SIMPLICITY OF EIGENVALUES IN THE ANDERSON MODEL

ABEL KLEIN AND STANISLAV MOLCHANOV

Abstract. We give a simple, transparent, and intuitive proof that all eigenvalues of the Anderson model in the region of localization are simple.

The Anderson tight binding model is given by the random Hamiltonian $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbf{Z}^d)$, where $\Delta(x,y) = 1$ if $|x-y| = 1$ and zero otherwise, and the random potential $V_\omega = \{V_\omega(x), x \in \mathbf{Z}^d\}$ consists of independent identically distributed random variables whose common probability distribution μ has a bounded density ρ. It is known to exhibit exponential localization at either high disorder or low energy [FMSS, DK, AM].

We prove a general result about eigenvalues of the Anderson Hamiltonian with fast decaying eigenfunctions, from which we conclude that in the region of exponential localization all eigenvalues are simple. We call $\varphi \in \ell^2(\mathbb{Z}^d)$ fast decaying if it has β-decay for some $\beta > \frac{5d}{2}$, that is, $|\varphi(x)| \le C_\varphi \langle x\rangle^{-\beta}$ for some $C_\varphi < \infty$, where $\langle x \rangle := \sqrt{1+|x|^2}$.

Theorem. *The Anderson Hamiltonian H_ω cannot have an eigenvalue with two linearly independent fast decaying eigenfunctions with probability one.*

We have the immediate corollary:

Corollary. *Let I be an interval of exponential localization for the Anderson Hamiltonian H_ω. Then, with probability one, every eigenvalue of H_ω in I is simple.*

This corollary was originally proved by Simon [S], who proved that in intervals of localization the vectors δ_x, $x \in \mathbb{Z}^d$, are cyclic for H_ω with probability one, and hence the pure point spectrum is simple. Simon's cyclicity result cannot be extended to Anderson-type Hamiltonians in the continuum.

In contrast, our proof is quite transparent and intuitive. The only step in the proof that cannot presently be done in the continuum is the use of Minami's estimate [M], stated below in (7). But some form of Minami's estimate must hold in the continuum. When Minami's estimate is extended to the continuum, our proof will give the simplicity of eigenvalues also for continuous Anderson-type Hamiltonians.

While the simplicity of eigenvalues for Anderson-type Hamiltonians in the continuum is not presently known, Germinet and Klein [GK] have recently

A.K. was supported in part by NSF Grant DMS-0200710.

proved that all eigenvalues in the region of complete localization (i.e., the region of applicability of the multiscale analysis) have finite multiplity. Their result does not require that the probability distribution of the potential at a single site has a bounded density.

The proof of the theorem is based on two lemmas regarding the finite volume operators, the first one a deterministic result.

We let Λ_L be the open box centered at the origin with side of length $L > 0$, and write χ_L for its characteristic function. Given $H = -\Delta + V$, we let H_L be the operator H restricted to $\ell^2(\Lambda_L)$ with zero boundary conditions outside Λ_L. We identify $\ell^2(\Lambda_L)$ with $\chi_L \ell^2(\mathbb{Z}^d)$, in which case $H_L = \chi_L H \chi_L$. We write $H_L^\perp = (1-\chi_L)H(1-\chi_L)$, and $\Gamma_L = H - H_L - H_L^\perp = -\Delta + \Delta_L + \Delta_L^\perp$. By $C_{a,b,\dots}$ we will always denote some finite constant depending only on $a, b, \dots$. We write χ_J for the charateristic function of the set J.

Lemma 1. *Let E be an eigenvalue for $H = -\Delta + V$ with two linearly independent eigenfunctions with β-decay for some $\beta > \frac{d}{2}$. Then there exists $C = C_{d,\beta,\varphi_1,\varphi_2}$, where φ_1 and φ_2 are the two eigenfunctions, such that if we set $\varepsilon_L = CL^{-\beta+\frac{d}{2}}$ and $J_L = [E - \varepsilon_L, E + \varepsilon_L]$, we have $\operatorname{tr} \chi_{J_L}(H_L) \geq 2$ for all sufficiently large L.*

Proof. Let $\varphi_i \in \ell^2(\mathbb{Z}^d)$, $i = 1, 2$, be orthonormal with β-decay such that $H\varphi_i = E\varphi_i$. Given $\varphi \in \ell^2(\mathbb{Z}^d)$ we set $\varphi_L = \chi_L \varphi$ and $\varphi_L^\perp = \varphi - \varphi_L$. We have

$$\|\varphi_{i,L}^\perp\| \leq \varepsilon_L \quad \text{and} \quad \|\varphi_{i,L}\| \geq \sqrt{1 - \varepsilon_L^2}, \quad i = 1, 2, \tag{1}$$

$$|\langle \varphi_{1,L}, \varphi_{2,L} \rangle| \leq \varepsilon_L^2, \tag{2}$$

$$\|(H_L - E)\varphi_{i,L}\| = \|\Gamma_L \varphi_{i,L}^\perp\| \leq C'_{d,\beta,\varphi_1,\varphi_2} L^{-\beta+\frac{d-1}{2}} \leq \varepsilon_L, \quad i = 1, 2, \tag{3}$$

for all large L (assumed from now on), with $\varepsilon_L = C_{d,\beta,\varphi_1,\varphi_2} L^{-\beta+\frac{d}{2}}$.

It follows that $\varphi_{1,L}$ and $\varphi_{2,L}$ are linearly independent, and hence their linear span V_L has dimension two. Moreover, we can check that

$$\|(H_L - E)\psi\| \leq 2\varepsilon_L \|\psi\| \quad \text{for all } \psi \in V_L. \tag{4}$$

Now let $J_L = [E - 3\varepsilon_L, E + 3\varepsilon_L]$, and set $P_L = \chi_{J_L}(H_L)$, $Q_L = I - P_L$. Then for all $\psi \in V_L$ we have, using (4),

$$\begin{aligned} \|Q_L\psi\| &\leq (3\varepsilon_L)^{-1} \|(H_L - E)Q_L\psi\| = (3\varepsilon_L)^{-1} \|Q_L(H_L - E)\psi\| \\ &\leq (3\varepsilon_L)^{-1} \|(H_L - E)\psi\| \leq \tfrac{2}{3} \|\psi\|, \end{aligned} \tag{5}$$

and hence

$$\|P_L\psi\|^2 = \|\psi\|^2 - \|Q_L\psi\|^2 \geq \tfrac{5}{9} \|\psi\|^2. \tag{6}$$

Thus P_L is injective on V_L and we conclude that $\operatorname{tr} P_L \geq \dim V_L = 2$.

Redefining the constant in the definition of ε_L we get the lemma. □

The second lemma is probabilistic; it says that the probability of two eigenvalues (perhaps equal) of the finite volume operator being close together is very small for large volumes. It depends crucially on the following beautiful estimate of Minami [M, Lemma 2 and proof of Eq. (2.48)]:

$$\mathbb{P}\left\{\operatorname{tr}\chi_J(H_{\omega,L}) \geq 2\right\} \leq \pi^2 \|\rho\|_\infty^2 |J|^2 L^{2d} \tag{7}$$

for all intervals J and length scales $L \geq 1$.

Lemma 2. *Let H_ω be the Anderson Hamiltonian. If I is a bounded interval and $q > 2d$, let $\mathcal{E}_{L,I,q}$ denote the event that $\operatorname{tr}\chi_J(H_{\omega,L}) \leq 1$ for all subintervals $J \subset I$ with length $|J| \leq L^{-q}$. Then*

$$\mathbb{P}\{\mathcal{E}_{L,I,q}\} \geq 1 - 8\pi^2 \|\rho\|_\infty^2 (|I|+1) L^{-q+2d}. \tag{8}$$

Proof. We can cover the interval I by $2\left(\left[\frac{L^q}{2}|I|\right]+1\right) \leq L^q|I|+2$ intervals of length $2L^{-q}$, in such a way that any subinterval $J \subset I$ with length $|J| \leq L^{-q}$ will be contained in one of these intervals. ($[x]$ denotes the largest integer $\leq x$.) Since the complementary event, $\mathcal{E}_{L,I,q}^c$, occurs if there exists an interval $J \subset I$ with $|J| \leq L^{-q}$ such that $\operatorname{tr}\chi_J(H_{\omega,L}) \geq 2$, its probability can be estimated, using (7), by

$$\mathbb{P}\{\mathcal{E}_{L,I,q}^c\} \leq \pi^2 \|\rho\|_\infty^2 (L^q|I|+2)(2L^{-q})^2 L^{2d} \leq 8\pi^2 \|\rho\|_\infty^2 (|I|+1) L^{-q+2d}, \tag{9}$$

and hence (8) follows. □

Proof of Theorem. Let I be a bounded open interval, and set $L_k = 2^k$ for $k = 1, 2, \ldots$. It follows from Lemma 2, applying the Borel-Cantelli Lemma, that if $q > 2d$, then for $\mathbb{P}$-a.e. ω there exists $k(q,\omega) < \infty$ such that the event $\mathcal{E}_{L_k,I,q}$ occurs for all $k \geq k(q,\omega)$. But if $E \in I$ is an eigenvalue for H_ω with two linearly independent eigenfunctions with β-decay for some $\beta > \frac{5d}{2}$, then Lemma 1 tells us that for all large k we have $\operatorname{tr}\chi_{J_k}(H_{\omega,L_k}) \geq 2$, where $J_k = J_{L_k}$ is a subinterval of I with $|J_k| \leq C L_k^{-(\beta-\frac{d}{2})}$, which is not possible since if $\beta > \frac{5d}{2}$ there exists $q > 2d$ such that $\beta - \frac{d}{2} > q$. □

Acknowledgement. We thank François Germinet for a critical reading.

References

[AM] Aizenman, M., Molchanov, S.: Localization at large disorder and extreme energies: an elementary derivation. Commun. Math. Phys. **157**, 245-278 (1993)

[DK] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. Commun. Math. Phys. **124**, 285-299 (1989)

[FMSS] Fröhlich, J., Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Commun. Math. Phys. **101**, 21-46 (1985)

[GK] Germinet, F, Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. Preprint

[M] Minami, N.: Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. Comm. Math. Phys. **177** 709–725 (1996)

[S] Simon, B.: Cyclic vectors in the Anderson model. Special issue dedicated to Elliott H. Lieb. Rev. Math. Phys. **6**, 1183-1185 (1994)

University of California, Irvine, Department of Mathematics, Irvine, CA 92697-3875, USA
E-mail address: aklein@uci.edu

University of North Carolina at Charlotte, Department of Mathematics, Charlotte, NC 28223-0001, USA
E-mail address: smolchan@uncc.edu