The Dynamics of Pseudographs in Convex Hamiltonian Systems

Patrick Bernard [1] [2]

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ABSTRACT. We study the evolution, under convex Hamiltonian flows on cotangent bundles of compact manifolds, of certain distinguished subsets of the phase space. These subsets are generalizations of Lagrangian graphs, we call them pseudographs. They emerge in a natural way from Fathi's weak KAM theory. By this method, we find various orbits which connect prescribed regions of the phase space. Our study is inspired by works of John Mather. As an application, we obtain the existence of diffusion in a large class of a priori unstable systems and provide a solution to the large gap problem. We hope that our method will have applications to more examples.

RÉSUMÉ. Nous étudions l'évolution, par le flot d'un Hamiltonien convexe sur une variété compacte, de certains ensembles de l'espace des phases. Nous appelons pseudographes ces ensembles, qui sont des généralisations de graphes Lagrangiens apparaissant de manière naturelle dans la théorie KAM faible de Fathi. Par cette méthode, nous trouvons diverses orbites qui joignent des domaines donnés de l'espace des phases. Notre étude s'inspire de travaux de John Mather. Nous obtenons l'existence de diffusion dans une large classe de systèmes à priori instables comme application de cette méthode, qui permet de résoudre le probleme de l'écart entre les tores invariants. Nous espérons que la méthode s'appliquera à d'autres exemples.

Contents

[1] Institut Fourier, BP 74, 38402, Saint Martin d'Hères cedex, France,
patrick.bernard@ujf-grenoble.fr, http://www-fourier.ujf-grenoble.fr/~pbernard/

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Introduction

In all this paper, M denotes a connected compact manifold without boundary, of dimension d, and TM and T^*M are its tangent and cotangent bundle. We shall consider the time-dependant Hamiltonian system generated by a function $H : \mathbb{R} \times T^*M \times \longrightarrow \mathbb{R}$, and denote by ϕ_s^t the flow from time s to time t.

(0.1) In order to motivate our discussion, we begin with a precise question: Given two Lagrangian manifolds $\mathcal{G}$ and $\mathcal{G}'$ in the cotangent bundle, which are graphs over the base M, does there exist a trajectory which connects $\mathcal{G}$ and $\mathcal{G}'$, or in other words does there exist times $s < t$ such that the Lagrangian manifold $\phi_s^t(\mathcal{G})$ intersects $\mathcal{G}'$?

(0.2) This question formulates some well known problems. As an exemple, let us suppose that $M = \mathbb{T}^d$, and identify the cotangent bundle $T^*\mathbb{T}^d$ with $\mathbb{T}^d \times \mathbb{R}^d$. Let us consider the Hamiltonian $H_0 = h(p)$. where $h : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a real function. Such Hamiltonians will be called fully integrable in the sequel. It is known that they leave invariant the tori $\mathbb{T}_p := \mathbb{T}^d \times \{p\}$, for $p \in \mathbb{R}^d$. As a consequence, the answer to the previous question is obviously negative for $\mathcal{G} = \mathbb{T}_p$ and $\mathcal{G}' = \mathbb{T}_{p'}$, when $p \neq p'$. What happens for Hamiltonians H which are close to H_0? For example, it is known that the solar system can be described by a fully integrable Hamiltonian H_0, if the interactions between planets are neglected. In this example, the variables $p \in \mathbb{R}^d$ encode the parameters of the elliptic trajectories of the planets. It is well known that these parameters would not change in time if the interaction between planets did not exist. Understanding for which values of p and p' the question (0.1) has a positive answer with $\mathcal{G} = \mathbb{T}_p$ and $\mathcal{G}' = \mathbb{T}_{p'}$, amounts to understand to what extent the elliptic trajectories will deform under the influence of mutual interactions. In other words, it amounts to understand the secular dynamics, and the stability of the solar system. We will not treat these specific examples in the present papers, although they are parts of our motivations. See [1] and [18] for beautiful and deep examples of perturbations of fully integrable systems.

(0.3) Our main goal in the present paper is to develop abstract tools to study question (0.1). By abstract, we mean tools involving hypotheses which may be hard to check on example. In order to demonstrate the usefulness of these tools, we shall then present applications to more explicit examples. First, we will easily verify that our theory is optimal for twist maps. We will obtain that two graphs can be connected by an orbit if they belong to the same region of instability. Then, we shall present a class of systems in higher dimension which can be fruitfully studied from our point of view. These systems are *a priori* unstable according to the terminology in use in the world of Arnold's diffusion. Shortly, this means that they contain an invariant submanifold which presents some kind of hyperbolicity (here, minimality). It appears clearly in the fundamental paper of Arnold, [1] that the presence of such a hyperbolic invariant manifold intersecting $\mathcal{G}$ and $\mathcal{G}'$ greatly favors a positive answer to question (0.1). *A priori* unstable systems have been widely studied because they appear naturally in the perturbation of completely integrable systems, and are easier to deal with.

In the work of Arnold, it is also assumed that the restriction of the dynamics to the hyperbolic manifold is integrable. This means that this invariant manifold is foliated by invariant tori which he called whiskered tori because of the presence of hyperbolicity. These whiskered tori are the building blocks of Arnold's construction, so that this second hypothesis was very important. The main point in our application is that we do not make this assumption.

We only assume that the restricted dynamics is generic, in a sense which will be clearly specified.

In the context of perturbations of fully integrable systems, the restriction of the flow to the hyperbolic manifold is close to integrable, and KAM theory implies the existence of many whiskered tori. However, when computing precisely the various quantities that appear in Arnold's construction, one observes that there does not exist enough tori in general. More precisely, the gap between tori is too big, this is the Large Gap problem, see for example [17] for a more precise explanation.

Overcoming this problem has long been considered as a major challenge. While the classical approaches based on refinements on the sheme of Arnold were worked out in that direction, new variational methods were introduced, by John Mather in [20]. It is also worth mentionaing the work of Bessi, [5], where the results sketched by Arnold are proved using variational methods. This paper contains one on the first relevant achievements of variational methods in these kind of questions, and it has been very influential. However, these variational methods were facing the same kind of difficulties as classical methods. In several special instances, the Large Gap problem can be bypassed because for specific reasons there exist more whiskered tori. This remark has been exploited to obtain many non-trivial results from Arnold's construction or variational methods. For example, orbits of unbounded speed where built in [6] using the scheme of Arnold. A similar result had previously been obtained by John Mather, [21], using variational methods, see also [15]. Other works exploit the same remark in different directions, see for example [4], which elaborates on [5], and many other texts.

Solutions to the Large Gap problem has recently been given by Delshams, de la Llave and Seara, see the announcement in [10], and by Treschev, see announcement in [25] using elaborations on Arnold's method. The details in these works are far from simple. Cheng and Yan have also proposed a solution using elaborations on the variational methods initiated by Mather, see [7], as well as Z. Xia, see [26] and [27]. The solution we give is close to these ones and relies on variational methods.

In the class of systems we will present, it is not assumed that the restricted dynamics is close to integrable. This makes the method of [10] and [25] inefficient. On the other hand, other hypotheses such as convexity are required, which are useless in [10]. The methods of [7], and [27] are closer to ours, and could perhaps be used to study similar examples. The method presented in [15] may also be efficient to treat these systems, and at least provides a nice geometric picture on the situation.

The influence of John Mather's published and unpublished works on the developpement of these variational approaches could not be overestimated. He has announced in [22] very deep results on the perturbation of fully integrable systems in dimension 2, and given indications on proofs in various talks and lectures. I hope that the tools developed in the present paper will contribute to clarify and extend these results.

(0.4) It is now clear that question (0.1) is especially interesting when the Lagrangian manifolds $\mathcal{G}$ and $\mathcal{G}'$ have different cohomologies. In this case, we have a problem of non exact Lagrangian intersection, and it seems that the powerful tools developed to deal with exact intersections provide no interesting insight. In order to study this problems, we make strong assumptions on the Hamiltonian H, namely that it is convex, super-linear, periodically time-dependent, and complete, see details in (1.1). The method we use strongly relies on the possibility of studying trajectories from the point of view of action minimization, that is on the convexity of the Hamiltonian and on the fibered structure of the phase space. The major drawback of this approach is that these construction are not natural from a symplectic point of view. However, several of the objects we define are symplectic invariants, see for instance [3] and [24]. This is very important because the most interesting applications will require the use of Hamiltonian normal form theory in conjuction with the theory exposed here. In order to keep the present paper to a reasonable lengh, we will not study these symplectic aspects, which are to a large extent independent from the ones discussed here.

(0.5) Given a Lipschitz function $u : M \longrightarrow \mathbb{R}$ and a smooth form η on M, we consider the subset $\mathcal{G}_{\eta,u}$ of T^*M defined by

$$\mathcal{G}_{\eta,u} = \big\{(x, \eta_x + du_x), x \in M \text{ such that } du_x \text{ exists}\big\}.$$

We call the subset $\mathcal{G} \subset T^*M$ an *overlapping pseudographs* if there exists a smooth form η and a semi-concave function u such that $\mathcal{G} = \mathcal{G}_{\eta,u}$. See Appendix A for the definition of semi-concave functions. Each pseudograph $\mathcal{G}$ has a well defined cohomology $c(\mathcal{G}) \in H^1(M, \mathbb{R})$, see (2.2). We denote by $\mathbb{P}$ the set of overlapping pseudographs. If $M = \mathbb{T}$ is a circle, then overlapping pseudographs are graphs of functions which have only discontinuities with downward jumps, or in other words functions which can be locally written as the sum of a continuous and a decreasing function. Such sets were introduced in [16], where they are used in very elegant proofs of many known properties of Twist maps.

(0.6) We define the equivalence relation $\lessdot\!\!\gtrdot$ on $H^1(M, \mathbb{R})$ as follows: We say that $c \lessdot\!\!\gtrdot c'$ if there exists an integer $N \in \mathbb{N}$ such that, for each pseudograph $\mathcal{G}$ of cohomology c (resp. c'), there exists a pseudograph $\mathcal{G}'$ of cohomology c' (resp. c) such that

$$\overline{\mathcal{G}'} \subset \bigcup_{1 \leqslant i \leqslant N} \phi^i(\mathcal{G}).$$

If $c \lessdot\!\!\gtrdot c'$, if $\mathcal{G}$ is a Lagrangian graph of cohomology c, and if $\mathcal{G}'$ is a Lagrangian graph of cohomology c', then clearly there exists a Hamiltonian orbit which connects $\mathcal{G}$ and $\mathcal{G}'$. This is why our main purpose in this article will be to understand the relation $\lessdot\!\!\gtrdot$. We will see later many more consequences of such an understanding.

(0.7) We shall define an operator $\Phi : \mathbb{P} \longrightarrow \mathbb{P}$ in (2.5), with the following fundamental properties:

$$\overline{\Phi(\mathcal{G})} \subset \phi(\mathcal{G}),$$

where ϕ is the time-one Hamiltonian flow, and $c(\Phi(\mathcal{G})) = c(\mathcal{G})$. Fathi's weak KAM theorem, [12] states that, for each $c \in H^1(M, \mathbb{R})$, the operator Φ has fixed points of cohomology c. We call $\mathbb{V}_c$ the set of these fixed points, see section 3 for details. The fixed points $\mathcal{G}$ satisfy

$$\overline{\mathcal{G}} \subset \phi(\mathcal{G}),$$

and give rise to compact invariant sets

$$\tilde{\mathcal{I}}(\mathcal{G}) := \bigcap_{i \in \mathbb{N}} \phi^{-i}(\bar{\mathcal{G}}).$$

This provides a new way, due to Albert Fathi, to define various invariant sets previously introduced by Mather in [19] and [20] using a different approach.

(0.8) More precisely, to each cohomology $c \in H^1(M, \mathbb{R})$ we associate the compact invariant sets

$$\tilde{\mathcal{M}}(c) \subset \tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{N}}(c),$$

where

$$\tilde{\mathcal{A}}(c) := \bigcap_{\mathcal{G} \in \mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G}) \text{ and } \tilde{\mathcal{N}}(c) := \bigcup_{\mathcal{G} \in \mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G}),$$

are respectively called the Aubry set and the Mañe set, and $\tilde{\mathcal{M}}(c)$, called the Mather set, is the closure of the union of the supports of the invariant measures of the action of ϕ on $\tilde{\mathcal{A}}(c)$ (or equivalently on $\tilde{\mathcal{N}}(c)$), see (3.5) for more details. A standing notation will be to denote by $\tilde{\mathcal{X}}$ subsets of T^*M, and by $\mathcal{X}$ their projection on M.

(0.9) Proposition.

(i) *If $c \lhd\!\rhd c'$, there exists a heteroclinic trajectory of the Hamiltonian flow between $\tilde{\mathcal{A}}(c)$ and $\tilde{\mathcal{A}}(c')$.*

(ii) *Let $c_i, i \in \mathbb{Z}$, be a sequence of cohomology classes. Assume that $c_i \lhd\!\rhd c_{i+1}$ for each i, and fix for each i a neighborhood U_i of $\tilde{\mathcal{M}}(c_i)$ in T^*M. There exists a trajectory of the Hamiltonian flow which visits in turn all the sets U_i. In addition, if the sequence stabilizes to $c-$ on the left, or to $c+$ on the right, the trajectory can be assumed negatively asymptotic to $\mathcal{A}(c-)$ or positively asymptotic to $\mathcal{A}(c+)$.*

(iii) *Let $\mathcal{G}_i, i \in \mathbb{Z}$, be a sequence of Lagrangian graphs whose cohomologies $c(\mathcal{G}_i)$ are $\lhd\!\rhd$-equivalent. Then there exists an orbits $(q(t), p(t))$ which crosses the Graphs $\mathcal{G}_i$ in any prescribed order.*

The proof is given in section 5. Let us now state our main results.

(0.10) For each $\mathcal{G} \in \mathbb{V}$, we define the subspace $R(\mathcal{G})$ of $H^1(M, \mathbb{R})$ as the set of cohomology classes of smooth closed one-forms whose support is disjoint from $\mathcal{I}(\mathcal{G})$. For each cohomology class $c \in H^1(M, \mathbb{R})$, we define the subspace $R(c)$ as

$$R(c) = \bigcap_{\mathcal{G} \in \mathbb{V}_c} R(\mathcal{G}) \subset H^1(M, \mathbb{R}).$$

The following Theorem reformulates and refines results of John Mather, see [20] and also [2] and [7]. It is proved in section 7.

Theorem. *For each $c_0 \in H^1(M, \mathbb{R})$, there exists a positive ϵ such that the following holds: Each class $c \in H^1(M, \mathbb{R})$ such that $c - c_0 \in R(c_0)$ and $\|c - c_0\| \leqslant \epsilon$ satisfies $c_0 \lhd\!\rhd c$.*

(0.11) There is a natural partition of the Aubry set $\tilde{\mathcal{A}}(c)$ into compact invariant subsets $\tilde{\mathcal{S}}$ called the static classes, see section 4. A generalized version of the following theorem is proved in section 8.

Theorem. *Assume that there exists only finitely many static classes in $\tilde{\mathcal{A}}(c)$, and that the set $\tilde{\mathcal{N}}(c) - \tilde{\mathcal{A}}(c)$ is not empty and contains finitely many orbits. Then the cohomology c is in the interior of its class of $\lhd\!\rhd$ equivalence.*

(0.12) Let us now give an explicit example. Proofs and more general statements are given in section 10. We take $M = \mathbb{T}\times\mathbb{T}^{d-1}$, and denote by $(q,p) = (q_1, q_2, p_1, p_2)$ the points of T^*M, where $q_1 \in \mathbb{T}$, $q_2 \in \mathbb{T}^{d-1}$, $p_1 \in \mathbb{R}$, $p_2 \in \mathbb{R}^{d-1}$. We consider the time-periodic Hamiltonian

$$H(t, q, p) = H_1(t, q_1, p_1) + |p_2|^2 - V(q_2)F(t, q)$$

and we assume that the conditions of convexity, superlinearity and completeness are satisfied. In addition, we assume that $F : \mathbb{T} \times \mathbb{T}^d \longrightarrow \mathbb{R}$ takes positive values, and that $V : \mathbb{T}^{d-1} \longrightarrow \mathbb{R}$ takes positive values except at a single point, say 0, where its takes the value 0. The manifold $\mathbb{T} \times \mathbb{R} := \{q_2 = 0, p_2 = 0\}$ is then invariant under the Hamiltonian Flow. The restricted flow is generated by the restricted Hamiltonian H_1. Under these hypotheses, it is not hard to prove (we will do it) that each rotational invariant circle of the restricted dynamics H_1 admits a homoclinic orbit. We make two additional non-degeneracy assumptions:

(H1) The Hamiltonian H_1 is generic in the sense that its irrotational invariant circles of rational rotation number are completely periodic. (We allow periodic circles in order to include the case where H_1 is integrable).

(H2) We assume a non-degeneracy hypothesis on the set of action minimizing homoclinic orbits to the invariant circles of H_1. This hypothesis is detailed in section 10, it should be seen as analogous to the classical hypothesis of transversality of the stable and unstable manifolds.

Under these hypotheses, our abstract results imply the following.

Theorem *If P and P' are given real numbers, there exists a Hamiltonian trajectory $(q(t), p(t))$ and an integer $n \in \mathbb{N}$ such that $p_1(0) = P$ and $p_1(n) = P'$. In addition, if $P(i) : \mathbb{Z} \longrightarrow \mathbb{R}$ is a sequence of real numbers, there exists an orbits $(q(t), p(t))$ and an increasing sequence $n(i) : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $p_1(n(i)) = P(i)$. Finally, given two invariant circles of the dynamics restricted to $T\mathbb{T}$, there exists a heteroclinic orbit which connects them.*

(0.13) Let us now present the content of the paper. The whole paper heavily relies on the notion of semi-concave function and of equi-semi-concave sets of functions. These notions are presented in Appendix A. In Appendix B, we prove some backgroud results essentially due to Fathi, about the propeties of the Action.

Mather-Fathi Theory.

In the first part of the paper, we introduce our main objects, overlapping pseudographs. We use them to present some salient facts on the theory of Mather, Mañe and Fathi of globally minimizing orbits. In section 1, we present the context, detail the standing hypotheses, and recall some known results of the calculus of variations which will be of constant use (proofs are given in Appendix B). Pseudographs are defined and their basic properties studied in Section2. In Section 3, we use these pseudographs to Mather's theory of globally minimizing orbits from a point of view which is essentially due to Fathi. This part does not contain new result, but recalls what will be needed, with a partly original point of view. The theory is continued in section 4, where we explain the decomposition in static classes of the Aubry set, and use this desomposition to construct homoclinic orbits, which will play a central role in section 8

Abstract mechanism.

This part contains our main results. In section 5 we define the relation $\lhd\!\rhd$, and gather some technical tools which will be needed to study this relation. In section 6 we explain how various orbits can be built once the relation $\lessdot\!\!\!\gtrdot$ is understood and prove Proposition (0.9). In section 7, we prove Theorem (0.10). In section 8 we study the case where there exist only finitely many static classes. We generalize and prove Theorem (0.11).

Applications.
In this short part, we detail some straightforward applications of the results obtained before. We hope that it is possible to obtain much more applications by applying our results with Hamiltonian methods such as normal form theory, but this aspect is not discussed here. Section 9 briefly mentions the application to twist maps. Section 10 details (0.12) above.

Mather-Fathi Theory

In this part, we introduce the main objects and present the theory of Mather and Fathi. Our point of view is close to the one of Fathi, Most of the material exposed here is a small deformations of results in [19], [11], [23], [9], or [8].

1 Calculus of variations

(1.1) We shall consider C^2 Hamiltonian functions $H : \mathbb{R} \times T^*M \longrightarrow \mathbb{R}$. We will denote by $P = (x, p)$ the points of T^*M. The Cotangent bundle is endowed with its standard symplectic structure. We denote by $X(t, P)$ or $X(t, x, p)$ the Hamiltonian vector-field of H, which is a time-dependent vector-field on T^*M. We fix once and for all a Riemann g metric on M, and use it to define norms of tangent vectors and tangent covectors of M. We will denote this norm indifferently by $|P|$ or by $|p|$ when $P = (x, p) \in T^*_xM$. We assume the following standard set of hypotheses.

1. Periodicity. $H(t + 1, P) = H(t, P)$ for each $(t, P) \in \mathbb{R} \times T^*M$.
2. Completeness. The Hamiltonian vector-field X generates a complete flow of diffeomorphisms on T^*M. We denote by $\phi^t_s : T^*M \longrightarrow T^*M$ the flow from time s to time t, and by ϕ the flow ϕ^1_0.
3. Convexity. For each $(t, x) \in \mathbb{R} \times M$, the function $p \longrightarrow H(t, x, p)$ is convex on T^*_xM, with positive definite Hessian. Shortly, $\partial^2_p H > 0$.
4. Superlinearity. For each $(t, x) \in \mathbb{R} \times M$, the function $p \longmapsto H(t, x, p)$ is superlinear, which means that $\lim_{|p| \longrightarrow \infty} H(t, x, p)/|p| = \infty$.

(1.2) We associate to the Hamiltonian H a Lagrangian function $L : \mathbb{R} \times M \longrightarrow \mathbb{R}$ defined by

$$L(t, x, v) = \sup_{p \in T^*_xM} p(v) - H(t, x, p).$$

The fiberwise differential $\partial_p H$ of H can be seen as a mapping

$$\partial_p H : \mathbb{R} \times T^*M \longrightarrow \mathbb{R} \times TM,$$

this mapping is a diffeomorphism, whose inverse is given by

$$\partial_v L : \mathbb{R} \times TM \longrightarrow \mathbb{R} \times T^*M.$$

We have the relations $L(t,x,v) = \partial_v L(t,x,v)(v) - H(t,x,\partial_v L(t,x,v))$ and $H(t,x,p) = \partial_p H(t,x,p)(p) - L(t,x,\partial_p H(t,x,v))$. The Lagrangian L satisfies the following properties, which follow from the analogous properties of H:

1. PERIODICITY. $L(t+1,V) = L(t,V)$ for each $(t,V) \in \mathbb{R} \times TM$.
2. CONVEXITY. For each $(t,x) \in \mathbb{R} \times M$, the function $v \longmapsto L(t,x,v)$ is a convex function on T_xM, with positive definite Hessian. Shortly, $\partial_v^2 L > 0$.
3. SUPERLINEARITY. For each $(t,x) \in \mathbb{R} \times M$, the function $v \longmapsto L(t,x,v)$ is superlinear on T_xM.

See Appendix B for comments related to these hypotheses. The hypotheses listed above are very suitable to use the calculus of variations.

(1.3) Let us fix two times $s > t$ in $\mathbb{R}$ and two points x and y in M. Let $\Sigma(t,x;s,y)$ be the set of absolutely continuous curves $\gamma : [t,s] \longrightarrow M$ such that $\gamma(t) = x$ and $\gamma(s) = y$. As usual, we define the action of the curve γ as $A(\gamma) = \int_t^s L(\sigma,\gamma(\sigma),\dot\gamma(\sigma))\,d\sigma$. It is known that, for each C, the set of curves γ in $\Sigma(t,x;s,y)$ which satisfy $A(\gamma) \leqslant C$ is compact for the topology of uniform convergence. As a consequence, there exist curves minimizing the action. Let us define the value

$$A(t,x;s,y) = \min_{\gamma \in \Sigma(t,x;s,y)} \int_t^s L(\sigma,\gamma(\sigma),\dot\gamma(\sigma))\,d\sigma,$$

and let $\Sigma_m(t,x;s,y)$ be the set of curves in Σ reaching the above minimum. The set $\Sigma_m(t,x;s,y)$ is not empty, and it is compact for the topology of uniform convergence. Each curve $\gamma(\sigma) \in \Sigma_m$ is C^2 and satisfies the Euler-Lagrange equation. Setting

$$p(\sigma) = \partial_v L(\sigma,\gamma(\sigma),\dot\gamma(\sigma))$$

these equations are

$$\dot p(\sigma) = \partial_x L(\sigma,\gamma(\sigma),\dot\gamma(\sigma)) = -\partial_x H(\sigma,\gamma(\sigma),p(\sigma))$$

Hence the curve $(\gamma(\sigma),p(\sigma))$ is a trajectory of the Hamiltonian flow.

(1.4) For each minimizing curve $\gamma \in \Sigma_m(t,x;s,y)$, we have

$$-p(t) = -\partial_v L(t,x,\dot\gamma(t)) \in \partial_x^+ A(t,x;s,y),$$

where $\partial_x^+ A(t,x;s,y)$ denotes the set of proximal super-differentials of $q \longmapsto A(t,q;s,y)$ at point $q = x$, see Appendix. We also have

$$p(s) = \partial_v L(s,y,\dot\gamma(s)) \in \partial_y^+ A(t,x;s,y).$$

For each $t' > t$, the set of functions $(x,y) \longmapsto A(t,x;s,y), s \geqslant t'$ is equi-semi-concave on $M \times M$, hence equi-Lipschitz, see Appendix. In addition, the three following properties are equivalent:

(*i*) The set $\Sigma_m(t,x;s,y)$ contains only one point.

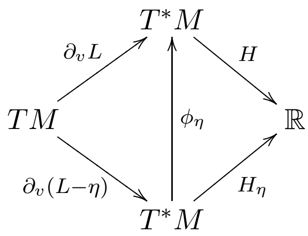
(*ii*) The function $A(t,.;s,y)$ is differentiable at x.

(iii) The function $A(t,x;s,.)$ is differentiable at y.

If these equivalent properties hold, and if $\gamma(\sigma)$ is the unique curve of $\Sigma_m(t,x;s,y)$, then setting $p(\sigma) = \partial_v L(\sigma, y, \dot\gamma(\sigma))$, we have

$$p(t) = -\partial_x A(t,x;s,y) \text{ and } p(s) = \partial_y A(t,x;s,y).$$

(1.5) Let η be a smooth one-form. We will see the form η sometimes as a section of the cotangent bundle $\eta : M \longrightarrow T^*M$ and sometimes as a fiberwise linear function of the tangent bundle $\eta : TM \longrightarrow \mathbb{R}$ in a natural way. If the form η is closed then the diffeomorphism $\phi_\eta : (x,p) \longmapsto (x, p+\eta(x))$ of T^*M is symplectic. The Hamiltonian $H_\eta(t,x,p) = H \circ \phi_\eta = H(t,x,p+\eta)$ satisfies our hypotheses. The associated Lagrangian is $(L-\eta)(t,x,v) = L(t,x,v) - \eta(x,v)$, where η is considered as a function of TM. The following diagram commutes for each t.



(1.6) We will also consider the modified action

$$A_\eta(t,x;s,y) = \inf_{\gamma \in \Sigma(t,x;s,y)} \int_t^s L(\sigma, \gamma(\sigma), \dot\gamma(\sigma)) - \eta_{\gamma(\sigma)}(\dot\gamma(\sigma))\, d\sigma,$$

which of course satisfies all the properties of (1.4).

(1.7) Let Ω be the set of closed smooth forms on M. It is useful to fix once and for all a linear section S of the projection $\Omega \longrightarrow H^1(M,\mathbb{R})$. In other words, S is a linear mapping from $H^1(M,\mathbb{R})$ to Ω such that $[S(c)] = c$. We shall abuse notations and denote by c the form $S(c)$, in such a way that the symbol c denotes either a cohomology class or a standard form representing this cohomology class.

(1.8) The following consequence of Appendix B will be useful. See appendix A for the definition of equi-semi-concave.

PROPOSITION. *If C is a bounded subset of $H^1(M,\mathbb{R})$, and ϵ is a positive number, the functions $A_c(s,.;t,.), c \in C, t \geqslant s+\epsilon$ are equi-semi-concave on $M \times M$.*

2 Overlapping pseudographs.

We present the main objects, overlapping pseudographs, and study some basic properties.

(2.1) Given a Lipschitz function $u : M \longrightarrow \mathbb{R}$ and a smooth form η on M, we define the subset $\mathcal{G}_{\eta,u}$ of T^*M by

$$\mathcal{G}_{\eta,u} = \big\{(x, \eta_x + du_x), x \in M \text{ such that } du_x \text{ exists}\big\}.$$

We call the subset $\mathcal{G} \subset T^*M$ an *overlapping pseudograph* if there exists a smooth form η and a semi-concave function u such that $\mathcal{G} = \mathcal{G}_{\eta,u}$. See Appendix A for the definition of semi-concave functions. We shall denote by $\mathbb{P}$ the set of overlapping pseudographs. Given a pseudograph $\mathcal{G}$, we will denote by $\mathcal{G}_U$ the set $\mathcal{G}_U := \mathcal{G} \cap \pi^{-1}(U)$.

(2.2) It is not hard to see that if an overlapping pseudograph $\mathcal{G}$ is represented in two ways as $\mathcal{G}_{\eta,u}$ and $\mathcal{G}_{\mu,v}$, then the closed forms η and μ have the same cohomology in $H^1(M,\mathbb{R})$. As a consequence, it is possible to associate to each pseudograph $\mathcal{G}$ a cohomology $c(\mathcal{G})$, in such a way that

$$c(\mathcal{G}_{\eta,u}) = [\eta].$$

We will denote by $\mathbb{P}_c$ the set of overlapping pseudographs of cohomology c. If $\mathcal{G}$ is an overlapping pseudograph of cohomology c, then $\mathcal{G}$ can be represented in the form $\mathcal{G} = \mathcal{G}_{c,u}$, where c is the standard form defined in (1.7). The function u is then uniquely defined up to an additive constant. As a consequence, denoting by $\mathbb{S}$ the set of semi-concave functions on M, and by $\mathbb{P}$ the set of overlapping pseudographs, we have the identification

$$\mathbb{P} = H^1(M,\mathbb{R}) \times \mathbb{S}/\mathbb{R}.$$

This identification endows $\mathbb{P}$ with the structure of a real vector space. Let us endow the factor $\mathbb{S}/\mathbb{R}$ with the norm $|u| = (\max u - \min u)/2$, which is the norm induced from the uniform norm on $\mathbb{S}$. More precisely, we have $|u| = \min_v \|v\|_\infty$, where the minimum is taken on functions v which represent the class u. We put on $\mathbb{P}$ the norm

$$\|\mathcal{G}_{c,u}\| = |c| + (\max u - \min u)/2 \leqslant |c| + \|u\|_\infty.$$

The set $\mathbb{P}$ is now a normed vector space. It is also useful to define, for each subset $U \in M$, the number

$$\|\mathcal{G}_{c,u}\|_U := |c| + (\max_U u - \min_U u)/2.$$

We define in the same way the set $\breve{\mathbb{P}}$ of *anti-overlapping* pseudographs $\breve{\mathcal{G}}$, which are the sets $\mathcal{G}_{\eta,-u}$, with η a smooth form and $u \in \mathbb{S}$. This set is similarly a normed vector space.

(2.3) LEMMA. *Let $\mathcal{G}$ be an overlapping pseudograph, and $\breve{\mathcal{G}}$ be an anti-overlapping pseudograph. If $\mathcal{G}$ and $\breve{\mathcal{G}}$ have the same cohomology, then they have nonempty intersection.*

PROOF. Let us write $\mathcal{G} = \mathcal{G}_{\eta,u}$ and $\breve{\mathcal{G}} = \mathcal{G}_{\eta,-v}$. The continuous function $u+v$ has a minimum at X. since they are semi-concave, both u and v are differentiable at X, and $du(X) = -dv(X)$. It follows that the point $(X, \eta(X) + du(X)) = (X, \eta(X) - dv(X))$ belongs both to $\mathcal{G}$ and to $\breve{\mathcal{G}}$. □

It is natural to introduce the following definition.

DEFINITION. *Let $\mathcal{G}$ be an overlapping pseudograph, and $\breve{\mathcal{G}}$ be an anti-overlapping pseudograph. If $\mathcal{G}$ and $\breve{\mathcal{G}}$ have the same cohomology c, write them $\mathcal{G} = \mathcal{G}_{c,u}$ and $\breve{\mathcal{G}} = \mathcal{G}_{c,\breve{u}}$. We denote by*

$$\mathcal{G} \wedge \breve{\mathcal{G}} \subset M$$

the set of points of minimum of the difference $u - \breve{u}$, and by $\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}} \subset \mathcal{G} \cap \breve{\mathcal{G}}$ the set

$$\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}} := \mathcal{G} \cap \pi^{-1}(\mathcal{G} \wedge \breve{\mathcal{G}}) = \breve{\mathcal{G}} \cap \pi^{-1}(\mathcal{G} \wedge \breve{\mathcal{G}}) = \mathcal{G} \cap \breve{\mathcal{G}} \cap \pi^{-1}(\mathcal{G} \wedge \breve{\mathcal{G}}) \subset T^*M.$$

This set is compact, not empty, and it is a Lipschitz graph over its projection $\mathcal{G} \wedge \breve{\mathcal{G}}$.

Proof. We have proved already that the set $\mathcal{G} \wedge \breve{\mathcal{G}}$ is not empty. Let $K > 0$ be such that the functions u and $-\breve{u}$ are K-semi concave. Let x be a point of minimum of $u - \breve{u}$. If f is a function with K-Lipschitz differential such that $f - \breve{u}$ has a maximum at x, then clearly, $u - f$ has a minimum at x. As a consequence, points of minimum of $u - \breve{u}$ are points where $\partial^{K-}u(x)$ is not empty. The super-differential $\partial^{K+}u(x)$ is of course also non-empty since u is K-semi-concave. It follows from Appendix (A.7) that u and $\breve{u}$ are differentiable on the set $\mathcal{G} \wedge \breve{\mathcal{G}}$, that they have the same differential, and that this differential is a Lipschitz section of the cotangent bundle over $\mathcal{G} \wedge \breve{\mathcal{G}}$. This makes the definition meaningful. The set $\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}}$ is compact because it is the image of the compact set $\mathcal{G} \wedge \breve{\mathcal{G}}$ by a Lipschitz map. □

(2.4) Let us fix a closed form η. We define the associated Lax-Oleinik mapping on $C^0(M, \mathbb{R})$ by the expression

$$T_\eta u(x) = \min_{q \in M} u(q) + A_\eta(0, q; 1, x)$$

Let us recall some important properties of the Lax-Oleinik mapping, which are all direct consequences of the properties of the function A given in (1.4). For each form η, The functions $T_\eta u, u \in C(M, \mathbb{R})$ are equi-semi-concave, see Appendix A. The mapping T_η is a contraction:

$$\|T_\eta u - T_\eta v\|_\infty \leqslant \|u - v\|_\infty.$$

To finish, the mapping T_η is non-decreasing, and it satisfies $T_\eta(a + u) = a + T_\eta(u)$ for all real a.

(2.5) There exists a unique mapping $\Phi : \mathbb{P} \longrightarrow \mathbb{P}$ such that

$$\Phi(\mathcal{G}_{\eta,u}) = \mathcal{G}_{\eta, T_\eta u}$$

for all smooth form η and all semi-concave function u. We have

$$c(\Phi(\mathcal{G})) = c(\mathcal{G}).$$

The mapping Φ is continuous (see (5.4) for the proof of a more general result). For each cohomology c, the image $\Phi(\mathbb{P}_c)$ is a relatively compact subset of $\mathbb{P}_c$, as follows directly from the properties of the Lax-Oleinik transformation recalled above. Note that this implies the existence of a fixed point of Φ in each $\mathbb{P}_c$, this is how Fathi proved the existence of fixed points. See (3.2) for another proof. We call $\mathbb{V}_c$ the set of these fixed points, and $\mathbb{V} = \cup_c \mathbb{V}_c$. We also define the sets

$$\mathbb{O} := \bigcap_{n \in \mathbb{N}} \overline{\Phi^n(\mathbb{P})} = \bigcap_{n \in \mathbb{N}} \Phi^n(\mathbb{P})$$

and $\mathbb{O}_c := \mathbb{O} \cap \mathbb{P}_c$. Note that $\mathbb{O}$ is compact and invariant under Φ, and that $\mathbb{V} \subset \mathbb{O}$. A pseudograph $\mathcal{G} \in \mathbb{P}_c$ belongs to $\mathbb{O}$ if and only if there exists a sequence $\mathcal{G}_n \in \mathbb{P}, n \in \mathbb{Z}$ of pseudographs such that $\Phi^{m-n}(\mathcal{G}_n) = \mathcal{G}_m$ for all $m \geqslant n$, and such that $\mathcal{G}_0 = \mathcal{G}$. Note that we then have $\mathcal{G}_n \in \mathbb{O}_c$ for each $n \in \mathbb{Z}$.

(2.6) The mapping Φ satisfies

$$\overline{\Phi(\mathcal{G})} \subset \phi(\mathcal{G}).$$

This inclusion is a consequence of the following Proposition, which will be central througrought the paper.

(2.7) Proposition *Let us fix a pseudograph $\mathcal{G}_{\eta,u} \in \mathbb{P}$, an open set $U \subset M$ and two times $s < t$. Let us set*

$$v(x) = \min_{q \in \bar{U}} u(q) + A_\eta(s, q; t, x),$$

where $\bar{U}$ is the closure of U. Let $V \subset M$ be an open set and let $N \subset M$ be the set of points where the minimum is reached in the definition of $v(x)$ for some $x \in V$. If $\bar{N} \subset U$, then

$$\overline{\mathcal{G}_{\eta,v|V}} = \phi_{s,t}\big(\mathcal{G}_{\eta,u|\bar{N}}\big)$$

and $\mathcal{G}_{\eta,u|\bar{N}}$ is a Lipschitz graph above $\bar{N}$. In other words, the function u is differentiable at each point of $\bar{N}$, and the mapping $x \longmapsto du_x$ is Lipschitz on $\bar{N}$.

Addendum. *In addition, the Hamiltonian trajectories $(x(\sigma), p(\sigma)) : [s,t] \longrightarrow T^*M$ which terminate in $\overline{\mathcal{G}_{\eta,v|V}}$, i. e. such that $(x(t), p(t)) \in \overline{\mathcal{G}_{\eta,v|V}}$ satisfy*

$$v(x(t)) = u(x(s)) + \int_s^t L(\sigma, x(\sigma), \dot{x}(\sigma)) - \eta_{x(\sigma)}(\dot{x}(\sigma)) d\sigma$$

$$= u(x(s)) + A_\eta(s, x(s); t, x(t))) = \min_{x \in U} u(x) + A_\eta(s, x; t, x(t))).$$

Conversely, if the curve $x(\sigma) : [s,t] \longrightarrow T^*M$ satisfies these equalities, with $x(s) \in U$ and $x(t) \in V$, then the curve $(x(\sigma), p(\sigma) = \partial_v L(\sigma, x(\sigma), \dot{x}(\sigma)))$ is a Hamiltonian trajectory terminating in $\overline{\mathcal{G}_{\eta,v}}$,

Proof. If $x \in V$ is a point of differentiability of v, then there exists a unique point $(x, \eta(x) + dv(x))$ of $\mathcal{G}_{\eta,v}$ above x. Let us consider a point $q \in N$ minimizing in the expression of $v(x)$. The function $A_\eta(s, q, t, .)$ is then differentiable at x because $v - A_\eta(s, q, t, .)$ has a maximum at x. As a consequence, see (1.4), the set $\Sigma_m(s, q, t, x)$ of minimizing curves contains one and only one curve $x(\sigma)$. This curve is such that the associated Hamiltonian trajectory

$$(x(\sigma), p(\sigma)) = (x(\sigma), \partial_v L(\sigma, x(\sigma), \dot{x}(\sigma))) : [s,t] \longrightarrow T^*M$$

terminates at $(x, \eta(x) + dv(x))$. It follows, still from (1.4), that the function u is differentiable at q and satisfies

$$(x(s), p(s)) = (q, du(q) + \eta(q)) \subset \mathcal{G}_{\eta,u}.$$

Let $K > 0$ be such that the functions u and $A_\eta(s, ., t, x), x \in M$ are K-semi-concave, see Appendix A. Then the function u has a K-super-differential and a K-sub-differential at q. Since this holds a every point of N, it follows from Appendix (A.7) that the function u is differentiable at each point of $\bar{N}$, and that the differential $du(q)$ is a Lipschitz section of the cotangent bundle over $\bar{N}$. As a consequence, we have

$$\mathcal{G}_{\eta,u|\bar{N}} = \overline{\mathcal{G}_{\eta,u|N}}.$$

We have also proved that $\phi_{t,s}(x, \eta(x)+dv(x)) \in \mathcal{G}_{\eta,u|N}$ for all point $(x, \eta(x)+dv(x))$ of $\mathcal{G}_{\eta,v|V}$, so that $\phi_{t,s}(\mathcal{G}_{\eta,v|V}) = \mathcal{G}_{\eta,u|N}$. Taking the closures, we get

$$\phi_{t,s}\big(\overline{\mathcal{G}_{\eta,v|V}}\big) = \mathcal{G}_{\eta,u|\bar{N}}.$$

□

(2.8) Let $\mathcal{G} = \mathcal{G}_{c,u}$ be a fixed point of Φ. And let $n < m$ be two relative integers. Following Fathi, we say that a curve $x(t) : [n, m] \longrightarrow M$ is *calibrated* by $\mathcal{G}$ if

$$u(x(n)) + \int_n^m L(t, x(t), \dot{x}(t)) - \eta_{x(t)}(\dot{x}(t))dt = \min_{x\in M} u(x) + A_c(n, x; m, x(m)).$$

The curve $x(t)$ is calibrated by $\mathcal{G}$ if and only if the curve $(x(t), p(t) = \partial_v L(t, x(t), \dot{x}(t)))$ is a Hamiltonian trajectory and satisfies $(x(k), p(k)) \in \bar{\mathcal{G}}$ for each integer $k \in [n, m]$. If $x(t)$ is calibrated by $\mathcal{G}$ then we also have $(x(k), p(k)) \in \mathcal{G}$ for each integer $k \in]n, m[$.

(2.9) The following Corollary is the reason why we have called the elements of $\mathbb{P}$ *overlapping.*

Corollary *All overlapping pseudographs $\mathcal{G} \in \mathbb{P}$ satisfy $\pi \circ \phi(\mathcal{G}) = M$.*
Proof. We have $\overline{\Phi(\mathcal{G})} \subset \phi(\mathcal{G})$, and $\pi(\Phi(\mathcal{G}))$ is dense in M, so that $\pi\big(\overline{\Phi(\mathcal{G})}\big) = M$. □

(2.10) It is useful to define "dual" concepts. We define the dual Lax-Oleinik operator associated to a closed form η by the expression

$$\breve{T}_\eta u(x) = \max_{q\in M} u(q) - A_\eta(0, x; 1, q), \quad u \in C(M, \mathbb{R})$$

and we associate to this operator a mapping $\breve{\Phi} : \breve{\mathbb{P}} \longrightarrow \breve{\mathbb{P}}$ by the expression $\breve{\Phi}(\mathcal{G}_{c,-u}) = \mathcal{G}_{c,\breve{T}_c(-u)} \in \breve{\mathbb{P}}$. We have

$$\overline{\breve{\Phi}(\breve{\mathcal{G}})} \subset \phi^{-1}(\breve{\mathcal{G}})$$

if $\breve{\mathcal{G}} \in \breve{\mathbb{P}}$. We denote by $\breve{\mathbb{V}}$ the set of fixed points of $\breve{\Phi}$. Let $\breve{\mathcal{G}} = \mathcal{G}_{c,-u}$ be a fixed point of $\breve{\Phi}$, and let $n < m$ be two relative integers. We say that a curve $x(t) : [n, m] \longrightarrow M$ is calibrated by $\breve{\mathcal{G}}$ if

$$u(x(m)) - A_c(n, x(n); m, x(m)) = \max_{x\in M} u(x) - A_c(n, x(n); m, x).$$

3 Aubry-Mather sets

We use the overlapping pseudographs to recover various invariant sets introduced by Mather, and to study their major properties. We also establish the equivalence between the different definitions of the same sets given in the literature.

(3.1) Proposition *There exists a function $\alpha : H^1(M, \mathbb{R}) \longrightarrow \mathbb{R}$ such that, for each continuous function u and each form η of cohomology c, the sequence $T^n_\eta u(x) + n\alpha(c), n \geqslant 1$ of continuous functions is equi-bounded and equi-Lipschitz. The function $c \longmapsto \alpha(c)$ is convex* *and superlinear.*

Proof. Let us fix a cohomology class c, and define the sequences

$$M_n(c) := \max_{x\in\mathbb{T}} T_c^n(0)(x) \text{ and } m_n(c) := \min_{x\in\mathbb{T}} T_c^n(0)(x),$$

where 0 is the zero function on M. Since the functions $T_c^n(0), n \geqslant 1$, are equi-semi-convex, see Appendix A, there exists a constant K such that

$$0 \leqslant M_n(c) - m_n(c) \leqslant K$$

for $n \geqslant 1$. We claim that $M_{n+m}(c) \leqslant M_n(c) + M_m(c)$. This follows from the inequalities

$$T_c^{m+n}(0)(x) = T_c^m(T_c^n(0))(x) \leqslant T_c^m(M_n(c))(x) \leqslant M_n(c) + T_c^m(0)(x).$$

Hence by a classical result on subadditive sequences, we have $\lim M_n(c)/n = \inf M_n(c)/n$. We denote by $-\alpha(c)$ this limit. In the same way, the sequence $-m_n(c)$ is subadditive, hence $m_n(c)/n \longrightarrow \sup m_n(c)/n$. This limit is also $-\alpha(c)$ because $0 \leqslant M_n(c) - m_n(c) \leqslant K$. As a consequence, we have, for all $n \geqslant 1$,

$$-K - n\alpha(c) \leqslant m_n(c) \leqslant -n\alpha(c) \leqslant M_n(c) \leqslant K - n\alpha(c).$$

Now far all $u \in C(M, \mathbb{R})$, $n \in \mathbb{N}$ and $x \in M$, we have

$$\min_{\mathbb{T}} u - K \leqslant \min_{\mathbb{T}} u + m_n(c) + n\alpha(c) \leqslant T_c^n u(x) + n\alpha(c) \leqslant \max_{\mathbb{T}} u + M_n(c) + n\alpha(c) \leqslant \max_{\mathbb{T}} u + K.$$

we obtain the first conclusion of the Proposition. The explicit definition of the value $m_n(c)$ is

$$m_n(c) = \min_{\gamma} \int_0^n L(s, \gamma(s), \dot\gamma(s)) - c_{\gamma(s)}(\dot\gamma(s))ds$$

where the minimum is taken on all absolutely continuous curves $\gamma : [0, n] \longrightarrow M$. As a consequence, the function $c \longmapsto m_n(c)$ is concave, as a minimum of linear functions. Hence each of the functions $c \longmapsto m_n(c)/n$ is concave, so that the limit $-\alpha(c)$ is concave, and the function $\alpha(c)$ is convex. Finally, there exists a superlinear function $l_0 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $L(s, x, v) \geqslant l_0(|v|)$, and a constant C such that $|c_x(p)| \leqslant C|c||p|$. As a consequence, we obtain from the expression above the estimate

$$\alpha(c) \geqslant m_n(c)/n \geqslant \min_{p\in\mathbb{R}^n} l_0(|p|) - C|c||p|$$

so that the function α is superlinear. □

(3.2) Proposition. *Let us fix a closed form η and a continuous function u. Let us set*

$$v := \liminf_{n\longrightarrow\infty} T_\eta^n(u) + n\alpha([\eta]),$$

then v is a fixed point of $T_\eta + \alpha$ hence $\mathcal{G}_{\eta,v}$ is a fixed point of Φ.

Proof. The one-form η is fixed once and for all in this proof, we omit the subscript η, and denote by α the number $\alpha([\eta])$. Let us first prove that $Tv + \alpha \leqslant v$. In order to do so, we fix $x \in M$ and consider an increasing sequence n_k of integers such that $T^{n_k}u(x) + n_k\alpha \longrightarrow v(x)$. Let q_k be a point such that $T^{n_k}u(x) = T^{n_k-1}u(q_k) + A(0, q_k; 1, x)$ or equivalently, $T^{n_k}u(x) + n_k\alpha = T^{n_k-1}u(q_k) + (n_k - 1)\alpha + \alpha + A(0, q_k; 1, x)$. We can suppose that the sequence q_k has a limit q. Taking the $\liminf$ in the equality above gives

$$v(x) \geqslant v(q) + A(0, q, 1, x) + \alpha \geqslant Tv(x) + \alpha$$

where we have used equi-continuity of the functions $T^n u, n \in \mathbb{N}$.

In order to prove that $Tv + \alpha \geqslant v$, just notice that $T^n u(x) \leqslant T^{n-1}u(q) + A(0, q; 1, x)$ for each q and x, or equivalently that $T^n u(x) + n\alpha \leqslant T^{n-1}u(q) + (n-1)\alpha + A(0, q; 1, x) + \alpha$ and take the liminf. □

(3.3) LEMMA *Let us fix a closed form η of cohomology c. Let $\mathbb{M} \subset C(M, \mathbb{R})$ be a family of fixed points of the Lax-Oleinik operator $T_\eta + \alpha(c)$. Assume that the minimum $v(x) = \min_{u \in \mathbb{M}} u(x)$ exists for each $x \in M$. Then the function v is a fixed point of $T_\eta + \alpha(c)$.*

PROOF. All functions $u \in \mathbb{M}$ satisfy $u(x) = \min_{y \in \mathbb{M}} u(y) + A_\eta(0, x; 1, y) + \alpha(c)$. It follows that

$$v(x) \leqslant \min_y v(y) + A_\eta(0, x; 1, y) + \alpha(c).$$

In order to prove the other inequality, let us take a function $u \in \mathbb{M}$ such that $u(x) = v(x)$, and consider a point $y \in M$ such that $u(x) = u(y) + A_\eta(0, x; 1, y) + \alpha(c)$. We obtain

$$v(x) = u(x) = u(y) + A_\eta(0, x; 1, y) + \alpha(c) \geqslant v(y) + A_\eta(0, x; 1, y) + \alpha(c).$$

□

(3.4) Fixed points of the Lax-Oleinik operator $T_c + \alpha(c)$ will be called weak KAM solutions, following Fathi. We denote by $\mathbb{V} \subset \mathbb{P}$ the set of fixed points of Φ, and $\mathbb{V}_c \subset \mathbb{P}_c$ the set of fixed points of Φ of cohomology c. Sometimes, we will also denote by $\mathbb{V}_C$ the set of fixed points of Φ whose cohomology belongs to the subset $C \subset H^1(M, \mathbb{R})$. The set $\mathbb{V}_c$ is non-empty for each c. If $\mathcal{G} \in \mathbb{V}$, then it follows from (2.6) that

$$\overline{\mathcal{G}} \subset \phi(\mathcal{G}).$$

It is then natural to define the set

$$\tilde{\mathcal{I}}(\mathcal{G}) = \bigcap_{n \in \mathbb{N}} \phi^{-n}(\bar{\mathcal{G}}),$$

which is a non-empty compact ϕ-invariant subset of T^*M. We also define

$$\mathcal{I}(\mathcal{G}) = \pi(\tilde{\mathcal{I}}(\mathcal{G})) \subset M.$$

More generally, for each $\mathcal{G} \in \mathbb{P}$, we define the set

$$\tilde{\mathcal{I}}(\mathcal{G}) := \bigcap_{n \in \mathbb{N}} \phi^{-n}\big(\overline{\Phi^n(\mathcal{G})}\big)$$

which is always compact, but may in general be empty.

(3.5) For each $\mathcal{G} \in \mathbb{V}$, we define the set $\tilde{\mathcal{M}}(\mathcal{G})$ as the closure of the union of the support of invariant measures of $\phi_{|\tilde{\mathcal{I}}(\mathcal{G})}$. If $\mathcal{G} \in \mathbb{V}$ and $\mathcal{G}' \in \mathbb{V}$ have the same cohomology c, then it is known that

$$\tilde{\mathcal{M}}(\mathcal{G}) \subset \tilde{\mathcal{I}}(\mathcal{G}')$$

hence $\tilde{\mathcal{M}}(\mathcal{G}) = \tilde{\mathcal{M}}(\mathcal{G}')$. As a consequence, the set $\tilde{\mathcal{M}}$, usually called the Mather set, depends only on the cohomology c. It will be denoted by

$$\tilde{\mathcal{M}}(c),$$

and as usual, we will denote by $\mathcal{M}(c)$ the projection $\pi(\tilde{\mathcal{M}}(c))$. We also define the Aubry set in a usual way by

$$\tilde{\mathcal{A}}(c) = \bigcap_{\mathcal{G} \in \mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G})$$

and $\mathcal{A}(c) = \pi(\tilde{\mathcal{A}}(c))$. The Mañe set is defined by

$$\tilde{\mathcal{N}}(c) = \bigcup_{\mathcal{G} \in \mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G})$$

and $\mathcal{N}(c) = \pi(\tilde{\mathcal{N}}(c))$. A bigger set will be useful in some occasions, defined by

$$\tilde{\mathcal{B}}(c) = \bigcup_{\mathcal{G} \in \mathbb{O}_c} \tilde{\mathcal{I}}(\mathcal{G}),$$

where $\mathbb{O}_c$ is as defined in (2.5). As an intersection of Lipschitz graphs, the Aubry set $\tilde{\mathcal{A}}(c)$ is a Lipschitz graph over $\mathcal{A}(c)$. Note however that the Mañe set is not a Graph in general. The sets

$$\tilde{\mathcal{M}}(c) \subset \tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{N}}(c) \subset \tilde{\mathcal{B}}(c)$$

are compact and invariant under ϕ. The compactness of $\tilde{\mathcal{N}}(c)$ and $\tilde{\mathcal{B}}(c)$ is mentioned here for completeness, it will be proved later in this sections, in lemma (3.12) and (3.13) below. These Lemma also prove that the Mañe set is indeed the set of orbits called c-minimizing by Mather and semi-static by Mañe, and that the set $\tilde{\mathcal{B}}$ is the set of minimizing orbits, called $\tilde{\mathcal{G}}$ in [2].

(3.6) It is possible to associate to each dual fixed point $\breve{\mathcal{G}} \in \breve{\mathbb{V}}$ the invariant set

$$\tilde{\mathcal{I}}(\breve{\mathcal{G}}) = \bigcap_{n \in \mathbb{N}} \phi^n\left(\overline{\breve{\mathcal{G}}}\right)$$

and its projection $\mathcal{I}(\breve{\mathcal{G}})$ on M. The following is essentially due to Fathi, [13].

PROPOSITION. *Let us fix a cohomology c, and consider pseudographs $\mathcal{G} \in \mathbb{V}_c$ and $\breve{\mathcal{G}} \in \breve{\mathbb{V}}_c$. The set $\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}}$ is non-empty, compact and invariant by ϕ. In addition, this set intersects the Aubry set $\tilde{\mathcal{A}}(c)$, and satisfies*

$$\mathcal{G} \tilde{\wedge} \breve{\mathcal{G}} \subset \tilde{\mathcal{I}}(\mathcal{G}) \cap \tilde{\mathcal{I}}(\breve{\mathcal{G}})$$

so that

$$\mathcal{G} \wedge \breve{\mathcal{G}} \subset \mathcal{I}(\mathcal{G}) \cap \mathcal{I}(\breve{\mathcal{G}}).$$

Furthermore, for each pseudograph $\mathcal{G} \in \mathbb{V}_c$, there exists a pseudograph $\breve{\mathcal{G}} \in \breve{\mathbb{V}}_c$ such that

$$\mathcal{G} \wedge \breve{\mathcal{G}} = \mathcal{I}(\mathcal{G}) = \mathcal{I}(\breve{\mathcal{G}}).$$

In a symmetric way, for each pseudograph $\breve{\mathcal{G}} \in \breve{\mathbb{V}}_c$, there exists a pseudograph $\mathcal{G} \in \mathbb{V}_c$ such that this relation holds. As a consequence, we have

$$\tilde{\mathcal{A}}(c) = \bigcap_{\mathcal{G}\in\mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G}) = \bigcap_{\breve{\mathcal{G}}\in\breve{\mathbb{V}}_c} \tilde{\mathcal{I}}(\breve{\mathcal{G}})$$

and

$$\tilde{\mathcal{N}}(c) = \bigcup_{\mathcal{G}\in\mathbb{V}_c} \tilde{\mathcal{I}}(\mathcal{G}) = \bigcup_{\breve{\mathcal{G}}\in\breve{\mathbb{V}}_c} \tilde{\mathcal{I}}(\breve{\mathcal{G}}).$$

PROOF. We have already proved that the set $\mathcal{G}\tilde{\wedge}\breve{\mathcal{G}}$ is compact and not empty, see (2.3). Let us prove that it is invariant. In order to do so, we consider a weak KAM solution u and a dual weak KAM solution $\breve{u}$ such that $\mathcal{G} = \mathcal{G}_{c,u}$ and $\breve{\mathcal{G}} = \mathcal{G}_{c,\breve{u}}$. Let $(x(t), p(t)) : \mathbb{R} \longrightarrow T^*M$ be an orbit of the Hamiltonian flow, such that $(x(0), p(0)) \in \mathcal{G}\tilde{\wedge}\breve{\mathcal{G}}$. Clearly, both u and $\breve{u}$ are differentiable at $x(0)$, and $p(0) = c(x(0)) + du(x(0))$. For each $m \leqslant n$ in $\mathbb{N}$, we have

$$u(x(n)) = \min_{x\in M} u(x) + A_c(m, x, n, x(n)) + (n-m)\alpha(c)$$

$$\leqslant u(x(m)) + A_c(m, x(m), n, x(n)) + (n-m)\alpha(c).$$

On the other hand, we have $(x(0), p(0)) \in \breve{\mathcal{G}}$ hence, in view of (2.7),

$$\breve{u}(x(m)) = \breve{u}(x(n)) + A_c(m, x(m), n, x(n)) + (n-m)\alpha(c).$$

As a consequence, the sequence $n \longmapsto (u - \breve{u})(x(n))$ is non-increasing on $\mathbb{N}$. Since its initial value $(u - \breve{u})(x(0))$ has been chosen to be a minimum of the function $u - \breve{u}$, the sequence must be constant, so that $x(n)$ is a point of $\mathcal{G} \wedge \breve{\mathcal{G}}$ for each $n \geqslant 0$. A symmetric argument shows that this is also true for $n \leqslant 0$. In addition, we obtain that the inequality $u(x(n)) \leqslant u(x(m)) + A_c(m, x(m), n, x(n)) + (n-m)\alpha(c)$ is in fact an equality for $0 \leqslant m \leqslant n$. Since this formula is true in view of (2.7) for $m \leqslant n \leqslant 0$ in $\mathbb{Z}$, we obtain that, for all $m \leqslant n$ in $\mathbb{Z}$,

$$u(x(n)) = u(x(m)) + A_c(m, x(m), n, x(n)) + (n-m)\alpha(c).$$

In other words, the curve $x(t)$ is calibrated by $\mathcal{G}$ and by $\breve{\mathcal{G}}$, see (2.8). This implies that $(x(n), p(n)) \in \mathcal{G} \cap \breve{\mathcal{G}}$ for each $n \in \mathbb{Z}$, and, since $x(n) \in \mathcal{G} \wedge \breve{\mathcal{G}}$, we get $(x(n), p(n)) \in \mathcal{G}\tilde{\wedge}\breve{\mathcal{G}}$. This proves that $\mathcal{G}\tilde{\wedge}\breve{\mathcal{G}}$ is invariant by ϕ and contained in $\mathcal{I}(\mathcal{G})$ and in $\mathcal{I}(\breve{\mathcal{G}})$.

Every compact invariant set of $\tilde{\mathcal{I}}(\mathcal{G})$ carries an invariant measure. As a consequence, every compact invariant set of $\tilde{\mathcal{I}}(\mathcal{G})$ intersects the Mather set $\tilde{\mathcal{M}}(c)$, see (3.5). Since $\tilde{\mathcal{M}}(c) \subset \tilde{\mathcal{A}}(c)$, the set $\mathcal{G}\tilde{\wedge}\breve{\mathcal{G}}$, which is a compact and invariant subset of $\tilde{\mathcal{I}}(\mathcal{G})$, intersects $\tilde{\mathcal{A}}(c)$. Let us now fix the Pseudograph $\mathcal{G}_{c,u} \in \mathbb{V}_c$, and prove the existence of a pseudograph $\breve{\mathcal{G}} \in \breve{\mathbb{V}}_c$ such that $\mathcal{G} \wedge \breve{\mathcal{G}} = \mathcal{I}(\mathcal{G}) = \mathcal{I}(\breve{\mathcal{G}})$. In order to do so, we set

$$\breve{u} := \liminf_{n\longrightarrow\infty} \breve{T}_c^n u - n\alpha(c).$$

We will prove that $\breve{u} \leqslant u$, with equality on $\mathcal{I}(\mathcal{G})$. For each $x \in M$ and each $n \in \mathbb{N}$, there exists $y_n \in M$ such that

$$\breve{T}_c^n u(x) = u(y_n) - A_c(0, x; n, y_n).$$

On the other hand, we have the inequality

$$u(y_n) - u(x) \leqslant A_c(0, x; n, y_n) + n\alpha(c),$$

hence $\breve{T}_c^n u(x) \leqslant u(x) + n\alpha(c)$, and finally $\breve{u} \leqslant u$.

Let $(x(t), p(t)) : \mathbb{R} \longrightarrow T^*M$ be a Hamiltonian orbit satisfying $(x(0), p(0)) \in \tilde{\mathcal{I}}(\mathcal{G})$. The orbit $x(t)$ is then calibrated by $\mathcal{G}$, see (2.8), so that the relation

$$u(x(n)) - u(x(m)) = A_c(m, x(m); n, x(n)) + (n - m)\alpha(c)$$

holds for all $m \leqslant n$ in $\mathbb{Z}$. It is clear from this relation that, for each $n \in \mathbb{N}$,

$$\breve{T}_c^n u(x(0)) \geqslant u(x(n)) - A_c(0, x(0); n, x(n)) = u(x(0)) + n\alpha(c),$$

So that $\breve{T}_c^n u(x(0)) - n\alpha(c) = u(x(0))$, hence $\breve{u} = u$ on $\mathcal{I}(\mathcal{G})$. As a consequence, the set of points minimizing $u - \breve{u}$ contains $\mathcal{I}(\mathcal{G})$. Since we have already proved that this set is contained in $\mathcal{I}(\mathcal{G})$, we can conclude, as desired, that

$$\mathcal{G}_{c,u} \wedge \mathcal{G}_{c,\breve{u}} = \mathcal{I}(\mathcal{G}).$$

Finally, we have to prove that $\mathcal{I}(\mathcal{G}_{c,\breve{u}}) = \mathcal{I}(\mathcal{G}_{c,u})$. It is enough to prove the inclusion $\mathcal{I}(\mathcal{G}_{c,\breve{u}}) \subset \mathcal{I}(\mathcal{G}_{c,u})$, the other inclusion follows from the fact that $\mathcal{I}(\mathcal{G}) = \mathcal{G}_{c,u} \wedge \mathcal{G}_{c,\breve{u}} \subset \mathcal{I}(\mathcal{G}_{c,\breve{u}})$. Let $(x(t), p(t)) : \mathbb{R} \longrightarrow T^*M$ be a Hamiltonian orbit in the invariant set $\tilde{\mathcal{I}}(\mathcal{G}_{c,\breve{u}})$. The orbit $x(t)$ is then calibrated by $\tilde{u}$, so that the relation

$$\breve{u}(x(n)) - \breve{u}(x(m)) = A_c(m, x(m); n, x(n)) + (n - m)\alpha(c)$$

holds for all $m \leqslant n$ in $\mathbb{Z}$. Both the α-limit and the ω-limit of the orbit $(x(t), p(t))$ intersect the Mather set $\tilde{\mathcal{M}}(c)$. Since the equality $\breve{u} = u$ holds on the Aubry set, so it holds on the Mather set, and we deduce the existence of two sequences $n_k \longrightarrow \infty$ and $m_k \longrightarrow -\infty$ of integers such that $u(x(n_k)) - \breve{u}(x(n_k)) \longrightarrow 0$ and $u(x(m_k)) - \breve{u}(x(m_k)) \longrightarrow 0$. It follows that

$$u(x(n_k)) - u(x(m_k)) - A_c(m_k, x(m_k); n_k, x(n_k)) - (n_k - m_k)\alpha(c) \longrightarrow 0$$

hence the curve $(x(t), p(t))$ is also calibrated by u. We have proved that $\mathcal{I}(\mathcal{G}_{c,\breve{u}}) \subset \mathcal{I}(\mathcal{G}_{c,u})$. □

(3.7) PROPOSITION. *The restriction to $\mathbb{V}$ of the function $c : \mathbb{P} \longrightarrow H^1(M, \mathbb{R})$ is continuous and proper.*

PROOF. Let us consider a compact subset C of $H^1(M, \mathbb{R})$. The Family of Hamiltonians $H(t, x, c_x + p), c \in C$, is clearly a uniform family of Hamiltonians, see Appendix B. As a consequence, the associated functions $A_c(0, .; 1, .), c \in C$ form an equi-semi-concave family of functions on $M \times M$. As a consequence, the functions $A_c(0, x; 1, .), c \in C, x \in M$ form an equi-semi-concave family of functions on M, see Appendix A. It follows that the functions $u(x) + A_c(0, x; 1, .), c \in C, x \in M$ also form an equi-semi-concave family, hence that the functions $\min_x u(x) + A_c(0, x; 1, .), c \in C$ form an equi-semi-concave family. As a consequence, the set $\Phi(\mathbb{P}_C)$ is relatively compact. Since the set $\mathbb{V}_C$ is obviously closed, and contained in $\Phi(\mathbb{P}_C)$, it is compact. □

We have proved the following Lemma, which is interesting in itself:

LEMMA. *If C is a compact subset of $H^1(M, \mathbb{R})$, the set $\Phi(\mathbb{P}_C)$ is equi-semi-concave.*

(3.8) Following Mather, we will use the function

$$h_c(x,y) := \liminf_{n \longrightarrow \infty} A_c^n(x,y) + n\alpha(c).$$

In view of (3.2), the function $h_c(x,.)$ is a fixed point of $T_c + \alpha(c)$. Similarly, the function $-h_c(.,y)$ is a fixed point of $\breve{T}_c - \alpha(c)$. Let us recall here some basic properties of the function h_c.

- For each $x, y \in M$ and $c \in H^1(M,\mathbb{R})$, we have $h_c(x,y) + h_c(y,x) \geqslant 0$, and $h_c(x,x) \geqslant 0$.
- For each $x, y, z \in M$ and $c \in H^1(M,\mathbb{R})$, we have the triangle inequality $h_c(x,y) + h_c(y,z) \geqslant h_c(x,z)$.
- For each compact set $C \in H^1(M,\mathbb{R})$, the set of functions $h_c : M \times M \longrightarrow \mathbb{R}, c \in C$, is equi-semi-concave.

(3.9) PROPOSITION. *If the pseudograph $\mathcal{G}_{c,u}$ is a fixed point of Φ, then we have*

$$u(y) - u(x) \leqslant h_c(x,y)$$

for each x and y. In addition,

$$u(x) = \min_{y \in M} u(y) + h_c(y,x) = \min_{a \in \mathcal{A}(c)} u(a) + h_c(a,x).$$

PROOF. We have, for each n, $u = T_c^n u + n\alpha(c)$. As a consequence, for each n,

$$u(x) = \min_{y \in M} u(y) + A_c(0,y;n,x) + n\alpha(c).$$

We obtain the inequality $u(x) \leqslant u(y) + A_c(0,y;n,x) + n\alpha(c)$ and, by taking the liminf, $u(x) \leqslant u(y) + h_c(y,x)$. In order to obtain the first equality, we consider a point $y_n \in M$ such that

$$u(x) = u(y_n) + A_c(0,y_n;n,x) + n\alpha(c).$$

We consider an increasing sequence n_k of integers such that the subsequence y_{n_k} has a limit y, and refine this subsequence in such a way that the subsequence $A_c(0,y,n_k,x)$ has a limit l. We have

$$u(x) = u(y) + l \geqslant u(y) + h_c(y,x).$$

Cumulated with the previously shown inequality, this proves the first equality in the statement. In order to prove the second equality, notice that the set of points y which minimize the function $u(.) + h_c(.,x)$ is precisely the set $\mathcal{G} \wedge \mathcal{G}_{c,-h_c(.,x)}$, and that $\mathcal{G}_{c,-h_c(.,x)} \in \breve{\mathbb{V}}_c$. By (3.6), the set $\mathcal{G} \wedge \mathcal{G}_{c,-h_c(.,x)}$ intersects $\mathcal{A}(c)$. □

Specializing the result to the case where $u(y) = h_c(x,y)$, we obtain the following refinement of a result of Albert Fathi:

(3.10) COROLLARY. *For each x and y in M and $c \in H^1(M,\mathbb{R})$, we have*

$$h_c(x,y) = \min_{z \in M} h_c(x,z) + h_c(z,y) = \min_{a \in \mathcal{A}(c)} h_c(x,a) + h_c(a,y).$$

The following result connects our definition of the Aubry set to the one of Mather.

(3.11) PROPOSITION. *The Aubry set $\mathcal{A}(c)$ is the set of points x such that $h_c(x,x)=0$.*

PROOF. Let us consider a Hamiltonian trajectory $(x(t),p(t)) : \mathbb{R} \longrightarrow T^*M$ such that $(x(0),p(0)) \in \tilde{\mathcal{A}}(c)$. This trajectory is calibrated by each fixed point of $T_c+\alpha(c)$, so in particular by $h_c(x(0),.)$. Consequently, we have

$$h_c(x(0),x(n)) - h_c(x(0),x(0)) = A_c(0,x(0);n,x(n)).$$

Taking a subsequence such that $x(n)$ has a limit x, and then a subsequence such that $A_c(0,x(0);n,x)$ is converging to a limit $l \geqslant h_c(x(0),x)$, we get, at the limit,

$$h_c(x(0),x) - h_c(x(0),x(0)) \geqslant h_c(x(0),x)$$

thus $h_c(x(0),x(0)) \leqslant 0$ and then $h_c(x(0),x(0)) = 0$. We have proved that the functions $h_c(x,x)$ vanishes on $\mathcal{A}(c)$.

Conversely, let us assume that $h_c(x,x)=0$. Then there exists an increasing sequence n_k of integers and a sequence of trajectories $(x_k(t),p_k(t)) : [0,n_k] \longrightarrow T^*M$ such that $x(0) = x(n_k) = x$ and

$$\int_0^{n_k} L(t,x_k(t),\dot{x}_k(t)) - c_{x_k(t)}(\dot{x}_k(t)) + \alpha(c)\,dt = A_c(0,x;n_k,0) + n_k\alpha(c) \longrightarrow 0.$$

Let $y_k : [-n_k,n_k] \longrightarrow M$ be the curve such that $y_k(t) = x_k(t+n_k)$ for $-n_k \leqslant t \leqslant 0$ and $y_k(t) = x_k(t)$ for $t \geqslant 0$. The curves y_k are equi-Lipschitz hence, by taking a subsequence, we can suppose that the sequence y_k is converging, uniformly on compact sets, to a limit $y(t) : \mathbb{R} \longrightarrow M$. Let u be a fixed point of $T_c+\alpha(c)$. We have, for each $n \in \mathbb{N}$ and k large enough,

$$u(y_k(n)) - u(x) - A_c(0,x;n,y_k(n)) - n\alpha(c)$$
$$\leqslant u(y_k(n_k)) - u(x) - A_c(0,x;n_k,y_k(n)) - n_k\alpha(c) \longrightarrow 0$$

hence $u(y(n)) - u(y(0)) = A_c(0,y(0);n,y(n)) + n\alpha(c)$. Similarly, we prove that $u(y(0)) - u(y(-n)) = A_c(-n,y(-n);0,y(0)) + n\alpha(c)$, for all $n \in \mathbb{N}$. Consequently, the curve $y(t)$ is calibrated by u. Since this holds for each weak KAM solution u, we have $x \in \mathcal{A}(c)$. □

The following result connects our definition of the Mañe set with the usual one, and implies its compactness.

(3.12) LEMMA *The following properties are equivalent for a continuous curve $P(t) = (x(t),p(t)) : \mathbb{R} \longrightarrow T^*M$.*

(i) The curve $P(t)$ is a Hamiltonian trajectory and $P(\mathbb{Z}) \subset \tilde{\mathcal{N}}(c)$.

(ii) The curve $P(t)$ satisfies $p(t) = \partial_v L(t,x(t),\dot{x}(t))$ and there exists $\mathcal{G}_{c,u} \in \mathbb{V}_c$ such that, for each $m \geqslant n$ in $\mathbb{Z}$, we have

$$u(x(m)) - u(x(n)) = \int_n^m L(t,x(t),\dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c).$$

(iii) The curve $P(t)$ satisfies $p(t) = \partial_v L(t, x(t), \dot{x}(t))$ and for each $m \geqslant n$ in $\mathbb{Z}$, we have

$$\int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c) = \min_{l \in \mathbb{N}, l>0} A_c(0, x(n); l, x(m)) + l\alpha(c).$$

PROOF. We shall prove that $(iii) \Rightarrow (ii)$. The other implications are left to the reader. Let $P(t)$ be a curve satisfying (iii). let n_k be an increasing sequence of integers such that $x(-n_k)$ has a limit α. Then we have, for $m \geqslant n$,

$$\int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c)$$

$$= \int_{-n_k}^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t)) + \alpha(c)dt - \int_{-n_k}^n L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t)) + \alpha(c)\, dt$$

$$= A_c(-n_k, x(-n_k); m, x(m)) + (m+n_k)\alpha(c) - A_c(-n_k, x(-n_k); n, x(n)) + (n+n_k)\alpha(c).$$

By (iii), we have

$$A_c(-n_k, x(-n_k); m, x(m)) + (m+n_k)\alpha(c) = \min_{l \in \mathbb{N}, l>0} A_c(0, x(-n_k); l, x(m)) + l\alpha(c)$$

$$\leqslant h_c(x(-n_k), x(m))$$

which implies that

$$A_c(-n_k, x(-n_k); m, x(m)) + (m+n_k)\alpha(c) \longrightarrow h_c(\alpha, x(m))$$

as $k \longrightarrow \infty$. Similarly,

$$A_c(-n_k, x(-n_k); n, x(n)) + (n+n_k)\alpha(c) \longrightarrow h_c(\alpha, x(n)),$$

so that

$$h_c(\alpha, x(m)) - h_c(\alpha, x(n)) = \int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c).$$

We have proved (ii) with $u = h_c(\alpha, .)$. □

We now give equivalent definitions for the set $\tilde{\mathcal{B}}$. The following Lemma shows that the set $\tilde{\mathcal{B}}$ is the set called $\tilde{\mathcal{G}}$ in [2], and implies its compactness.

(3.13) LEMMA *The following properties are equivalent for a continuous curve $P(t) = (x(t), p(t)) : \mathbb{R} \longrightarrow T^*M$.*

(i) The curve $P(t)$ is a Hamiltonian trajectory and $P(\mathbb{Z}) \subset \tilde{\mathcal{B}}(c)$.

(ii) The curve $P(t)$ satisfies $p(t) = \partial_v L(t, x(t), \dot{x}(t))$, and there exists a sequence u_n of functions such that, for each $m \geqslant n$, we have $T_c^{m-n} u_n = u_m$ and

$$u_m(x(m)) - u_n(x(n)) = \int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt.$$

(iii) The curve $P(t)$ satisfies $p(t) = \partial_v L(t, x(t), \dot{x}(t))$, and for each $m \geqslant n$, we have

$$\int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt = A_c(n, x(n); m, x(m))$$

PROOF. *(ii)* $\Rightarrow$ *(i)*. Then for each pair $m \geqslant n$ of integers, the curve $x(t) : [n, m] \longrightarrow M$ is minimizing the action between its endpoints. Hence the curve $P(t)$ is a Hamiltonian trajectory. It follows from (2.7) that, for each $n \geqslant 0$, we have $P(n) \in \overline{\mathcal{G}_{c,u_n}}$ and since $P(n) = \phi^n(P(0))$, we have

$$P(0) \in \phi^{-n}(\overline{\Phi^n(\mathcal{G}_{c,u_0})}).$$

This inclusion holds for all n, so that $P(0) \in \tilde{\mathcal{I}}(\mathcal{G}_{c,u_0})$. Now *(i)* follows from the fact $\mathcal{G}_{c,u_0} \in \mathbb{O}$ and that $\tilde{\mathcal{I}}(\mathcal{G}_{c,u_0})$ is invariant under ϕ.

(i) $\Rightarrow$ *(ii)*. There exists a pseudograph $\mathcal{G}_{c,u_0} \in \mathbb{O}$ such that $P(0) \in \tilde{\mathcal{I}}(\mathcal{G}_{c,u_0})$. There exists a sequence $u_n, n \in \mathbb{Z}$ of functions on M such that $T_c^{m-n} u_n = u_m$ for $m \geqslant n$. For each $m \geqslant 0$, since $P(m) \in \overline{\mathcal{G}_{c,u_m}}$, we have

$$u_m(x(m)) - u_0(x(0)) = \int_0^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt.$$

On the other hand, for each $n \leqslant 0$, we can find by minimization a curve $y_n(t) : [n, 0] \longrightarrow M$ such that

$$u_0(y_n(0)) - u_n(y_n(n)) = \int_n^0 L(t, y_n(t), \dot{y}_n(t)) - c_{y_n(t)}(\dot{y}_n(t))dt.$$

There exists a subsequence n_k such that the curves $y_{n_k}(t)$ converge, uniformly on compact sets, to a limit $y(t) : (-\infty, 0] \longrightarrow M$. By (1.3), this curve satisfies, for all $n \leqslant 0$,

$$u_0(y(0)) - u_n(y(n)) = \int_n^0 L(t, y(t), \dot{y}(t)) - c_{y(t)}(\dot{y}(t))dt.$$

Hence, for $n \leqslant 0 \leqslant m$, we have

$$u_m(x(m)) - u_n(y(n)) = \int_n^0 L(t, y(t), \dot{y}(t)) - c_{y(t)}(\dot{y}(t))dt + \int_0^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt.$$

As a consequence, the curve obtained by gluing y on $\mathbb{R}^-$ and x on $\mathbb{R}^+$ is the projection of a Hamiltonian trajectory, which, by Cauchy-Lipschitz uniqueness, has to be $P(t)$. In other words, we have proved that $x(t) = y(t)$ on $\mathbb{R}^-$. The relation of calibration is now established.

(iii) $\Rightarrow$ *(ii)*. Let $P(t)$ be a curve satisfying *(iii)*. Then we have, for $m \geqslant n \geqslant k$,

$$\int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m - n)\alpha(c)$$

$$= A_c(k, x(k); m, x(m)) + (m - k)\alpha(c) - A_c(k, x(k); n, x(n)) + (n - k)\alpha(c).$$

Let us denote by u_n^k the function

$$u_n^k(x) = A_c(k, x(k); n, x) + (n - k)\alpha(c),$$

we obviously have $T_c^{m-n} u_n^k + (m - n)\alpha(c) = u_m^k$ for $m \geqslant n \geqslant k$. By diagonal extraction, we find an increasing sequence of integers n_k such that $u_n^{-n_k}$ has a limit u_n for each n as

$k \longrightarrow \infty$. We then have $T_c^{m-n}u_n + (m-n)\alpha(c) = u_m$ for each $m \geqslant n$, so that $\mathcal{G}_{c,u_n} \in \mathbb{O}$. In addition, we have

$$\int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c) = u_m(x(m)) - u_n(x(n)).$$

$(ii) \Rightarrow (iii)$ is clear. □

(3.14) Lemma *For each $P \in \tilde{\mathcal{B}}(c)$, the orbit $\phi^n(P)$ is α-asymptotic and ω-asymptotic to the Aubry set $\tilde{\mathcal{A}}(c)$. As a consequence, the Mather set $\tilde{\mathcal{M}}(c)$ is the closure of the union of the supports of the invariant measures of the action of ϕ on $\tilde{\mathcal{B}}(c)$*

Proof. Let $P(t) = (x(t), p(t))$ be the Hamiltonian orbit of P. Let $u_n, n \in \mathbb{Z}$ be a sequence of continuous functions such that, for $m \geqslant n$, we have $u_m = T_c^{m-n}u_n + (m-n)\alpha(c)$ and

$$u_m(x(m)) - u_n(x(n)) = \int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c)$$

If v is a weak KAM solution, that is a fixed point of $T_c - \alpha(c)$, we have, for $m \geqslant n$,

$$u_m(x(m)) - u_n(x(n)) \leqslant \int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c)$$

It follows that the sequence $n \longmapsto u_n(x(n)) - v(x(n))$ is non-decreasing. In view of (3.1), this sequence is bounded, so that it has a limit l as $n \longrightarrow -\infty$. Let us now consider an increasing sequence n_k of integers such that the sequence $P(t - n_k)$ converges, uniformly on compact set, to a limit $Z(t) = (y(t), z(t)$ which is a Hamiltonian trajectory. Extracting a subsequence in n_k, we can suppose that there exists a sequence w_n of continuous functions on M such that $u_{n-n_k} \longrightarrow w_n$ uniformly, for each n, as $k \longrightarrow \infty$. Then, the sequence w_n satisfies $T_c^{m-n}w_n = w_m$ for $m \geqslant n$. In addition, we have $w_n(y(n)) - v(y(n)) = l$ and, for $m \geqslant n$,

$$w_m(x(m)) - w_n(x(n)) = \int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c).$$

It follows that, for $m \geqslant n$,

$$v(x(m)) - v(x(n)) = \int_n^m L(t, x(t), \dot{x}(t)) - c_{x(t)}(\dot{x}(t))dt + (m-n)\alpha(c)$$

which implies that $Z(\mathbb{Z}) \in \tilde{\mathcal{I}}(\mathcal{G}_{c,v})$. Since this holds for all weak KAM solutions v, we obtain that $Z(\mathbb{Z}) \in \tilde{\mathcal{A}}(c)$. We have proved that the trajectory $P(t)$ is α-asymptotic to $\tilde{\mathcal{A}}(c)$. Similarly, one can prove that it is also ω-asymptotic to $\tilde{\mathcal{A}}(c)$. This implies that the invariant measures of the action of ϕ on $\tilde{\mathcal{B}}(c)$ are supported on $\tilde{\mathcal{A}}(c)$. □

4 Static classes and heteroclinics

In this section, we see that there is a natural partition of the Aubry set in compact invariant subsets, which we call static classes, following the terminology of Mañe. We also discuss the existence of heteroclinic orbits between these static classes.

(4.1) LEMMA. *Let x and y be two points in M. The following properties are equivalent:*

(i) The points x and y are in $\mathcal{A}(c)$ and the function $z \longmapsto h_c(x,z) - h_c(y,z)$ is constant on $\mathcal{A}(c)$.

(ii) $h_c(x,y) + h_c(y,x) = 0$.

(iii) The points x and y are in $\mathcal{A}(c)$ and, for each pair $(\mathcal{G}, \breve{\mathcal{G}}) \in \mathbb{V}_c \times \breve{\mathbb{V}}_c$, either the set $\mathcal{G} \wedge \breve{\mathcal{G}}$ contains x and y or it contains neither x nor y.

If x and y satisfy these properties, we have, for each $z \in M$, $h_c(x,z) = h_c(x,y) + h_c(y,z)$.

PROOF. $i \Rightarrow ii$. Assuming i, we evaluate the constant function at x and y and get $h_c(x,x) - h_c(y,x) = h_c(x,y) - h_c(y,y)$, hence $h_c(x,y) + h_c(y,x) = 0$.

$ii \Rightarrow iii$. We have $h_c(x,x) \leqslant h_c(x,y) + h_c(y,x) = 0$ hence $x \in \mathcal{A}(c)$. Similarly, $y \in \mathcal{A}(c)$. Let us consider $\mathcal{G} = \mathcal{G}_{c,u} \in \mathbb{V}_c$ and $\breve{\mathcal{G}} = \mathcal{G}_{c,\breve{u}} \in \breve{\mathbb{V}}_c$ such that $x \in \mathcal{G} \wedge \breve{\mathcal{G}}$ (such a pair exists because $x \in \mathcal{A}(c)$) We have to prove that $y \in \mathcal{G} \wedge \breve{\mathcal{G}}$. We have the inequalities $u(y) \leqslant u(x) + h_c(x,y)$ and $\breve{u}(y) \geqslant \breve{u}(x) - h_c(y,x)$. We obtain the inequality

$$(u - \breve{u})(y) \leqslant (u - \breve{u})(x) + h_c(x,y) + h_c(y,x).$$

As a consequence, if $h_c(x,y) + h_c(y,x) = 0$, then y is also a point of minimum of $u - \breve{u}$, which is the desired result.

$iii \Rightarrow ii$. The point x is a point of minimum of the function $h_c(x,.) + h_c(.,x)$. As a consequence, the point y is also a point of minimum for this function, so that $h_c(x,y) + h_c(y,x) = h_c(x,x) + h_c(x,x) = 0$.

$ii \Rightarrow i$. We have the inequalities

$$h_c(x,z) \leqslant h_c(x,y) + h_c(y,z) \text{ and } h_c(y,z) \leqslant h_c(y,x) + h_c(x,z).$$

If $h_c(x,y) + h_c(y,x) = 0$, then these inequalities sum to an equality, hence they are both equalities. □

(4.2) The equivalent properties of the Lemma define an equivalence relation on $\mathcal{A}(c)$. We call *static classes* the classes of equivalence for this relation. In other worlds, we say that the points x and y of $\mathcal{A}(c)$ belong to the same static class if they satisfy (i), (ii) or (iii) of the lemma. We usually denote by $\mathcal{S}$ a static class, and by $\mathcal{S}(x)$ the static class containing x. If $\mathcal{S}$ is a static class, we denote by $\tilde{\mathcal{S}}$ the set of points of $\tilde{\mathcal{A}}(c)$ whose projection on M belong to $\mathcal{S}$. We will also call static classes the sets of the form $\tilde{\mathcal{S}}$. The static classes $\mathcal{S}$ are compact and partition $\mathcal{A}(c)$, the static classes $\tilde{\mathcal{S}}$ are compact, invariant, and they partition $\tilde{\mathcal{A}}(c)$. The invariance is a direct consequence of the caracterisation (iii) of the equivalence relation. To each point x of the Aubry set $\mathcal{A}(c)$, we associate the weak KAM solution $h_c(x,.)$, and we denote by $E_{c,x}$ the associated element of $\mathbb{V}_c$. The pseudographs of this form are called elementary solutions of $\mathbb{V}_c$. Two points of a same static class give rise to the same elementary solution, we will denote by $E_{c,\mathcal{S}}$ the elementary solution induced by points of $\mathcal{S}$. There is a one to one correspondence between the set of static classes and the set of elementary solutions. We will denote by $\mathbb{E}_c \subset \mathbb{V}_c$ this set. We endow it with the induced metric, it is clearly a compact set for this metric. We also denote by $\breve{E}_{c,\mathcal{S}}$ the fixed point of $\breve{\Phi}$ associated to the dual weak KAM solution $-h(.,x)$ for $x \in \mathcal{S}$.

(4.3) PROPOSITION. *Let $\mathcal{G} \in \mathbb{V}_c$ be a fixed point, and let P be a point of $\bar{\mathcal{G}}$. The α-limit of the orbit $\phi^n(P)$ is contained in one static class $\tilde{\mathcal{S}} \subset \tilde{\mathcal{A}}(c)$. We then have $P \in \bar{E}_{c,\mathcal{S}}$. In a similar way, if $P \in \breve{\mathcal{G}} \in \breve{\mathbb{V}}_c$, then the ω-limit of the orbit of P is contained in one static class of $\tilde{\mathcal{A}}(c)$.*

PROOF. Let $\alpha \subset M$ be the α-limit of the orbit of $P \in \mathcal{G}_{c,u}$. We claim that, for each weak KAM solution or backward weak KAM solution v, the function $u-v$ is constant on α. Clearly, this implies that α is contained in a static class. In order to prove the claim, we consider the projection $x(t)$ on M of the orbit of P. The curve $x(t)$ is calibrated by u on $(-\infty, 0]$, hence the equality

$$u(x(-m)) - u(x(-n)) = A_c(-n, x(-n), -m, x(-m))$$

holds for all $n, m \in \mathbb{N}$ such that $m \leqslant n$. On the other hand, if v is a weak KAM solution or a backward weak KAM solution, we have the inequality

$$v(x(-m)) - v(x(-n)) \leqslant A_c(-n, x(-n), -m, x(-m))$$

for all $n, m \in \mathbb{N}$ such that $m \leqslant n$. We deduce that the sequence $(u-v)(x(-n)) : \mathbb{N} \longrightarrow \mathbb{R}$ is non-increasing. Now let $y = \lim_{k\to\infty} x(-n_k)$ and $z = \lim_{k\to\infty} x(-m_k)$ be two points of α. We can suppose that $n_k \leqslant m_k \leqslant n_{k+1}$ by extracting subsequences. We obtain $(u-v)(x(-n_k)) \geqslant (u-v)(x(-m_k)) \geqslant (u-v)(x(-n_{k+1}))$, and at the limit $(u-v)(y) \geqslant (u-v)(z) \geqslant (u-v)(y)$. Hence the function $u-v$ is constant on α. This proves the first statement of the proposition.

Taking $v = h_c(\alpha, .)$, we obtain that the sequence $u(x(-n)) - h_c(\alpha, x(-n)) : \mathbb{N} \longrightarrow \mathbb{R}$ is non-increasing. On the other hand, we have $u(x(0)) - h_c(\alpha, x(0)) \leqslant u(\alpha)$. It follows that the sequence is in fact constant, so that the curve $x(t)$ is calibrated by $E_{c,\mathcal{S}(\alpha)}$ on $(-\infty, 0]$, and, be (2.8), $((x(0), p(0)) \in \bar{E}_{c,\mathcal{S}(\alpha)}$. □

COROLLARY. *Let $P \in \tilde{\mathcal{N}}(c)$ be a point whose α-limit is contained in $\tilde{\mathcal{S}}$ and whose ω-limit is contained in $\tilde{\mathcal{S}}'$. We have*

$$P \in E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}'}.$$

PROOF. Let $(x(t), p(t))$ be the orbit of P. Let α be an α-limit point of the curve $x(t)$, and let ω be an ω-limit point. It follows from the proposition that $(x(m), p(m)) \in \bar{E}_{c,\mathcal{S}}$ for each $m \in \mathbb{Z}$. Applying the discussions in the proof of the proposition with the point $P = (x(m), p(m))$ and the functions $u = h_c(\alpha, .)$ and $v = -h_c(., \omega)$, we get that the sequence $h_c(\alpha, x(n)) + h_c(x(n), \omega)$ is non-decreasing on $n \leqslant m$. Since we can take any $m \in \mathbb{Z}$, this sequence is non-decreasing on $\mathbb{Z}$. Taking a sequence $n_k \longrightarrow -\infty$ such that $x(-n_k) \longrightarrow \alpha$ we obtain the inequality $h_c(\alpha, \omega) \leqslant h_c(\alpha, x(n)) + h_c(x(n), \omega)$. Taking a sequence $m_k \longrightarrow \infty$ such that $x(m_k) \longrightarrow \omega$ we obtain the inequality $h_c(\alpha, \omega) \geqslant h_c(\alpha, x(n)) + h_c(x(n), \omega)$. It follows that, for each n,

$$h_c(\alpha, x(n)) + h_c(x(n), \omega) = h_c(\alpha, \omega) = \min_M h_c(\alpha, .) + h_c(., \omega).$$

This is precisely saying that

$$x(n) \in E_{c,\mathcal{S}(\alpha)} \tilde{\wedge} \breve{E}_{c,\mathcal{S}(\omega)}.$$

□

(4.4) LEMMA. *If the static class $\mathcal{S}$ is isolated in $\mathcal{A}(c)$, then there exists a neighborhood V of $\mathcal{S}$ in M such that the α-limit of every point $P \in E_{c,\mathcal{S}}$ satisfying $\pi(P) \in V$ is contained in $\mathcal{S}$.*

PROOF. If the result did not hold true, we could find a sequence $(x_n, p_n) \in E_{c,\mathcal{S}}$ that has a limit $(x, p) \in \tilde{\mathcal{S}}$ and a sequence α_n of α-limit points of (x_n, p_n) that has a limit α in $\tilde{\mathcal{A}}(c) - \tilde{\mathcal{S}}$. Note that the orbit $(x_n(t), p_n(t)) : (-\infty, 0] \longrightarrow T^*M$ of (x_n, p_n) is contained in $E_{c,\mathcal{S}}$. Hence it is calibrated by the function $h_c(x, .)$, that is

$$h_c(x, x_n(0)) = h_c(x, x_n(-k)) + A_c(-k, x_n(-k); 0, x_n(0)) + k\alpha(c)$$

for all $k \in \mathbb{N}$. At the liminf $k \longrightarrow \infty$, for fixed n, we obtain the inequality $h_c(x, x_n) \geqslant h_c(x, \alpha_n) + h_c(\alpha_n, x_n)$ hence the equality $h_c(x, x_n) = h_c(x, \alpha_n) + h_c(\alpha_n, x_n)$. At the limit $n \longrightarrow \infty$ we get $0 = h_c(x, x) = h_c(x, \alpha) + h_c(\alpha, x)$. This is in contradiction with the fact that α and x do not belong to the same static class. □

(4.5) Let $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$ be two different static classes in $\tilde{\mathcal{A}}(c)$. The set $E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}'}$ contains $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$ as well as other orbits of $\tilde{\mathcal{N}}(c)$. The following result is similar to Theorem A of [8].

PROPOSITION. *The set $E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}'} - (\mathcal{S} \cup \mathcal{S}')$ is not empty and contains points in every neighborhood of $\mathcal{S}$, as well as in every neighborhood of $\mathcal{S}'$. More precisely, if $\mathcal{S}$ and $\mathcal{S}'$ are two, possibly equal, static classes, and if $\tilde{\mathcal{K}} \subset \tilde{\mathcal{S}}$ and $\tilde{\mathcal{K}}' \subset \tilde{\mathcal{S}}'$ are two disjoint compact invariant sets, then the set $E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}'} - (\mathcal{K} \cup \mathcal{K}')$ contains points in every neighborhood of $\mathcal{K}$, as well as in every neighborhood of $\mathcal{K}'$.*

PROOF. Let V be an open neighborhood of $\mathcal{K}$ in M which does not intersect $\mathcal{S}'$. Let us fix a recurrent orbit $(y(t), z(t)) : \mathbb{R} \longrightarrow T^*M$ such that $(y(0), z(0)) = (y, z) \in \mathcal{K}$ and a recurrent orbit $(y'(t), z'(t)) : \mathbb{R} \longrightarrow T^*M$ such that $(y'(0), z'(0)) = (y', z') \in \mathcal{K}'$ Consider a sequence $n_k \in \mathbb{N}$ of integers and a sequence $(x_k(t), p_k(t)) : [0, n_k] \longrightarrow T^*M$ of Hamiltonian trajectories such that $x_k(0) = y$ and $x_k(n_k) = y'$ and

$$\int_0^{n_k} L(t, x_k(t), \dot{x}_k(t)) - c_{x_k(t)}(\dot{x}_k(t)) + \alpha(c)\, dt = A_c(0, y; n_k, y') + n_k\alpha(c) \longrightarrow h_c(y, y').$$

We extend the curve $x_k : [0, n_k] \longrightarrow M$ to a curve $x_k : \mathbb{R} \longrightarrow \mathbb{R}$ by setting $x_k(t) = y(t)$ for $t \leqslant 0$ and $x_k(t) = y'(t - n_k)$ for $t \geqslant n_k$. Let a_k and b_k be two increasing sequences of integers such that $y(-a_k) \longrightarrow y$ and $y'(b_k) \longrightarrow y'$. The existence of such sequences follows from the fact that the curves $y(t)$ and $y'(t)$ are recurrent. Since the curve $y(t)$ is calibrated by $h_c(y, .)$, we have, as $k \longrightarrow \infty$,

$$A_c(-a_k, y(-a_k); 0, y) + a_k\alpha(c) = -h_c(y, y(-a_k)) \longrightarrow 0$$

and similarly

$$A_c(0, y'; b_k, y_k(b_k)) + b_k\alpha(c) = h_c(y', y(b_k)) \longrightarrow 0.$$

As a consequence, we have, as $k \longrightarrow \infty$,

$$A_c(-a_k, x_k(-a_k); b_k, x_k(b_k)) + (b_k + a_k)\alpha(c) \longrightarrow h_c(y, y').$$

For each k, let T_k be the maximum of all times $i \in \mathbb{N}$ such that $x_k(i) \in V$. Note that $x_k(T_k+1)$ does not belong to V. We can assume, taking a subsequence, that the curve $x_k(t+T_k)$ is converging, uniformly on compact sets to a limit $x(t) : \mathbb{R} \longrightarrow M$. Clearly, the continuous curve $x(t)$ does not belong to $\mathcal{K} \cup \mathcal{K}'$. Let us now fix $m \leqslant n$ in $\mathbb{Z}$. Summing the inequalities

$$\liminf_{k \longrightarrow \infty} \Big(A_c(-a_k, x_k(-a_k); T_k + m, x_k(T_k + m)) + (T_k + m + a_k)\alpha(c) \Big) \geqslant h_c(y, x(m)),$$

$$\liminf_{k \longrightarrow \infty} \Big(A_c(T_k + m, x_k(T_k + m); T_k + n, x_k(T_k + n)) \Big) = A_c(m, x(m); n, x(n))$$

and

$$\liminf_{k \longrightarrow \infty} \Big(A_c(T_k + n, x_k(T_k + n); b_k, x_k(b_k))) + (b_k - T_k - n)\alpha(c) \Big) \geqslant h_c(x(n), y'),$$

we get

$$\begin{aligned} h(y, y') &= \liminf_{k \longrightarrow \infty} A_c(-a_k, x_k(-a_k); b_k, x_k(b_k)) + (b_k + a_k)\alpha(c) \\ &\geqslant h_c(y, x(n)) + A_c(m, x(m); n, x(n)) + (n - m)\alpha(c) + h_c(x(n), y'). \end{aligned}$$

Since the converse inequality obviously holds, we obtain the equality

$$h_c(y, y') = h_c(y, x(m)) + A_c(m, x(m); n, x(n)) + (m - n)\alpha(c) + h_c(x(n), y'),$$

for all $m \leqslant n$. It follows that all the inequalities above are in fact equalities, so that we also have

$$h_c(y, x(m)) + A_c(m, x(m); n, x(n)) + (m - n)\alpha(c) = h_c(y, x(n))$$

so that the orbit $x(t)$ is calibrated by the weak KAM solution h_c on $\mathbb{R}$. Hence it is the projection of a Hamiltonian trajectory $(x(t), p(t))$. Moreover, we have the equality

$$h_c(y, y') = h_c(y, x(n)) + h_c(x(n), y'),$$

so that the point $x(n)$ is a point of minimum of the function $h_c(y, .) + h_c(., y')$. Hence it belongs to $E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}'}$. We have proved that the sequence $(x(n), p(n)), n \in \mathbb{Z}$ is an orbit of ϕ which is contained in the invariant graph $E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}'}$. Since the point $x(1)$ is not a point of $\mathcal{K}$, this orbit does not intersect the invariant set $\tilde{\mathcal{K}}$. As a consequence, the point $x(0)$ belongs to $\bar{V} - \mathcal{K}$. We have proved that the set $E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}'} - (\mathcal{K} \cup \mathcal{K}')$ contains points in each neighborhood of $\mathcal{K}$. One can prove in a similar way that this set contains points in every neighborhood of $\mathcal{K}'$. □

(4.6) Corollary. *A static class $\tilde{\mathcal{S}}$ can not be decomposed as the union of two disjoint invariant compact subsets.*

Proof. Assume, by contradiction, that there exists a static class $\tilde{\mathcal{S}} = \tilde{\mathcal{K}}_1 \cup \tilde{\mathcal{K}}_2$, with $\tilde{\mathcal{K}}_i$ invariant, compact and disjoint. In view of (4.5), the set

$$E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}} - (\mathcal{K}_1 \cup \mathcal{K}_2)$$

is not empty. On the other hand, we have $E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}} = \mathcal{S}$ and $\mathcal{K}_1 \cup \mathcal{K}_2 = \mathcal{S}$, so that

$$E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}} - ((\mathcal{S} \cap \mathcal{K}_1) \cup (\mathcal{S} \cap \mathcal{K}_2))$$

is empty, which is a contradiction. □

(4.7) Let $(x(t), p(t)) : \mathbb{R} \longrightarrow T^*M$ be an orbit of the Mañe set, that is an orbit satisfying $(x(0), p(0)) \in \tilde{\mathcal{N}}(c)$. This orbit is α-asymptotic to a static class $\tilde{\mathcal{S}}$, and ω-asymptotic to a static class $\tilde{\mathcal{S}}'$.

LEMMA. *The inclusion $(x(0), p(0)) \in \tilde{\mathcal{A}}(c)$ holds if and only if $\mathcal{S} = \mathcal{S}'$. In this case, we have $(x(0), p(0)) \in \tilde{\mathcal{S}}$*

PROOF. Let us first assume that $\mathcal{S} = \mathcal{S}'$. In this case, we see from Corollary (4.3) that $(x(0), p(0)) \in E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}}$. But is is clear from the definition of static classes that $E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}} = \tilde{\mathcal{S}}$. Consequently, we have $(x(0), p(0)) \in \tilde{\mathcal{S}} \subset \tilde{\mathcal{A}}(c)$. Conversely, assume that $(x(0), p(0)) \in \tilde{\mathcal{A}}(c)$. Then this point is contained in one static class $\tilde{\mathcal{S}}_0$. Since this static class is compact and invariant, it contains the α and the ω-limits of the orbit $(x(t), p(t))$, so that we have $\tilde{\mathcal{S}} = \tilde{\mathcal{S}}_0 = \tilde{\mathcal{S}}'$. □

COROLLARY *We have the equality $\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c)$ if and only if there is exactly one static class in $\tilde{\mathcal{A}}(c)$.*

(4.8) Let $\tilde{\mathcal{H}}_c(\tilde{\mathcal{S}}, \tilde{\mathcal{S}}')$ be the set of orbits of $\tilde{\mathcal{N}}(c)$ which are heteroclinic orbits between the static classes $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$, we denote by $\mathcal{H}_c(\mathcal{S}, \mathcal{S}')$ its projection on M. We have

$$\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c) \cup \bigcup_{\mathcal{S}, \mathcal{S}'} \tilde{\mathcal{H}}_c(\mathcal{S}, \mathcal{S}'),$$

where the union is taken on all pairs $(\mathcal{S}, \mathcal{S}')$ of different static classes. Recall, from Corollary (4.3), that

$$\tilde{\mathcal{H}}_c(\tilde{\mathcal{S}}, \tilde{\mathcal{S}}') \subset E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}'}.$$

The following result is from [14] and [8].

(4.9) PROPOSITION. *If the static class $\tilde{\mathcal{S}}$ is properly contained and isolated in $\tilde{\mathcal{A}}(c)$, then there exists an orbit of ϕ in $\tilde{\mathcal{N}}(c) - \tilde{\mathcal{A}}(c)$ which is α-asymptotic to $\tilde{\mathcal{S}}$. This orbit is then ω-asymptotic to another static class $\tilde{\mathcal{S}}'$.*

PROOF. Let us chose, according to (4.4), a neighborhood V of $\mathcal{S}$ such that every orbit of $E_{c,\mathcal{S}}$ starting above V has its α-limit contained in $\mathcal{S}$. Now let us choose any static class $\mathcal{S}''$ different from $\mathcal{S}$. In view of (4.5), the set $E_{c,\mathcal{S}} \wedge \breve{E}_{c,\mathcal{S}''}$ intersects $V - \mathcal{S}$. Let $P(t) = (x(t), p(t)) : \mathbb{R} \longrightarrow T^*M$ be an orbit such that $P(0) \in E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}''}$ and $x(0) \in V - \mathcal{S}$. The α-limit of the orbit $P(t)$ is contained in $\tilde{\mathcal{S}}$. On the other hand, this orbits belongs to $\tilde{\mathcal{N}}(c)$, hence its ω-limit is contained in some static class $\tilde{\mathcal{S}}'$. □

(4.10) We have treated so far the case where there exist several static classes. We recall, however that the existence of a single static class in $\mathcal{A}(c)$, is, for c fixed, a generic property of the Lagrangian, see [8]. In order to treat this case, it is useful to recall here a device due to Fathi, as well as Contreras and Paternain, see [14] and [8]. Let $P : M_0 \longrightarrow M$ be a finite connected covering, and $P^* : H^1(M, \mathbb{R}) \longrightarrow H^1(M_0, \mathbb{R})$ the induced mapping. Let us also denote by $L \circ TP : \mathbb{R} \times TM_0 \longrightarrow \mathbb{R}$ the lifted Lagrangian

$$L \circ TP(t, x, v) = L(t, P(x), dP_x(v)),$$

and by $T^*P : T^*M_0 \longrightarrow T^*M$ the covering

$$(x,p) \longmapsto (P(x), p \circ dP_x^{-1}).$$

The lifted Hamiltonian $H \circ T^*P$ is in natural duality with the Lagrangian $L \circ T^*P$. As a consequence, the Hamiltonian flow associated to the Lagrangian $L \circ TP$ is the Hamiltonian flow of $H \circ T^*P$, which is the lifting of the Hamiltonian flow of H. Each overlapping pseudograph $\mathcal{G} = \mathcal{G}_{c,u}$ on M lifts to a pseudograph

$$P^*\mathcal{G} := T^*P^{-1}(\mathcal{G}) = \mathcal{G}_{P^*c, u\circ P}$$

on M_0. Note that $c(P^*\mathcal{G}) = P^*(c(\mathcal{G}))$. As a consequence, it is not hard to see that the Aubry set $\tilde{\mathcal{A}}_{L\circ TP}(P^*(c))$ associated to $L \circ TP$ on M_0 is precisely

$$\tilde{\mathcal{A}}_{L\circ TP}(P^*(c)) = T^*P^{-1}(\tilde{\mathcal{A}}_L(c)),$$

while we only have the inclusion

$$\tilde{\mathcal{N}}_{L\circ TP}(P^*(c)) \supset T^*P^{-1}(\tilde{\mathcal{N}}_L(c)).$$

Finally, if $\mathcal{S}_{L\circ TP}$ is a static class of $\mathcal{A}_{L\circ TP}(P^*(c))$, then $P(\mathcal{S}_{L\circ TP})$ is a static class of $\mathcal{A}_L(c)$. Note however that the lifting $P^{-1}(\mathcal{S}_L)$ of a static class of $\mathcal{A}_L(c)$ can contain several static classes of $\mathcal{A}_{L\circ TP}(P^*(c))$. This is illustrated by the following result which, in conjunction with (4.9), allows to prove the existence of heteroclinic orbits in the case where there is only one static class, see [14] and [8]. We need first another definition. If $\tilde{\mathcal{X}} \subset TM$ is an invariant set of the time-one flow ϕ, then we denote by $s\tilde{\mathcal{X}} \subset M \times \mathbb{T}$ the set $\cup_{t\in\mathbb{R}, x\in\tilde{\mathcal{X}}}(\phi_0^t(x), t)$ and by $s\mathcal{X}$ its projection on $M \times \mathbb{T}$.

Proposition. *Assume that the set $\mathcal{A}(c)$ contains finitely many static classes, and that there exists an open neighborhood $U \subset M \times \mathbb{T}$ of the compact set $s\mathcal{A}(c)$ such that the mapping $h : H_1(U,\mathbb{Z}) \longrightarrow H_1(M,\mathbb{Z})$ is not surjective, where h is the composition of the mappings*

$$H_1(U,\mathbb{Z}) \xrightarrow{i_*} H_1(M\times\mathbb{T},\mathbb{Z}) \xrightarrow{p_*} H_1(M,\mathbb{Z})$$

induced from the inclusion and the projection. Then there exists a finite connected Galois covering $P : M_0 \longrightarrow M$ with k sheets, $k \geqslant 2$, such that, for each static class $\tilde{\mathcal{S}}$ of $\tilde{\mathcal{A}}(c)$, the lifting $T^*P^{-1}(\tilde{\mathcal{S}})$ is the union of exactly k different static classes of $\tilde{\mathcal{A}}_{L\circ TP}(P^*(c))$.

Proof. Let N be the number of static classes in $\mathcal{A}(c)$. First, we claim that for each static class $\mathcal{S}$, the set $s\mathcal{S}$ is connected. This follows easily from (4.6). As a consequence, we can suppose that the neighborhood U is a union of finitely many connected open sets $U_i, 1 \leqslant i \leqslant N$, each of which contains exactly one of the sets $s\mathcal{S}$. Since the group $H^1(M,\mathbb{Z})$ is Abelian and of finite type, and since the mapping $h : H_1(U,\mathbb{Z}) \longrightarrow H_1(M,\mathbb{Z})$ is not surjective, there exists an integer $k \geqslant 2$ and a surjective morphism $g : H^1(M,\mathbb{Z}) \longrightarrow \mathbb{Z}/k\mathbb{Z}$ whose kernel contains the subgroup $h(H_1(U,\mathbb{Z}))$. There is a connected Galois covering $P : M_0 \longrightarrow M$ with k sheets associated to this morphism. This means that, if $\chi : \pi_1(M) \longrightarrow H_1(M,\mathbb{Z})$ is the Hurewitz map, then the image $P_*(\pi_1(M_0))$ in $\pi_1(M)$ is precisely the kernel of $g \circ \chi$. As a consequence, the image $(P \times Id)_*(\pi_1(M_0 \times \mathbb{T}))$ in $\pi_1(M \times \mathbb{T})$ is precisely the kernel of the morphism $g \circ \chi \circ p_* : \pi_1(M \times \mathbb{T}) \longrightarrow \mathbb{Z}/k\mathbb{Z}$. Hence we have the inclusion

$$i_*(\pi_1(U)) \subset (P \times Id)_*(\pi_1(M_0 \times \mathbb{T})).$$

As a consequence, the covering $P \times Id$ is trivial above U. It follows that each connected component U_i of U has k disjoint connected preimages $V_i^j \subset M_0 \times \mathbb{T}$. Now it is not hard to see that the static classes of $\mathcal{A}_{L\circ TP}(P^*(c))$ are precisely the intersections

$$TP^{-1}(\mathcal{A}(c)) \cap V_i^j = \mathcal{A}_{L\circ TP}(P^*(c)) \cap V_i^j, 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant k.$$

□

Abstract mechanisms

This part contains the main new results of the paper.

5 The relation

In this section, we describe the basic tools which are necessary to understand the relation $\lhd\!\rhd$.

(5.1) Let us introduce some useful notations. Given two subsets $\mathcal{G}$ and $\mathcal{G}'$ of T^*M, we define the relation $\mathcal{G} \lhd_N \mathcal{G}'$ as follows:

$$\mathcal{G} \lhd_N \mathcal{G}' \Longleftrightarrow \bar{\mathcal{G}}' \subset \bigcup_{n=0}^{N} \phi_n(\mathcal{G}),$$

where as usual $\bar{\mathcal{G}}$ is the closure of $\mathcal{G}$. We write $\mathcal{G} \lhd \mathcal{G}'$ if there exists an integer N such that $\mathcal{G} \lhd_N \mathcal{G}'$. If $\mathcal{G}$ is a subset of T^*M and if $c \in H^1(M, \mathbb{R})$, the relations

$$\mathcal{G} \lhd c \text{ and } \mathcal{G} \lhd_N c$$

mean that there exists an overlapping pseudograph $\mathcal{G}'$ of cohomology c' and such that $\mathcal{G} \lhd \mathcal{G}'$ (resp. $\mathcal{G} \lhd_N \mathcal{G}'$). To finish for c and c' two cohomology classes, the relation

$$c \lhd_N c'$$

means that, for each pseudograph $\mathcal{G} \in \mathbb{P}_c$, we have $\mathcal{G} \lhd_N c'$. As the reader may have guessed, we will then write $c \lhd c'$ if there exists an integer N such that $c \lhd_N c'$. The relation $\lhd$ (between subsets as well as between cohomology classes) is obviously transitive. We will be concerned in this paper with understanding the relation $\lhd$ between cohomology classes. For this purpose, it is useful to introduce the symmetric relation

$$c \lhd\!\rhd c' \Longleftrightarrow c \lhd c' \text{ and } c' \lhd c.$$

Proposition *The relation $\lhd\!\rhd$ is an equivalence relation on $H^1(M, \mathbb{R})$.*

(5.2) Let us give a first result of a (negative) result about this relation. If the pseudograph $\mathcal{G}$ is the graph of a continuous section of T^*M, then $\mathcal{G} \in \mathbb{V} \cap \breve{\mathbb{V}}$ is in fact an invariant Lipschitz graph, and the relation $c(\mathcal{G}) \lhd c$ holds if and only if $c = c(\mathcal{G})$. Note that, if $C \subset H^1(M, \mathbb{R})$ is bounded, it is possible to chose a uniform constant K such that all the invariant Lipschitz Graphs $\mathcal{G}$ whose cohomology satisfies $c \in C$ are K-Lipschitz. In other words, the elements of $\mathbb{V}_C \cap \breve{\mathbb{V}}$ are equi-Lipschitz graphs. Of course, we would like to be able to prove that the relation $\lhd\!\rhd$ has non-trivial classes. For this purpose, we now introduce several useful tools.

(5.3) Given two integers $N' \geqslant N \geqslant 1$, and a cohomology c, we define the function $A_c^{N,N'}$: $M \times M \longrightarrow \mathbb{R}$ by the expression

$$A_c^{N,N'}(x,y) = \min_{k \in \mathbb{N}, N \leqslant k \leqslant N'} A_c(0,x;k,y) + k\alpha(c).$$

Since each of the mappings

$$H^1(M,\mathbb{R}) \longrightarrow C(M \times M, \mathbb{R})$$
$$c \longmapsto A_c(0,.;k,.)$$

is continuous (see appendix B), it is easy to see that, for fixed $N' \geqslant N$, the mapping

$$H^1(M,\mathbb{R}) \longrightarrow C(M \times M, \mathbb{R})$$
$$c \longmapsto A_c^{N,N'}$$

is continuous.

Proposition *Let c be a fixed cohomology class. For each $\epsilon > 0$, there exist integers $N' \geqslant N \geqslant 1$ such that*

$$\|A_c^{N,N'} - h_c\|_\infty \leqslant \epsilon$$

This proposition is an obvious consequence of the following general Lemma.

Lemma *Let K be a compact metric space, let $h_n : K \longrightarrow \mathbb{R}$ be a sequence of functions which is equi-bounded and equi-continuous. Let us set $h = \liminf h_n$ and, for each $(N,N') \in \mathbb{N}^2$, $h_{N,N'} = \min_{N \leqslant n \leqslant N'} h_n$. For each $\epsilon > 0$, there exists $(N,N') \in \mathbb{N}^2$ such that*

$$\|h_{N,N'} - h\| \leqslant \epsilon.$$

Proof. It is easy to see that all the functions $h_{N,N'}$ satisfy the same bound and the same modulus of continuity as the function h_n. We have $h = \lim_{N \longrightarrow \infty} h_{N,\infty}$, this limit is uniform and the function h also satisfies the bound and the modulus of continuity. Assuming now by contradiction that the conclusion of the Lemma does not hold, we can find an $\epsilon > 0$ and a family $x_{N,N'}$ of points such that

$$|h_{N,N'}(x_{N,N'}) - h(x_{N,N'})| \geqslant \epsilon.$$

Let us fix N and let N' go to infinity in such a way that the sequence $x_{N,N'}$ has a limit $x_{N,\infty}$. We have

$$|h_{N,\infty}(x_{N,\infty}) - h(x_{N,\infty})| \geqslant \epsilon.$$

But this is in contradiction with the fact that the functions $h_{N,\infty}$ converge uniformly to h. □

(5.4) It is useful to generalize the operators $\Phi^N : \mathbb{P} \longrightarrow \mathbb{P}$. Given two interegers $N \leqslant N'$, and an open set $U \subset M$, we define the operator

$$\Phi_U^{N,N'} : \mathbb{P} \longrightarrow \mathbb{P}$$

by the relation $\Phi_U^{N,N'}(\mathcal{G}_{c,u}) = \mathcal{G}_{c,T_{c,U}^{N,N'}u}$ where

$$T_{c,U}^{N,N'}u(x) = \min_{y\in\bar{U}, N\leqslant k\leqslant N'} T_c^k u(y) + k\alpha(c) = \min_{y\in\bar{U}} u(y) + A_c^{N,N'}(y,x).$$

For simplicity we will denote by $\Phi^{N,N'}$ the operator $\Phi_M^{N,N'}$. For each $\mathcal{G} \in \mathbb{P}$, we have

$$\mathcal{G} \triangleleft_{N'} \Phi^{N,N'}(\mathcal{G}).$$

Lemma. *For each integers $1 \leqslant N \leqslant N'$ and each open set $U \subset M$, the operator $\Phi_U^{N,N'} : \mathbb{P} \longrightarrow \mathbb{P}$ is continuous, when the source is endowed with the seminorm $\|.\|_U$ and the image with the norm $\|.\|$, see (2.2).*

Proof. Let $\mathcal{G}_n \in \mathbb{P}$ be a sequence converging to $\mathcal{G}$. It is possible to write the pseudographs $\mathcal{G}_n$ and $\mathcal{G}$ on the form $\mathcal{G}_{c_n,u_n}$ and $\mathcal{G}_{c,u}$ with $c_n \longrightarrow c$ and $u_n \longrightarrow u$ uniformly. We then have

$$\|\Phi_U^{N,N'}(\mathcal{G}_n) - \Phi_U^{N,N'}(\mathcal{G})\| \leqslant |c_n - c| + \|T_{c_n,U}^{N,N'}u_n - T_{c,U}^{N,N'}u\|.$$

So it is enough to prove that $\|T_{c_n,U}^{N,N'}u_n - T_{c,U}^{N,N'}u\| \longrightarrow 0$. Let us write

$$T_{c,U}^{N,N'}u = u(y) + A_c^{N,N'}(y,x)$$

with $y \in \bar{U}$. Then, we have

$$T_{c_n,U}^{N,N'}u_n - T_{c,U}^{N,N'}u \leqslant u_n(y) + A_{c_n}^{N,N'}(y,x) + u(y) - A_c^{N,N'}(y,x)$$

and by symmetry

$$\|T_{c_n,U}^{N,N'}u_n - T_{c,U}^{N,N'}u\| \leqslant \sup_{y\in U}|u_n(y) - u(y)| + \|A_{c_n}^{N,N'} - A_c^{N,N'}\|$$

The conclusion follows from the continuity of the mapping $c \longmapsto A_c^{N,N'}$, see (5.3) □

(5.5) Similarly, we define the operator

$$\Phi_U^\infty : \mathbb{P} \longrightarrow \mathbb{V}$$

by the relation $\Phi_U^\infty(\mathcal{G}_{c,u}) = \mathcal{G}_{c,T_{c,U}^\infty u}$ where

$$T_{c,U}^\infty u(x) = \min_{y\in\bar{U}} u(y) + h_c(y,x).$$

Proposition *Let c be a fixed cohomology class. For each $\epsilon > 0$ there exist integers $N \geqslant N'$ such that, for each pseudograph $\mathcal{G} = \mathcal{G}_{c,u} \in \mathbb{P}_c$ and each open set $U \subset M$, we have*

$$\|\Phi_U^{N,N'}(\mathcal{G}) - \Phi_U^\infty(\mathcal{G})\| \leqslant \epsilon.$$

PROOF. It is not hard to see that, for each continuous function u,

$$\|T^{N,N'}_{c,U}u - T^{\infty}_{c,U}u\| \leqslant \|A^{N,N'}_c - h_c\|.$$

The proposition follows from (5.3). $\square$

(5.6) PROPOSITION *Let $\mathcal{G}_0 = \mathcal{G}_{c_0,u_0} \in \mathbb{P}$ be a pseudograph, and let $\epsilon > 0$ be fixed. Assume that there exists an open sets $U \subset M$ and two compact sets $\mathcal{K} \subset U$ and $\mathcal{K}_1 \subset M$ such that, for each $x \in \mathcal{K}_1$, the minimum in the expression $T^{\infty}_{c_0,U}u_0(x) = \min_{y\in\bar{U}} u_0(y) + h_{c_0}(y,x)$ is never reached outside of $\mathcal{K}$. Then there exists integers $N \leqslant N'$, a positive number δ and an open neighborhood U_1 of $\mathcal{K}_1$ such that, for each pseudograph $\mathcal{G} \in \mathbb{P}$ satisfying $\|\mathcal{G} - \mathcal{G}_0\|_U \leqslant \delta$ we have*

$$\mathcal{G}_{|U} \triangleleft_{N'} \Phi^{N,N'}_U(\mathcal{G})_{|U_1}$$

and

$$\|\Phi^{N,N'}_U(\mathcal{G}) - \Phi^{\infty}_U(\mathcal{G})\| \leqslant \epsilon.$$

PROOF. Let us denote by ∂U the boundary of U. There exists a positive number ϵ and a neighborhood U_1 of $\mathcal{K}_1$ such that, for each $x \in \bar{U}_1$,

$$\min_{y\in\partial U} u_0(y) + h_{c_0}(y,x) \geqslant \min_{y\in U} u_0(y) + h_{c_0}(y,x) + 7\epsilon.$$

In view of (5.3), there exist integers N and N' such that

$$\|A^{N,N'}_{c_0} - h_{c_0}\| \leqslant \epsilon.$$

For fixed N and N', the function $A^{N,N'}_c$ depends continuously on $c \in H^1(M,\mathbb{R})$, see (5.3). As a consequence, if c is sufficiently close to c_0, we have

$$\|A^{N,N'}_{c_0} - A^{N,N'}_c\| \leqslant \epsilon.$$

For these values of N and N', if $u \in C(M,\mathbb{R})$ is such that $\sup_U |u - u_0| \leqslant \epsilon$, we have, for each $x \in M$ and $y \in \bar{U}$,

$$|u_0(y) + h_{c_0}(y,x) - u(y) - A^{N,N'}_c(y,x)| \leqslant 3\epsilon.$$

Hence we have the inequality

$$\min_{y\in\partial U} u(y) + A^{N,N'}_c(y,x) \geqslant \min_{y\in U} u(y) + A^{N,N'}_c(y,x) + \epsilon.$$

As a consequence, if $\|\mathcal{G}_{c,u} - \mathcal{G}_0\|_U$ is sufficiently small, then there exists a compact set $\mathcal{K}_1 \subset U$ such that the minimum in the expression

$$T^{N,N'}_{c,U}u(x) = \min_{y\in\bar{U}} u(y) + A^{N,N'}_c(y,x)$$

is reached in $\mathcal{K}_1$ for all $x \in \bar{U}_1$. Now let us set $v = T_{c,U}^{N,N'} u(x)$ and consider a point

$$(x,p) \in \overline{\mathcal{G}_{c,v|U_1}}.$$

The point (x,p) is the limit of a sequence $(x_n, p_n) \in \mathcal{G}_{c,v|U_1}$. In other words, the points $x_n \in U_1$ are points of differentiability of v, and we have $dv_{x_n} + c_{x_n} = p_n$. Let $y_n \in \mathcal{K}_1$ and $k_n \in \mathbb{N}, N \leqslant k_n \leqslant N'$, satisfy

$$v(x_n) = u(y) + A_c(0, y_n; k_n, x_n) + k_n \alpha(c).$$

By extracting a subsequence, we can suppose that the sequence k_n is a constant k. By arguments similar to those of (2.7), recalling that the function u is semi-concave, we conclude that the function u is differentiable at y_n, and, setting $z_n = c_{y_n} + du_{y_n}$, that $\phi_0^k(y_n, z_n) = (x_n, p_n)$. By extracting another subsequence, we can suppose that the sequence y_n has a limit $y \in \mathcal{K}_1$. We then have

$$v(x) = u(y) + A_c(0, y; k, x) + k\alpha(c),$$

so that the function u is differentiable at x. Since the function u is semi-concave, we then have $du_y = \lim du_{y_n}$, see Appendix (A.7). Passing at the limit in $\phi_0^k(y_n, z_n) = (x_n, p_n)$, we get $\phi_0^k(y, z) = (x, p)$, where $z := dv_y + c_y$. We have proved that

$$\overline{\mathcal{G}_{c,v|U_1}} \subset \bigcup_{k=N}^{N'} \phi_0^k(\mathcal{G}_{c,u|U}).$$

□

(5.7) PROPOSITION *Assume that, for each weak KAM solution $\mathcal{G}_0 \in \mathbb{V}_c$, there exists a positive number $\epsilon > 0$ and an integer N with the following property : For each pseudograph $\mathcal{G} \in \mathbb{P}_c$ such that $\|\mathcal{G} - \mathcal{G}_0\| \leqslant \epsilon$, there exists a pseudograph $\mathcal{G}' \in \mathbb{P}_{c'}$ such that $\mathcal{G} \lhd_N \mathcal{G}'$. Then $c \lhd c'$.*

PROOF. By compactness of $\mathbb{V}_c$, there exists a neighborhood $\mathbb{U}$ of $\mathbb{V}_c$ in $\mathbb{P}_c$ and an integer N such that, for all $\mathcal{G} \in \mathbb{U}$, we have $\mathcal{G} \lhd_N c'$. In view of Proposition (5.5), there exist integers $k \leqslant k'$ such that $\Phi^{k,k'}(\mathcal{G}) \in \mathbb{U}$ for each $\mathcal{G} \in \mathbb{P}_c$. We obtain, for each $\mathcal{G} \in \mathbb{P}_c$, the existence of a $\mathcal{G}' \in \mathbb{P}_{c'}$ such that

$$\mathcal{G} \lhd_{k'} \Phi^{k,k'}(\mathcal{G}) \lhd_N \mathcal{G}'$$

so that $\mathcal{G} \lhd_{k'+N} \mathcal{G}'$.

□

(5.8) It will be useful to consider liftings to finite Galois coverings $P : M_0 \longrightarrow M$, see (4.10).

LEMMA. *Let c and c' be two cohomology classes in $H^1(M, \mathbb{R})$. If $P^*(c) \lhd_N P^*(c')$ for the relation $\lhd$ associated to the Lagrangian $L \circ TP$ on M_0, then $c \lhd_N c'$.*

PROOF. Let us consider a pseudopgraph $\mathcal{G} \subset \mathbb{P}_c$. If $P^*(c) \lhd_N P^*(c')$ then there exists a pseudograph $\mathcal{G}'$ on M_0 of cohomology $P^*(c')$ and such that $P^*\mathcal{G} \lhd_N \mathcal{G}'$. Let $\mathbb{D}$ be the group of deck transformations of the covering P. The elements of $\mathbb{D}$ are the diffeomorphisms D of

M_0 such that $P \circ D = P$. To each element D of $\mathbb{D}$ we associate the fibered diffeomorphism T^*D of T^*M defined by

$$T^*D(x,p) = (D(x), p \circ dD_x^{-1}).$$

This diffeomorphism is a Deck transformation of the covering T^*P. Let us prove that there exists a pseudograph $\mathcal{G}''$ on M_0 which is invariant by deck transformations, which has cohomology $P^*(c')$, and such that $P^*\mathcal{G} \triangleleft \mathcal{G}''$. Let η be a form on M with cohomology c', and let $P^*\eta$ be its lifting to M_0. We write $\mathcal{G}'$ on the form $\mathcal{G}_{P^*\eta,u}$. Since the flow of $H \circ T^*P$ commutes with Deck transformations, and since the pseudograph $P^*\mathcal{G}$ is invariant by deck transformations, we have

$$P^*\mathcal{G} \triangleleft T^*D(\mathcal{G}')$$

for each deck transformation D. It is easy to check that $T^*D(\mathcal{G}') = \mathcal{G}_{P^*\eta, u\circ D^{-1}}$. Setting

$$v := \min_{D \in \mathbb{D}} u \circ D^{-1},$$

and $\mathcal{G}'' = \mathcal{G}_{\eta,v}$, we have the desired properties for $\mathcal{G}''$. Since P is a Galois covering, functions on M_0 which are invariant by deck transformations are liftings of functions on M. As a consequence, there exists a continuous function $w : M \longrightarrow \mathbb{R}$ such that $v = w \circ P$. Hence the pseudograph $\mathcal{G}''$ is the lifting of the pseudograph $\mathcal{G}_{\eta,w}$ on M, which satisfies $\mathcal{G} \triangleleft_N \mathcal{G}_{\eta,w}$, and whose cohomology is c'. Since this construction can be done for all $\mathcal{G}$, we have $c \triangleleft_N c'$ □

6 Consequences on the dynamics.

We prove Proposition (0.9). Let us first slightly generalize the statement.

(6.1) PROPOSITION

(*i*) *If $c \triangleleft c'$, there exists a heteroclinic trajectory of the Hamiltonian flow between $\tilde{\mathcal{A}}(c)$ and $\tilde{\mathcal{A}}(c')$. Other kind of trajectories can be built: For any closed forms η of cohomology c and η' of cohomology c', there exists a positive integer N and a trajectory $(q(t), p(t)) : [0, N] \longrightarrow T^*M$ of the Hamiltonian flow such that $p(0) = \eta(q(0))$ and $p(N) = \eta'(q(N))$.*

(*ii*) *Let $c_i, i \in \mathbb{Z}$, be a sequence of cohomology classes. Assume that $c_i \triangleleft c_{i+1}$ for each i, and fix for each i a neighborhood U_i of $\tilde{\mathcal{M}}(c_i)$ in T^*M. There exists a trajectory of the Hamiltonian flow which visits in turn all the sets U_i. In addition, if the sequence stabilizes to $c-$ on the left, or to $c+$ on the right, the trajectory can be assumed negatively asymptotic to $\mathcal{A}(c-)$ or positively asymptotic to $\mathcal{A}(c+)$. If $\mathcal{G}_i \in \mathbb{P}_{c_i} \cap \breve{\mathbb{P}}_{c_i}$ is a sequence of Lagrangian graphs, then there exists an orbit $P(t)$ and an increasing sequence $t_i : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $P(t_i) \in \mathcal{G}_i$.*

PROOF. Let us first assume that $c \triangleleft c'$. Take a fixed point $\mathcal{G}_c \in \mathbb{V}_c$. There exists a graph $\mathcal{G} \in \mathbb{P}_{c'}$ such that $\mathcal{G}_c \triangleleft \mathcal{G}$. Now, consider a pseudograph $\breve{\mathcal{G}}_{c'} \in \breve{\mathbb{V}}_{c'}$. It follows form Lemma (2.3) that $\mathcal{G}$ intersects $\breve{\mathcal{G}}_{c'}$. The points of intersection are α-asymptotic to $\tilde{\mathcal{A}}(c)$ and ω-asymptotic to $\tilde{\mathcal{A}}(c')$. In the same way, we can take for $\mathcal{G}_c$ the graph of the closed form η, choose $\mathcal{G} \in \mathbb{P}_{c'}$ such that $\mathcal{G}_c \triangleleft \mathcal{G}$, and take for $\breve{\mathcal{G}}_{c'}$ the graph of η'. The points of the intersection $\mathcal{G} \cap \breve{\mathcal{G}}_{c'}$ have trajectories from $\mathcal{G}_c$ to $\breve{\mathcal{G}}_{c'}$. This proves (*i*).

(6.2) Lemma *Let us fix a cohomology c.*

(*i*) *For each neighborhood U of $\tilde{\mathcal{B}}(c)$, there exists $N \in \mathbb{N}$ such that, for all $l \geqslant N$ and all $\mathcal{G} \in \mathbb{P}$, we have*

$$\phi^{-l}\left(\overline{\Phi^{2l}(\mathcal{G})}\right) \subset U.$$

(*ii*) *If V is an open neighborhood of $\tilde{\mathcal{M}}(c)$ in T^*M, there exists $N \in \mathbb{N}$ such that, for each $\mathcal{G} \in \mathbb{P}_c$ and each $P \in \overline{\Phi^N(\mathcal{G})}$, one of the points $\phi^{-i}(P), 1 \leqslant i \leqslant N-1$ belongs to V.*

Proof. In order to prove (i), it is sufficient to prove that, if $\mathcal{G}_n \in \Phi\mathbb{P}_c$ is a sequence of pseudographs, if m_n is an increasing sequence of integers, and if $(x_n(t), p_n(t)) : [0, m_n] \longrightarrow T^*M$ is a Hamiltonian trajectory which satisfies

$$(x_n(m_n), p_n(m_n)) \in \overline{\Phi^{2m_n}(\mathcal{G}_n)}$$

and which converges uniformly on compact sets to a limit $(x(t), p(t)) : \mathbb{R}^+ \longrightarrow T^*M$, then $(x(0), p(0)) \in \tilde{\mathcal{B}}(c)$.

Let us write the pseudographs $\Phi^{m_n}(\mathcal{G}_n)$ on the form $\mathcal{G}_{c,u_n}$. For each $k, n \in \mathbb{N}$, we have

$$T_c^k u_n(x_n(k)) = u_n(x_n(0)) + \int_0^k L(t, x_n(t), \dot{x}_n(t)) + c_{x_n(t)}(\dot{x}_n(t))dt$$

Since the functions u_n lie in the image of the operator $T_c^{m_n}$, they are equi-Lipschitz, and there exists a real sequence λ_n such that the sequence of functions $\lambda_n + T_c^{m_n} u_n$ has an accumulation points in $C(M, \mathbb{R})$. As a consequence, we can assume, taking a subsequence if necessary, that the functions $\lambda_n + T_c^{m_n} u_n$ converge uniformly to a limit u. We have $\mathcal{G}_{c,u} = \lim \Phi^{m_n}(\mathcal{G}_n) \in \mathbb{O}_c$. For each fixed $k \in \mathbb{Z}$, taking the limit as $n \longrightarrow \infty$, we get

$$T_c^k u(x(k)) = u(x(-k)) + \int_0^k L(t, x_n(t), \dot{x}_n(t)) + c_{x_n(t)}(\dot{x}_n(t))dt.$$

Hence we have $P(0) \in \phi^{-k}(\Phi^k(\overline{\mathcal{G}_{c,u}}))$, and, since this holds for all $k \in \mathbb{N}$, we conclude that

$$P(0) \in \tilde{\mathcal{I}}(\mathcal{G}_{c,u}) \subset \tilde{\mathcal{B}}(c).$$

In order to prove (ii), it is useful to recall that $\tilde{\mathcal{B}}(c)$ is a compact set, invariant under the time-one flow ϕ, and that the Mather set $\tilde{\mathcal{M}}(c)$ is the closure of the union of the supports of the invariant measures of the action of ϕ on $\tilde{\mathcal{B}}(c)$. The claim below follows from general facts about topological dynamics on compact spaces: For each neighborhood W of $\tilde{\mathcal{M}}(c)$ in $\tilde{\mathcal{B}}(c)$, there exists an integer k such that, for each point P of $\tilde{\mathcal{B}}(c)$, one of the points $\phi^i(P), 1 \leqslant i \leqslant k$, belongs to W. As a consequence, if V is a neighborhood of $\tilde{\mathcal{M}}(c)$ in T^*M, there exists a neighborhood U of $\tilde{\mathcal{B}}(c)$ in T^*M such that, for each $P \in U$, one of the points $\phi^i(P), 1 \leqslant i \leqslant k$, belongs to V. Now let us take $l \geqslant k$ such that (i) holds for this neighborhood U, and set $N = 2l$. For each $\mathcal{G} \in \mathbb{P}_c$ and each $P \in \overline{\Phi^N}(\mathcal{G})$, we have $\phi^{-l}(P) \in U$. Hence one of the points $\phi^{l-i}(P), 1 \leqslant i \leqslant k$ is in V, which proves (ii). □

(6.3) Let us now prove (ii). Let $M_i \in \mathbb{N}, i \in \mathbb{Z}$ be a sequence of integers such that $c_i \triangleleft_{M_i} c_{i+1}$, and let $W_i \subset V_i$ be closed neighborhoods of $\tilde{\mathcal{M}}(c_i)$. In view of lemma (6.2), there exists a sequence N_i of integers such that, for each $\mathcal{G} \in \mathbb{P}_{c_i}$ and each

$$P \in \phi^{-N_i}(\overline{\Phi^{N_i}(\mathcal{G})}),$$

one of the points $\phi^l(P)$, $0 \leqslant l < N_i$ belongs to W_i. We denote by $\mathbb{K}_i$ the compact set $\overline{\Phi(\mathbb{P}_{c_i})}$.

Let us first fix an integer $k \in \mathbb{Z}$, and choose a pseudograph $\mathcal{G}^k_{-k} \in \mathbb{K}_{-k}$. Since $c_{-k} \triangleleft_{M_{-k}} c_{1-k}$, there exists a pseudograph $G^k_{1-k} \in \mathbb{P}_{c_{1-k}}$ such that $\Phi^{N_k}(\mathcal{G}^k_{-k}) \triangleleft_{M_{-k}} G^k_{1-k}$. We set $\mathcal{G}^k_{1-k} := \Phi(G^k_{1-k})$, and build, by induction, a sequence $\mathcal{G}^k_i \in \mathbb{K}_i, i \geqslant k$ of pseudographs such that

$$\mathcal{G}^k_i \triangleleft_{(M_i+1)} \mathcal{G}^k_{i+1}$$

for each $i \geqslant k$.

Let us now take a point $P^k_k \in \mathcal{G}^k_k$. There exists a positive integer $l^k_{k-1} \leqslant 1+M_{k-1}$ such that $\phi^{-l^k_k}(P^k_k) \in \Phi^{N_{k-1}}(\mathcal{G}^k_{k-1})$. We then set $P^k_{k-1} = \phi^{-(l^k_{k-1}+N_{k-1})}(P^k_k)$. We can build a sequence $P^k_i, -k \leqslant i \leqslant k$ of points of $\mathcal{G}^k_i$ and a sequence $l^k_i, -k \leqslant i \leqslant k-1$ of integers satisfying $0 \leqslant l^k_i \leqslant M_i + 1$ and such that

$$\phi^{N_i+l^k_i}(P^k_i) = P^k_{i+1}$$

for each i. In addition, one of the points $\phi^j(P^k_i), 0 \leqslant j < N_i$ belongs to W_i.

There exists an increasing sequence k_n of integers such that each of the sequences $n \longmapsto l^{k_n}_i$, for fixed i, is the constant l_i after a certain rank, and each of the sequences $n \longmapsto P^{k_n}_i$, for fixed i, is converging to P_i. Clearly, we have $\phi^{l_i+N_i}(P_i) = P_{i+1}$ for each $i \in \mathbb{Z}$, and one of the points $\phi^j(P_i), 0 \leqslant j < N_i$ belongs to W_i. This proves the main part of the statement.

If the sequence c_i stabilizes to $c-$ on the right, then it is possible to build a sequence $\mathcal{G}_i \in \mathbb{P}_{c_i}$ as above which stabilizes to $\mathcal{G}- \in \mathbb{V}_{c-}$ on the right, and we obtain by the above method an orbit which is α-asymptotic to $\tilde{\mathcal{A}}(c-)$ and then visits in turn all the sets W_i. If the sequence c_i stabilizes to c^+ on the right, say for $i \geqslant I$, then it is possible to impose that $P_I \in \breve{\mathcal{G}}+ \in \breve{\mathbb{V}}_{c+}$ in the construction above, and we then obtain an orbit which is ω-asymptotic to $\tilde{\mathcal{A}}(c+)$. □

7 Mather mechanism

We comment and prove Theorem (0.10) in this section. Let us first give a useful property of the subspace $R(c)$.

(7.1) Lemma. *The space $R(c) \subset H^1(M, \mathbb{R})$ depends semi-continuously of c in the following sense: For each $c_0 \in H^1(M, \mathbb{R})$, there exists a neighborhood V of c_0 in $H^1(M, \mathbb{R})$ such that, for each $c \in V$, we have $R(c_0) \subset R(c)$.*

Proof. Let $c_n \longrightarrow c_0$ be a sequence of cohomology classes, and let $\mathcal{G}_n \in \mathbb{V}_{c_n}$ be a sequence of pseudographs. In view of (3.7), we can assume, taking a subsequence, that the sequence $\mathcal{G}_n$ is converging to a limit $\mathcal{G} \in \mathbb{V}_{c_0}$. Now let $\mathcal{U} \subset M$ be an open neighborhood of $\mathcal{I}(\mathcal{G})$ which is such that each cohomology class of $R(\mathcal{G})$ can be represented by a closed form which vanishes on $\mathcal{U}$. Now for n large enough, we have $\mathcal{I}(\mathcal{G}_n) \subset \mathcal{U}$, hence each cohomology class of $R(\mathcal{G})$ can be represented by a class whose support is disjoint from $\mathcal{I}(\mathcal{G}_n)$, so that $R(\mathcal{G}) \subset R(\mathcal{G}_n)$. □

The following proposition is the main step in the proof of Theorem (0.10). We denote by $B_E(r)$ the open ball of radius r centered at the origin in the normed vector space E.

(7.2) PROPOSITION. *For each $\mathcal{G}_0 \in \mathbb{V}_{c_0}$, there exists a positive number ϵ_0, and an integer N such that the following holds: For each pseudograph $\mathcal{G} \in \mathbb{P}$ satisfying $\|\mathcal{G} - \mathcal{G}_0\| < \epsilon_0$ and $c(\mathcal{G}) - c_0 \in R(c_0)$, for each cohomology class c satisfying $c - c_0 \in B_{R(c_0)}(\epsilon_0) \subset R(c_0)$, there exists a pseudograph $\mathcal{G}_1$ such that*

$$\mathcal{G} \lhd_N \mathcal{G}_1.$$

(7.3) PROOF OF THEOREM (0.10). We assume the proposition. For Each $\mathcal{G}_0 \in \mathbb{V}_{c_0}$, we consider the number ϵ_0 given by the proposition, and the open ball $B_{\mathbb{P}}(\mathcal{G}_0, \epsilon_0)$ of center $\mathcal{G}_0$ and radius ϵ_0 in $\mathbb{P}$. Since $\mathbb{V}_{c_0}$ is compact, it can be covered by a finite number of these balls, we denote $\mathcal{G}_i$ and $\epsilon_i, 1 \leqslant i \leqslant N$ the associated centers and radii. Since the function c restricted to $\mathbb{V}$ is proper, (3.7), there exists a positive number δ such that $\mathbb{V}_c \subset U_i B_{\mathbb{P}}(\mathcal{G}_i, \epsilon_i)$ when $|c - c_0| \leqslant \delta$. Consider two cohomology classes c and c' in $c_0 + B_{R(c_0)}(\epsilon)$, with $\epsilon = \min\{\delta, \epsilon_i\}$. It follows from (5.7) that the relation $c \lhd c'$ holds. The theorem clearly follows. □ THEOREM □

(7.4) PROOF OF THE PROPOSITION. Let us fix a $\mathcal{G}_0 \in \mathbb{V}_{c_0}$ and choose a neighborhood U of $\mathcal{I}(\mathcal{G}_0)$ in such a way that $R(\mathcal{G}_0)$ is the set of cohomology classes of smooth closed one-forms vanishing on U.

LEMMA *There exist $\delta > 0$ and $N \in \mathbb{N}$ such that, for all overlapping pseudographs $\mathcal{G}$ satisfying $\|\mathcal{G} - \mathcal{G}_0\| \leqslant \delta$, we have*

$$\mathcal{G}_{|U} \lhd_N \Phi_U^{N,N}(\mathcal{G})$$

PROOF. Let us write the pseudograph $\mathcal{G}_0$ on the form $\mathcal{G}_{c_0,u_0}$. We have seen in (3.9) that

$$u_0(x) = \min_{y \in M} u_0(y) + h_{c_0}(y, x) = \min_{y \in \mathcal{A}(c_0)} u_0(y) + h_{c_0}(x, y).$$

As a consequence, we have $T_U^\infty u_0 = T_M^\infty u_0 = u_0$, and the minimum in the definition of $T_U^\infty u_0(x)$ is not reached outside of $\mathcal{I}(\mathcal{G})$, which is a compact set contained in U. The lemma now follows from proposition (5.6). □

(7.5) LEMMA. *Let us fix a $\delta > 0$. There exists $\epsilon_0 > 0$ such that, if we take :*
One one hand a cohomology class c satisfying $c - c_0 \in R(\mathcal{G}_0)$ and $\|c - c_0\| \leqslant \epsilon_0$;
On the other hand a pseudograph $\mathcal{G} \in \mathbb{P}$ satisfying $\|\mathcal{G} - \mathcal{G}_0\| \leqslant \epsilon_0$ and $c(\mathcal{G}) \in c_0 + R(\mathcal{G}_0)$;
Then there exists a pseudograph $\mathcal{G}' \in \mathbb{P}_c$ with the following properties: $\|\mathcal{G}' - \mathcal{G}_0\| \leqslant \delta$ and $\mathcal{G}_{|U} = \mathcal{G}'_{|U}$.

PROOF. Let us write $\mathcal{G}_0 = \mathcal{G}_{\eta_0,u_0}$. In view of the definition of $R(\mathcal{G}_0)$, it is possible to associate to each cohomology class $d \in R(\mathcal{G}_0)$ a closed one-form μ_d which is null on U. In addition, we can impose that the correspondence $d \longmapsto \mu_d$ is linear. Given a pseudograph $\mathcal{G} \in \mathbb{P}$ and a cohomology c satisfying the hypotheses of the Lemma, we consider the pseudograph

$$\mathcal{G}' = \mathcal{G} + \mathcal{G}_{\mu_{(c-c(\mathcal{G}))},0} \in \mathbb{P}_c.$$

It is clear that $\mathcal{G}'_{|U} = \mathcal{G}_{|U}$, and that $\|\mathcal{G}' - \mathcal{G}_0\| \leqslant \delta$ if ϵ_0 is small enough. □

(7.6) We are now in a position to end the proof of the proposition. Let us consider δ given by Lemma (7.4), and the associated ϵ_0 as given by Lemma (7.5). If $\mathcal{G}$ and c satisfy the hypotheses of the Proposition with this value of ϵ_0, then, by Lemma (7.5), there exists a pseudograph $\mathcal{G}'$ such that $c(\mathcal{G}') = c$ and $\mathcal{G}'_{|U} = \mathcal{G}_{|U}$ and $\|\mathcal{G}' - \mathcal{G}_0\| \leqslant \delta$. In view of Lemma (7.4), we have $\mathcal{G}_{|U} \lhd_N \Phi_U^{N,N}(\mathcal{G}')$, so that $\mathcal{G} \lhd_N \Phi_U^{N,N}(\mathcal{G}')$. □ PROPOSITION □

8 Systems with finitely many static classes

We prove and generalize Theorem (0.11).

(8.1) Let $\tilde{\mathcal{H}}_c(\tilde{\mathcal{S}}, \tilde{\mathcal{S}}')$ be the set of orbits of the Mañe set $\tilde{\mathcal{N}}(c)$ which are heteroclinic orbits between the static classes $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{S}}'$, we denote by $\mathcal{H}_c(\mathcal{S}, \mathcal{S}')$ its projection on M. We have, from section 4,

$$\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c) \cup \bigcup_{\mathcal{S},\mathcal{S}'} \tilde{\mathcal{H}}_c(\mathcal{S}, \mathcal{S}'),$$

where the union is taken on all pairs $(\mathcal{S}, \mathcal{S}')$ of different static classes. In addition, it is useful to recall that

$$\tilde{\mathcal{H}}_c(\tilde{\mathcal{S}}, \tilde{\mathcal{S}}') \subset E_{c,\mathcal{S}} \tilde{\wedge} \breve{E}_{c,\mathcal{S}'}.$$

We say that the set $\tilde{\mathcal{H}}_c(\mathcal{S}, \mathcal{S}')$ is *neat* if it admits a compact subset $\tilde{\mathcal{K}}$ which contains one and only one point in each orbit of $\phi_{|\tilde{\mathcal{H}}_c(\mathcal{S},\mathcal{S}')}$ and whose projection $\mathcal{K}$ on M is acyclic. This means that $\mathcal{K}$ has a neighborhood U whose inclusion i into M induces the null map $i_* : H_1(U, \mathbb{R}) \longrightarrow H_1(M, \mathbb{R})$.

(8.2) THEOREM *Let c_0 be a cohomology class such that the number of static classes in $\mathcal{A}(c_0)$ is finite and greater than one. Assume in addition that all the sets $\tilde{\mathcal{H}}_{c_0}(\mathcal{S}, \mathcal{S}')$ are neat. Then the class c_0 is in the interior of its class of $\lhd\!\rhd$-equivalence.*

Let us gather some preliminary consequences of the hypotheses.

(8.3) LEMMA *We assume the hypotheses of the theorem. Let $\mathcal{S}_0$ be a static class and V_0 be a neighborhood of $\mathcal{S}_0$.*

- *(i) There exists an open neighborhood V of $\mathcal{S}_0$, contained in V_0, such that the boundary of V does not intersect $\mathcal{I}(E_{c_0,\mathcal{S}_0})$.*
- *(ii) There exists an acyclic open set $U \subset V_0 - \mathcal{S}_0$ and a static class $\mathcal{S}_1$ such that the intersection $U \cap \mathcal{I}(E_{c_0,\mathcal{S}_0})$ is not empty, compact, and contained in $\mathcal{H}(\mathcal{S}_0, \mathcal{S}_1)$.*

PROOF. Let V_0 be a neighborhood of $\mathcal{S}_0$, sufficiently small for lemma (4.4) to apply, so that we have

$$V_0 \cap \mathcal{I}(E_{c_0,\mathcal{S}_0}) = \mathcal{S}_0 \cup \bigcup_{\mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0} \big(\mathcal{H}(\mathcal{S}_0, \mathcal{S}) \cap V_0\big),$$

where the union is taken on all static classes $\mathcal{S} \neq \mathcal{S}_0$. We shall also assume that $\bar{V}_0 \cap \mathcal{A}(c_0) = \mathcal{S}_0$.

For each static class $\mathcal{S}$, let us consider an acyclic compact set $\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})$ which contains one and only one point in every orbit of $\tilde{\mathcal{H}}(\mathcal{S}_0, \mathcal{S})$, and denote by $\mathcal{K}(\mathcal{S}_0, \mathcal{S})$ its projection on the base. Clearly, the sets $\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S}), \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$, are pairwise disjoint and they all belong to the Lipschitz graph $\tilde{\mathcal{I}}(E_{c_0,\mathcal{S}_0})$, so that their projections $\mathcal{K}(\mathcal{S}_0, \mathcal{S})$ on the base are also pairwise disjoint. Let us consider a static class $\mathcal{S} \neq \mathcal{S}_0$. For n large enough, we have $\pi \circ \phi^{-n}(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})) \subset V_0$. In addition, since $\mathcal{K}(\mathcal{S}_0, \mathcal{S})$ is acyclic in M, the compact $\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})$ is acyclic in T^*M. As a consequence, the compact set $\phi^{-n}(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S}))$ is acyclic in T^*M, so that $\pi \circ \phi^{-n}(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S}))$ is acyclic in M. Consequently, there is no loss of generality in supposing that the sets $\mathcal{K}(\mathcal{S}_0, \mathcal{S}), \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$, are all contained in V_0.

Let us prove that each of the sets $\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})$ is isolated in $\tilde{\mathcal{I}}(E_{c_0,\mathcal{S}_0})$. Let F be a compact neighborhood of $\mathcal{S}_0$ which does not intersect any of the sets $\mathcal{K}(\mathcal{S}_0, \mathcal{S}), \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$. Since the points of $\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})$ are α-asymptotic to $\tilde{\mathcal{S}}_0$ and ω asymptotic to $\tilde{\mathcal{S}}$, there exists an integer N such that all the sets $\pi \circ \phi^n(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})), n \in \mathbb{N}, \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$, are contained in F for $n \leqslant -N$, and do not intersect $\bar{V}_0$ for $n \geqslant N$. The set $(V_0 - F) \cap \mathcal{I}(E_{c_0,\mathcal{S}_0})$ is thus covered by finitely many pairwise disjoint compact sets of the form $\pi \circ \phi^n(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})), n \in \mathbb{N}, \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$. As a consequence, each of the sets $\mathcal{K}(\mathcal{S}_0, \mathcal{S})$ is isolated in $(V_0 - F) \cap \mathcal{I}(E_{c_0,\mathcal{S}_0})$, and then also in $\mathcal{I}(E_{c_0,\mathcal{S}_0})$. Let us fix a static class $\mathcal{S}_1 \neq \mathcal{S}_0$ such that $\mathcal{K}(\mathcal{S}_0, \mathcal{S}_1)$ is not empty. Such a static class exists by (4.9). Then, we can find an open neighborhood $U \in V_0$ of $\mathcal{K}(\mathcal{S}_0, \mathcal{S}_1)$ such that U is acyclic and such that $U \cap \mathcal{I}(E_{c_0,\mathcal{S}_0}) = \mathcal{K}(\mathcal{S}_0, \mathcal{S}_1)$ is a non-empty compact set contained in $\mathcal{H}(\mathcal{S}_0, \mathcal{S}_1)$. We have proved (ii).

Let us consider again the finite family of pairwise disjoint compact sets $\pi \circ \phi^n(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S})), n \in \mathbb{N}, |n| \leqslant N, \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$. There exists a finite family of pairwise disjoint compact sets $\mathcal{K}'_n(\mathcal{S}_0, \mathcal{S}), n \in \mathbb{N}, |n| \leqslant N, \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0$ such that $\mathcal{K}'_n(\mathcal{S}_0, \mathcal{S})$ is a neighborhood of $\pi \circ \phi^n(\tilde{\mathcal{K}}(\mathcal{S}_0, \mathcal{S}))$. We can clearly assume in addition that the sets $\mathcal{K}'_n(\mathcal{S}_0, \mathcal{S})$ do not intersect $\mathcal{S}_0$. The set

$$V = V_0 - \bigcup_{n \in \mathbb{N}, |n| \leqslant N, \mathcal{S} \in \mathbb{E}(c_0) - \mathcal{S}_0} \mathcal{K}'_n(\mathcal{S}_0, \mathcal{S})$$

is an open neighborhood of $\mathcal{S}_0$ which is contained in V_0, and its boundary does not intersect $\mathcal{I}(E_{c_0,\mathcal{S}_0})$. We have proved (i). □

The following proposition is the main step in the proof of the theorem.

(8.4) PROPOSITION *Let c_0 satisfy the hypotheses of Theorem (8.2). For each weak KAM solution $\mathcal{G}_0 \in \mathbb{V}_{c_0}$, there exists a number $\epsilon > 0$ and an integer N such that, if $\mathcal{G} \in \mathbb{P}$ and $c \in H^1(M, \mathbb{R})$ satisfy*

$$\|\mathcal{G} - \mathcal{G}_0\| \leqslant \epsilon \text{ and } |c - c_0| \leqslant \epsilon$$

then $\mathcal{G} \lhd_N c$.

(8.5) Proof of the Theorem. We assume the Proposition. Let us cover the compact set $\mathbb{V}_{c_0}$ by a finite number of balls $B(\mathcal{G}_i, \epsilon_i)$, where ϵ_i is given by the Proposition applied to $\mathcal{G}_i$. Since the function c restricted to $\mathbb{V}$ is proper, the union of these finite balls covers the sets $\mathbb{V}_c$ for c sufficiently close to c_0. The Theorem holds by Proposition (5.7). □ THEOREM □

We now prove the Proposition in three steps.

(8.6) Step 1. *Let $\mathcal{G} \in \mathbb{V}_{c_0}$ be a fixed point. If there exist only finitely many static classes in $\mathcal{A}(c_0)$, then there exists an elementary solution E_0 and a neighborhood U_0 of the corresponding static class $\mathcal{S}_0$ such that $\mathcal{G}_{|U_0} = E_{0|U_0}$.*
Proof. Let us fix the solution $\mathcal{G} = \mathcal{G}_{c_0,u}$. We define a partial order on the set of static classes by saying that $\mathcal{S} \leqslant \mathcal{S}'$ if, for each $x \in \mathcal{S}$ and $x' \in \mathcal{S}'$, we have $h_{c_0}(x, x') = u(x') - u(x)$. It is easy to check that this relation satisfies the following three axioms of order relations:

- $\mathcal{S} \leqslant \mathcal{S}$,
- $\mathcal{S} \leqslant \mathcal{S}'$ and $\mathcal{S}' \leqslant \mathcal{S}'' \Longrightarrow \mathcal{S} \leqslant \mathcal{S}''$,
- $\mathcal{S} \leqslant \mathcal{S}'$ and $\mathcal{S}' \leqslant \mathcal{S} \Longrightarrow \mathcal{S} = \mathcal{S}'$.

As a consequence, there exists an initial element $\mathcal{S}_0$, that is an element which is not greater than any other element. Let us write

$$u(x) = \min_{y \in M} u(y) + h_{c_0}(y, x),$$

and consider, for each point x, the set $\mathcal{G}_{c,u} \wedge \breve{E}_{c,x}$ of points y where the minimum is reached. For each x this set contains a static class. It is equivalent to say that the class $\mathcal{S}_0$ is initial for the order above, and to say that

$$\mathcal{A}(c_0) \cap \mathcal{G}_{c,u} \wedge \breve{E}_{c,x} = \mathcal{S}_0$$

In other words, for $x \in \mathcal{S}_0$, the compact set $\mathcal{G}_{c,u} \wedge \breve{E}_{c,x}$ does not intersect other static classes than $\mathcal{S}_0$. Since the set $\mathcal{G}_{c,u} \wedge \breve{E}_{c,x}$ has to contain a static class for all x, it contains $\mathcal{S}_0$ when x is close to $\mathcal{S}_0$. As a consequence, we have, if x is sufficiently close to $\mathcal{S}_0$,

$$u(x) = u(y) + h_{c_0}(y, x),$$

for each $y \in \mathcal{S}_0$. In other words, the difference $x \longmapsto h_{c_0}(y, x) - u(x)$ is the constant $u(y)$ in a neighborhood of $\mathcal{S}_0$. □ STEP 1 □

(8.7) step 2. *Let $\mathcal{S}_0$ be a static class of $\mathcal{A}(c_0)$ and let U_0 be a neighborhood of $\mathcal{S}_0$ satisfying (ii) of (8.3). There exists a static class $\mathcal{S}_1$, an open neighborhood U_1 of $\mathcal{S}_1$, an integer N and, for each $\delta > 0$, a number $\epsilon > 0$ with the following property : If $\mathcal{G} \in \mathbb{P}$ satisfies $\|\mathcal{G} - E_{c_0,\mathcal{S}_0}\|_{U_0} \leqslant \epsilon$ and $c \in H^1(M, \mathbb{R})$ satisfies $|c - c_0| \leqslant \epsilon$, then there exists a pseudograph $\mathcal{G}' \in \mathbb{P}_c$ such that $\|\mathcal{G}' - E_{c_0,\mathcal{S}_1}\|_{U_1} \leqslant \delta$ and*

$$\mathcal{G}_{|U_0} \lhd_N \mathcal{G}'_{|U_1}$$

PROOF. There exists a static class $\mathcal{S}_1$ and an acyclic open set $U \subset U_0 - \mathcal{A}(c_0)$ such that

$$\mathcal{I}(E_{c_0,\mathcal{S}_0}) \cap \bar{U} = \mathcal{I}(E_{c_0,\mathcal{S}_0}) \cap U$$

is a compact set $\mathcal{K} \subset \mathcal{H}(\mathcal{S}, \mathcal{S}_1)$. Let us fix a point $x_0 \in \mathcal{S}_0$, and denote by u_0 the function $h_{c_0}(x_0, .)$.

(8.8) LEMMA. *There exists a neighborhood U_1 of $\mathcal{S}_1$ such that the equality*

$$T^{\infty}_{c_0,U} u_0(y) = h_{c_0}(x_0, x_1) + h_{c_0}(x_1, y) = h_{c_0}(x_0, x) + h_{c_0}(x, x_1) + h_{c_0}(x_1, y)$$

holds for all $x \in \mathcal{K}$, $y \in U_1$, and $x_1 \in \mathcal{S}_1$. As a consequence, we have

$$\Phi^{\infty}_{U}\left(E_{c_0,\mathcal{S}_0}\right)_{|U_1} = E_{c_0,\mathcal{S}_1|U_1},$$

and the minimum in the definition of $T^{\infty}_{c_0,U} u_0(y)$ is not reached outside of $\mathcal{K}$ when $y \in U_1$.

PROOF. Let us set $v = T^{\infty}_{c_0,U} u_0$ for simplicity. Recall, from (3.9), that all weak KAM solutions $v \in \mathbb{V}_{c_0}$ satisfy $v(y) = \min_{a \in \mathcal{A}(c_0)} v(a) + h_{c_0}(a, y)$. Here, we obtain

$$v(y) = \min_{x \in \bar{U}, a \in \mathcal{A}(c_0)} h_{c_0}(x_0, x) + h_{c_0}(x, a) + h_{c_0}(a, y). \qquad (\diamond)$$

We claim that, for $y \in \mathcal{S}_1$, the set of minimizing pairs (x, a) is $\mathcal{K} \times \mathcal{S}_1$. Indeed, if $(x, a) \in \mathcal{K} \times \mathcal{S}_1$, then $x \in E_{c_0,x_0} \wedge \breve{E}_{c_0,a}$, so that $h_{c_0}(x_0, x) + h_{c_0}(x, a) = h_{c_0}(x_0, a)$, and

$$h_{c_0}(x_0, x) + h_{c_0}(x, a) + h_{c_0}(a, y) = h_{c_0}(x_0, y) = \min_{(z,z') \in M \times M} h_{c_0}(x_0, z) + h_{c_0}(z, z') + h_{c_0}(z', y).$$

Hence we have

$$h_{c_0}(x_0, x) + h_{c_0}(x, a) + h_{c_0}(a, y) = \min_{(z,z') \in \bar{U} \times \mathcal{A}(c)} h_{c_0}(x_0, z) + h_{c_0}(z, z') + h_{c_0}(z', y).$$

We have proved that the pairs of $\mathcal{K} \times \mathcal{S}_1$ are minimizing in the equation $(\diamond)$ for $y \in \mathcal{S}_1$. Let us now prove that they are the only minimizing pairs. A pair (x, a) is minimizing if and only if $h_{c_0}(x_0, a) + h_{c_0}(a, y) = h_{c_0}(x_0, y)$ and $h_{c_0}(x_0, x) + h_{c_0}(x, a) = h_{c_0}(x_0, a)$. The second equality implies

$$x \in E_{c_0,\mathcal{S}_0} \wedge \breve{E}_{c_0,\mathcal{S}(a)} \subset \mathcal{I}(E_{c_0,\mathcal{S}_0}).$$

Since $\mathcal{I}(E_{c_0,\mathcal{S}_0}) \cap \bar{U} = \mathcal{K}$, this implies $x \in \mathcal{K}$.

If $x \in \mathcal{K}$ and $a \in \mathcal{A}(c_0)$, then the equality $h_{c_0}(x, a) = h_{c_0}(x, y) + h_{c_0}(y, a)$ holds for all $y \in \mathcal{S}_1$. Indeed, let x_n be the projection of the orbit of $\tilde{\mathcal{N}}(c_0)$ such that $x_0 = x$. We have, for each $n \in \mathbb{N}$, the equality of calibration by $-h_{c_0}(., a)$:

$$A_{c_0}(0, x, n, x_n) + n\alpha(c_0) = h_{c_0}(x, a) - h_{c_0}(x_n, a).$$

Let n_k be an increasing sequence of integers such that the subsequence x_{n_k} has a limit $\omega \in \mathcal{S}_1$. Taking the liminf as $k \longrightarrow \infty$, we get $h_{c_0}(x, \omega) \leqslant h_{c_0}(x, a) - h_{c_0}(\omega, a)$, which implies the desired equality for ω, and then for all points of $\mathcal{S}_1$.

Since (x,a) is a minimizing pair for $v(y)$, we get, by decomposing $h_{c_0}(x,a)$ in the expression of v,

$$v(y) = h_{c_0}(x_0,x) + h_{c_0}(x,y) + h_{c_0}(y,a) + h_{c_0}(a,y)$$

and, since $v(y) \leqslant h_{c_0}(x_0,x) + h_{c_0}(x,y)$, we finally obtain that $h_{c_0}(x_1,a) + h_{c_0}(a,x_1) \leqslant 0$ so that $a \in \mathcal{S}_1$. We have proved the claim. In addition, we have proved, for $x_1 \in \mathcal{S}_1$ and $x \in \mathcal{K}$, the equality

$$v(x_1) = h_{c_0}(x_0,x) + h_{c_0}(x,x_1) = h_{c_0}(x_0,x_1).$$

As a consequence, for $y \in \mathcal{S}_1$, each point $a \in \mathcal{A}(c)$ which is minimizing in the equation

$$v(y) = \min_{a \in \mathcal{A}(c_0)} v(a) + h_{c_0}(a,y)$$

belong to $\mathcal{S}_1$. Since $\mathcal{S}_1$ is isolated in $\mathcal{A}(c)$, the conclusion holds also for y sufficiently close to $\mathcal{S}_1$. We then have the equality

$$v(y) = v(x_1) + h_{c_0}(x_1,y) = h_{c_0}(x_0,x) + h_{c_0}(x,x_1) + h_{c_0}(x_1,y)$$

for all $x_1 \in \mathcal{S}_1$ and $x \in \mathcal{K}$ (and no other x in $\bar{U}$). □

(8.9) Applying (5.6), we get the existence of a positive ϵ' and of integers $N \leqslant N'$ such that each $\mathcal{G} \in \mathbb{P}$ which satisfies $\|\mathcal{G} - E_{c_0,\mathcal{S}_0}\|_U \leqslant \epsilon'$ also satisfies

$$\mathcal{G}_{|U} \triangleleft_{N'} \Phi_U^{N,N'}(\mathcal{G})_{|U_1}$$

and

$$\|\Phi_U^{N,N'}(\mathcal{G}) - \Phi_U^{\infty}(\mathcal{G})\| \leqslant \delta/2.$$

Note as a consequence, since $\Phi_U^{\infty}\big(E_{c_0,\mathcal{S}_0}\big)_{|U_1} = E_{c_0,\mathcal{S}_1|U_1}$ in view of the lemma, that

$$\|\Phi_U^{N,N'}(E_{c_0,\mathcal{S}_0}) - E_{c_0,\mathcal{S}_1}\|_{U_1} \leqslant \delta/2.$$

By continuity of $\Phi_U^{N,N'}$, we can assume in addition that ϵ' is sufficiently small for the following inequality to hold when $\|\mathcal{G} - E_{c_0,\mathcal{S}_0}\|_U \leqslant \epsilon'$:

$$\|\Phi_U^{N,N'}(\mathcal{G}) - \Phi_U^{N,N'}(E_{c_0,\mathcal{S}_0})\| \leqslant \delta/2.$$

As a consequence, we also have

$$\|\Phi_U^{N,N'}(\mathcal{G}) - E_{c_0,\mathcal{S}_1}\|_{U_1} \leqslant \delta.$$

Since U is acyclic, for each cohomology c and each pseudograph $\mathcal{G}$, there exists a pseudograph $\mathcal{G}(c)$ which has cohomology c and such that $\mathcal{G}_{|U} = \mathcal{G}(c)_{|U}$. There exists a positive ϵ such that, if $|c - c_0| \leqslant \epsilon$ and if $\|\mathcal{G} - E_{c_0,\mathcal{S}_0}\|_U \leqslant \epsilon$, then we have

$$\|\mathcal{G}(c) - E_{c_0,\mathcal{S}_0}\|_U \leqslant \epsilon'.$$

Note that this norm does not depend on the choice of $\mathcal{G}(c)$. As a consequence, setting $\mathcal{G}' = \Phi_U^{N,N'}(\mathcal{G}(c))$, we have $c(\mathcal{G}') = c$,

$$\mathcal{G}_{|U} = \mathcal{G}(c)_{|U} \triangleleft_{N'} \mathcal{G}'_{|U_1}$$

and

$$\|\mathcal{G}' - E_{c_0,\mathcal{S}_1}\|_{U_1} \leqslant \delta.$$

□ STEP 2 □

(8.10) STEP 3. *Let $\mathcal{S}_1$ be a static class in $\mathcal{A}(c_0)$ satisfying (i) of (8.3), and let U_1 be a fixed neighborhood of $\mathcal{S}_1$. There exists a number $\delta > 0$ and an integer N such that, if $\mathcal{G}' \in \mathbb{P}$ satisfies $\|\mathcal{G}' - E_{c_0,\mathcal{S}_1}\|_{U_1} \leqslant \delta$, then $\mathcal{G}'_{|U_1} \lhd_N c(\mathcal{G}')$*

PROOF. There exists an open neighborhood $V_1 \subset U_1$ of $\mathcal{S}_1$ such that $\mathcal{I}(E_{c_0,\mathcal{S}_1}) \cap V_1 = \mathcal{I}(E_{c_0,\mathcal{S}_1}) \cap \bar{V}_1$. Let x_1 be a point of $\mathcal{S}_1$ and set $u_1 = h_{c_0}(x_1, .)$ (this is (i) of (8.3)). Recall that

$$T^{\infty}_{c_0,V_1} u_1(x) = \min_{y \in \bar{V}_1} h_{c_0}(x_1, y) + h_{c_0}(y, x).$$

By taking $y = x_1$ in this expression, we obtain the inequality $T^{\infty}_{c_0,V_1} u_1(x) \leqslant u_1(x)$. On the other hand, we have the triangle inegality $u(x) \leqslant h_{c_0}(x_1, y) + h_{c_0}(y, x)$ for each y, so that $T^{\infty}_{c_0,V_1} u_1(x) = u_1(x)$, and

$$\min_{y \in \bar{V}_1} h_{c_0}(x_1, y) + h_{c_0}(y, x) = h_{c_0}(x_1, x) = \min_{y \in M} h_{c_0}(x_1, y) + h_{c_0}(y, x).$$

Now the points y where this last minimum is reached belong to $\mathcal{I}(E_{c_0,\mathcal{S}_1})$. As a consequence, for each $x \in M$, the points where the minimum is reached in the definition of $T^{\infty}_{c_0,V_1} u_1(x)$ belong to $\mathcal{I}(E_{c_0,\mathcal{S}_1}) \cap V_1$, which is a compact set contained in V_1. In view of (5.6), there exist integers N and N' and a positive real number δ such that, if $\mathcal{G} \in \mathbb{P}$ satisfies $\|\mathcal{G} - E_{c_0,\mathcal{S}_1}\|_{V_1} \leqslant \delta$, then

$$\mathcal{G}_{|V_1} \lhd_N \Phi^{N,N'}_{V_1}(\mathcal{G}).$$

□ STEP 3 □

The proposition obviously follows from the three steps above. □ PROPOSITION □

APPLICATIONS

9 Twist Maps

The case where $M = \mathbb{T}$ is well known and have been studied many times. The resulting time-one flow is then a finite composition of right twist maps of the biinfinite annulus $T^*\mathbb{T}$. In view of (0.9), much of what is known on the existence of orbits with prescribed behavior is summed up in the following discussion.

(9.1) Let $G \in H^1(\mathbb{T}, \mathbb{R})$ be the set of cohomology classes of invariant curves which are Lipschitz graphs. The set G is closed, and every point $c \in G$ is alone in its class of $\lhd\!\rhd$-equivalence, as follows from (5.2). It follows easily from (0.10) that the classes of $\lhd\!\rhd$-equivalence are the points of G and the connected components of the complement of G.

(9.2) For completeness, we recall without proof some of the special properties of Aubry sets in dimension one. The function α is differentiable, and its differential $\alpha'(c)$ is the rotation number of every orbit of $\tilde{\mathcal{N}}(c)$. If $\alpha'(c)$ is irrational, then there is only one element in $\mathbb{V}_c$. If $\alpha'(c)$ is rational, then the Mather set $\tilde{\mathcal{M}}(c)$ is made of periodic orbits.

10 Generalized Arnold Example

(10.1) In this application, we take

$$M = \mathbb{T} \times N,$$

where N is a compact manifold of dimension $d-1$, and denote by $q = (q_1, q_2)$ the points of M. We assume that the homology group $H_1(N, \mathbb{Z})$ is not trivial. We denote the points of TM by $(q, v) = (q_1, q_2, v_1, v_2)$, where $(q_1, v_1) \in T\mathbb{T}$ and $(q_2, v_2) \in TN$. In the same way, we denote by $(q, p) = (q_1, q_2, p_1, p_2)$ the points of T^*M. We will consider the projection $\pi_1 : \mathbb{T} \times N \longrightarrow \mathbb{T}$ and the induced mapping

$$\pi_1^* : H^1(\mathbb{T}, \mathbb{R}) \longrightarrow H^1(\mathbb{T} \times N, \mathbb{R}).$$

(10.2) Let us fix a point 0 in N. We will consider Lagrangian systems which satisfy

$$L(t, q_1, q_2, v_1, v_2) > L(t, q_1, 0, v_1, 0)$$

for all $(q_2, v_2) \neq (0, 0)$, all $t \in \mathbb{R}$ and all $(q_1, v_1) \in T\mathbb{T}$. Let $\partial_v L : TM \longrightarrow \mathbb{T}^*M$ be the Legendre transform associated to L. We denote by $\mathbb{T}_1$ the submanifold $\mathbb{T} \times \{0\}$ of M, by $T^*\mathbb{T}_1$ the submanifold $\{q_2 = 0, p_2 = 0\}$ of T^*M, and $T\mathbb{T}_1$ the submanifold $\{q_2 = 0, v_2 = 0\}$ of TM. We have

$$\partial_v L(T\mathbb{T}_1) = T^*\mathbb{T}_1,$$

and this manifold is invariant under the Hamiltonian flow. Moreover, the restriction of the flow to $T^*\mathbb{T}_1$ is the Hamiltonian flow of the restriction $H_1(t, q_1, p_1) := H(t, q_1, p_1, 0, 0)$ of H. Setting $L_1(t, q_1, v_1) = L(t, q_1, 0, v_1, 0)$, we see that L_1 is the Lagrangian associated to H_1. We denote by ϕ_1 the restriction of ϕ to $T^*\mathbb{T}$.

(10.3) Theorem *Under the non-degeneracy conditions (10.4) and (10.5) to be specified below, the image of π_1^* is contained in one class of $\lhd\!\rhd$-equivalence.*

(10.4) Genericity property for ϕ_1. We assume that every rotational invariant circle of ϕ_1 which contains a periodic orbit is completely periodic (every orbit of this circle is periodic). We could, more simply, require that the map ϕ_1 does not have any invariant circle containing a periodic orbit. This property is known to be generic in any reasonable sense of the term. However, allowing periodic circles includes the important case where ϕ_1 is integrable, as in the original Arnold's example.

(10.5) nondegeneracy of external homoclinics. We assume that, for each $c \in \pi_1^*(H^1(\mathbb{T}, \mathbb{R}))$, there exists a finite Galois covering $P : M_0 \longrightarrow M$ such that the set

$$\tilde{\mathcal{N}}_{L \circ TP}(P^*(c)) - T^*P^{-1}(T^*\mathbb{T}_1)$$

is not empty and contains finitely many orbits. Note that, since $H^1(N, \mathbb{Z})$ is not zero, it follows from (4.10), (4.5), and lemma (10.6) below, that there exists a finite Galois covering $P : M_0 \longrightarrow M$ such that the set under consideration is not empty. So the important point of our assumption is finiteness. As the reader will see it in the proof, this assumption could easily be weakened.

(10.6) Lemma *For each cohomology $c = \pi_1^*(c_1)$, with $c_1 \in H^1(\mathbb{T}, \mathbb{R})$, we have $\mathcal{N}(c) \subset \mathbb{T}_1$. As a consequence, the restriction to $\mathbb{T}_1$ gives a bijection between the set $\mathbb{V}_c$ and the set $\mathbb{V}_{c_1}$ associated to the Lagrangian L_1 on $T\mathbb{T}_1$.*

Proof. Let us fix a cohomology $c_1 \in H^1(\mathbb{T}, \mathbb{R})$ and its image $c := \pi_1^*(c_1)$. Let μ be a form on $\mathbb{T}$ which represents c_1, and η be its pull back on $M = \mathbb{T} \times N$. Consider a pseudograph $\mathcal{G} \in \mathbb{V}_c$, and write it $\mathcal{G} = \mathcal{G}_{\eta,u}$. We want to prove that $\tilde{\mathcal{I}}(\mathcal{G}) \subset T^*\mathbb{T}_1$. Let $(q(t), p(t))$ be the trajectory of the Hamiltonaina flow starting in $\tilde{\mathcal{I}}(\mathcal{G})$. We have, for $k < l$ in $\mathbb{Z}$,

$$u(q(l)) - u(q(k)) = \int_k^l L(\sigma, q(\sigma), \dot{q}(\sigma)) - \mu_{q_1(\sigma)}(\dot{q}_1(\sigma)) + \alpha(c)\, d\sigma$$

and

$$u(q_1(t), 0) - u(q_1(s), 0) \leqslant \int_k^l L(\sigma, (q_1(\sigma), 0, \dot{q}_1(\sigma), 0) - \mu_{q_1(\sigma)}(\dot{q}_1(\sigma)) + \alpha(c)\, d\sigma.$$

It follows that

$$\int_k^l L(\sigma, q(\sigma), \dot{q}(\sigma)) - L(\sigma, (q_1(\sigma), 0), (\dot{q}_1(\sigma), 0))\, d\sigma \leqslant 2(\max u - \min u)$$

Let us denote by $\tilde{L}$ the function

$$\tilde{L}(t, q, v) = L(t, q, v) - L(t, (q_1, 0), (v_1, 0))$$

which is positive except on $T\mathbb{T}_1$. Since the integral $\int_{\mathbb{R}} \tilde{L}(\sigma, q(\sigma), \dot{q}(\sigma))$ is finite, we have

$$\liminf_{|\sigma| \longrightarrow \infty} \tilde{L}(\sigma, q(\sigma), \dot{q}(\sigma)) = 0,$$

and consequently $\liminf_{|\sigma| \longrightarrow \infty} (q_2(t), v_2(t)) = 0$. We now return to the inequality

$$\int_k^l \tilde{L}(\sigma, q(\sigma), \dot{q}(\sigma)) d\sigma \leqslant u(q(t)) - u(q_1(t), 0) - u(q(s)) + u(q_1(s), 0),$$

from which we get

$$\int_{-\infty}^{\infty} \tilde{L}(\sigma, q(\sigma), \dot{q}(\sigma)) d\sigma = 0,$$

which implies that $(q_2, v_2) \equiv 0$. We have proved that $\tilde{\mathcal{I}}(\mathcal{G}) \subset T^*\mathbb{T}_1$. □

(10.7) Let us fix cohomologies $c = \pi_1^*(c_1), c_1 \in H^1(\mathbb{T}, \mathbb{R})$, such that there exists an invariant Lipschitz Graph $\mathcal{G}$ in $\mathbb{V}_{c_1}$. If the rotation number of $\phi_{1|\mathcal{G}}$ is irrational, then $\mathbb{V}_{c_1}$ contains only one element. As a consequence, $\mathbb{V}_c$ also contains only one element, so that $\tilde{\mathcal{N}}(c) = \tilde{\mathcal{A}}(c) = \mathcal{G}$, and there is only one static class in $\tilde{\mathcal{A}}(c)$. If the rotation number is rational, then in view of (10.4) the graph $\mathcal{G}$ is a union of periodic orbits, so that $\mathcal{G} = \tilde{\mathcal{M}}(c)$. As a consequence, we have $\mathcal{A}(c) = \mathbb{T}_1$, and there is only one static class.

In view of (10.5), there exists a finite Galois covering $P : M_0 \longrightarrow M$ such that the Lagrangian $L \circ TP$ satisfies the hypotheses of (8.2). As a consequence, the cohomology $P^*(c)$ is in the interior of its $\lhd\!\rhd$-equivalence class for $L \circ TP$. It follows from (5.8) that the cohomology c is in the interior of its class of $\lhd\!\rhd$ -equivalence for L.

(10.8) Let $c = \pi_1^*(c_1)$ be such that each set $\mathcal{I}(\mathcal{G}), \mathcal{G} \in \mathbb{V}_c$ is properly contained in $\mathbb{T}_1$. Applying (0.10), we observe that $R(c) = H^1(M, \mathbb{R})$, and c is in the interior of its class of $\lhd\!\rhd$-equivalence.

(10.9) We have proved that each $c \in \pi_1^*(\mathbb{T}, \mathbb{R})$ is in the interior of its class of $\lhd\!\rhd$-equivalence. Since the subspace $\pi_1^*(H^1(\mathbb{T}, \mathbb{R}))$ is obviously connected, it is contained in one class of $\lhd\!\rhd$-equivalence. $\square$

Appendix

A semi-concave functions

We review some standard facts on semi-concave functions. In this section, the compact manifold M, of dimension d, is endowed with a Riemannian metric. The cotangent bundle is then in a natural way endowed with a metric. A function $f : M \longrightarrow \mathbb{R}$ will be called differentiable with K-Lipschitz differential if the differential df is K-Lipschitz as a section $M \longrightarrow T^*M$ for the distances associated to these metrics.

(A.1) Let $u : M \longrightarrow \mathbb{R}$ be a continuous function. We say that $p \in T_x^*M$ is a proximal super-differential of u at x if there exists a smooth function $f : M \longrightarrow \mathbb{R}$ such that $df(x) = p$ and such that $f - u$ has a minimum at x. Proximal sub-differentials are defined in the same way. The set of proximal super-differential of the function u at x is denoted $\partial^+ u(x)$. The set of proximal sub-differential is denoted $\partial^- u(x)$. It is useful to give a more quantitative version of this definition. We say that p is a K-super-differential of u at x if there exists a function $f : M \longrightarrow \mathbb{R}$ with K-Lipschitz differential such that $f - u$ has a minimum at x and $df(x) = p$. It is known that K-super-differentials are proximal super-differentials, see for instance [11], Lemma 7.3.5. We denote by $\partial^{K+} u(x)$ the set of K-super-differentials. It is not hard to see that $\partial^+ u(x) = \cup_{K \geqslant 0} \partial^{K+} u(x)$.

(A.2) We say that a function u is K-semi-concave if, for each x, the set $\partial^{K+}(x)$ is not empty. A function u is called semi-concave if it is K-semi-concave for some $K \geqslant 0$. A set of functions is called equi-semi-concave if there exists a positive constant K such that each function of the set is K-semi-concave. The semi-concave functions and the equi-semi-concave sets of functions do not depend on the choice of the metrics used to define them.

A function u is semi-concave if and only if there exists a compact set $\mathbb{K} \subset C^1(M)$ with equi-Lipschitz differentials and such that

$$u(x) = \inf_{f \in \mathbb{K}} f(x).$$

Proof. Note first that we then have

$$u(x) = \min_{f \in \mathbb{K}} f(x).$$

If this holds, we can choose, for each point x, a function f in $\mathbb{K}$ such that $f(x) = u(x)$. If K is such that all the differentials of functions of $\mathbb{K}$ are K-Lipschitz, then $df(x) \in \partial^{K+} u(x)$, hence the set $\partial^{K+} u(x)$ is non-empty for each x, hence the function u is K-semi-concave. Conversely, Assume that the function u is K-semi-concave. Let $\mathbb{K}$ be the set formed by functions f with K-Lipschitz differential and such that $u \leqslant f \leqslant K\mathrm{diam}^2(M) + \max u$, where $\mathrm{diam}^2(M)$ is the square of the diameter $\mathrm{diam}(M)$ of M. Clearly, the set $\mathbb{K}$ is compact for the C^1 topology.

In addition, we have $u(x)=\min_{f\in\mathbb{K}} f(x)$. Indeed, the inequality $u(x)\leqslant \min_{f\in\mathbb{K}} f(x)$ always holds by definition of $\mathbb{K}$. On the other hand, for each point $x\in M$, there exists a function f with K-Lipschitz differential which satisfies $f\geqslant u$ and $f(x)=u(x)$. Since the differential of f vanishes at one point (the minimum of f) and is K-Lipschitz, we have $\|df\|_\infty\leqslant K\mathrm{diam}(M)$, hence $\|f\|_\infty\leqslant u(x)+K\mathrm{diam}^2(M)$, so that $f\in\mathbb{K}$. Recalling that $f(x)=u(x)$, we obtain that $u(x)\geqslant \min_{f\in\mathbb{K}} f(x)$. □

Consequently, we have the following property:

(A.3) *An equi-semi-concave set of functions is equi-Lipschitz.*

(A.4) *If $\mathbb{U}$ is a set of K-semi-concave functions on M, and if the infimum $\inf_{u\in\mathbb{U}} u(x_0)$ is finite for some $x_0\in M$, then the function $v(x)=\inf_{u\in\mathbb{U}} u(x)$ is finite and K-semi-concave.*

Proof. Since the functions in $\mathbb{U}$ are equi-Lipschitz, it is not hard to see that the infimum is finite at each point if it is finite at one point. In this case, we can choose, for each point $x_1\in M$, a sequence u_n of functions of $\mathbb{U}$ such that $u_n(x_1)\longrightarrow v(x_1)$, and then a sequence f_n of C^1 functions with K-Lipschitz differential such that $f_n\geqslant u_n$ and $f_n(x_1)=u_n(x_1)$. Taking a subsequence, we can assume that f_n converges to a limit f for the C^1 topology. This limit clearly satisfies $f\geqslant v$ and $f(x_1)=v(x_1)$, and it has a K-Lipschitz differential. We have proved that the set $\partial^{K+}v(x_1)$ is not empty, hence the function v is K-semi-concave. □

(A.5) *Let $\mathbb{U}$ be an equi-semi-concave set of functions on $N\times M$, where N is another compact manifold. Then the functions $u(x,.):M\longrightarrow\mathbb{R}, x\in N, u\in\mathbb{U}$ form an equi-semi-concave set.*

Proof. Let us assume that the manifold N is endowed with a Riemannian metric, end that the functions of $\mathbb{U}$ are K-semi-concave for the product metric on $N\times N$. We shall prove that the functions $u(x,.)$ are K-semi-concave. Let $f:N\times M\longrightarrow\mathbb{R}$ be a function with K-Lipschitz differential such that $f-u$ has a minimum at (x,y). The function $f(x,.)-u(x,.)$ has a minimum at y, so that it is enough to prove that the function $g=f(x,.)$ has a K-Lipschitz differential. Let us identify the tangent bundle $T^*(N\times M)$ with the product $T^*N\times T^*M$, the metric is the product metric. We have $df_{(x,z)}=\big(d_1f_{(x,z)},d_2f_{(x,z)}\big)$, and $d_2f_{(x,z)}=dg_z$. As a consequence,

$$\mathrm{dist}\big(dg_{z_1},dg_{z_2}\big)\leqslant \mathrm{dist}\big(df_{(x,z_1)},df_{(x,z_2)}\big)\leqslant K\mathrm{dist}\big((x,z_1),(x,z_2)\big)=K\mathrm{dist}\big(z_1,z_2\big).$$

□

(A.6) *The set of K-semi-concave functions is closed for the topology of uniform convergence. In addition, if u_n is a sequence of K-semi-concave functions converging uniformly to a limit u, then we have the following additional convergence property: If x_n is a sequence of points of differentiability of u_n, converging to a point of differentiability x of u, then $du_n(x_n)\longrightarrow du(x)$.*

Let u_n be a sequence of K-semi-concave functions, and let $x_n\longrightarrow x$ be a sequence of points of M. Consider a sequence f_n of functions with K-Lipschitz differential such that $f_n - u_n$ has a minimum at x_n and $f_n(x_n) = 0$. If f is any accumulation point of the sequence f_n in the C^1 topology, then $df(x)$ is a K-super-differential of u at x. So in particular if x_n and x are points of differentiability of u, we get $du_n(x_n) = df_n(x_n) \longrightarrow df(x) = du(x)$.

(A.7) *Let u be a continuous function on M, and let Let A be the compact subset of M formed by points $x \in M$ such that $\partial^{K+}u(x)$ and $\partial^{K-}(x)$ are both non-empty. Then the function u is differentiable at each point of A, and the mapping $x \longmapsto du(x)$ is Lipschitz on A, with a Lipschitz constant that depends only on K.*

This follows from Proposition 4.5.3 in Fathi's book [11].

(A.8) Let Φ be a finite atlas of M composed of charts $\varphi : B_3 \longrightarrow M$, where B_r is the open ball of radius r centered at zero in $\mathbb{R}^d$. Assume that the sets $\varphi(B_1), \varphi \in \Phi$ cover M.

Let $\mathbb{U}$ be a set of functions on M. The following statements are equivalent:

- *(i) The set $\mathbb{U}$ is equi-semi-concave.*
- *(ii) There exists a constant $K > 0$ such that the function $x \longmapsto u \circ \varphi - K\|x\|^2$ is concave on B_2 for each function $u \in \mathbb{U}$ and each chart $\varphi \in \Phi$.*
- *(iii) There exists a constant $K > 0$ such that, for each function $u \in \mathbb{U}$, each chart $\varphi \in \Phi$, and each point $x \in B_2$, there exists a linear form l_x on $\mathbb{R}^d$ such that*

$$u \circ \varphi(y) - u \circ \varphi(x) \leqslant l_x(y - x) + K\|y - x\|^2$$

for each $y \in B_2$.

We leave as an easy exercise for the reader to prove the equivalence between (ii) and (iii). It is not hard either to see that $(i) \Rightarrow (iii)$. We shall prove that $(iii) \Rightarrow (i)$. Let $\mathbb{U}$ be the set of functions satisfying both (iii) and (ii). We shall prove that the set $\mathbb{U}$ is equi-semi-concave.

Let us first prove that there exists a constant L, which depends only on K, and such that each linear form l satisfying (iii) for some chart $\varphi \in \Phi$, some function $u \in \mathbb{U}$ and some point $x \in B_1$ satisfies $\|l\| \leqslant L$, where $\|l\|$ is the standard Euclidean norm of l. Note that this result easily implies that the functions of $\mathbb{U}$ are equi-Lipschitz. We need a lemma.

LEMMA. *Let us consider a chart $\varphi \in \Phi$, a point $x_0 \in B_1$, and a linear form l satisfying (iii) at x_0. If $\|l\| \geqslant 11K$, then there exists a point $y \in B_2$ which is a point of differentiability of $u \circ \varphi$ and satisfies*

$$\|d(u \circ \varphi)_y\| \geqslant (\|l\| - 11K)/3.$$

and

$$u \circ \varphi(y) < \inf_{B_1} u \circ \varphi.$$

PROOF OF THE LEMMA. Let us prove first that the infimum of $u \circ \varphi$ in B_2 is not reached in $\bar{B}_1$.

Assume, by contradiction, that there exists a point $m \in \bar{B}_1$ such that $u \circ \varphi(m) = \inf_{B_2} u \circ \varphi$. Then clearly the function $u \circ \varphi$ is differentiable at m, its differential is zero, and the inequality

$$u \circ \varphi(x_0) \leqslant u \circ \varphi(m) + K\|x_0 - m\|^2$$

holds. On the other hand, we have

$$u \circ \varphi(m) \leqslant u \circ \varphi(x) \leqslant u \circ \varphi(x_0) + l(x - x_0) + K\|x - x_0\|^2$$

for all $x \in B_2$. Combining these inequalities gives

$$l(x_0 - x) \leqslant K\|x - x_0\|^2 + K\|x_0 - m\|^2$$

for all $x \in \bar{B}_2$. Hence $\|l\| \leqslant 5K$, which is in contradiction with the hypothesis.

Let us now consider a vector $v \in \mathbb{R}^d$ of norm 1 and such that $l(v) = -\|l\|$. We get

$$u \circ \varphi(x_0 + v) - u \circ \varphi(x_0) \leqslant l(v) + K\|v\|^2 = K - \|l\|.$$

Hence the infimum of $u \circ \varphi$ on B_2 is not greater than $u \circ \varphi(x_0) + K - \|l\|$. It is then possible to choose a point y in B_2 such that

$$u \circ \varphi(y) < \min\big(\inf_{B_1} u \circ \varphi, u \circ \varphi(x_0) + 2K - \|l\|\big).$$

In addition, since the function $u \circ \varphi$ is differentiable almost everywhere, we can assume that the function $u \circ \varphi$ is differentiable at y. We have the inequality

$$u \circ \varphi(x_0) \leqslant u \circ \varphi(y) + d(u \circ \varphi)_y(y - x_0) + K\|y - x_0\|^2$$

from which follow

$$d(u \circ \varphi)_y(x_0 - y) \leqslant u \circ \varphi(y) - u \circ \varphi(x_0) + K\|y - x_0\|^2 \leqslant 11K - \|l\|.$$

Hence $\|d(u \circ \varphi)_{x_1}\| \geqslant (\|l\| - 11K)/3$. This ends the proof of the lemma. □

Let us consider a function $u \in \mathbb{U}$, a chart $\varphi_0 \in \Phi$, a point $x_0 \in B_1$, and a linear form l_0 satisfying (iii) for these data. Let $y_0 \in B_2$ be the point given by the lemma. Let us consider a chart $\varphi_1 \in \Phi$ such that $y_1 \in \varphi_1(B_1)$, and the point $x_1 \in B_1$ such that $\varphi_1(x_1) = \varphi_0(y_0)$. Note that $u \circ \varphi_1$ is differentiable at x_1, and define

$$l_1 := d(u \circ \varphi_1)_{x_1} = d(u \circ \varphi_0)_{x_0} \circ d(\varphi_0^{-1} \circ \varphi_1)_{x_1}.$$

In view of the lemma, there exists a constant $C > 1$, which depends only on the atlas Φ, and such that

$$\|l_1\| \geqslant (\|l_0\| - 11K)/C.$$

If l_0 is large enough, then we have $\|l_1\| \geqslant 11K$, hence we can apply the lemma again, and find a chart ϕ_2, a point x_2 and a linear form l_2. In addition, we have

$$u \circ \varphi_2(x_2) < \inf_{\varphi_0(B_1) \cup \varphi_1(B_1)} u,$$

so that the charts φ_0, φ_1 and φ_2 are different. Now if $\|l_0\|$ is sufficiently large, the process can be continued further and we can build inductively, for $0 \leqslant i \leqslant N$, a sequence $x_i \in B_1$

of points, a sequence $\varphi_i \in \Phi$ of different charts, and a sequence l_i of linear forms such that $\|l_{i+1}\| \geqslant (\|l_i\| - 11K)/C$. The process can be continued as long as $\|l_i\| \geqslant 11K$. Recall that the cardinal of Φ is finite, and denote it by $|\Phi|$. Since all the charts involved in the construction above are different, at most $|\Phi|$ steps can be performed. Hence there exists an integer $N \leqslant |\Phi|$ such that $\|l_i\| \geqslant 11K$ for $i < N$, and $\|l_N\| \leqslant 11K$. This gives a bound to $\|l_0\|$.

We have proved the existence of a bound L, which depends only on the atlas Φ and the number K, such that all linear forms l satisfying (iii) at some point $x \in B_1$, for some chart $\varphi \in \Phi$, and for some function $u \in \mathbb{U}$ satisfies $\|l\| \leqslant L$. Clearly, it follows that the set $\mathbb{U}$ is equi-Lipschitz, and that there exists a constant Δ such that

$$\max u - \min u \leqslant \Delta$$

for all $u \in \mathbb{U}$.

Now we have obtained the desired uniform bounds, let us consider a smooth function $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that $0 \leqslant g \leqslant 1$, and such that $g = 0$ outside of B_2 and $g = 1$ inside B_1. Let us associate, to each chart $\varphi \in \Phi$, each point $x \in B_1$, and each linear form l on $\mathbb{R}^d$, the function $f_{l,x,\varphi} : M \longrightarrow \mathbb{R}$ defined by

$$f_{l,x,\varphi} \circ \varphi(y) := g(y)\big(l(y-x) + K\|y-x\|^2\big) + (1 - g(y))\Delta$$

for $y \in B_2$, and $f_{x,\varphi} = \Delta$ outside of $\varphi(B_2)$. The functions $f_{l,x,\varphi}$ have equi-Lipschitz differentials. On the other hand, if $u \in \mathbb{U}$ and $z \in M$ are given, let us consider a chart $\varphi \in \Phi$ and a point $x \in B_1$ such that $\varphi(x) = z$, and let l be an associated linear form given by (iii). We claim that $f_{l,x,\varphi} - u$ has a minimum at z. Indeed, we have the inequalities

$$u \circ \varphi(y) \leqslant u(z) + l(y-x) + K\|y-x\|^2$$

for $y \in B_2$ and $u \leqslant u(z) + \Delta$. Hence we have $f_{l,x,\varphi} - u \geqslant -u(z)$, with equality at point z. We have proved that the set $\mathbb{U}$ is equi-semi-concave.

B Uniform families of Hamiltonians

Let us fix once and for all a Riemann metric on the compact manifold M. We use this metric to define a norm $|v|$ for tangent vectors, and a norm $|p|$ for tangent covectors.

(B.1) A family of pairs (H, L) of dual Hamiltonians and Lagrangians satisfying the hypotheses (1.1) and (1.2) is called uniform if:

- (i) There exist two superlinear functions h_0 and $h_1 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that each Hamiltonian H of the family satisfies $h_0(p) \leqslant H(t,x,p) \leqslant h_1(p)$.
- (ii) There exists an increasing function $K(k) : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that, if ϕ is the flow of a Hamiltonian of the family and if the times t and s satisfy $t-1 \leqslant s \leqslant t+1$, then

$$\phi_t^s\big(\{|p| \leqslant k\}\big) \subset \{|p| \leqslant K(k)\} \subset T^*M.$$

- (iii) There exists a finite atlas Ψ of M such that, for each chart $\psi \in \Psi$ and each Lagrangian L of the family, we have $\|d^2(L \circ T\psi)_{(t,x,v)}\| \leqslant K(k)$ for $|v| \leqslant k$.

Note that condition (i) could have equivalently been replaced by the following:

- (i') There exist two superlinear functions l_0 and $l_1 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that each Lagrangian L of the family satisfies $l_0(v) \leqslant L(t,x,v) \leqslant l_1(v)$.

(B.2) The Uniform families of highest use in the present paper are the following ones. Let Ω be the set of smooth closed one-forms on M. Let $\Omega_1 \subset \Omega$ be a finite dimensional vector subspace, and let $\Omega_2 \subset \Omega_1$ be a bounded part of Ω_1. Then it is easy to check that, for a fixed H satisfying (1.1), the Hamiltonians $H(t, x, p + \omega_x), \omega \in \Omega_2$ form a uniform family. Equivalently, the Lagrangians $L(t, x, v) - \omega_x(v)$ form a uniform family.

(B.3) *In a uniform family, we have*

$$|\partial_p H(t, x, p)| \geqslant \frac{h_0(|p|) - h_1(0)}{|p|}$$

and

$$|\partial_v L(t, x, v)| \geqslant \frac{l_0(|v|) - l_1(0)}{|v|}.$$

In other words, the Legendre transforms are uniformly proper.

Proof. In view of the convexity of H, we have

$$|\partial_p H_{(t,x,p)}| \geqslant \frac{H(t, x, p) - H(t, x, 0)}{|p|}.$$

□

(B.4) Given a Lagrangian L satisfying the hypotheses of (1.2), we define the function $A_L(t, x; s, y) : \mathbb{R} \times M \times \mathbb{R} \times M \longrightarrow \mathbb{R}$ by

$$A_L(t, x; s, y) = \inf_{\gamma \in \Sigma(t,x;s,y)} \int_t^s L(\sigma, \gamma(\sigma), \dot{\gamma}(\sigma))\, d\sigma,$$

Where $\Sigma(t, x; s, y)$ is the set of absolutely continuous curves $\gamma : [s, t] \longrightarrow M$ satisfying $\gamma(t) = x$ and $\gamma(s) = y$. We denote by $\Sigma_m^L(t, x; s, y)$ the set of curves of $\Sigma(t, x; s, y)$ which realize the minimum.

(B.5) *For each uniform family of Lagrangians, there exists a decreasing function $K_1(\epsilon) :]0, \infty) \longrightarrow \mathbb{R}^+$ such that, If L is a Lagrangian of the family and if t and s are two real times satisfying $t \geqslant s + \epsilon$, then each curve $\gamma \in \Sigma_m^L(t, x; s, y)$ is C^2 and satisfies $|\dot{\gamma}(\sigma)| \leqslant K(\epsilon)$ for each $\sigma \in [s, t]$.*

Proof. By comparing the action of γ with that of a geodesic with the same endpoints, we get

$$\int_s^t l_0(|\dot{\gamma}(\sigma)|)d\sigma \leqslant \int_s^t L(\sigma, \gamma(\sigma), \dot{\gamma}(\sigma))d\sigma \leqslant (t - s) l_1 \left(\frac{\operatorname{diam}(M)}{t - s} \right)$$

The right hand side is clearly bounded by a constant which depends only of the parameters of the uniform family and of ϵ. We obtain

$$(t - s) \min l_0(|\dot{\gamma}(\sigma)|) \leqslant C,$$

from which follows, with another constant C, that $\min|\dot\gamma(\sigma)| \leqslant C$. But then in view of (B.3), we have

$$\min_{\sigma\in[s,t]} |\partial_v L(\sigma,\gamma(\sigma),\dot\gamma(\sigma))| \leqslant C,$$

then in view of (*ii*),

$$\max_{\sigma\in[s,t]} |\partial_v L(\sigma,\gamma(\sigma),\dot\gamma(\sigma))| \leqslant C,$$

so that finally, using (B.3) again, we get $\max|\dot\gamma| \leqslant C$. We have used the symbol C for different constants which depend only of ϵ and of the parameters of the family. □

(B.6) *For each times $s<t$, the mapping which, to a Lagrangian L, associates the function*

$$(x,y) \longmapsto A_L(s,x,t,y)$$

of $C(M\times M,\mathbb{R})$, is continuous on each uniform family of Lagrangians endowed with the topology of uniform convergence on compact sets.

Proof. Let L_0 and L_1 be two Lagrangians of the family. Let $\gamma(\sigma):[s,t]\longrightarrow M$ be such that

$$A_{L_0}(s,\gamma(s);t,\gamma(t)) = \int_s^t L_0(\sigma,\gamma(\sigma),\dot\gamma(\sigma))\,d\sigma.$$

We have

$$A_{L_1}(s,\gamma(s);t,\gamma(t)) \leqslant \int_s^t L_1(\sigma,\gamma(\sigma),\dot\gamma(\sigma))\,d\sigma,$$

so that

$$A_{L_1}(s,\gamma(s);t,\gamma(t)) - A_{L_0}(s,\gamma(s);t,\gamma(t)) \leqslant (t-s)\max_{|v|\leqslant K_1(t-s)} L_1 - L_0,$$

where K_1 is defined in (B.5). By symmetry, we get that

$$\|A_{L_0}(s,.;t,.) - A_{L_1}(s,.;t,.)\|_\infty \leqslant (t-s)\max_{|v|\leqslant K_1(t-s)} |L_1-L_0|.$$

□

(B.7) Theorem. *For each uniform family of Lagrangians and each $\epsilon>0$, consider the set $\mathbb{U}_\epsilon$ of continuous functions $M\times M\longrightarrow\mathbb{R}$ given by*

$$(x,y)\longmapsto A_L(s,x,t,y)$$

where $t\geqslant s+\epsilon$ and L is a Lagrangian of the family. This set is equi-semi-concave, hence equi-Lipschitz on $M\times M$. In addition, for each curve $\gamma\in\Sigma^L_m(t,x;s,y)$, the covector

$$\big(-\partial_v L(s,\gamma(s),\dot\gamma(s)),\partial_v L(t,\gamma(t),\dot\gamma(t))\big)$$

is a proximal super-differential.

Proof. Let us consider a finite atlas Ψ of M formed by charts $\psi: B^d_6\longrightarrow M$, where B^d_r is the Euclidean ball of radius r in $\mathbb{R}^d$. Assume that the open sets $\psi(B^d_{1/2})$, $\psi\in\Psi$, cover M. Let Φ be the atlas of $M\times M$ composed of products $\psi\times\psi'$, with $\psi\in\Psi$ and $\psi'\in\Psi'$. The charts φ of Φ are defined on B^{2d}_3, and the images $\varphi(B^{2d}_1)$, $\varphi\in\Phi$, cover $M\times M$. So this atlas satisfies the conditions of (A.8). In order to prove that the set $\mathbb{U}_\epsilon$ is equi-semi-concave, we shall prove that (iii) of (A.8) holds. So from now on we shall work in a fixed chart $\varphi=\psi_0\times\psi_1$.

Let (x_0,x_1) be a point in $\psi_0(B_2)\times\psi_1(B_2)$, and let y_0 and y_1 be the preimages in B_2. Let $\gamma(t):[s,t]\longrightarrow M$ be a curve in $\Sigma_m(s,x_0;t,x_1)$. In view of (B.5), we have $|\dot\gamma|\leqslant K_1(\epsilon)$. As a consequence, there exists a constant $a>0$, which depends only on the atlas, on the parameters of the family, and of ϵ, such that the curve $\psi_0^{-1}\circ\gamma:[s,s+1/a]\longrightarrow B^d_4$ is well defined and a-Lipschitz, as well as the curve $\psi_1^{-1}\circ\gamma:[t-1/a,t]\longrightarrow B^d_4$. Let us call $y_0(\sigma)$ and $y_1(\sigma)$ these curves, note that $y_0(s)=y_0$ and $y_1(t)=y_1$. Let us now define, for each points z_0 and z_1 in B_4, the curves

$$y_0(\sigma,z_0):=y_0(\sigma)+(1+a(s-\sigma))(z_0-y_0)$$

and

$$y_1(\sigma,z_1):=y_1(\sigma)+(1+a(\sigma-t))(z_1-y_1).$$

For simplicity we define the Lagrangians L_0 and L_1 on $\mathbb{R}\times B^d_4\times\mathbb{R}^d$ by the expression $L_i(\sigma,x,v)=L(\sigma,\psi_i(x),d\psi_{ix}(v))$, shortly, $L_i=L\circ T\psi_i$. We have

$$\begin{aligned}&A(s,\psi_0(z_0);t,\psi_1(z_1))\\ \leqslant A(s,x_0;t,x_1)&+\int_s^{s+1/a}L_0(\sigma,y_0(\sigma,z_0),\dot y_0(\sigma,z_0))-L_0(\sigma,y_0(\sigma),\dot y_0(\sigma))\,d\sigma\\ &+\int_{t-1/a}^{t}L_1(\sigma,y_1(\sigma,z_1),\dot y_1(\sigma,z_1))-L_1(\sigma,y_1(\sigma),\dot y_1(\sigma))\,d\sigma.\end{aligned}$$

There exists a constant $C>0$, which depends only on the atlas, on the parameters of the family, of ϵ, and of a, such that, for $(t,x,v)\in\mathbb{R}\times B^d_4\times B^d_a$ and $(y,w)\in\mathbb{R}\times B^d_4\times B^d_a$, we have

$$L_i(\sigma,y,w)-L_i(\sigma,x,v)\leqslant\partial_xL_{i(\sigma,x,v)}(y-x)+\partial_vL_{i(\sigma,x,v)}(w-v)+C(\|y-x\|^2+\|w-v\|^2).$$

We get

$$\begin{aligned}&A(s,\psi_0(z_0);t,\psi_1(z_1))\leqslant A(s,x_0;t,x_1)\\ &+\int_s^{s+1/a}\partial_xL_{0(\sigma,y_0(\sigma),\dot y_0(\sigma))}(y_0(\sigma,z_0)-y_0(\sigma))+\partial_vL_{0(\sigma,y_0(\sigma),\dot y_0(\sigma))}(\dot y_0(\sigma,z_0)-\dot y_0(\sigma))\,d\sigma\\ &+\int_{t-1/a}^{t}\partial_xL_{0(\sigma,y_0(\sigma),\dot y_0(\sigma))}(y_0(\sigma,z_0)-y_0(\sigma))+\partial_vL_{0(\sigma,y_0(\sigma),\dot y_0(\sigma))}(\dot y_0(\sigma,z_0)-\dot y_0(\sigma))\,d\sigma\\ &+C\int_s^{s+1/a}\|y_0(\sigma)-y_0(\sigma,z_0)\|^2+\|\dot y_0(\sigma)-\dot y_0(\sigma,z_0)\|^2\,d\sigma\\ &+C\int_{t-1/a}^{t}\|y_1(\sigma)-y_1(\sigma,z_1)\|^2+\|\dot y_1(\sigma)-\dot y_1(\sigma,z_1)\|^2\,d\sigma.\end{aligned}$$

Taking advantage of the Euler-Lagrange equations, this simplifies to

$$\begin{aligned}A(s,\psi_0(z_0);t,\psi_1(z_1))\leqslant A(s,x_0;t,x_1)&-\partial_vL_{0(\sigma,y_0,\dot y_0(s))}(z_0-y_0)+\partial_vL_{1(\sigma,y_1,\dot y_1(t))}(z_1-y_1)\\ &+C(1+a^2)(\|y_0-z_0\|^2+\|y_1-z_1\|^2).\end{aligned}$$

□

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