5 (massless case)**.** *Let* $(\mathbb{D}_\omega(0,c))_{\omega\in\Omega}$ *be as in (9) and* $V \in (0,c]$, $V \neq c/\sqrt{2}$. *Then, for* $q > 0$ *and* Ψ *with only one nonzero component, there exists* $0 < C_q(\omega) < \infty$ *such that, for* $T > 0$,

(i) $A^{(q)}_{\Psi,\omega}(0,T) \geq C_q(\omega) T^{q-1/2}$ $\quad$ $\boldsymbol{P} -$ *a.s.*,

(ii) $A^{(q)}_{\Psi,\omega}(0,T) \geq C_q(\omega) T^{q-1}$ $\quad$ *for every* ω,

i.e., $\mathbb{D}_\omega(0,c)$ *is not dynamically localized on its spectrum.*

The following theorem establishes very general upper bounds for the dynamical moments of the position operator; notice that it holds for any potential sequence $\tilde{V}$ and is not restricted to the Bernoulli case.

Theorem 6. *Let* $\mathbb{D}(m,c)$ *be as in (2),* $m \geq 0$ *and* Ψ *with only one nonzero component (so in the domain of* $|X|^q$ *for all* $q > 0$*). Then for any* $q \in \mathbb{N}$ *there exists* $0 < K_q(\tilde{V},m,c) < \infty$ *such that*

$$M^{(q)}_\Psi(m,t) \leq K_q(\tilde{V},m,c)\, t^q, \quad t \geq 1.$$

Remark. It is possible to adjust the constant K_q so that the above upper bound holds for $t \geq \varepsilon$ for any given $\varepsilon > 0$, instead of just $t \geq 1$.

Since $M_\Psi^{(q)}(m,t) \geq M_\Psi^{(q')}(m,t)$ for $q \geq q'$, it is evident that $M_\Psi^{(q)}(m,t) \leq K_{\lceil q \rceil}(\tilde{V}, m, c)\, t^{\lceil q \rceil}$ for real q.

4. Tools

In this section some tools and notations that will be used in the proofs of the results presented in Section 3 are collected. For studying the positivity of the Lyapunov exponent γ_m, $m \geq 0$, the following particular form of Furstenberg Theorem [5] will be used:

Lemma 2. *Let $\mathcal{G}_m(E)$ be the smallest closed subgroup of $SL(2, \mathbb{R})$ generated by the matrices $T_m^V(E)$ and $T_m^{-V}(E)$. Then $\gamma_m(E) > 0$ if*

- *$\mathcal{G}_m(E)$ is not compact, and*
- *there is no probability measure on $P(\mathbb{R}^2)$ (the set of all the directions of $\mathbb{R}^2$) that is invariant under the action of $\mathcal{G}_m(E)$, which is equivalent to the statement: the orbit $\mathcal{G}_m(E) \cdot \tilde{x} := \{T \cdot \tilde{x},\ T \in \mathcal{G}_m(E)\}$ of each direction $\tilde{x} \in P(\mathbb{R}^2)$ contains at least three elements.*

If $L > 0$, $n \in \mathbb{Z}$, consider the finite subset of $\mathbb{Z}$

$$\Lambda_L(n) = \left\{ k \in \mathbb{Z} :\ |k - n| \leq \frac{L}{2} \right\}$$

with boundary

$$\partial \Lambda_L(n) = \{(k, k') :\ k \in \Lambda_L(n),\ k' \notin \Lambda_L(n),\ |k - k'| = 1\}.$$

Denote by $\mathbb{D}_\omega^{\Lambda_L(n)}(m,c)$ the operator $\mathbb{D}_\omega(m,c)$ restricted to $\ell^2(\Lambda_L(n); \mathbb{C}^2)$ with zero boundary conditions outside $\Lambda_L(n)$.

The matrix elements of an operator $\mathcal{O}$ on $\ell^2(\mathbb{Z}; \mathbb{C}^2)$ are given by

$$\mathcal{O}_{nk} = \begin{pmatrix} \langle \delta_n^+, \mathcal{O}\delta_k^+ \rangle & \langle \delta_n^+, \mathcal{O}\delta_k^- \rangle \\ \langle \delta_n^-, \mathcal{O}\delta_k^+ \rangle & \langle \delta_n^-, \mathcal{O}\delta_k^- \rangle \end{pmatrix}$$

with "norm"

$$\|\mathcal{O}_{nk}\|^2 = |\langle \delta_n^+, \mathcal{O}\delta_k^+ \rangle|^2 + |\langle \delta_n^+, \mathcal{O}\delta_k^- \rangle|^2 + |\langle \delta_n^-, \mathcal{O}\delta_k^+ \rangle|^2 + |\langle \delta_n^-, \mathcal{O}\delta_k^- \rangle|^2.$$

Now two important results required for the multiscale analysis are described. The first one is the Wegner's estimate, adapted from [6] to the discrete Dirac operator (details will be omitted, since they are long and very similar to the Schrödinger case):

Lemma 3. *Let $\mathbb{D}_\omega(m,c)$ be as in (9) and I a compact energy interval. For any $\theta \in (0,1)$ and $\tau > 0$ there exist $L_0 = L_0(I, \theta, \tau, m) > 0$ and*

$a = a(I, \theta, \tau, m) > 0$ such that

$$\boldsymbol{P}\left\{\omega : \mathrm{dist}\left(E, \sigma\big(\mathbb{D}_\omega^{\Lambda_L(0)}(m,c)\big)\right) \le e^{-\tau L^\theta}\right\} \le e^{-aL^\theta}$$

for all $E \in I$ and $L \ge L_0$.

The second result is the initial estimate for localization, adapted from [28] (details omitted):

Lemma 4. *Let $\mathbb{D}_\omega(m,c)$ be as in (9), $\epsilon > 0$ and $\theta \in (0,1)$. For each $E_0 \in \mathbb{R}$, $E_0 \notin \sigma\big(\mathbb{D}_\omega^{\Lambda_L(0)}(m,c)\big)$ with $\gamma_m(E_0) > \epsilon$, there exist $L_0 = L_0(E_0, \epsilon, \theta, m) > 0$ and $r = r(E_0, \epsilon, \theta, m) > 0$ such that*

$$\boldsymbol{P}\left\{\omega : \left\|\left(\mathbb{D}_\omega^{\Lambda_L(0)}(m,c) - E_0\right)^{-1}_{0k}\right\| \le e^{-(\gamma_m(E_0)-\epsilon)L/2} \quad \forall\, k \in \partial\Lambda_L(0)\right\} \ge\; 1 - e^{-rL^\theta}$$

for all $L \ge L_0$.

In order to obtain dynamical localization from the multiscale analysis, the following properties of $\mathbb{D}_\omega(m,c)$ are useful:

(P1) With respect to the spectral measure of $\mathbb{D}_\omega(m,c)$, almost every energy is a generalized eigenvalue, i.e., with polynomially bounded eigenvector (see [3, 22]).

(P2) If $E \notin \sigma\big(\mathbb{D}_\omega^{\Lambda_L(n)}(m,c)\big)$ and $\Psi \in \ell^2(\mathbb{Z}; \mathbb{C}^2)$ so that $\mathbb{D}_\omega(m,c)\Psi = E\Psi$, then

$$\begin{aligned}\Psi(n) \;&=\; -\left(\mathbb{D}_\omega^{\Lambda_L(n)}(m,c) - E\right)^{-1}_{nl_1} \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \Psi(l_1 - 1) \\ &\quad - \left(\mathbb{D}_\omega^{\Lambda_L(n)}(m,c) - E\right)^{-1}_{nl_2} \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \Psi(l_2 + 1),\end{aligned}$$

with $\{(l_1, l_1 - 1), (l_2, l_2 + 1)\} = \partial\Lambda_L(n)$.

Property **(P2)** follows after defining the *boundary operator* $\mathcal{F}_{\Lambda_L(n)}$ by its matrix elements

$$\left(\mathcal{F}_{\Lambda_L(n)}\right)_{jk} = \begin{cases} -\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} & \text{if } j-1 = k,\ j \in \Lambda_L(n),\ k \notin \Lambda_L(n); \\ -\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} & \text{if } j+1 = k,\ j \in \Lambda_L(n),\ k \notin \Lambda_L(n); \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise,} \end{cases}$$

noting that $l^2(\mathbb{Z}; \mathbb{C}^2) = l^2(\Lambda_L(n); \mathbb{C}^2) \oplus l^2(\mathbb{Z} \setminus \Lambda_L(n); \mathbb{C}^2)$ and

$$\mathbb{D}_\omega(m,c) = \mathbb{D}_\omega^{\Lambda_L(n)}(m,c) + \mathbb{D}_\omega^{\mathbb{Z}\setminus\Lambda_L(n)}(m,c) - \mathcal{F}_{\Lambda_L(n)}.$$

In the zero mass case ($m = 0$) the operators $\mathbb{D}_\omega(0, c)$, $\omega \in \Omega$, presents critical energies $E_V = \pm V$ for $V \in (0, c]$, $V \neq c/\sqrt{2}$, as defined in [18], since either $T_0^V(V) = Id_2$ and $T_0^{-V}(V)$ is elliptic (that is, $|\text{trace } T_0^{-V}(V)| < 2$) or $T_0^{-V}(-V) = Id_2$ and $T_0^V(-V)$ is elliptic. Thus there exists a real invertible matrix Q such that

$$Q\ T_0^{\pm V}(E_V)\ Q^{-1} = \begin{pmatrix} \cos(\eta_\pm) & -\sin(\eta_\pm) \\ \sin(\eta_\pm) & \cos(\eta_\pm) \end{pmatrix}.$$

Since the eigenvalues of this matrix are $e^{i\eta_\pm}$ and $e^{-i\eta_\pm}$, for both of the above cases one has $\eta_+ - \eta_- \neq k\pi,\ k \in \mathbb{Z}$ (a condition required in [18]). By using the modified Prüfer variables, phase shifts, oscillatory sums, large deviation estimates as in [18], one obtains the following result

Lemma 5 (massless case)**.** *Let $\lambda > 0$ be arbitrary. Then there are $b > 0$ and $C < \infty$ such that for every $N \in \mathbb{N}$, there exists a set $\Omega_N(\lambda) \subset \Omega$ with $\boldsymbol{P}(\Omega_N(\lambda)) \leq Ce^{-bN^\lambda}$ and*

$$\left\|T_0^\omega(E; n, k)\right\| \leq C$$

for all $\omega \in \Omega \backslash \Omega_N(\lambda)$, $0 \leq k \leq n \leq N$ and $E \in [E_V - N^{-\lambda-1/2}, E_V + N^{-\lambda-1/2}]$.

On the other hand, since $\|Q\ T_0^{\pm V}(E_V)\ Q^{-1}\| = 1$, expanding $T_0^{\pm V}(E_V + \epsilon)$ into powers of ϵ one obtains

$$\|Q\ T_0^{\pm V}(E_V + \epsilon)\ Q^{-1}\| \leq 1 + a|\epsilon|$$

for $|\epsilon| \leq \delta$, $0 < a < \infty$, and one deduces the following

Lemma 6 (massless case)**.** *For $\delta > 0$ there exists $C < \infty$ such that for all $n, k \in \mathbb{Z}$ and $E \in [E_V - \delta, E_V + \delta]$,*

$$\left\|T_0^\omega(E; n, k)\right\| \leq Ce^{C\delta|n-k|}.$$

An inductive argument shows that, for $\zeta \in \mathbb{C}$ and $m \geq 0$,

(11)

$$T_m^\omega(E + \zeta; n, k) = T_m^\omega(E; n, k) - \zeta \sum_{l=k}^{n-1} T_m^\omega(E + \zeta; n, l + 1)\ S_\zeta^\omega(E; l)\ T_m^\omega(E; l, k),$$

where

$$S_\zeta^\omega(E; l) = \frac{\zeta}{c^2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{c} \begin{pmatrix} \frac{2}{c}(E - V_\omega(l)) & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, for $z \in \mathbb{C}\backslash\mathbb{R}$ and $m \geq 0$, introduce the two-components Green's function

$$\begin{pmatrix} G^{+}_{m,\omega}(z;n) \\ G^{-}_{m,\omega}(z;n) \end{pmatrix} = \begin{pmatrix} \left\langle \delta^{+}_{n}, (\mathbb{D}_\omega(m,c)-z)^{-1}\,\delta^{+}_{0} \right\rangle \\ \left\langle \delta^{-}_{n}, (\mathbb{D}_\omega(m,c)-z)^{-1}\,\delta^{+}_{0} \right\rangle \end{pmatrix},$$

so that

$$(\mathbb{D}_\omega(m,c)-z)\begin{pmatrix} G^{+}_{m,\omega}(z;n) \\ G^{-}_{m,\omega}(z;n) \end{pmatrix} = \delta^{+}_{0}(n) \ .$$

By using transfer matrices, one obtains for $n \leq 0$,

$$\begin{pmatrix} G^{+}_{m,\omega}(z;n) \\ G^{-}_{m,\omega}(z;n-1) \end{pmatrix} = T^{\omega}_{m}(z;n,0)\begin{pmatrix} G^{+}_{m,\omega}(z;0) \\ G^{-}_{m,\omega}(z;-1) \end{pmatrix} \tag{12}$$

and for $n \geq 1$,

$$\begin{pmatrix} G^{+}_{m,\omega}(z;n) \\ G^{-}_{m,\omega}(z;n-1) \end{pmatrix} = T^{\omega}_{m}(z;n,1)\begin{pmatrix} G^{+}_{m,\omega}(z;1) \\ G^{-}_{m,\omega}(z;0) \end{pmatrix}. \tag{13}$$

For $z = E + i/T$ ($T > 0$) and $m \geq 0$, it is also valid the following identity (adapted from Lemma 3.2 in [19]):

$$A^{(q)}_{\delta^{+}_{0},\omega}(m,T) = \frac{1}{2\pi T}\sum_{n\in\mathbb{Z}} |n|^{q} \int_{\mathbb{R}} (|G^{+}_{m,\omega}(z;n)|^{2} + |G^{-}_{m,\omega}(z;n)|^{2})\ dE. \tag{14}$$

5. Localization Proofs

In this section the proofs of Theorems 2 - 6 are presented.

Proof. **(Theorems 2 and 3)**

The strategy of the proof is based on reference [10], where the random dimer Schrödinger operator was studied. Since for the discrete Dirac operator there are the particular role played by the mass and some different possibilities for the transfer matrices, a rather detailed proof will be presented. The idea is to show that given $\epsilon > 0$, $I \subset \sigma(\mathbb{D}_\omega(m,c))$ a compact energy interval not containing the excluded V values, then for each $0 < \gamma < \gamma_m(I) := \inf\{\gamma_m(E) : E \in I\}$ there exist a constant $C(\omega,\epsilon,\gamma)$ and, for each eigenfunction $\varphi_{j,\omega} = \begin{pmatrix} \varphi^{+}_{j,\omega} \\ \varphi^{-}_{j,\omega} \end{pmatrix}$ with energy $E_{j,\omega} \in I$, a "center" $z_{j,\omega} \in \mathbb{Z}$, such that

$$\|\varphi_{j,\omega}(n)\| \leq C(\omega,\epsilon,\gamma)e^{\gamma|z_{j,\omega}|^{\epsilon}}e^{-\gamma|n-z_{j,\omega}|}, \quad \forall n \in \mathbb{Z}. \tag{15}$$

If Ψ decays exponentially with rate $\theta_0 > 0$ and if $q > 0$, it is known that (15) (that is, SULE condition) implies the existence of a constant $C_\Psi(m,I,\omega)$ so that

$$\sup_{t} M^{(q)}_{\Psi,I,\omega}(m,t) \leq C_\Psi(m,I,\omega), \quad \mathbf{P}-a.s.,$$

i.e., $\mathbb{D}_\omega(m,c)$ is dynamically localized on I (see Section 2 in [16]).

To prove (ii) and (15), it is sufficient to show strict positivity of the Lyapunov exponent, because in this case Lemmas 3 and 4 hold. By using the multiscale analysis [28] together with (P1) and (P2), one can then follow the proof of Theorem 3.1 in [16] (properly adapted to $\mathbb{D}_\omega(m,c)$) to obtain (ii) and (15) (details will be omitted).

Now the proof of (i). It follows from Furstenberg and Kesten Theorem [5] that, **P**-a.s. the Lyapunov exponent γ_m exists and is independent of ω.

Consider first the energies $E \neq \pm V$ and it will be proven that $\gamma_m(E \neq \pm V) > 0$ for all $m \geq 0$ and for all $E \in \sigma(\mathbb{D}_\omega(m,c))$. Let $\mathcal{G}_m(E)$ be as in the Lemma 2. Put $\alpha = E - V$, $\beta = E + V$ and rename $T_m^V(E) = T_m^{(\alpha)}$, $T_m^{-V}(E) = T_m^{(\beta)}$. In the present case $\alpha \neq 0$ and $\beta \neq 0$.

Since the problem is symmetric in α and β, the proof is reduced to the study of three cases:

- **a)** $T_m^{(\alpha)}$ and $T_m^{(\beta)}$ are elliptic ($|\mathrm{trace}\, T_m^{(\alpha)}| < 2,\ |\mathrm{trace}\, T_m^{(\beta)}| < 2$);
- **b)** $T_m^{(\alpha)}$ is parabolic ($|\mathrm{trace}\, T_m^{(\alpha)}| = 2$);
- **c)** $T_m^{(\alpha)}$ is hyperbolic ($|\mathrm{trace}\, T_m^{(\alpha)}| > 2$).

Note that in cases **b)** and **c)** the group $\mathcal{G}_m(E)$ is not compact.

Case a) Since $T_m^{(\alpha)}$ and $T_m^{(\beta)}$ are both elliptic, then $|\alpha|, |\beta| \in (mc^2, c\sqrt{4+m^2c^2})$. In this case such matrices do not commute. Since the operator

$$T_m^{(\alpha)} T_m^{(\beta)}\ (T_m^{(\alpha)})^{-1} (T_m^{(\beta)})^{-1}$$

built from two noncommuting elliptic elements is hyperbolic, it follows that $\mathcal{G}_m(E)$ is not compact. Moreover, note that

$$\mathrm{trace}\, \big(T_m^{(\alpha)}\big)^2 = \frac{\alpha^4}{c^4} - \left(2m^2 + \frac{4}{c^2}\right)\alpha^2 + m^2c^2(4 + m^2c^2) + 2$$

$\big($analogous for $T_m^{(\beta)}\big)$. Hence, if $\alpha^2 \neq 2c^2 + m^2c^4$ or $\beta^2 \neq 2c^2 + m^2c^4$, then $T_m^{(\alpha)}$ and $\big(T_m^{(\alpha)}\big)^2$ or $T_m^{(\beta)}$ and $\big(T_m^{(\beta)}\big)^2$ are elliptic. Since elliptic elements have no fixed points in $P(\mathbb{R}^2)$, it follows that for any $\tilde{x} \in P(\mathbb{R}^2)$, $\mathcal{G}_m(E)\cdot\tilde{x}$ contains at least the three elements $\tilde{x},\ T_m^{(\alpha)}\cdot\tilde{x},\ \big(T_m^{(\alpha)}\big)^2\cdot\tilde{x}$ or $\tilde{x},\ T_m^{(\beta)}\cdot\tilde{x},\ \big(T_m^{(\beta)}\big)^2\cdot\tilde{x}$. Therefore, by Lemma 2, $\gamma_m(E) > 0$. If, on the other hand, $\alpha^2 = 2c^2 + m^2c^4$ and $\beta^2 = 2c^2 + m^2c^4$, then $E = 0$ and $V = c\sqrt{2+m^2c^2}$, which is one of the excluded pairs described in Theorem 4.

Case b) Suppose $T_m^{(\alpha)}$ is parabolic, that is, $|\alpha| = mc^2$ or $|\alpha| = c\sqrt{4+m^2c^2}$. First the possibility $\alpha = mc^2$ will be discussed (the case $\alpha = -mc^2$ is similar). In this case

$$T_m^{(\alpha)} = \begin{pmatrix} 1 & 2mc \\ 0 & 1 \end{pmatrix}, \quad \text{and so} \quad \big(T_m^{(\alpha)}\big)^n = \begin{pmatrix} 1 & 2nmc \\ 0 & 1 \end{pmatrix}.$$

Denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$. By taking a vector $x = x_1 e_1 + x_2 e_2$, and setting $\tilde{x}$ for its direction, one concludes that $\lim_{n\to\infty} \left(T_m^{(\alpha)}\right)^n \cdot \tilde{x} = \tilde{e}_1$. If ν is a probability measure that is invariant under the action of $\mathcal{G}_m(E)$, and if $f \in C_0^\infty(P(\mathbb{R}^2))$, by Lebesgue's dominated convergence Theorem one has

$$f(\tilde{e}_1) = \lim_{n\to\infty} \int f\left(\left(T_m^{(\alpha)}\right)^n \cdot \tilde{x}\right)\, d\nu(\tilde{x}) \ .$$

This means that $\nu = \delta_{\tilde{e}_1}$. But the matrix $T_m^{(\beta)}$ does not leave invariant the direction $\tilde{e}_1$ since

$$T_m^{(\beta)}\, e_1 = \left(1 + \frac{m^2c^4 - \beta^2}{c^2}\right)e_1 + \frac{-\beta + mc^2}{c}\, e_2 \quad \text{and} \quad \beta \neq mc^2 \ .$$

Thus it is proven that there is no invariant measure under the action of $\mathcal{G}_m(E)$. Therefore, by Lemma 2 one gets $\gamma_m(E) > 0$.

Consider now the possibility $\alpha = c\sqrt{4 + m^2c^2}$ (the case $\alpha = -c\sqrt{4 + m^2c^2}$ is similar). In this case an eigenvector of

$$T_m^{(\alpha)} = \begin{pmatrix} -3 & mc + \sqrt{4 + m^2c^2} \\ mc - \sqrt{4 + m^2c^2} & 1 \end{pmatrix}$$

is given by $v_1 = \left(\dfrac{mc + \sqrt{4 + m^2c^2}}{2}, 1\right)$. Picking $v_2 = \left(\dfrac{-mc + \sqrt{4 + m^2c^2}}{2}, -1\right)$ a vector orthogonal to v_1, the matrix $T_m^{(\alpha)}$ in the basis $\{v_1, v_2\}$ is

$$\begin{pmatrix} -1 & -4 - m^2c^2 + mc\sqrt{4 + m^2c^2} \\ 0 & -1 \end{pmatrix} .$$

Repeating the previous calculation for this case, one obtains $\nu = \delta_{\tilde{v}_1}$. But $T_m^{(\beta)}$ does not leave invariant the direction $\tilde{v}_1$ except for $\beta = 0$ or $\beta = c\sqrt{4 + m^2c^2} = \alpha$, which are excluded since the first condition yields $E = -V$ and the second one $V = 0$. Thus it is proven that there is no invariant measure and, by Lemma 2, $\gamma_m(E) > 0$.

Case c) Suppose now that $T_m^{(\alpha)}$ is hyperbolic (so $|\alpha| < mc^2$ or $|\alpha| > c\sqrt{4 + m^2c^2}$). It is sufficient to study the orbit of the eigendirections of $T_m^{(\alpha)}$, namely

$$e_m^\epsilon = \begin{pmatrix} \alpha^2 - m^2c^4 + \epsilon\sqrt{(\alpha^2 - m^2c^4)(\alpha^2 - m^2c^4 - 4c^2)} \\ 2c(\alpha - mc^2) \end{pmatrix}, \quad \epsilon = \pm 1.$$

If $T_m^{(\beta)}$ is hyperbolic then the orbit of e_m^ϵ is infinite. Hence $\gamma_m(E) > 0$ by Lemma 2. If $T_m^{(\beta)}$ is parabolic, it is again case b). Finally, suppose that $T_m^{(\beta)}$

is elliptic. If $T_m^{(\beta)}\tilde{e}_m^{\epsilon} \neq \tilde{e}_m^{-\epsilon}$, then $T_m^{(\beta)}\tilde{e}_m^{\epsilon}$ cannot belong to the eigendirections of $T_m^{(\alpha)}$ and its orbit is infinite. Hence $\gamma_m(E) > 0$ by Lemma 2. If $T_m^{(\beta)}\tilde{e}_m^{\epsilon} = \tilde{e}_m^{-\epsilon}$, then simple calculations lead to the equations

$$\left(1 + m^2c^2 - \frac{\beta^2}{c^2}\right)(\alpha^2 - m^2c^4 + \epsilon u) = 4(m^2c^4 - \beta\alpha) + (\alpha^2 - m^2c^4 - \epsilon u)$$

with $\epsilon = \pm 1$ and $u = \sqrt{(\alpha^2 - m^2c^4)(\alpha^2 - m^2c^4 - 4c^2)} \neq 0$. It implies $\beta^2 = 2c^2 + m^2c^4$ and $\alpha = \beta \pm c\sqrt{2}$, which means $V = c/\sqrt{2}$ and $E = \pm c\sqrt{2 + m^2c^2} - c/\sqrt{2}$. The symmetric case where one assumes that $T_m^{(\beta)}$ is hyperbolic leads naturally to $\alpha^2 = 2c^2 + m^2c^4$ and $\beta = \alpha \pm c\sqrt{2}$, which means $V = c/\sqrt{2}$ and $E = \pm c\sqrt{2 + m^2c^2} + c/\sqrt{2}$. Those are excluded pairs that will be discussed in the proof of Theorem 4.

Consider now the energy $E = V$ (the case $E = -V$ is analogous). Note that $\alpha = 0$ and $\beta = 2V$. First the case $m > 0$ will be discussed. The two possible transfer matrices are

$$T_m^{(\alpha)} = \begin{pmatrix} 1 + m^2c^2 & mc \\ mc & 1 \end{pmatrix} \quad \text{and} \quad T_m^{(\beta)} = \begin{pmatrix} 1 + \dfrac{m^2c^4 - 4V^2}{c^2} & \dfrac{mc^2 + 2V}{c} \\ \dfrac{mc^2 - 2V}{c} & 1 \end{pmatrix}.$$

Observe that $T_m^{(\alpha)}$ and $T_m^{(\beta)}$ do not commute, and that $T_m^{(\alpha)}$ is hyperbolic. It is enough to study this case for $\beta = c\sqrt{4 + m^2c^2}$ $\big(T_m^{(\beta)}$ is parabolic$\big)$. The eigendirections of $T_m^{(\alpha)}$ are

$$e_m^{\delta} = \begin{pmatrix} \dfrac{mc + \delta\sqrt{4 + m^2c^2}}{2} \\ 1 \end{pmatrix}, \quad \delta = \pm 1.$$

The matrices $T_m^{(\alpha)}$ and $T_m^{(\beta)}$ in the basis $\{e_m^1, e_m^{-1}\}$ are given, respectively, by

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 4 + m^2c^2 - mc\sqrt{4 + m^2c^2} \\ 0 & -1 \end{pmatrix},$$

with

$$\lambda_1\lambda_{-1} = \left(1 + \frac{m^2c^2}{2} + \frac{mc\sqrt{4 + m^2c^2}}{2}\right)\left(1 + \frac{m^2c^2}{2} - \frac{mc\sqrt{4 + m^2c^2}}{2}\right) = 1.$$

Suppose that $T_m^{(\alpha)}$ occurs with probability $0 < p < 1$ and $T_m^{(\beta)}$ occurs with probability $1 - p$. Denote by n_α (resp. n_β) the number of times that $T_m^{(\alpha)}$ (resp. $T_m^{(\beta)}$) occurs in the product $T_m^{\omega}(E; n, 1)$. Supposing, without loss of generality, that $T_m^{V_\omega(1)}(E) = T_m^{(\alpha)}$, one has

$$T_m^\omega(E;n,1) = \begin{pmatrix} \lambda_1^{n_\alpha} & C_n P(\lambda_1,\lambda_{-1}) \\ 0 & \lambda_{-1}^{n_\alpha} \end{pmatrix}$$

P-a.s., where C_n is a constant and $P(\lambda_1,\lambda_{-1})$ is a polynomial in λ_1 and λ_{-1}. Thus,

$$\left\| T_m^\omega(E;n,1) \right\| \geq \left\| \begin{pmatrix} \lambda_1^{n_\alpha} \\ 0 \end{pmatrix} \right\| = \lambda_1^{n_\alpha}, \ \lambda_1 > 1,$$

and therefore **P**-a.s.

$$\gamma_m(E=V) = \lim_{n\to\infty} \frac{1}{|n|} \ln \|T_m^\omega(E;n,1)\| \geq (\ln \lambda_1) \lim_{n\to\infty} \frac{n_\alpha}{|n|} = (\ln \lambda_1) p > 0.$$

Now the case $m=0$ will be treated. In this case

$$T_0^{(\alpha)} = Id_2 \quad \text{and} \quad T_0^{(\beta)} = \begin{pmatrix} 1 - 4V^2/c^2 & 2V/c \\ -2V/c & 1 \end{pmatrix}.$$

One then finds

$$\lim_{n\to\infty} \left\| \left(T_0^{(\beta)}\right)^n \right\|^{1/n} = 1$$

if $V \in (0,c]$ and

$$\lim_{n\to\infty} \left\| \left(T_0^{(\beta)}\right)^n \right\|^{1/n} > 1$$

if $V > c$. Hence, if $V \in (0,c]$, $V \neq c/\sqrt{2}$,

$$\gamma_0(E=V) = \lim_{n\to\infty} \frac{n_\beta}{|n|} \ln \left\| \left(T_0^{(\beta)}\right)^{n_\beta} \right\|^{1/n_\beta} = (1-p)\ln 1 = 0,$$

and $\gamma_0(E=V) > 0$ if $V > c$. □

Proof. **(Theorem 4)**

By analyzing the proof of Theorems 2 and 3 observe that for $V = c/\sqrt{2}$ (resp. $V = c\sqrt{2+m^2c^2}$) one has $\gamma_m(E_V \neq \pm c\sqrt{2+m^2c^2} \pm c/\sqrt{2}) > 0$ (resp. $\gamma_m(E_V \neq 0) > 0$) and then the same conclusions of Theorem 2 (resp. Theorem 3) hold. It remains to show that γ_m vanishes at the pairs $\left(V = c\sqrt{2+m^2c^2},\ E_V = 0\right)$ and $\left(V = c/\sqrt{2},\ E_V = \pm c\sqrt{2+m^2c^2} \pm c/\sqrt{2}\,\right)$. Note that in all cases $E_V \in \sigma(\mathbb{D}_\omega(m,c))$ **P**-a.s. .

First it will be treated the case $\left(V = c/\sqrt{2},\ E_V = -c\sqrt{2+m^2c^2} - c/\sqrt{2}\,\right)$ (the others three excluded cases with $V = c/\sqrt{2}$ are similar). In this case one has $\beta = -c\sqrt{2+m^2c^2}$ and $\alpha = \beta - c\sqrt{2}$. The eigenvectors of $T_m^{(\alpha)}$ are given by

$$\begin{pmatrix} \dfrac{2c-\sqrt{2}\beta+\epsilon\sqrt{4c^2+2m^2c^4-2\sqrt{2}c\beta}}{\beta-c\sqrt{2}-mc^2} \\ \\ 1 \end{pmatrix}, \quad \epsilon=\pm 1,$$

and by looking at the matrices in the basis given by these two vectors, the study is reduced to products of matrices of the following two types:

$$\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \kappa_- \\ \kappa_+ & 0 \end{pmatrix}$$

with $\lambda_+\lambda_- = 1$ and $\kappa_+\kappa_- = -1$, where

$$\lambda_\pm = -1 + \frac{\sqrt{2}\beta}{c} \pm \frac{\sqrt{4c^2+2m^2c^4-2\sqrt{2}c\beta}}{c}$$

and

$$\kappa_\pm = \Big(-\frac{\beta}{c}+mc\Big)\Big(\frac{2c-\sqrt{2}\beta\pm\sqrt{4c^2+2m^2c^4-2\sqrt{2}c\beta}}{\beta-c\sqrt{2}-mc^2}\Big)+1.$$

Moreover,

$$\big(T_m^{(\beta)}\big)^2 = \begin{pmatrix} -1 & mc+\beta/c \\ \\ mc-\beta/c & 1 \end{pmatrix}^2 = -Id_2 \ .$$

Therefore the proof that $\gamma_m\big(E_V=-c\sqrt{2+m^2c^2}-c/\sqrt{2},\ V=c/\sqrt{2}\ \big)=0$ is analogous to the Schrödinger case (see the proof of Theorem 2.4 in [10]).

Now consider the excluded case $\big(V=c\sqrt{2+m^2c^2},\ E_V=0\big)$. In this case $\alpha^2=\beta^2=2c^2+m^2c^4$. Since $\alpha\neq\beta$ (otherwise $V=0$), then $\alpha=-\beta=\pm c\sqrt{2+m^2c^2}$. Noting that $\big(T_m^{(\alpha)}\big)^2=\big(T_m^{(\beta)}\big)^2=-Id_2$ and $T_m^{(\alpha)}T_m^{(\beta)}$ is hyperbolic, the proof that $\gamma_m\big(E_V=\ 0,\ V=c\sqrt{2+m^2c^2}\ \big)=0$ is again similar to the corresponding Schrödinger case (see the proof of Theorem 2.4 in [10]). □

Proof. **(Theorem 5)**

(i) It is sufficient to prove the Theorem for $\Psi=\delta_0^+$. Given $\lambda>0$, there exists $b'>0$ and by Lemma 5 there are $b>0$ and $C<\infty$ such that, by applying Lemma 5 for $N=[b'T]$, together with the relation (11) for $m=0$ and $\zeta=i/T$, one concludes that there exists a set $\Omega_N(\lambda)\subset\Omega$ with $\mathbf{P}(\Omega_N(\lambda))\le Ce^{-bN^\lambda}$ and

$$\big\|T_0^\omega\big(E+i/T;n,1\big)\big\|\le C \tag{16}$$

for all $\omega\in\Omega\backslash\Omega_N(\lambda)$, $1\le n\le N$ and $E\in I_V=[E_V-N^{-\lambda-1/2},E_V+N^{-\lambda-1/2}]$.

Supposing that

$$|G^{+}_{0,\omega}(E+i/T;1)|^2+|G^{-}_{0,\omega}(E+i/T;0)|^2\geq B_1(\omega)>0,$$

it follows from (13) and

$$\|T_0^{\omega}(E+i/T;n,1)^{-1}\|=\|T_0^{\omega}(E+i/T;n,1)\|$$

that

(17)

$$\max\{|G^{+}_{0,\omega}(E+i/T;n)|^2,|G^{-}_{0,\omega}(E+i/T;n-1)|^2\}\geq\frac{B_1(\omega)}{2\|T_0^{\omega}(E+i/T;n,1)\|^2}\ .$$

Thus, replacing (16) and (17) into (14), $\mathbf{P}$-a.s. one has

$$A^{(q)}_{\delta_0^+,\omega}(0,T)\geq\frac{1}{2\pi T}\sum_{0\leq n\leq[b'T]}n^q\int_{I_V}\frac{B_1(\omega)}{2C^2}\,dE\geq B_q(\omega)T^qN^{-\lambda-1/2}\geq C_q(\omega)T^{q-1/2-\lambda}$$

for some constant $C_q(\omega)>0$.

If, on the other hand,

$$|G^{+}_{0,\omega}(E+i/T;0)|^2+|G^{-}_{0,\omega}(E+i/T;-1)|^2\geq B_2(\omega)>0,$$

then one gets this estimate in the same way, but based on (12) instead of (13). Since $\lambda>0$ is arbitrary, this finishes the proof.

(ii) It follows from the above arguments by using Lemma 6. □

Proof. **(Theorem 6)**

The arguments will be a variation of [23]. Define the operator

$$p:=i\,[\mathbb{D}(m,c),X]=ci\begin{pmatrix}0&-d^*-1\\d+1&0\end{pmatrix}$$

with $[\cdot,\cdot]$ denoting the commutator. Note that p is self-adjoint and bounded. Set

$$X(t)=e^{i\mathbb{D}(m,c)t}Xe^{-i\mathbb{D}(m,c)t}\quad\text{and}\quad p(t)=e^{i\mathbb{D}(m,c)t}pe^{-i\mathbb{D}(m,c)t},$$

so that

$$\frac{d}{dt}X(t)=i\,[\mathbb{D}(m,c),X(t)]=e^{i\mathbb{D}(m,c)t}i\,[\mathbb{D}(m,c),X]\,e^{-i\mathbb{D}(m,c)t}=p(t).$$

Hence

$$X(t)=X+\int_0^t p(s)\,ds.$$

Using this relation, the boundedness $\|p(t)\|=\|p\|<\infty$ for all t, Cauchy-Schwarz inequality, and keeping only the dominant terms for large t, it follows that for $t \geq 1$ and $q \in \mathbb{N}$, there exists $C_q(\tilde{V}, m, c) > 0$ so that

$$\begin{aligned} M_\Psi^{(q)}(m,t) &= \langle \Psi, |X(t)|^q \Psi \rangle \\ &\leq C_q(\tilde{V}, m, c) \int_0^t \cdots \int_0^t \langle \Psi, p(s_1) \cdots p(s_q) \Psi \rangle \, ds_1 \cdots ds_q \\ &\leq C_q(\tilde{V}, m, c) \|p\|^q \, t^q = K_q(\tilde{V}, m, c) \, t^q. \end{aligned}$$

□

6. Dynamical Comparison

The aim of this section is to compare the dynamical moments $M_\Psi^{(q)}(m,t)$, as in Definition 1, for different masses and general potentials, in particular for the massless and massive Bernoulli cases.

Theorem 7. *Let $\mathbb{D}(m,c)$ and $\mathbb{D}(m',c)$ be Dirac operators on $l^2(\mathbb{Z}, \mathbb{C}^2)$ defined as in (2) with the same potential $\tilde{V}$, and let Ψ be with only one nonzero component. Given $T > 0$, there exists a constant $B_q > 0$ so that*

$$\sup_{0 \leq t \leq T} \left| M_\Psi^{(q)}(m,t) - M_\Psi^{(q)}(m',t) \right| \leq B_q \, |m - m'| c^2 \, T^{q+2}. \tag{18}$$

Proof. Observe that for $m = m'$ the result is immediate. Suppose $m \neq m'$. For the proof it will be assumed that $m > 0$, $m' = 0$ and $\Psi = \delta_0^+$ (the case $m > 0$, $m' > 0$ and Ψ as in the hypotheses is similar).

For fixed $\alpha > 0$ consider the Banach space

$$B_\alpha := \left\{ \Phi \in l^2(\mathbb{Z}, \mathbb{C}^2) : \ \|\Phi\|_\alpha = \sup_{k \in \mathbb{Z}} e^{\alpha |k|} \Big(|\langle \delta_k^+, \Phi \rangle| + |\langle \delta_k^-, \Phi \rangle| \Big) < \infty \right\}.$$

Since $\mathbb{D}(m,c)$ is a bounded operator on B_α, it follows that

$$\begin{aligned} |\langle \delta_n^+, e^{-i\mathbb{D}(m,c)t} \delta_0^+ \rangle| + |\langle \delta_n^-, e^{-i\mathbb{D}(m,c)t} \delta_0^+ \rangle| &\leq \|e^{-i\mathbb{D}(m,c)t} \delta_0^+\|_\alpha \, e^{-n\alpha} \\ &\leq e^{-n\alpha + t\|\mathbb{D}(m,c)\|_\alpha}. \end{aligned} \tag{19}$$

For $k \in \mathbb{N}$ denote by X^k the restriction of the position operator X to the set $\{n \in \mathbb{Z} : \ |n| \leq k\}$ and by $M_{\delta_0^+}^{(q),k}(m,t)$ the corresponding dynamical moments. Then, for all times $t \leq \dfrac{\alpha \, k}{2\|\mathbb{D}(m,c)\|_\alpha}$, using (19) one has

$$\left| M_{\delta_0^+}^{(q)}(m,t) - M_{\delta_0^+}^{(q),k}(m,t) \right| =$$

$$= \sum_{|n| > k} |n|^q \left(\left| \langle \delta_n^+, e^{-i\mathbb{D}(m,c)t} \delta_0^+ \rangle \right|^2 + \left| \langle \delta_n^-, e^{-i\mathbb{D}(m,c)t} \delta_0^+ \rangle \right|^2 \right)$$

$$\leq C_1(q) k^q e^{-k\alpha + 2t\|\mathbb{D}(m,c)\|_\alpha} \leq \ C_1(q) k^q. \tag{20}$$

Furthermore, it follows by DuHamel's formula that

$$M_{\delta_0^+}^{(q),k}(m,t) - M_{\delta_0^+}^{(q),k}(0,t) =$$

$$-i\int_0^t \left\langle \delta_0^+, e^{i\mathbb{D}(m,c)t}|X^k|^q e^{-i\mathbb{D}(m,c)(t-s)}\left(\mathbb{D}(m,c)-\mathbb{D}(0,c)\right)e^{-i\mathbb{D}(0,c)s}\delta_0^+\right\rangle ds$$

$$+i\int_0^t \left\langle \delta_0^+, e^{i\mathbb{D}(m,c)(t-s)}(\mathbb{D}(m,c)-\mathbb{D}(0,c))e^{i\mathbb{D}(0,c)s}|X^k|^q e^{-i\mathbb{D}(0,c)t}\delta_0^+\right\rangle ds.$$

Hence, for $t \le \dfrac{\alpha\, k}{2\|\mathbb{D}(m,c)\|_\alpha}$, using (19), the fact of the operator $e^{i\mathbb{D}(m,c)t}$ on $\ell^2(\mathbb{Z};\mathbb{C}^2)$ be unitary and Cauchy-Schwarz, it is found that

$$\begin{aligned}(21)\quad \left|M_{\delta_0^+}^{(q),k}(m,t)-M_{\delta_0^+}^{(q),k}(0,t)\right| &\le C_2(q)mc^2k^{q+1}t\; e^{-k\alpha+t\|\mathbb{D}(m,c)\|_\alpha}\\ &\le C_2(q)mc^2\frac{\alpha}{2\|\mathbb{D}(m,c)\|_\alpha}\;k^{q+2}.\end{aligned}$$

Thus, by (20) and (21),

$$\left|M_{\delta_0^+}^{(q)}(m,t)-M_{\delta_0^+}^{(q)}(0,t)\right| \le B_q mc^2\frac{\alpha}{2\|\mathbb{D}(m,c)\|_\alpha}\;k^{q+2},$$

for all times $t \le \dfrac{\alpha\, k}{2\|\mathbb{D}(m,c)\|_\alpha}$.

Now, for each $T>0$ choose k to be the smallest integer such that

$$k \ge \frac{2\|\mathbb{D}(m,c)\|_\alpha}{\alpha}\;T.$$

Therefore, for all $t \le T$,

$$\left|M_{\delta_0^+}^{(q)}(m,t)-M_{\delta_0^+}^{(q)}(0,t)\right| \le B_q mc^2T^{q+2}.$$

□

With respect to the Bernoulli-Dirac model (9), the relation (18) with $m>0$ and $m'=0$ gives an estimate of how, for small times and/or sufficiently small mass, the dynamics of the localized regime follows the delocalized one. In terms of the average dynamical moments $A_{\Psi,\omega}^{(q)}(m,T)$ defined in (10) one has

Corollary 1. *Let* $(\mathbb{D}_\omega(m,c))_{\omega\in\Omega}$ *be as in (9),* $m\ge 0$ *and* $V\in(0,c], V\neq c/\sqrt{2}$*, and let* Ψ *be with only one nonzero component. Then, for each* $q>0$*,* $\boldsymbol{P}$*-a.s. there is* $\tilde{C}_{q,\omega}>0$ *so that*

$$\left|1-\frac{A_{\Psi,\omega}^{(q)}(m,t)}{A_{\Psi,\omega}^{(q)}(0,t)}\right| \le \tilde{C}_{q,\omega}\, mc^2t^{5/2},\quad t>0.$$

Notice that the power exponent on the r.h.s. of this expression does not depend on q*.*

Proof. By Theorem 7 with $m > 0$ and $m' = 0$, it follows that for all $t > 0$

$$\left| A^{(q)}_{\Psi,\omega}(m,t) - A^{(q)}_{\Psi,\omega}(0,t) \right| \leq \Gamma(q+3) C_{q,\omega}\, mc^2 t^{q+2},$$

with Γ the usual gamma function. By Theorem 5(i) there is $0 < B_q(\omega) < \infty$ such that

$$A^{(q)}_{\Psi,\omega}(0,t) \geq B_q(\omega) t^{q-1/2} \qquad \mathbf{P} - a.s.,$$

and the result follows with $\tilde{C}_{q,\omega} = \Gamma(q+3) C_{q,\omega}/B_q(\omega)$. □

Remark. By using Theorem 6 one gets (for $q \in \mathbb{N}$)

$$\left| M^{(q)}_{\Psi}(m,t) - M^{(q)}_{\Psi}(m',t) \right| \leq \tilde{K}_q(\omega, m, m', c)\ t^q,$$

but with no expression for the constant $\tilde{K}_q(\omega, m, m', c)$. The price paid for the explicit dependence on the masses and light speed c in Theorem 7 is the larger exponent $q+2$ instead of just q. In the same way, the exponent $5/2$ in Corollary 1 could be replaced by $3/2$, but with no precise dependence of the resulting multiplicative constant on m and c.

References

[1] P. W. Anderson, Absence of Diffusion in Certain Random Lattices, Phys. Rev. **109** (1958) 1492–1505.

[2] C. Basu, C. L. Roy, E. Maciá, F. Domínguez-Adame and A. Sánchez, Localization of Relativistic Electrons in a One-Dimensional Disordered System, J. Phys. A **27** (1994) 3285–3291.

[3] J. M. Berezanskii, Expansions in Eigenfunctions of Self-Adjoint Operators, Providence: Amer. Mat. Soc., 1968.

[4] S. D. Bjorken and J. D. Drell, Relativistic Quantum Mechanics, McGraw-Hill, New York, 1965.

[5] P. Bougerol and J. Lacroix, Products of Random Matrices with Applications to Schrödinger Operators, Birkhäuser, Boston, 1985.

[6] R. Carmona, A. Klein and F. Martinelli, Anderson Localization for Bernoulli and other Singular Potentials, Commun. Math. Phys. **108** (1987) 41–66.

[7] T. O. Carvalho and C. R. de Oliveira, Critical Energies in Random Palindrome Models, J. Math. Phys. **44** (2003) 945–961.

[8] D. Damanik, D. Lenz and G. Stolz, Lower Transport Bounds for One-Dimensional Continuum Schrödinger Operators, Preprint (2004).

[9] D. Damanik, A. Sütő and S. Tcheremchantsev, Power-Law Bounds on Transfer Matrices and Quantum Dynamics in One Dimension II, J. Funct. Anal. **216** (2004) 362–387.

[10] S. De Bièvre and F. Germinet, Dynamical Localization for Random Dimer Schrödinger Operator, J. Stat. Phys. **98** (2000) 1135–1148.

[11] C. R. de Oliveira and R. A. Prado, Dynamical Delocalization for the 1D Bernoulli Discrete Dirac Operator, J. Phys. A: Math. Gen. **38** (2005) L115–L119.

[12] R. del Rio, S. Jitomirskaya, Y. Last and B. Simon, Operators with Singular Continuous Spectrum IV: Hausdorff Dimensions, Rank One Perturbations and Localization, J. d'Analyse Math. **69** (1996) 153–200.

[13] D. H. Dunlap, H.-L. Wu, and P. W. Phillips, Absence of Localization in a Random Dimer Model, Phys. Rev. Lett. **65** (1990) 88–91.

[14] J. Fröhlich and T. Spencer, Absence of Diffusion with Anderson Tight Binding Model for Large Disorder or Low Energy, Commun. Math. Phys. **88** (1983) 151–184.

[15] J. Fröhlich, F. Martinelli, E. Scoppola and T. Spencer, Constructive Proof of Localization in the Anderson Tight Binding Model, Commun. Math. Phys. **101** (1985) 21–46.

[16] F. Germinet and S. De Bièvre, Dynamical Localization for Discrete and Continuous Random Schrödinger Operators, Commun. Math. Phys. **194** (1998) 323–341.

[17] F. Germinet, A. Kiselev and S. Tcheremchantsev, Transfer Matrices and Transport for 1D Schrödinger Operators with Singular Spectrum, Ann. Inst. Fourier **54** (2004) 787–830.

[18] S. Jitomirskaya, H. Schulz-Baldes and G. Stolz, Delocalization in Random Polymer Models, Commun. Math. Phys. **233** (2003) 27–48.

[19] R. Killip, A. Kiselev and Y. Last, Dynamical Upper Bounds on Wavepacket Spreading, Amer. J. Math. **125** (2003) 1165–1198.

[20] C. L. Roy, Some Special Features of Relativistic Tunnelling through Multi-Barrier Systems with $\delta-$function Barriers, Phys. Lett. A **189** (1994) 345–350.

[21] C. L. Roy and C. Basu, Relativistic Study of Electrical Conduction in Disordered Systems, Phys. Rev. B **45** (1992) 14293–14301.

[22] B. Simon, Schrödinger Semigroups, Bull. Amer. Math. Soc. **7** (1982) 447–526.

[23] B. Simon, Absence of Ballistic Motion, Commun. Math. Phys. **134** (1990) 209–212.

[24] C. M. Soukoulis and E. N. Economou, Off-diagonal disorder in one-dimensional systems, Phys. Rev. **B 24** (1981) 5698–5702.

[25] A. A. Stahlhofen, Supertransparent Potentials for the Dirac Equation, J. Phys. A **27** (1994) 8279–8290.
[26] B. Thaller, The Dirac Equation, Springer-Verlag, Berlin, 1991.
[27] G. Theodorou and M. H. Cohen, Extended States In an One-Dimensional System With Off-Diagonal Disorder, Phys. Rev. **B 13** (1976) 4597–4601.
[28] H. von Dreifus and A. Klein, A New Proof of Localization in the Anderson Tight Binding Model, Commun. Math. Phys. **124** (1989) 285–299.

Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil
E-mail address: oliveira@dm.ufscar.br

Departamento de Matemática – UFSCar, São Carlos, SP, 13560-970 Brazil,
E-mail address: rap@dm.ufscar.br