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Models with Recoil for Bose-Einstein Condensation and Superradiance

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Abstract

In this paper we consider two models which exhibit *equilibrium* BEC superradiance. They are related to two different types of superradiant scattering observed in recent experiments. The first one corresponds to the amplification of matter-waves due to Raman superradiant scattering from a cigar-shaped BE condensate, when the recoiled and the condensed atoms are in different internal states. The main mechanism is stimulated Raman scattering in two-level atoms, which occurs in a superradiant way. Our second model is related to the superradiant Rayleigh scattering from a cigar-shaped BE condensate. This again leads to a matter-waves amplification but now with the recoiled atoms in the same state as the atoms in the condensate. Here the recoiling atoms are able to interfere with the condensate at rest to form a matter-wave grating (interference *fringes*) which is observed experimentally.

Keywords: Bose-Einstein Condensation, Raman/Rayleigh Superradiance, Optic Lattice, Matter-Wave Grating
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1 Introduction

This paper is the third in a series about models for equilibrium Bose-Einstein Condensation (BEC) superradiance motivated by the discovery of the Dicke superradiance and BEC matter waves amplification [1]-[5]. In these experiments the condensate is illuminated with a laser beam, the so called *dressing beam*. The BEC atoms then scatter photons from this beam and receive the corresponding recoil momentum producing coherent *four-wave mixing* of light and atoms [5]. The aim of our project is the construction of soluble statistical mechanical models for these phenomena.

In the first paper [6], motivated by the principle of *four-wave mixing* of light and atoms [5], we considered two models with a linear interaction between Bose atoms and photons, one with a global gauge symmetry and another one in which this symmetry is broken. In both cases we provided a rigorous proof for the emergence of a cooperative effect between BEC and superradiance. We proved that there is equilibrium superradiance and also that there is an enhancement of condensation compared with that occurring in the case of the free Bose gas.

In the second paper [7] we formalized the ideas described in [4, 5] by constructing a thermodynamically stable model whose main ingredient is the *two-level* internal states of the Bose condensate atoms. We showed that our model is equivalent to a *bosonized* Dicke maser model. Besides determining its equilibrium states, we computed and analyzed the thermodynamic functions, again finding the existence of a cooperative effect between BEC and superradiance. Here the phase diagram turns out to be more complex due to the two-level atomic structure.

In the present paper we study the effect of *momentum recoil* which was omitted in [6] and [7]. Here we consider two models motivated by two different types of superradiant scattering observed in recent experiments carried out by the MIT group, see e.g. [1]-[3]. Our first model (*Model 1*) corresponds to the Raman superradiant scattering from a cigar-shaped BE condensate considered in [1]. This leads to the amplification of matter waves (recoiled atoms) in the situation when amplified and condensate atoms are in *different* internal states. The main mechanism is stimulated Raman scattering in two-level atoms, which occurs in a way similar to Dicke superradiance [7].

Our second model (*Model 2*) is related to the superradiant Rayleigh scattering from a cigar-shaped BE condensate [2], [3]. This again leads to a matter-wave amplification but now with recoiled atoms in the *same* state as the condensate at rest. This is because the condensate is now illuminated by an off-resonant pump laser beam, so that for a long-pulse the atoms remain in their lower level states. In this case the (*non-Dicke*) superradiance is due to self-stimulated Bragg scattering [3].

From a theoretical point of view both models are interesting as they describe homogeneous systems in which there is *spontaneous breaking of translation invariance*. In the case of the Rayleigh superradiance this means that the phase transition corresponding to BEC is at the same time also a transition into a *matter-wave grating* i.e. a “frozen” spatial density wave structure, see Section 4. The fact that *recoiling* atoms are able to interfere with the condensate *at rest* to form a matter-wave grating (interference *fringes*) has been recently observed experimentally, see [3]-[5], and discussion in [8] and [9].

In the case of the Raman superradiance there is an important difference: the internal atomic states for condensed and recoiled bosons are *orthogonal*. Therefore these bosons are *different* and consequently cannot interfere to produce a matter-wave grating as in the first case. Thus the observed spatial modulation is not in the atomic density of interfering recoiled and condensed bosons, but in the *off-diagonal* coherence and photon condensate producing a one-dimensional (*corrugated*) optical lattice, see Sections 2 and discussion in Section 4.

Now let us make the definition of our models more exact. Consider a system of identical bosons of mass m enclosed in a cube $\Lambda \subset \mathbb{R}^\nu$ of volume $V = |\Lambda|$ centered at the origin. We impose periodic boundary conditions so that the momentum dual set is $\Lambda^* = \{2\pi p/V^{1/\nu} | p \in \mathbb{Z}^\nu\}$.

In *Model 1* the bosons have an internal structure which we model by considering them as two-level atoms, the two levels being denoted by $\sigma = \pm$. For momentum k and level σ, $a^*_{k,\sigma}$ and $a_{k,\sigma}$ are the usual boson creation and annihilation operators with $[a_{k,\sigma}, a^*_{k',\sigma'}] = \delta_{k,k'}\delta_{\sigma,\sigma'}$. Let $\epsilon(k) = \|k\|^2/2m$ be the single particle kinetic energy and $N_{k,\sigma} = a^*_{k,\sigma}a_{k,\sigma}$ the operator for the number of particles with momentum k and level σ. Then the total kinetic energy is

$$T_{1,\Lambda} = \sum_{k\in\Lambda^*} \epsilon(k)(N_{k,+} + N_{k,-}) \tag{1.1}$$

and the total number operator is $N_{1,\Lambda} = \sum_{k\in\Lambda^*}(N_{k,+} + N_{k,-})$. We define the Hamiltonian $H_{1,\Lambda}$ for *Model 1* by

$$H_{1,\Lambda} = T_{1,\Lambda} + U_{1,\Lambda} \tag{1.2}$$

where

$$U_{1,\Lambda} = \Omega\, b^*_q b_q + \frac{g}{2\sqrt{V}}(a^*_{q+}a_{0-}b_q + a_{q+}a^*_{0-}b^*_q) + \frac{\lambda}{2V}N^2_{1,\Lambda}, \tag{1.3}$$

$g > 0$ and $\lambda > 0$. Here b_q, b^*_q are the creation and annihilation operators of the photons, which we take as a one-mode boson field with $[b_q, b^*_q] = 1$ and a frequency Ω. g is the coupling constant of the interaction of the bosons with the photon external field which, without loss of generality, we can take to be positive as we can always incorporate the sign of g into b. Finally the λ-term is added in (1.2) to obtain a thermodynamical stable system and to ensure the right thermodynamic behaviour. This is explained in Section 2.

In *Model 2* we consider the situation when the excited atoms have already irradiated photons, i.e. we deal only with de-excited atoms $\sigma = -$. In other words, we neglect the atom excitation and consider only *elastic* atom-photon scattering. This is close to the experimental situation [3]-[5], in which the atoms in the BE condensate are irradiated by *off-resonance* laser beam. Assuming that detuning between the optical fields and the atomic two-level resonance is much larger that the natural line of the atomic transition (superradiant Rayleigh regime [2, 3]) we get that the atoms always remain in their *lower* internal energy state. We can then ignore the internal structure of the atoms and let a^*_k and a_k be the usual boson creation and annihilation operators for momentum k with $[a_k, a^*_{k'}] = \delta_{k,k'}$, $N_k = a^*_k a_k$ the operator for the number of particles with momentum k,

$$T_{2,\Lambda} = \sum_{k\in\Lambda^*} \epsilon(k)N_k \tag{1.4}$$

the total kinetic energy, and $N_{2,\Lambda} = \sum_{k\in\Lambda^*} N_k$ the total number operator. We then define the Hamiltonian $H^{(2)}_\Lambda$ for *Model 2* by

$$H_{2,\Lambda} = T_{2,\Lambda} + U_{2,\Lambda} \tag{1.5}$$

where
$$U_{2,\Lambda} = \Omega\, b_q^* b_q + \frac{g}{2\sqrt{V}}(a_q^* a_0 b_q + a_q a_0^* b_q^*) + \frac{\lambda}{2V} N_{2,\Lambda}^2 \,. \tag{1.6}$$

In this paper we provide a full mathematical analysis of *Model 1* for the Raman superradiance. This done in Section 2. The analysis of *Model 2* for the Rayleigh superradiance is very similar and therefore we do not repeat it but simply state the results in Section 3. In Section 4 we study the possible existence of spatial modulation of matter-waves (*matter-wave grating*) in these models and conclude with several remarks.

We close this introduction with the following comments:
- In our models we *do not* use for effective photon-boson interaction the *four-wave mixing principle*, see [5], [6], [10]. The latter seems to be important for the geometry, when a linearly polarized pump laser beam is incident in a direction perpendicular to the long axis of a cigar-shaped BE condensate, inducing the "45°- recoil pattern" picture [1]-[3]. Instead as in [7], we consider a *minimal* photon-atom interaction only with superradiated photons, cf [11]. This corresponds to superradiance in a "one-dimensional" geometry, when a pump laser beam is collimated and aligned along the long axis of a cigar-shaped BE condensate, see [9], [12].
- In this geometry the superradiant photons and recoiled matter-waves propagate in the same direction as the incident pump laser beam. If one considers it as a classical "source" (see [5]), then we get a *minimal* photon-atom interaction [7] generalized to take into account the effects of recoil. Notice that the further approximation of the BEC operators by c-numbers leads to a bilinear photon-atom interaction studied in [5], [6].
- In this paper we study *equilibrium* BEC superradiance while the experimental situation (as is the case with Dicke superradiance [13]) is more accurately described by non-equilibrium statistical mechanics. However we believe that for the purpose of understanding the quantum coherence interaction between light and the BE condensate our analysis is as instructive and is in the same spirit as the rigorous study of the Dicke model in thermodynamic equilibrium, see e.g. [14]-[16].
- In spite of the simplicity of our exactly soluble *Models 1* and *2* they are able to demonstrate the main features of the BEC superradiance with recoil: the photon-boson condensate *enhancement* with formation of the *light corrugated optical lattice* and the *matter-wave grating*. The corresponding phase diagrams are very similar to those in [7]. However, though the type of behaviour is similar, this is now partially due to the momentum recoil and not entirely to the internal atomic level structure.

2 Model 1

2.1 The effective Hamiltonian

We start with the *stability* of the Hamiltonian (1.2). Consider the term $U_{1,\Lambda}$ in (1.3). This gives

$$\begin{aligned} U_{1,\Lambda} &= \Omega\,(b_q^* + \frac{g}{2\Omega\sqrt{V}} a_{q+} a_{0-}^*)(b_q + \frac{g}{2\Omega\sqrt{V}} a_{q+}^* a_{0-}) - \frac{g^2}{4\Omega^2 V} N_{0-}(N_{q+}+1) + \frac{\lambda}{2V} N_{1,\Lambda}^2 \\ &\geq \frac{\lambda}{2V} N_{1,\Lambda}^2 - \frac{g^2}{4\Omega^2 V} N_{0-}(N_{q+}+1). \end{aligned} \tag{2.1}$$

On the basis of the trivial inequality $4ab \leq (a+b)^2$, the last term in the lower bound in (2.1) is dominated by the first term if $\lambda > g^2/8\Omega$, that is if the stabilizing coupling λ is large with respect to the coupling constant g or if the external frequency is large enough. We therefore assume the *stability condition*: $\lambda > g^2/8\Omega$.

Since we want to study the equilibrium properties of the model (1.2) in the grand-canonical ensemble, we shall work with the Hamiltonian

$$H_{1,\Lambda}(\mu) = H_{1,\Lambda} - \mu N_{1,\Lambda} \tag{2.2}$$

where μ is the chemical potential. Since $T_{1,\Lambda}$ and the interaction $U_{1,\Lambda}$ conserve the quasi-momentum, Hamiltonian (1.2) describes a *homogeneous* (translation invariant) system. To see this explicitly, notice that the external laser field possesses a natural quasi-local structure as the Fourier transform of the field operator $b(x)$:

$$b_q = \frac{1}{\sqrt{V}} \int_\Lambda dx\, e^{iq\cdot x} b(x). \tag{2.3}$$

If for $z \in \mathbb{R}^\nu$, we let τ_x be the translation automorphism $(\tau_z b)(x) = b(x+z)$, then $\tau_z b_q = e^{iq\cdot z} b_q$ and similarly $\tau_z a_{k,\sigma} = e^{iq\cdot z} a_{q,\sigma}$. Therefore, the Hamiltonian (1.2) is translation invariant. Consequently, in the thermodynamic limit, it is natural to look for translation invariant or homogeneous equilibrium states at all inverse temperatures β and all values of the chemical potential μ.

Because the interaction (1.3) is not bilinear or quadratic in the creation and annihilation operators the system cannot be diagonalized by a standard symplectic or Bogoliubov transformation. Therefore at the first glance one is led to conclude that the model is not soluble. However on closer inspection one notices that all the interaction terms contain space averages, namely, either

$$\frac{a_{0-}}{\sqrt{V}} = \frac{1}{V} \int_\Lambda dx\, a_-(x), \tag{2.4}$$

and its adjoint, or

$$\frac{1}{V} \int_\Lambda dx\, a^*_\sigma(x) a_\sigma(x). \tag{2.5}$$

Without going into all the mathematical details it is well-known [17] that space averages tend weakly to a multiple of the identity operator for all space-homogeneous *extremal* or *mixing* states. Moreover as all the methods of characterizing the equilibrium states (e.g. the variational principle, the KMS-condition, the characterization by correlation inequalities etc. [17]) involve only affine functionals on the states, we can limit ourselves to looking for the extremal or mixing equilibrium states and in so doing we can exploit the above mentioned property for space averages. One way of accomplishing this is by applying the so-called *effective Hamiltonian* method, which is based on the fact that an equilibrium state is not determined by the Hamiltonian but by its *Liouvillian.* The best route to realize this is to use the characterization of the equilibrium state by means of the *correlation inequalities* [18], [17]:

A state ω is an equilibrium state for $H_{1,\Lambda}(\mu)$ at inverse temperature β, if and only if for all local observables A, it satisfies

$$\lim_{V\to\infty} \beta\omega\left([A^*, [H_{1,\Lambda}(\mu), A]]\right) \geq \omega(A^*A) \ln \frac{\omega(A^*A)}{\omega(AA^*)}. \tag{2.6}$$

Clearly only the *Liouvillian* $[H_{1,\Lambda}(\mu), \cdot]$ of the Hamiltonian enters into these inequalities and therefore we can replace $H_{1,\Lambda}(\mu)$ by a simpler Hamiltonian, the *effective Hamiltonian*, which gives in the limiting state ω the same Liouvillian as $H_{1,\Lambda}(\mu)$ and then look for the equilibrium states corresponding to it. Now in our case for an extremal or mixing state we define the effective Hamiltonian $H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho)$ such that for all local observables A and B

$$\lim_{V\to\infty} \omega\left(A, [H_{1,\Lambda}(\mu), B]\right) = \lim_{V\to\infty} \omega\left(A, [H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho), B]\right). \tag{2.7}$$

The significance of the *parameters* η and ρ will become clear below. One can then replace (2.6) by

$$\lim_{V\to\infty} \beta\omega\left([A^*, [H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho), A]]\right) \geq \omega(A^*A) \ln \frac{\omega(A^*A)}{\omega(AA^*)}. \tag{2.8}$$

We choose $H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho)$ so that it can be diagonalized and thus (2.8) can be solved explicitly. For a given chemical potential μ, the inequalities (2.8) can have *more than one* solution. We determine the physical solution by minimizing the free energy density with respect to the set of states or equivalently by maximizing the grand canonical pressure on this set.
Let

$$\begin{aligned} H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho) &= (\lambda\rho - \mu + \epsilon(q))a^*_{q+}a_{q+} + (\lambda\rho - \mu)a^*_{0-}a_{0-} + \frac{g}{2}(\eta a^*_{q+}b_q + \bar{\eta} a_{q+}b^*_q) \\ &\quad + \Omega\, b^*_q b_q + \frac{g\sqrt{V}}{2}\left(\zeta a_{0-} + \bar{\zeta} a^*_{0-}\right) + T'_{1,\Lambda} + (\lambda\rho - \mu)N'_{1,\Lambda} \end{aligned} \tag{2.9}$$

where

$$T'_{1,\Lambda} = \sum_{k\in\Lambda^*,\, k\neq q} \epsilon(k)N_{k,+} + \sum_{k\in\Lambda^*,\, k\neq 0} \epsilon(k)N_{k,-}, \tag{2.10}$$

$$N'_{1,\Lambda} = \sum_{k\in\Lambda^*,\, k\neq q} N_{k,+} + \sum_{k\in\Lambda^*,\, k\neq 0} N_{k,-}, \tag{2.11}$$

η and ζ are complex numbers and ρ is a positive real number. Then one can easily check that (2.7) is satisfied if

$$\eta = \frac{\omega(a_{0-})}{\sqrt{V}}, \qquad \zeta = \frac{\omega(a^*_{q+}b_q)}{V} \qquad \text{and} \qquad \rho = \frac{\omega(N_{1,\Lambda})}{V}, \tag{2.12}$$

where the state ω coincides with the equilibrium state $\langle\cdot\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)}$ defined by the effective Hamiltonian $H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho)$. By virtue of (2.7) and (2.12) we then obtain the *self-consistency* equations

$$\eta = \frac{1}{\sqrt{V}}\langle a_{0-}\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)}, \qquad \zeta = \frac{1}{V}\langle a^*_{q+}b_q\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)}, \qquad \rho = \frac{1}{V}\langle N_{1,\Lambda}\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)}. \tag{2.13}$$

Note that since ζ is a function of η and ρ through (2.13), we do not need to label the effective Hamiltonian by ζ. The important simplification here is that $H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho)$ can be diagonalized:

$$\begin{aligned} H^{\text{eff}}_{1,\Lambda}(\mu, \eta, \rho) &= E_{+,\Lambda}(\mu, \eta, \rho)\alpha^*_1\alpha_1 + E_{-,\Lambda}(\mu, \eta, \rho)\alpha^*_2\alpha_2 \\ &\quad + (\lambda\rho - \mu)\alpha^*_3\alpha_3 + T'_{1,\Lambda} + (\lambda\rho - \mu)N'_{1,\Lambda} + \frac{g^2V|\zeta|^2}{4(\mu - \lambda\rho)}, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
E_{+,\Lambda}(\mu,\eta,\rho) &= \frac{1}{2}(\Omega-\mu+\lambda\rho+\epsilon(q))+\frac{1}{2}\sqrt{(\Omega+\mu-\lambda\rho-\epsilon(q))^2+g^2|\eta|^2},\\
E_{-,\Lambda}(\mu,\eta,\rho) &= \frac{1}{2}(\Omega-\mu+\lambda\rho+\epsilon(q))-\frac{1}{2}\sqrt{(\Omega+\mu-\lambda\rho-\epsilon(q))^2+g^2|\eta|^2},
\end{aligned} \tag{2.15}$$

$$\alpha_1 = a_{q+}\cos\theta + b_q\sin\theta, \qquad \alpha_2 = a_{q+}\sin\theta - b_q\cos\theta, \qquad \alpha_3 = a_{0-} + \frac{g\sqrt{V}\zeta}{2(\lambda\rho-\mu)}, \tag{2.16}$$

and

$$\tan 2\theta = -\frac{g|\eta|}{\Omega+\mu-\lambda\rho-\epsilon(q)}. \tag{2.17}$$

Note that the correlation inequalities (2.8) (see [18]) imply that

$$\lim_{V\to\infty}\omega\left(A^*, [H_{1,\Lambda}(\mu), A]\right) \geq 0 \tag{2.18}$$

for all observables A. Applying (2.18) with $A = a^*_{0+}$, one gets $\lambda\rho-\mu \geq 0$. Similarly, one obtains the condition $\lambda\rho+\epsilon(q)-\mu \geq 0$ by applying (2.18) to $A = a^*_{q+}$. We also have that $E_{+,\Lambda}(\mu,\eta,\rho) \geq E_{-,\Lambda}(\mu,\eta,\rho)$ and $E_{-,\Lambda}(\mu,\eta,\rho) = 0$ when $|\eta|^2 = 4\Omega(\lambda\rho+\epsilon(q)-\mu)/g^2$ and then $E_{+,\Lambda}(\mu,\eta,\rho) = \Omega-\mu+\lambda\rho+\epsilon(q)$. Thus we have the constraint: $|\eta|^2 \leq 4\Omega(\lambda\rho+\epsilon(q)-\mu)/g^2$. We shall need this information to make sense of the thermodynamic functions below.

The equations (2.13) can be made explicit using (2.14):

$$\eta = \frac{g}{2(\mu-\lambda\rho)}\zeta, \tag{2.19}$$

$$\zeta = \frac{1}{2}\frac{g|\eta|}{V(E_+-E_-)}\left\{\frac{1}{e^{\beta E_+}-1} - \frac{1}{e^{\beta E_-}-1}\right\} \tag{2.20}$$

and

$$\begin{aligned}
\rho &= |\eta|^2 + \frac{1}{V}\frac{1}{e^{-\beta(\mu-\lambda\rho)}-1} + \frac{1}{2V}\left\{\frac{1}{e^{\beta E_+}-1}+\frac{1}{e^{\beta E_-}-1}\right\}\\
&\quad -\frac{(\mu-\lambda\rho-\epsilon(q)+\Omega)}{2V(E_+-E_-)}\left\{\frac{1}{e^{\beta E_+}-1}-\frac{1}{e^{\beta E_-}-1}\right\}\\
&\quad +\frac{1}{V}\sum_{k\in\Lambda^*,\,k\neq q}\frac{1}{e^{\beta(\epsilon(k)-\mu+\lambda\rho)}-1} + \frac{1}{V}\sum_{k\in\Lambda^*,\,k\neq 0}\frac{1}{e^{\beta(\epsilon(k)-\mu+\lambda\rho)}-1}.
\end{aligned} \tag{2.21}$$

Combining (2.19) and (2.20) we obtain the consistency equation:

$$\eta = \frac{g^2|\eta|}{4(\mu-\lambda\rho)V(E_+-E_-)}\left\{\frac{1}{e^{\beta E_+}-1} - \frac{1}{e^{\beta E_-}-1}\right\}. \tag{2.22}$$

It is now clear that the equilibrium states are determined by the limiting form of the consistency equations (2.21) and (2.22). We shall now solve these equations and obtain the corresponding pressure so that we can determine the physical state when there are several solutions for a particular chemical potential.

Clearly if E_- does not tend to zero as $V \to \infty$ then the right-hand side of (2.22) tends to zero and $\eta = 0$. For $\eta \neq 0$ we must have $E_- \to 0$, that is $|\eta|^2 \to 4\Omega(\lambda\rho + \epsilon(q) - \mu)/g^2$. In fact for a finite limit, the large-volume asymptotic should to be

$$|\eta|^2 \approx \frac{4\Omega}{g^2}(\lambda\rho + \epsilon(q) - \mu - \frac{1}{\beta V \tau}), \tag{2.23}$$

where $\tau > 0$. This implies that

$$E_+ \to \Omega - \mu + \lambda\rho + \epsilon(q), \qquad E_- \approx \frac{\Omega}{\beta V \tau(\Omega - \mu + \lambda\rho + \epsilon(q))} \tag{2.24}$$

and (2.22) becomes in the limit:

$$\eta\left(1 - \frac{g^2\tau}{4(\lambda\rho - \mu)\Omega}\right) = 0. \tag{2.25}$$

The last equation has solutions:

$$\eta = 0, \qquad \text{or} \qquad \tau = \frac{4(\lambda\rho - \mu)\Omega}{g^2}. \tag{2.26}$$

For $\mu < 0$, let

$$\varepsilon_0(\mu) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{\epsilon(k) - \mu}{e^{\beta(\epsilon(k)-\mu)} - 1}, \tag{2.27}$$

$$\rho_0(\mu) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \frac{1}{e^{\beta(\epsilon(k)-\mu)} - 1} \tag{2.28}$$

and

$$p_0(\mu) = -\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} d^3k \ln(1 - e^{-\beta(\epsilon(k)-\mu)}), \tag{2.29}$$

that is the grand-canonical energy density, the particle density and the pressure for the *free Bose-gas*. Let

$$s_0(\mu) = \beta(\varepsilon_0(\mu) + p_0(\mu)), \tag{2.30}$$

and note that $s_0(\mu)$ is an increasing function of μ. We shall denote the free Bose-gas *critical* density by ρ_c, i.e. $\rho_c = \rho_0(0)$. Recall that ρ_c is infinite for $\nu < 3$ and finite for $\nu \geq 3$.

Now we analyze in detail the solutions of (2.25) and compute their thermodynamic functions. We consider three cases:

Case 1: Suppose that in the thermodynamic limit one has: $\eta = 0$ and $\lambda\rho - \mu > 0$. By virtue of (2.20) in this case we have $\zeta = 0$, i.e. there is *no condensation*:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{0-}a_{0-}\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}a_{q+}\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)} = 0, \tag{2.31}$$

and the photon and boson subsystems are *decoupled*. Now equation (2.21) takes the form

$$\rho = 2\rho_0(\mu - \lambda\rho), \tag{2.32}$$

the energy density is given by

$$\lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\text{eff}}_{1,\Lambda}(\mu,\eta,\rho)} = 2\varepsilon_0(\mu-\lambda\rho) - 2\lambda\rho\rho_0(\mu-\lambda\rho) + \frac{1}{2}\lambda\rho^2 = 2\varepsilon_0(\mu-\lambda\rho) - \frac{1}{2}\lambda\rho^2 \tag{2.33}$$

and the entropy density is equal to

$$s(\mu) = 2s_0(\mu - \lambda\rho). \tag{2.34}$$

Since the grand-canonical pressure is given by

$$p(\mu) = \frac{1}{\beta}s(\mu) - \lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)}, \tag{2.35}$$

then

$$p(\mu) = 2p_0(\mu - \lambda\rho) + \frac{1}{2}\lambda\rho^2. \tag{2.36}$$

Case 2: Now suppose that in this case the thermodynamic limit gives: $\eta = 0$, $\lambda\rho - \mu = 0$. Then the solution of equation (2.21) has the asymptotic form:

$$\rho_V = \frac{\mu}{\lambda} + \frac{1}{V\lambda\tau'} + o(\frac{1}{V}), \tag{2.37}$$

with some $\tau' \geq 0$. So, in the thermodynamic limit equation (2.21) yields the identity

$$\rho = \tau' + 2\rho_c, \tag{2.38}$$

which implies that $\rho \geq 2\rho_c$. Note that this case is possible only if $\nu \geq 3$. By explicit calculations one gets:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{0-}a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = \tau' = \rho - 2\rho_c, \tag{2.39}$$

i.e., there is a possibility of the *mean-field* condensation of *zero-mode non-excited* bosons $\sigma = -$. Notice that the gauge invariance implies $\lim_{V\to\infty}\langle a^\sharp_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)}/\sqrt{V} = 0$. Furthermore, we get also:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{q+}a_{q+}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,0,\rho)} = 0, \tag{2.40}$$

i.e., there is *no condensation* in the $q \neq 0$ modes and the laser boson field. Hence again the contribution from the interaction term vanishes, i.e. the photon and boson subsystems are *decoupled*.
In this case the energy density is given by:

$$\lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = 2\varepsilon_0(0) - \frac{\mu^2}{2\lambda} \tag{2.41}$$

and the entropy density has the form:

$$s(\mu) = 2s_0(0) = 2\beta(\varepsilon_0(0) + p_0(0)). \tag{2.42}$$

Thus for the pressure one gets:

$$p(\mu) = 2p_0(0) + \frac{\mu^2}{2\lambda}. \tag{2.43}$$

Case 3: Suppose that $\eta \neq 0$. Then by diagonalization of (2.9) we obtain a *simultaneous* condensation of the excited/non-excited bosons and the laser photons in the q-mode:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_{0-}a_{0-}\rangle_{H^{\rm eff}_{1,\Lambda}(\mu,\eta,\rho)} = |\eta|^2 = \frac{4\Omega(\lambda\rho + \epsilon(q) - \mu)}{g^2}, \tag{2.44}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{q+}^* a_{q+}\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)} = \tau = \frac{4(\lambda\rho-\mu)\Omega}{g^2}, \tag{2.45}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle b_q^* b_q\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)} = \frac{4(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}{g^2}. \tag{2.46}$$

Equation (2.21) becomes:

$$\rho = \frac{8\Omega}{g^2}(\lambda\rho-\mu+\epsilon(q)/2) + 2\rho_0(\mu-\lambda\rho). \tag{2.47}$$

Using the diagonalization of (2.9) one computes also

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{q+}^* b_q\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)} = \frac{4(\lambda\rho-\mu)\sqrt{\Omega(\lambda\rho+\epsilon(q)-\mu)}}{g^2}. \tag{2.48}$$

Then using (2.45) and (2.46), one obtains

$$\lim_{V\to\infty}\left|\frac{1}{V}\langle a_{q+}^* b_q\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)}\right|^2 = \lim_{V\to\infty}\frac{1}{V}\langle a_{q+}^* a_{q+}\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)}\lim_{V\to\infty}\frac{1}{V}\langle b_q^* b_q\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)}. \tag{2.49}$$

In this case the energy density is given by:

$$\begin{aligned}\lim_{V\to\infty}\frac{1}{V}\langle H_{1,\Lambda}(\mu)\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)} &= (\epsilon(q)-\mu)\frac{4(\lambda\rho-\mu)\Omega}{g^2} - \mu\frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)}{g^2}\\ &\quad -\frac{8\Omega(\lambda\rho-\mu)(\lambda\rho+\epsilon(q)-\mu)}{g^2} + \Omega\frac{4(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}{g^2}\\ &\quad +2\varepsilon_0(\mu-\lambda\rho) - 2\lambda\rho\rho_0(\mu-\lambda\rho) + \frac{1}{2}\lambda\rho^2\\ &= \frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}{g^2} + 2\varepsilon_0(\mu-\lambda\rho) - \frac{1}{2}\lambda\rho^2. \end{aligned}\tag{2.50}$$

The entropy density is again given by

$$s(\mu) = 2s_0(\mu-\lambda\rho) \tag{2.51}$$

and the pressure becomes

$$p(\mu) = 2p_0(\mu-\lambda\rho) + \frac{1}{2}\lambda\rho^2 - \frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}{g^2}. \tag{2.52}$$

Remark 2.1 The relation (2.49) in Case 3 requires some explanation. For simplicity we take $q = (2\pi/\gamma)\mathbf{e}_1$, where $\mathbf{e}_1 = (1,0,\ldots,0) \in \mathbb{R}^\nu$ and $\gamma > 0$. Let $\omega(\cdot) = \lim_{V\to\infty}\langle\cdot\rangle_{H_{1,\Lambda}^{\rm eff}(\mu,\eta,\rho)}$ denote our equilibrium state determined by the parameter η, the chemical potential μ and the density ρ, where in fact η and ρ are functions of μ through the self-consistency equations. Since the initial Hamiltonian (1.2) is translation invariant, so is the state ω. Note that (2.44), (2.45), (2.46) and (2.48) are expectation values in this state of *translation invariant* operators $a_{0-}^* a_{0-}$, $a_{q+}^* a_{q+}$, $b_q^* b_q$ and $a_{q+}^* b_q$. However the single operators $a_{q+}^\sharp$, and $b_q^\sharp$ are not translation invariant and their averages in the state ω are modulated with period γ in the $\mathbf{e}_1$ direction. Let ω_γ denote an equilibrium state periodic in the $\mathbf{e}_1$ direction with period γ. Then (2.49) suggests that for ω_γ the analogue of the mixing property for ω takes the following form:

$$\lim_{V\to\infty}\frac{1}{V}\,\omega(a_{q+}^* b_q) = \lim_{V\to\infty}\frac{1}{V}\,\omega_\gamma(a_{q+}^* b_q) = \lim_{V\to\infty}\frac{1}{\sqrt{V}}\,\omega_\gamma(a_{q+}^*)\lim_{V\to\infty}\frac{1}{\sqrt{V}}\,\omega_\gamma(b_q), \tag{2.53}$$

implying that both factors on right-hand side are non-zero. Similarly, we get, for example, that

$$\lim_{V\to\infty}\frac{1}{V}\,\omega(a_{0-}b_q)=\lim_{V\to\infty}\frac{1}{V}\,\omega_\gamma(a_{0-}b_q)=\lim_{V\to\infty}\frac{1}{\sqrt{V}}\,\omega_\gamma(a_{0-})\lim_{V\to\infty}\frac{1}{\sqrt{V}}\,\omega_\gamma(b_q). \tag{2.54}$$

To get decoupling in (2.53) and (2.54) note that the quasi-local structure (2.3) implies that

$$\frac{1}{\sqrt{V}}b_q=\frac{1}{V}\int_\Lambda dx\,e^{iq\cdot x}b(x) \tag{2.55}$$

is an operator space-average, which in the limit is a c-number in the periodic state ω_γ. Therefore the emergence of *macroscopic occupation* of the laser q-mode (2.46) is accompanied by the creation of a one-dimensional *optical lattice* in the $\mathbf{e}_1$ direction with period γ. We can then reconstruct the translation invariant state ω and thus recover (2.49), by averaging ω_γ over an elementary interval of of length γ:

$$\omega=\frac{1}{\gamma}\int_0^\gamma d\xi\;\;\omega_\gamma\circ\tau_{\xi\mathbf{e}_1}. \tag{2.56}$$

Having discussed the three cases we conclude by giving the values of the chemical potential μ when they occur. The analysis, which is given in the next subsection, involves a detailed study of the pressure.

Let $\kappa=8\Omega\lambda/g^2-1$ and $\alpha=\epsilon(q)(\kappa+1)/2$. From the condition for thermodynamic stability we know that $\kappa>0$. Let x_0 be the unique value of $x\in[0,\infty)$ such that $2\lambda\rho_0'(-x)=\kappa$ and let $\mu_0=2\lambda\rho_0(-x_0)+\kappa x_0$. Note that $\mu_0<2\lambda\rho_c$.
The case when $\mu_0+\alpha\geq 2\lambda\rho_c$ is easy. In this situation Case 1 applies for $\mu\leq 2\lambda\rho_c$ and there exists $\mu_1>\mu_0+\alpha$ such that Case 2 applies for $2\lambda\rho_c<\mu<\mu_1$ and Case 3 for $\mu\geq\mu_1$.
When $\mu_0+\alpha<2\lambda\rho_c$ the situation is more subtle. In Subsection 2.2 we shall show that there exists $\mu_1>\mu_0+\alpha$ such that Case 3 applies for $\mu\geq\mu_1$. However we are not able to decide on which side of $2\lambda\rho_c$, the point μ_1 lies. If $\mu_1>2\lambda\rho_c$ the situation is as in the previous subcase, while if $\mu_0+\alpha<\mu_1<2\lambda\rho_c$ the *intermediate phase* where Case 2 obtains is *eliminated*. This the situation is similar to [7], where one has $\alpha=0$.
Note that for $\nu<3$, Case 1 applies when $\mu<\mu_1$ and Case 3 when $\mu\geq\mu_1$.

2.2 The Pressure for Model 1

This subsection is devoted to a detailed study of the pressure for Model 1 as a function of the chemical potential μ.

Let $x=\lambda\rho-\mu$ and recall that $\kappa=8\Omega\lambda/g^2-1$ and $\alpha=\epsilon(q)(\kappa+1)/2$. In terms of x and η the above classification takes the form:

Case 1: $\eta=0$ and $x>0$. Then (2.32) becomes

$$2\lambda\rho_0(-x)-x=\mu. \tag{2.57}$$

For $\mu\leq 2\lambda\rho_c$ this has a unique solution in x, denoted by $x_1(\mu)$, while for $\mu>2\lambda\rho_c$ it has no solutions. Let

$$p_1(x,\mu)=2p_0(-x)+\frac{(x+\mu)^2}{2\lambda}. \tag{2.58}$$

Then

$$p(\mu) = p_1(x_1(\mu), \mu). \tag{2.59}$$

Case 2: $\eta = 0$ and $x = 0$. For $\mu > 2\lambda\rho_c$

$$p(\mu) = p_3(\mu) := 2p_0(0) + \frac{\mu^2}{2\lambda}. \tag{2.60}$$

Case 3: $\eta \neq 0$. Then (2.47) becomes

$$2\lambda\rho_0(-x) + \kappa x + \alpha = \mu. \tag{2.61}$$

Recall that x_0 is the unique value of $x \in [0, \infty)$ such that $2\lambda\rho_0'(-x) = \kappa$, $\mu_0 = 2\lambda\rho_0(-x_0) + \kappa x_0$ and that $\mu_0 < 2\lambda\rho_c$.

Then for $\mu < \mu_0 + \alpha$, equation (2.61) has *no* solutions. For $\mu_0 + \alpha \leq \mu \leq 2\lambda\rho_c + \alpha$ this equation has *two* solutions: $\tilde{x}_3(\mu)$ and $x_3(\mu)$, where $\tilde{x}_3(\mu) < x_3(\mu)$ if $\mu \neq \mu_0 + \alpha$, and $\tilde{x}_3(\mu_0 + \alpha) = x_3(\mu_0 + \alpha)$. Finally for $\mu > 2\lambda\rho_c + \alpha$ it has a *unique* solution $x_3(\mu)$. Let

$$p_3(x, \mu) = 2p_0(-x) + \frac{\{(x + \mu)^2 - (\kappa + 1)x^2 - 2\alpha x\}}{2\lambda}. \tag{2.62}$$

$$\frac{dp_3(\tilde{x}_3(\mu), \mu)}{d\mu} = \frac{\tilde{x}_3(\mu) + \mu}{\lambda} < \frac{x_3(\mu) + \mu}{\lambda} = \frac{dp_3(x_3(\mu), \mu)}{d\mu} \tag{2.63}$$

for $\mu \neq \mu_0 + \alpha$. Since $p_3(\tilde{x}_3(\mu_0 + \alpha), \mu_0 + \alpha) = p_3(x_3(\mu_0 + \alpha), \mu_0 + \alpha)$,

$$p_3(\tilde{x}_3(\mu), \mu) < p_3(x_3(\mu), \mu) \tag{2.64}$$

for $\mu_0 + \alpha < \mu \leq 2\lambda\rho_c + \alpha$. Therefore

$$p(\mu) = p_3(x_3(\mu), \mu) \tag{2.65}$$

for all $\mu \geq \mu_0 + \alpha$.

Note that $\tilde{x}_3(2\lambda\rho_c + \alpha) = 0$ so that

$$p_3(\tilde{x}_3(2\lambda\rho_c + \alpha), 2\lambda\rho_c + \alpha) = p_1(x_1(2\lambda\rho_c), 2\lambda\rho_c) = p_2(2\lambda\rho_c) = 2p_0(0) + 2\lambda\rho_c^2. \tag{2.66}$$

therefore

$$p_3(x_3(\mu_0 + \alpha), \mu_0 + \alpha) = p_3(\tilde{x}_3(\mu_0 + \alpha), \mu_0 + \alpha) < 2p_0(0) + 2\lambda\rho_c^2. \tag{2.67}$$

Also for large μ, $p_3(x_3(\mu), \mu) \approx (\mu^2/2\lambda)((\kappa + 1)/\kappa)$ while $p_2(\mu) \approx (\mu^2/2\lambda)$, so that $p_3(x_3(\mu), \mu) > p_2(\mu)$ eventually. We remark finally that the slope of $p_3(x_3(\mu), \mu)$ is greater than that of $p_2(\mu)$,

$$\frac{dp_3(x_3(\mu), \mu)}{d\mu} = \frac{x_3(\mu) + \mu}{\lambda} > \frac{\mu}{\lambda} = \frac{dp_2(\mu)}{d\mu}, \tag{2.68}$$

so that the corresponding curves intersect at most once.

The case $\alpha = 0$, i.e. $\epsilon(q = 0) = 0$, has been examined in [7].

For the case $\alpha > 0$ we have two *subcases*, Remark 2.1:
The subcase $\mu_0 + \alpha \geq 2\lambda\rho_c$ is easy. In this situation Case 1 applies for $\mu \leq 2\lambda\rho_c$. From (2.67) we see that

$$p_3(x_3(\mu_0 + \alpha), \mu_0 + \alpha) < 2p_0(0) + 2\lambda\rho_c^2 < p_2(\mu_0 + \alpha) \tag{2.69}$$

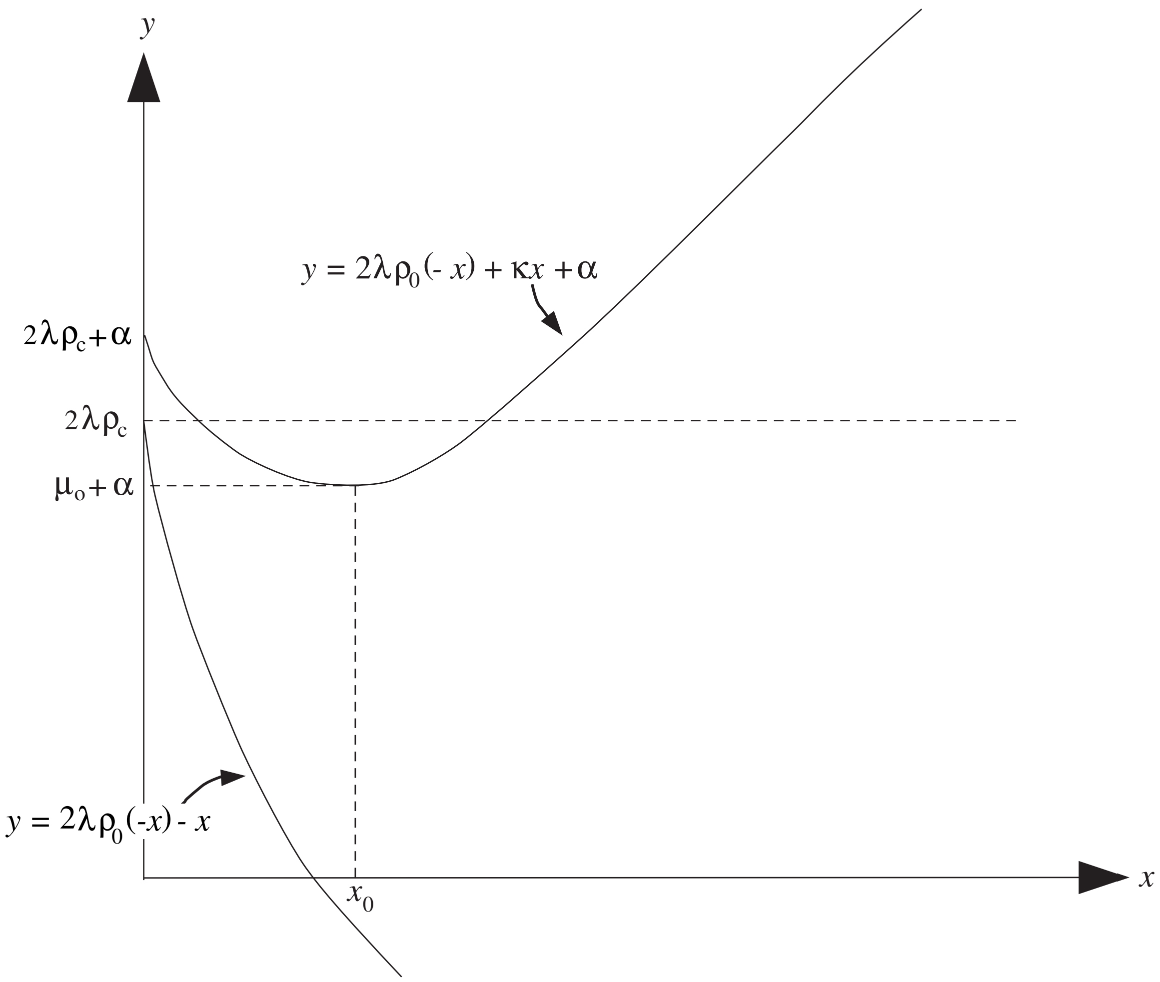

Figure 1: The density equation for $\nu \geq 3$

and therefore from the behaviour for large μ we can deduce that there exists $\mu_1 > \mu_0 + \alpha$ such that Case 2 applies for $2\lambda\rho_c < \mu < \mu_1$ and Case 3 for $\mu \geq \mu_1$.

The subcase $\mu_0+\alpha < 2\lambda\rho_c$ is more complicated. In Figure 1 we have drawn $y = 2\lambda\rho_0(-x)-x$ and $y = 2\lambda\rho_0(-x) + \kappa x + \alpha$ for this subcase. We know that

$$p_3(\tilde{x}_3(2\lambda\rho_c), 2\lambda\rho_c) < p_3(\tilde{x}_3(2\lambda\rho_c + \alpha), 2\lambda\rho_c + \alpha) = p_1(x_1(2\lambda\rho_c), 2\lambda\rho_c). \tag{2.70}$$

Therefore since the slope of $p_3(\tilde{x}_3(\mu), \mu)$ is greater than the slope of $p_1(x_1(\mu), \mu)$ for $\mu_0+\alpha < \mu < 2\lambda\rho_c$, (see Figure 1):

$$\frac{dp_3(\tilde{x}_3(\mu), \mu)}{d\mu} = \frac{\tilde{x}_3(\mu) + \mu}{\lambda} > \frac{x_1(\mu) + \mu}{\lambda} = \frac{dp_1(x_1(\mu), \mu)}{d\mu}, \tag{2.71}$$

we can conclude that

$$p_3(x_3(\mu_0 + \alpha), \mu_0 + \alpha) = p_3(\tilde{x}_3(\mu_0 + \alpha), \mu_0 + \alpha) < p_1(x_1(\mu_0 + \alpha), \mu_0 + \alpha). \tag{2.72}$$

We also know by the arguments above that there exists $\mu_1 > \mu_0+\alpha$ such that Case 3 applies for for $\mu \geq \mu_1$. However we do know on which side of $2\lambda\rho_c$, the point μ_1 lies. If $\mu_1 > 2\lambda\rho_c$ the situation is as in the previous subcase while if $\mu_0 + \alpha < \mu_1 < 2\lambda\rho_c$ the intermediate phase where Case 2 obtains is eliminated.

3 Model 2

As we said in the introduction the analysis for this model is very similar to that of Model 1. Therefore we briefly summarize the results without repeating the details. For Model 2 the effective Hamiltonian is

$$\begin{aligned} H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho) &= (\lambda\rho-\mu+\epsilon(q))a^*_q a_q + (\lambda\rho-\mu)a^*_0 a_0 + \frac{g}{2}(\eta a^*_q b_q + \bar{\eta} a_q b^*_q) \\ &\quad + \Omega\, b^*_q b_q + \frac{g\sqrt{V}}{2}\left(\zeta a_0 + \bar{\zeta} a^*_0\right) + T'_{2,\Lambda} + (\lambda\rho-\mu)N'_{2,\Lambda} \end{aligned} \tag{3.1}$$

where

$$T'_{2,\Lambda} = \sum_{k\in\Lambda^*,\, k\neq 0\, k\neq q} \epsilon(k)N_k, \tag{3.2}$$

$$N'_{2,\Lambda} = \sum_{k\in\Lambda^*,\, k\neq 0\, k\neq q} \epsilon(k)N_k. \tag{3.3}$$

The parameters η, ζ and ρ satisfy the *self-consistency* equations:

$$\eta = \frac{1}{\sqrt{V}}\langle a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)}, \qquad \zeta = \frac{1}{V}\langle a^*_q b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)}, \qquad \rho = \frac{1}{V}\langle N_{2,\Lambda}\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)}. \tag{3.4}$$

Solving these equations for this model we again have three cases:

Case 1: $\zeta = \eta = 0$ and $\lambda\rho - \mu > 0$. In this case there is *no condensation*:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_0 a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle a^*_q a_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = 0, \tag{3.5}$$

the density equation is

$$\rho = \rho_0(\mu - \lambda\rho) \tag{3.6}$$

and the pressure is

$$p(\mu) = p_0(\mu-\lambda\rho) + \frac{1}{2}\lambda\rho^2. \tag{3.7}$$

Case 2: $\eta = 0$, $\lambda\rho - \mu = 0$. Here $\rho \geq \rho_c$ and

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_0 a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \rho - \rho_c. \tag{3.8}$$

There is condensation in the $k=0$ mode but there is *no condensation* in the $k=q$ mode and the photon laser field:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_q a_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \lim_{V\to\infty}\frac{1}{V}\langle b^*_q b_q\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = 0. \tag{3.9}$$

The pressure density is given by

$$p(\mu) = p_0(0) + \frac{\mu^2}{2\lambda}. \tag{3.10}$$

Case 3: $\eta \neq 0$. There is simultaneous condensation of the zero-mode and the q-mode bosons as well as the laser q-mode photons:

$$\lim_{V\to\infty}\frac{1}{V}\langle a^*_0 a_0\rangle_{H^{\rm eff}_{2,\Lambda}(\mu,\eta,\rho)} = \frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)}{g^2}, \tag{3.11}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a_q^*a_q\rangle_{H_{2,\Lambda}^{\rm eff}(\mu,\eta,\rho)} = \frac{4(\lambda\rho-\mu)\Omega}{g^2}, \tag{3.12}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle b_q^*b_q\rangle_{H_{2,\Lambda}^{\rm eff}(\mu,\eta,\rho)} = \frac{4(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}{g^2}, \tag{3.13}$$

$$\lim_{V\to\infty}\frac{1}{V}\langle a_q^*b_q\rangle_{H_{2,\Lambda}^{\rm eff}(\mu,\eta,\rho)} = \frac{4(\lambda\rho-\mu)\sqrt{\Omega(\lambda\rho+\epsilon(q)-\mu)}}{g^2} \tag{3.14}$$

and

$$\lim_{V\to\infty}\left|\frac{1}{V}\langle a_q^*b_q\rangle_{H_{2,\Lambda}^{\rm eff}(\mu,\eta,\rho)}\right|^2 = \lim_{V\to\infty}\frac{1}{V}\langle a_q^*a_q\rangle_{H_{2,\Lambda}^{\rm eff}(\mu,\eta,\rho)}\lim_{V\to\infty}\frac{1}{V}\langle b_q^*b_q\rangle_{H_{2,\Lambda}^{\rm eff}(\mu,\eta,\rho)}. \tag{3.15}$$

The density is given by

$$\rho = \frac{8\Omega}{g^2}(\lambda\rho-\mu+\epsilon(q)/2)+\rho_0(\mu-\lambda\rho) \tag{3.16}$$

and pressure is

$$p(\mu) = p_0(\mu-\lambda\rho)+\frac{1}{2}\lambda\rho^2-\frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}{g^2}. \tag{3.17}$$

Note that relations between the values of μ and the three cases above are exactly the same as for Model 1 apart from the fact that $2\rho_0$ is now replaced by ρ_0 and $2\rho_c$ by ρ_c. For that one has to compare the kinetic energy operators (1.1) and (1.4).

4 The Matter-Wave Grating. Conclusion

The recently observed phenomenon of *periodic spacial variation in the boson-density* is responsible for the light and matter-wave *amplification* in superradiant condensation, see [2]-[4], [12]. This so called *matter-wave grating* is produced by the interference of two different macroscopically occupied momentum states: the first corresponds to a macroscopic number of *recoiled* bosons and the second to *residual* BE condensate at rest.

To study the possibility of such interference in *Model 1* we recall that for system (1.2), with *two* species of boson atoms the local particle density operator has the form

$$\rho(x) := \frac{1}{V}\sum_{k\in\Lambda^*,\,\sigma=\pm}\rho_{k,\sigma}e^{ikx}\,, \tag{4.1}$$

where the Fourier transforms of the local particle densities for two species are

$$\rho_{k,\sigma} := \sum_{p\in\Lambda^*} a^*_{p+k,\sigma}a_{p,\sigma}\,. \tag{4.2}$$

For the (limiting) translation invariant equilibrium states generated by the Hamiltonian (1.2) the momentum conservation law yields:

$$\lim_{V\to\infty}\omega(a^*_{p+k,\sigma}a_{p,\sigma}) = \delta_{k,0}\lim_{V\to\infty}\omega(a^*_{p,\sigma}a_{p,\sigma}). \tag{4.3}$$

So, in this state the equilibrium expectation of the local density

$$\lim_{V\to\infty} \omega(\rho(x)) = \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^*,\sigma} e^{ikx} \sum_{p\in\Lambda^*} \omega(a^*_{p+k,\sigma} a_{p,\sigma}) = \lim_{V\to\infty} \omega(\rho(0)) \tag{4.4}$$

is a constant.

Since condensation *breaks* the translation invariance (see Remark 2.1) in one direction, we would expect a corresponding non-homogeneity (*grating*) of the equilibrium total particle density in the extremal ω_γ state. This means that in the integral sum (4.4) over k, the $\pm q$-mode terms may survive in the thermodynamic limit. By (2.53) and (2.54) we know that condensation occurs only in the *zero* mode for the $\sigma = -$ bosons and in the q-mode for the $\sigma = +$ bosons. and therefore we have the following relations:

$$\lim_{V\to\infty} \frac{1}{V}\omega_\gamma(a^*_{q,+} a_{0,+}) = \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a^*_{q,+}) \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a_{0,+}) \,, \tag{4.5}$$

$$\lim_{V\to\infty} \frac{1}{V}\omega_\gamma(a^*_{q,-} a_{0,-}) = \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a^*_{q,-}) \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a_{0,-}) \,. \tag{4.6}$$

To get decoupling in (4.5) and (4.6) one has to note that (as in (2.53), (2.54)) the operators $a_{0,\pm}/\sqrt{V}$, see (2.4), or $a^*_{q,\pm}/\sqrt{V}$, are space-averages, which in the limit are c-numbers in the periodic state ω_γ.
Since there is no condensation of $\sigma = \pm$ bosons except in those two modes, the right-hand sides of both equations (4.5) and (4.6) are equal to zero. Noting that

$$\lim_{V\to\infty} \frac{1}{V}\omega_\gamma(a^*_{p+k,\sigma} a_{p,\sigma}) = 0 \tag{4.7}$$

for any other mode, one gets the *space homogeneity* of the equilibrium total particle density in the extremal ω_γ state: $\lim_{V\to\infty} \omega_\gamma(\rho(x)) = const.$ So we have no particle density space variation even in the presence of the *light corrugated lattice of condensed photons*, see Remark 2.1.

Let us now look at the corresponding situation in *Model 2*. The important difference is that in this model in Case 3 the *same* boson atoms condense in two states:

$$\lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a^*_q) \neq 0 \quad \text{and} \quad \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a_0) \neq 0. \tag{4.8}$$

Then (4.6) implies

$$\lim_{V\to\infty} \frac{1}{V}\omega_\gamma(a^*_q a_0) = \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a^*_q) \lim_{V\to\infty} \frac{1}{\sqrt{V}}\omega_\gamma(a_0) \equiv \xi \neq 0. \tag{4.9}$$

Therefore, the bosons of these two condensates may interfere. By virtue of (4.4) and (4.9) this gives the matter-wave grating formed by two macroscopically occupied momentum states:

$$\begin{aligned} \lim_{V\to\infty} \omega_\gamma(\rho(x)) &= \lim_{V\to\infty} \frac{1}{V} \sum_{k\in\Lambda^*,\sigma} e^{ikx} \sum_{p\in\Lambda^*} \omega_\gamma(a^*_{p+k,\sigma} a_{p,\sigma}) \qquad (4.10) \\ &= (\xi e^{iqx} + \overline{\xi} e^{-iqx}) + \lim_{V\to\infty} \frac{1}{V} \sum_{k\neq\pm q} e^{ikx} \omega_\gamma(\rho^\circ_k) \,, \end{aligned}$$

where

$$\rho_k^\circ := \sum_{p\neq 0,q} a_{p+k}^* a_p \,. \tag{4.11}$$

Notice that by (4.9) and by (4.10) there is no matter-wave grating in the Case 2, when one of the condensates (*q-condensate*) is empty, see (2.40).

We conclude this section with few remarks concerning the importance of the *matter-wave grating* for the amplification of light and matter waves observed in recent experiments.
It is clear that the absence of the matter-wave grating in Model 1 and its presence in Model 2 provides a physical distinction between Raman and Rayleigh superradiance. Note first that matter-wave amplification differs from light amplification in one important aspect: a matter-wave amplifier has to possess a *reservoir* of atoms. In Models 1 and 2 this is the BE condensate. In both models the superradiant scattering transfers atoms from the condensate at rest to a recoil mode.
The *gain* mechanism for the Raman amplifier is superradiant Raman scattering in a two-level atoms, transferring bosons from the condensate into the recoil state [1]
The Rayleigh amplifier is in a sense even more effective. Since now the atoms in a recoil state interfere with the BE condensate at rest, the system exhibits a space *matter-wave grating* and the quantum-mechanical amplitude of transfer into the recoil state is proportional to the product $N_0(N_q+1)$. Each time the momentum imparted by photon scattering is absorbed by the matter-wave grating by the coherent transfer of an atom from the condensate into the recoil mode. Thus, the variance of the grating grows, since the quantum amplitude for scattered atom to be transferred into recoiled state is increasing [2]-[4], [12]. At the same time the dressing laser beam prepares from the BE condensate a gain medium able to amplify the light. The matter-wave grating diffracts the dressing beam into the path of the probe light resulting in the amplification of the latter [5].

In the case of equilibrium BEC superradiance the amplification of the light and the matter waves manifests itself in Models 1 and 2 as a *mutual enhancement* of the BEC and the photons condensations, see Cases 3 in Sections 2 and 3. Note that the corresponding formulæ for condensation densities for Model 1 (2.44)-(2.46) and for Model 2 (3.11), (3.12), (3.13) are *identical*. The same is true for the boson-photon correlations (entanglements) between recoiled bosons and photons, see (2.48), (3.14), as well as between photons and the BE condensate at rest:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{0-}^* b_q\rangle_{H_{1,\Lambda}^{\mathrm{eff}}(\mu,\eta,\rho)} = \frac{1}{V}\langle a_0^* b_q\rangle_{H_{2,\Lambda}^{\mathrm{eff}}(\mu,\eta,\rho)} = \frac{4\Omega(\lambda\rho+\epsilon(q)-\mu)\sqrt{\lambda\rho-\mu}}{g^2}, \tag{4.12}$$

and the *off-diagonal coherence* between recoiled atoms and the condensate at rest:

$$\lim_{V\to\infty}\frac{1}{V}\langle a_{0-} a_{q+}^*\rangle_{H_{1,\Lambda}^{\mathrm{eff}}(\mu,\eta,\rho)} = \frac{1}{V}\langle a_0 a_q^*\rangle_{H_{2,\Lambda}^{\mathrm{eff}}(\mu,\eta,\rho)} = \frac{4\Omega\sqrt{(\lambda\rho+\epsilon(q)-\mu)(\lambda\rho-\mu)}}{g^2}. \tag{4.13}$$

As we have shown above, the *difference* between Models 1 and 2 becomes visible only on the level of the wave-grating or spatial modulation of the *local particle density* (4.10).

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