Stochastic heat and Burgers equations and their singularities II - Analytical Properties and Limiting Distributions

Ian M Davies, Aubrey Truman*, Huaizhong Zhao†

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Abstract

We study the inviscid limit, $\mu \to 0$, of the stochastic viscous Burgers equation, for the velocity field $v^\mu(x,t)$, $t > 0$, $x \in \mathbb{R}^d$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla) v^\mu = -\nabla c(x,t) - \epsilon \nabla k(x,t) \dot{W}_t + \frac{\mu^2}{2} \Delta v^\mu, \text{for small } \epsilon,$$

with $v^\mu(x,0) \equiv \nabla S_0(x)$ for some given S_0, $\dot{W}_t$ representing White Noise. Here we use the Hopf-Cole transformation, $v^\mu = -\mu^2 \nabla \ln u^\mu$, where u^μ satisfies the stochastic heat equation of Stratonovich type and the Feynmac-Kac Truman-Zhao formula for u^μ, where

$$du_t^\mu(x) = \left[\frac{\mu^2}{2} \Delta u_t^\mu(x) + \mu^{-2} c(x,t) u_t^\mu(x) \right] dt + \epsilon \mu^{-2} k(x,t) u_t^\mu(x) \circ dW_t,$$

with $u_0^\mu(x) = T_0(x) \exp\left(-S_0(x)/\mu^2\right)$, S_0 as before and T_0 a smooth positive function.

In an earlier paper, Davies, Truman and Zhao [10], an exact solution of the stochastic viscous Burgers equation was used to show how the formal "blow-up" of the Burgers velocity field occurs on *random shockwaves* for the $v^{\mu=0}$ solution of Burgers equation coinciding with the caustics of a corresponding Hamiltonian system with classical flow map Φ. Moreover, the $u^{\mu=0}$ solution of the stochastic heat equation

*Department of Mathematics, University of Wales Swansea, Singleton Park, Swansea, SA2 8PP, Wales, UK. Email: I.M.Davies@swansea.ac.uk, A.Truman@swansea.ac.uk

†Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire, LE11 3TU, England, UK. Email: H.Zhao@lboro.ac.uk

has its *wavefront* determined by the behaviour of the Hamilton principal function of the corresponding stochastic mechanics. This led in particular to the level surface of the minimizing Hamilton - Jacobi function developing cusps at points corresponding to points of intersection of the corresponding pre-level surface with the pre-caustic, "pre-" denoting the preimage under Φ determined algebraically. These results were primarily of a geometrical nature.

In this paper we consider small ϵ and derive the shape of the random shockwave for the inviscid limit of the stochastic Burgers velocity field and also give the equation determining the random wavefront for the stochastic heat equation both correct to first order in ϵ.

In the case $c(x,t) = \frac{1}{2}x^T\Omega^2 x$, $\nabla k(x,t) = -a(t)$, we obtain the exact random shockwave and prove that its shape is unchanged by the addition of noise, it merely being displaced by a random brownian vector $N(t)$. By exploiting the Jacobi fields for this problem we obtain the large time limit of the distribution of the Burgers fluid velocity for noises which have infinite time averages, such as almost periodic ones. Here resonance with the underlying $\epsilon = 0$ classical problem has an important effect. Imitating these results for the case of a periodic underlying classical problem perturbed by small noise, arming ourselves with some detailed estimates for Greens functions enables us to make generalisations.

In the stochastic case we have also the possibility of "infinitely rapid" changes in the number of cusps on the minimizing level surface of the Hamilton - Jacobi function. This will engender stochastic turbulence in the Burgers velocity field and, due to its stochasticity, may be of an "intermittent" nature. There is no analogue of this in the deterministic case.

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1 Introduction

Stochastic Burgers equations have attracted a considerable amount of attention in recent years, e.g. [2], [4], [7], [11], [12], [20], [21], [22], [23], [24], [35], [37], [39], [40]. See also [3], [5], [18], [30] for related works. Burgers equations have been used to give models of turbulence (see especially [11]) and to model the large scale structure of the universe [34]. Here we shall be interested in what has come to be called Burgulence. Primarily we show how a knowledge of Jacobi fields, and the geometry of the level surfaces of a Hamilton - Jacobi function and the associated caustic surface can be used in determining the behaviour of the velocity field of the viscous Burgers fluid in the inviscid limit. The presence of viscosity provides access to a range of powerful analytical methods.

Consider the stochastic viscous Burgers equation for the velocity field $v^\mu = v^\mu(x,t)$, $x \in \mathbb{R}^d$, $t > 0$, small $\epsilon \in \mathbb{R}$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla) v^\mu = \frac{\mu^2}{2}\Delta v^\mu - \nabla c(x) - \epsilon \nabla k(x,t) \dot{W}_t,$$

with initial velocity $v^\mu(x,0) = \nabla S_0(x) + O(\mu^2)$ where μ^2 is the coefficient of viscosity. Here c, k and S_0 are C^2 functions and W_t is a Wiener process on the probability space $\{\Omega, \mathcal{F}, P\}$. The corresponding heat equation for $u^\mu = u^\mu(x,t)$ is the Stratonovich equation

$$\begin{aligned} \frac{\partial u^\mu}{\partial t} &= \frac{\mu^2}{2}\Delta u^\mu + \frac{1}{\mu^2} c(x) u^\mu + \frac{\epsilon}{\mu^2} k(x,t) u^\mu \circ \dot{W}_t, \\ u^\mu(x,0) &= T_0(x) \exp\left(-S_0(x)/\mu^2\right), \end{aligned}$$

where T_0 is a smooth positive function, the square root of the initial Burgers fluid density.

The connection between u^μ and v^μ is the Hopf-Cole logarithmic transformation $v^\mu = -\mu^2 \nabla \ln u^\mu$. Our studies are driven by an interest in the 'blow-up' of $v^0(x,t)$ where

$$v^0(x,t) = \lim_{\mu \to 0} v^\mu(x,t).$$

We seek an understanding of the advent of discontinuities in v^0 and the large time limit of its probability density. In particular we show how for small, almost periodic noise resonating with the underlying classical problem there is no invariant measure.

It is this correspondence with the stochastic heat equation that enables us to appeal to asymptotic methods in our study of the viscous Burgers fluid.

Moreover, these methods highlight the importance of the stochastic dynamical flow and the stochastic Hamilton - Jacobi function in determining the behaviour of solutions. With this in mind we would expect, from the work of Donsker, Freidlin et al [17, 19, 37], that we have as $\mu \to 0$,

$$-\mu^2 \ln u^\mu(x,t) \to \inf_{x_0} [A(x_0,x,t) + S_0(x_0)] = S(x,t), \tag{1.1}$$

with

$$A(x_0,x,t) = \inf_{\substack{X(s)\\ X(0)=x_0 \\ X(t)=x}} A[X],$$

where $A[X]$ is the stochastic action

$$A[X] = \frac{1}{2}\int_0^t \left|\dot{X}^2(s)\right| ds - \int_0^t c(X(s))ds - \epsilon \int_0^t k(X(s),s)dW_s$$

and

$$\mathcal{A}(x_0,x,t) = A(x_0,x,t) + S_0(x_0).$$

We shall assume that $\tilde{x}_0(x,t)$ the minimiser of $\mathcal{A}(x_0,x,t)$ is unique. Note that we require absolute continuity of X (c.f. Davies and Truman [9] and references therein) and we will have $\dot{X}$ continuous almost surely. $S(x,t)$ is the minimising solution of the stochastic Hamilton-Jacobi equation

$$dS_t + \frac{|\nabla S|^2}{2} dt + c(x)\, dt + \epsilon k(x,t)\, dW_t = 0, \quad S(x,0) = S_0(x).$$

So $S(x,t)$ is Hamilton's principal function for a stochastic classical mechanical path $X(s)$, satisfying the second order stochastic differential equation

$$d\dot{X}(s) + \nabla c(X(s))\, ds + \epsilon \nabla k(X(s),s)\, dW_s = 0, \qquad s \in (0,t),$$

with

$$X(0) = x_0, \qquad \dot{X}(0) = \nabla S_0(x_0),$$

where $x_0 = x_0(x,t)$ is determined by the boundary condition $X(t) = x$. Here we finally have to set $x_0 = \tilde{x}_0(x,t)$ to get the minimising S.
When $x_0(x,t)$ above is unique, the solutions of the stochastic heat equation and the viscous stochastic Burgers equation, c.f. [37], for each $m \geq 0$ are respectively;

$$u_t^\mu(x) = \exp\left\{-\mu^{-2}\sum_{j=0}^{m} \mu^{2j} S_j(x,t)\right\} \mathbb{E}\exp\left\{ - \frac{\mu^{2m}}{2}\int_0^t \Delta S_m(y_s^\mu, t-s)\, ds \right.$$
$$\left. + \frac{1}{2}\sum_{j=m+1}^{2m} \mu^{2(j-1)} \sum_{\substack{0\leq i_1,i_2\leq m\\ i_1+i_2=j}} \int_0^t \nabla S_{i_1}\cdot\nabla S_{i_2}(y_s^\mu, t-s)\, ds\right\},$$

$$v^\mu(x,t) = \sum_{j=0}^{m} \mu^{2j} v_j(x,t) - \mu^2 \nabla \ln \mathbb{E}\Bigg\{ \exp\Bigg\{ -\frac{\mu^{2m}}{2} \int_0^t \nabla \cdot v_m(y_s^\mu, t-s)\, ds$$

$$+ \frac{1}{2} \sum_{j=m+1}^{2m} \mu^{2(j-1)} \sum_{\substack{0\le i_1,i_2\le m \\ i_1+i_2=j}} \int_0^t v_{i_1} \cdot v_{i_2}(y_s^\mu, t-s)\, ds \Bigg\}\Bigg\},$$

where $v_j(x,t) = \nabla S_j(x,t)$ is known explicitly for $j = 0, 1, 2, \dots$ (see Truman and Zhao [37]).

Note that each second factor in the expectation is of the form $(1 + O(\mu^{2m}))$ and the first factor gives the expansion up to μ^{2m-2} for each m. The S_j satisfy the stochastic Hamilton - Jacobi equations

$$\frac{\partial S_j}{\partial t} + \frac{1}{2} \sum_{\substack{i_1,i_2\ge 0 \\ i_1+i_2=j}} \nabla S_{i_1} \cdot \nabla S_{i_2} = \frac{1}{2} \Delta S_{j-1},$$

for $j = 0, 1, 2, \dots$, with the convention $\frac{1}{2}\Delta S_{-1} = -c - \epsilon k \dot{W}_t$, $\dot{W}_t$ being white noise and the Nelson diffusion process y_s^μ satisfying

$$dy_s^\mu = \mu\, dB_s - \nabla \sum_{j=0}^{m} \mu^{2j} S_j(y_s^\mu, t-s)\, ds, \qquad y_0^\mu = x, \qquad B_s \text{ is BM.}$$

It is important to note that S_1, and hence all S_j, are T_0 dependent, c.f. [37]. Formally, we are using $S^\mu \sim \sum_{j=0}^{\infty} \mu^{2j} S_j$ where S^μ is the solution of the viscous stochastic Hamilton - Jacobi equation,

$$dS^\mu + \frac{1}{2} |\nabla S^\mu|^2\, dt + c\, dt + \epsilon k\, dW_t = \frac{1}{2} \mu^2 \Delta S^\mu\, dt.$$

Changes in the degree of non-uniqueness of $x_0(x,t)$ is associated with discontinuities in $v^0(x,t)$ and $u^0(x,t)$ and this occurs when infinitely many paths $X(s)$ focus in zero volume centred at x. For non-degenerate critical paths $X(\cdot)$, when the multiplicity of $x_0(x,t)$, $n = n(x,t)$, is finite so that for a given x and t the set of possible initial positions x_0 is $\{x_0^1(x,t), x_0^2(x,t), \dots, x_0^n(x,t)\}$, we can deduce that

$$u^\mu(x,t) \sim \sum_{i=1}^{n} \theta_i \exp\left\{ -S_0^i(x,t)/\mu^2 \right\},$$

where

$$S_0^i(x,t) = S_0(x_0^i(x,t)) + A(x_0^i(x,t), x, t)$$

for $i = 1, 2, \dots, n$ and θ_i is an asymptotic series in μ^2 associated with $x_0^i(x,t)$ as above. (The detailed structure of θ_i may be developed by drawing on the papers of Davies and Truman [8, 9], Ellis and Rosen [14, 15, 16] and Truman and Zhao [37].) Needless to say the dominant term in the above comes from $\tilde{x}_0(x,t)$ the minimising $x_0(x,t)$ so that

$$S(x,t) = \min_{i=1,2,\dots,n} S_0^i(x,t)$$

in line with the results of Freidlin et al. [17, 19]. Here we assume, unless stated otherwise, that the minimiser $\tilde{x}_0(x,t)$ is unique. The caustic, where the focusing of the paths $X(s)$ occurs, is important since $u^0(x,t)$ can switch discontinuously from being exponentially large to exponentially small as we cross parts of the caustic. This is because two of the $x_0^i(x,t)$ can coalesce and then disappear causing the minimising S_0^i to disappear.

In this paper we develop inequalities showing how closely related the stochastic caustic and wavefront are to their classical dynamical counterparts for small noise. Once again the classical dynamical structures guide the arguments and we state our results for the caustic in terms of the classical path X^0, the stochastic path X^ϵ and their near neighbour $\tilde{X}^\epsilon$. We also derive inequalities relating ∇X^0, ∇X^ϵ and $\nabla \tilde{X}^\epsilon$. For small noise, the stochastic wavefront is shown to lie within an ϵ neighbourhood of the classical wavefront by utilizing the solution of the matrix Jacobi equation, the Green's function, to construct the appropriate estimates. Our main Theorems (3.5) and (4.6) relate to the non-existence of an invariant measure for the small noise stochastic Burgers equation when there is resonance between the underlying classical mechanical problem and the almost periodic noise. It is here that we use the detailed properties of Jacobi fields, developed by imitating more or less standard results for the linear harmonic oscillator.

2 Stochastic Dynamics, H_t and C_t

We now define our main structures and state without proof some of the key results from our earlier paper [10]. The familiar objects from classical dynamics are easily recognisable.

Define $A(x_0, p_0, t)$, the stochastic action, to be

$$\frac{1}{2}\int_0^t \left|\dot{X}(s)\right|^2 ds - \int_0^t [c(X(s))ds + \epsilon k(X(s), s)dW_s], \qquad \text{a.s.,}$$

with $X(s) = X(s, x_0, p_0)$ satisfying

$$d\dot{X}(s) = -\nabla c(X(s))\, ds - \epsilon \nabla k(X(s), s)\, dW_s,$$

$s \in [0, t]$, $X(0) = x_0$, $\dot{X}(0) = p_0$, $x_0, p_0 \in \mathbb{R}^d$. We shall assume that the minimising $X(s)$ satisfying $X(t) = x$ and $\dot{X}(0) = \nabla S_0(X(0))$ is unique in any space with $\dot{X}$ continuous, where we assume as usual that $X(s)$ is $\mathcal{F}_s$ measurable. Later on we shall have to consider Poisson brackets $\{X(s), X(u)\}_{x_0, p_0}$ with respect to x_0, p_0 variables which we suppress. We also allow for p_0 to be an as yet unspecified function of x_0 such as $\nabla S_0(x_0)$.

Then, for ∇c, ∇k Lipschitz, with Hessians $\nabla^2 c$, $\nabla^2 k$ and all second derivatives with respect to space variables of c and k bounded, according to Kunita [27], $\partial X(s)/\partial x_0^\alpha$ satisfies

$$\frac{d}{ds}\left(\frac{\partial X(s)}{\partial x_0^\alpha}\right) = \frac{\partial \dot{X}(0)}{\partial x_0^\alpha} - \int_0^s \left[\nabla^2 c(X(r))\frac{\partial X(r)}{\partial x_0^\alpha}\, dr + \epsilon \nabla^2 k(X(r), r)\frac{\partial X(r)}{\partial x_0^\alpha}\, dW_r\right].$$

Moreover,

$$\dot{X}(s) = \dot{X}(0) - \int_0^s \left[\nabla c(X(r))\, dr + \epsilon \nabla k(X(r), r)\, dW_r\right]. \tag{2.1}$$

Define the random map $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ corresponding to the classical flow by the second order stochastic differential equation

$$d_s\dot{\Phi}_s = -\nabla c(\Phi_s)\, ds - \epsilon \nabla k(\Phi_s, s)\, dW_s,$$

with $\Phi_0 = I$ and $\dot{\Phi}_0 = \nabla S_0$.

We then have $X(s) = \Phi_s \Phi_t^{-1} x$, where we accept that $x_0(x, t) = \Phi_t^{-1} x$ may not be necessarily unique. Given some regularity, the global inverse function theorem gives a caustic time $T(\omega)$ (> 0) such that, for $s < T(\omega)$, Φ_s is a random diffeomorphism [41]. So for $t < T(\omega)$, $x_0(x,t)$ is unique. Therefore, as we shall see,

$$v^0(x,t) = \dot{\Phi}_t \Phi_t^{-1} x = \nabla S(x,t)$$

is a formal solution of Burgers equation with $\mu = 0$, which is well defined up to the caustic time $T(\omega)$.

Lemma 2.1. *Assume $S_0, c \in C^2$ and $k \in C^{2,0}$, ∇c, ∇k are Lipschitz, with Hessians $\nabla^2 c$, $\nabla^2 k$ and all second derivatives with respect to space variables of c and k bounded. If $\dot{X}(s)$ satisfies Equation (2.1) and we have p_0, possibly x_0 dependent, then almost surely*

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \dot{X}(t) \cdot \frac{\partial X(t)}{\partial x_0^\alpha} - \dot{X}_\alpha(0).$$

Remark 2.1. Observe that, if we fix $X(t)$ independently of x_0, we obtain almost surely

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = -\dot{X}_\alpha(0),$$

for $\alpha = 1, 2, \ldots, d$.

Let $X(s, x_0, x) = X(s, x_0, p_0)|_{p_0 = p(x_0, x, t)}$ where $p_0 = p(x_0, x, t)$ is the (assumed unique) random minimiser of $A(x_0, p_0, t)$ with $X(t, x_0, p_0) = x$.

Remark 2.2. Here we need the maps $\mathbb{R}^d \to \mathbb{R}^d$ defined by $p_0 \mapsto X(x_0, p_0, t)$, for fixed t, for each $x_0 \in \mathbb{R}^d$ to be onto with probability one (c.f Kolokoltsov et al [25, 26]).

Set $A(x_0, x, t) = A(x_0, p_0, t)|_{p_0 = p(x_0, x, t)}$ and define Hamilton's principal function corresponding to the initial momentum $\nabla S_0(x_0)$ to be

$$\mathcal{A}(x_0, x, t) = A(x_0, x, t) + S_0(x_0).$$

Then

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \text{for } \alpha = 1, 2, \ldots, d \quad \Longrightarrow \quad \dot{X}(0) = \nabla S_0(X(0))$$

which defines the classical flow map Φ.

We define a prelevel surface of Hamilton's principal function for real constant c by eliminating x between the equations

$$\mathcal{A}(x_0, x, t) = c \qquad \text{and} \qquad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0,$$

$\alpha = 1, 2, \ldots, d$, and a level surface H_t by eliminating x_0. We denote the prelevel surface by $\Phi_t^{-1} H_t$. Similarly we define the caustic C_t and the pre-caustic $\Phi_t^{-1} C_t$ by eliminating x_0 or x between

$$\operatorname{Det}\left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}(x_0, x, t)\right) = 0 \qquad \text{and} \qquad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0,$$

$\alpha = 1, 2, \ldots, d.$

Remark 2.3. Note that $\Phi_t^{-1} H_t$ (and $\Phi_t^{-1} C_t$) are determined by taking algebraic inverse images i.e. by eliminating x above. Those are not necessarily the same as the topological inverse images $\Phi_t^{-1}(H_t)$ and $\Phi_t^{-1}(C_t)$. In fact as we shall demonstrate $\Phi_t^{-1} C_t = \Phi_t^{-1}(C_t)$.

We shall need.

Lemma 2.2. *The classical flow map $x = \Phi_t(x_0)$ is a differentiable map from $\Phi_t^{-1} H_t$ to H_t with Fréchet derivative*

$$D\Phi_t(x_0) = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}(x_0, x, t)\right)^{-1} \left(\frac{\partial^2 \mathcal{A}}{\partial {x_0}^2}(x_0, x, t)\right),$$

if $\mathcal{A}$ is C^3 in space derivatives.

In the next section we investigate the linear harmonic oscillator case in detail. This will provide us with the main ideas for the proofs of our general results.

3 Harmonic Oscillator Wells with Noise

The stochastic harmonic oscillator was studied by Albeverio et al [1], Markus and Weerasinghe [28] and McKean [29]. The stochastic Mehler formula was firstly obtained by Truman and Zhao [39], [38], [40] and later using different techniques by Truman and Zastawniak [36].
Consider $v = v(x,t)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$, satisfying

$$dv + (v \cdot \nabla)v\, dt = -\Omega^2 x\, dt - \epsilon \nabla k(x,t)\, dW_t, \tag{3.1}$$

with $v(x,0) = \nabla S_0(x)$, i.e. v is Burgers velocity field for the perturbed harmonic oscillator potential

$$c(x) = \frac{1}{2} x^T \Omega^2 x,$$

where Ω^2 is a positive definite, real, symmetric matrix. Here the perturbing potential is the White noise term $k_t(x)\dot{W}_t$, with real valued $k_t(x) = k(x,t) \in C^{2,1}(\mathbb{R}^d, \mathbb{R}^+)$ and W_t is one dimensional brownian motion on the probability space $\{\Omega, \mathcal{F}, P\}$. The corresponding stochastic mechanics is

$$d_s \dot{X}^\epsilon(s) + \Omega^2 X^\epsilon(s)\, ds = -\epsilon \nabla k(X^\epsilon(s), s)\, dW_s, \tag{3.2}$$

with

$$X^\epsilon(0)(x_0) = x_0,$$
$$\dot{X}^\epsilon(0)(x_0) = \nabla S_0(x_0).$$

Here x_0 has to be chosen such that

$$X^\epsilon(t)(x_0) = x,$$

for fixed t and x in $v^\mu(x,t)$ and to minimise $\mathcal{A}(x_0, x, t)$, the choice being unique.

When $\epsilon = 0$, we have a harmonic oscillator, which defines the classical flow map Φ_s^0

$$X^0(s) = \Phi_s^0 x_0 = \cos(\Omega s) x_0 + \sin(\Omega s) \Omega^{-1} \nabla S_0(x_0), \tag{3.3}$$

Ω being the obvious positive square root of Ω^2.

Note that the solution $X^\epsilon(s)$ of (3.2) can be found explicitly as follows, provided $\nabla k(\cdot, s) = -a(s)$, is independent of spatial variables. We obtain in matrix notation

$$X^\epsilon(s) = \cos(\Omega s)x_0 + \sin(\Omega s)\Omega^{-1}\nabla S_0(x_0) + \epsilon \int_0^s \sin(\Omega(s-u))\Omega^{-1}a(u)\,dW_u. \tag{3.4}$$

It therefore turns out that

$$X^\epsilon(s) - \epsilon \int_0^s \sin(\Omega(s-u))\Omega^{-1}a(u)\,dW_u = \Phi_s^0 x_0. \tag{3.5}$$

The following simple result for shockwaves in a random environment is an easy corollary of the above computations.

Theorem 3.1. *At time t, let $\Delta_t^0(x_0) = \mathrm{Det}\left(\frac{\partial X^0(t)}{\partial x_0}\right) = \mathrm{Det}\left(\nabla_{x_0}\Phi_t^0 x_0\right) = 0$ be the equation of the pre-caustic in x_0, where Φ_t^0 is the classical flow map for the harmonic oscillator without noise and let $x = \Phi_t x_0$. Eliminating y_0, let the corresponding equation at time t in $y = \Phi_t y_0$ be*

$$\Delta_t^0\left((\Phi_t^0)^{-1}(y)\right) = 0.$$

Then, if $\nabla k(x,t) = -a(t)$, independent of x, the random shock wave in the presence of noise at time t has equation

$$\Delta_t^0\left((\Phi_t^0)^{-1}(y + \epsilon N(t))\right) = 0,$$

where $N(t)$ is the random vector

$$N(t) = \int_0^t \Omega^{-1}\sin(\Omega(t-s))a(s)\,dW_s.$$

Remark 3.1. In this case the noise leaves the shape of the shockwaves unchanged merely displacing them by a random vector $-N(t)$.

Remark 3.2. Needless to say $N(t)$ satisfies the second order stochastic differential equation

$$d\dot{N}_t + \Omega^2 N_t\,dt = a(t)\,dW_t,$$

where $N(0) = 0$, $\dot{N}(0) = 0$.

We need the following lemma, a simple one-dimensional result.

Lemma 3.2. *Let $a(\cdot) \in C^1(\mathbb{R}^+)$ be bounded, with $\lim_{t\to\infty} t^{-1/2}a(t) = 0$, and let the infinite time average*

$$\lim_{t\to\infty} t^{-1}\int_0^t a^2(u)\,du = M(a^2) > 0.$$

Then, if $d(a^2(u))/du \in L^1(\mathbb{R}^+)$, the leading behaviour of the noise term

$$N(t) = \int_0^t \Omega^{-1} \sin(\Omega(t-s))a(s)\, dW_s,$$

as $t \to \infty$ is given almost surely by

$$N(t) \sim \Omega^{-1} M^{\frac{1}{2}}(a^2) \left(\sin(\Omega t) B^{\mathrm{c}}(t) - \cos(\Omega t) B^{\mathrm{s}}(t)\right),$$

where B^{c} and B^{s} are normalised BM($\mathbb{R}$) processes B, with $\mathbb{E}(B^2(t)) = t$, correlated by

$$\mathbb{E}\left(B^{\mathrm{c}}(t) B^{\mathrm{s}}(u)\right) = \int_0^{\tau_{\mathrm{c}}(t)\wedge\tau_{\mathrm{s}}(u)} a^2(r) \cos(\Omega r) \sin(\Omega r)\, dr,$$

with

$$\tau_{\mathrm{c}}(t) = \inf\left\{ s > 0 : \int_0^s a^2(u) \cos^2(\Omega u)\, du = t \right\},$$
$$\tau_{\mathrm{s}}(t) = \inf\left\{ s > 0 : \int_0^s a^2(u) \sin^2(\Omega u)\, du = t \right\}.$$

Proof. We need McKean's result on time changed Brownian motion (see page 29 of McKean [29]). Now

$$\int_0^t a^2(u) \cos^2(\Omega u)\, du = \frac{1}{2} \int_0^t \left(1 + \cos(2\Omega u)\right) a^2(u)\, du$$

and

$$t^{-1} \int_0^t \cos(2\Omega u) a^2(u)\, du$$
$$= \frac{1}{2\Omega t} \left(a^2(t) \sin(2\Omega t) - \int_0^t \sin(2\Omega u) \frac{d}{du}(a^2(u))\, du \right) \to 0,$$

as $t \to \infty$. □

If we now work in cartesian axes relative to which Ω is diagonal, $\Omega_{ij} = \Omega_i \delta_{ij}$, $\Omega_i > 0$, for $i, j = 1, 2, \ldots, d$, we can deduce a vector version of the last lemma.

Lemma 3.3. *Working in coordinates in which Ω is diagonal and $N(t) = (N_1(t), \ldots, N_d(t))$, we obtain the behaviour as $t \to \infty$*

$$N_i(t) \sim \Omega_i^{-1} M^{\frac{1}{2}}(a_i^2) \left(\sin(\Omega_i t) B^{\mathrm{c}_i}(t) - \cos(\Omega_i t) B^{\mathrm{s}_i}(t)\right),$$

for $i = 1, \ldots, d$, where the $B^{\mathrm{c}_i}, B^{\mathrm{s}_i}$ are normalised correlated BM($\mathbb{R}$) processes with

$$\mathbb{E}\left(B^{\mathrm{c}_i}(t) B^{\mathrm{s}_j}(u)\right) = \int_0^{\tau_{\mathrm{c}_i}(t)\wedge\tau_{\mathrm{s}_j}(u)} a_i(r) a_j(r) \cos(\Omega_i r) \sin(\Omega_j r)\, dr,$$

together with similar results for $\mathbb{E}\left(B^{\mathrm{c}_i}(t) B^{\mathrm{c}_j}(u)\right)$ and $\mathbb{E}\left(B^{\mathrm{s}_i}(t) B^{\mathrm{s}_j}(u)\right)$.

Remark 3.3. The above results hold for any finite value of ϵ, no matter how large. This gives the leading behaviour of $N(t)$ as $t \to \infty$ in terms of the Jacobi fields of the zero noise problem and thence the large time behaviour of $v^0(x,t)$ and $X^\epsilon(x,t)$ and their sample paths. The Jacobi fields are also important in understanding the small noise processes, as we see in the next section.

Example. We now illustrate the development of the Semicubical Parabolic shockwave with respect to time for the velocity field of a Burgers' fluid with initial velocity $(xy, x^2/2)$ in a harmonic well with $V(x,y) = (x^2 + \omega^2 y^2)/2$. The effect of noise here is to superimpose on the deterministic movement of the Cusp caustic a brownian motion. Let us just consider the deterministic case. It is easy to see

$$\Phi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 \cos t + x_0 y_0 \sin t \\ y_0 \cos \omega t + \frac{1}{2\omega} x_0^2 \sin \omega t \end{pmatrix}. \tag{3.6}$$

So

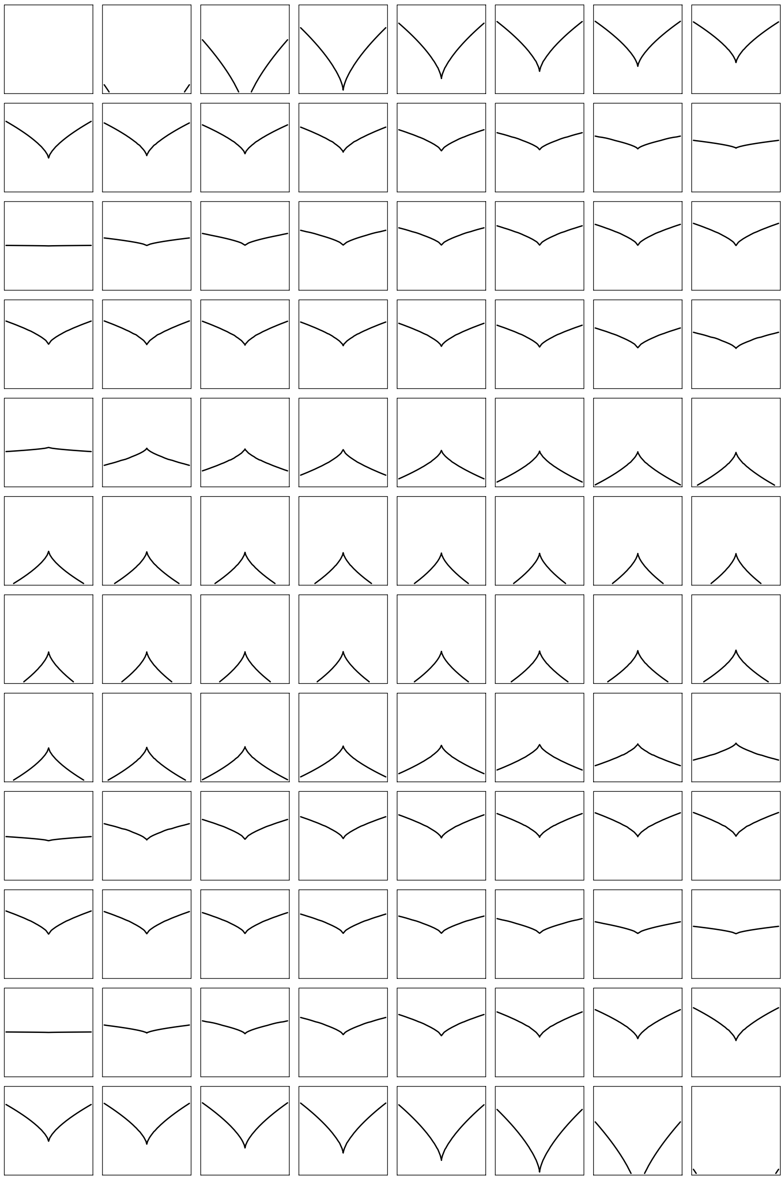
$$\begin{aligned} \Delta_t^0(x_0, y_0) &= \det \begin{pmatrix} \cos t + y_0 \sin t, & x_0 \sin t \\ \omega^{-1} x_0 \sin \omega t, & \cos \omega t \end{pmatrix} \\ &= \cos \omega t \cos t + y_0 \cos \omega t \sin t - \omega^{-1} x_0^2 \sin t \sin \omega t. \end{aligned} \tag{3.7}$$

The precaustic is then given by $\Delta_t^0(x_0, y_0) = 0$. The caustic is obtained by mapping the precaustic under the map Φ_t. If ω is rational then the deterministic motion of the cusp is periodic as is the deterministic part of v_t^0. The equation is then given by

$$8\left(y \csc \omega t + \cot t \cot \omega t\right)^3 \omega = 27 x^2 \cot^2 \omega t \csc^2 t. \tag{3.8}$$

The periodic motion of the cusp can be seen in the next figure where we plot a succession of caustics for the case $\omega = 3/2$. Time increases from left to right and top to bottom. Many more illustrations of this periodic behaviour can be found in the PhD thesis of B.T. Reynolds [31].

Figure 1. Periodic Motion of Caustic



We continue to investigate Burgers equation in $\mathbb{R}^d$, Equations (3.1) and (3.2), where $\nabla k(x,u) = -a(u)$ is independent of x and $\int_0^t |a(u)|^2\,du < \infty$. The key elementary lemma here is:-

Lemma 3.4. *Let $\dot{X}^\epsilon(t) - \dot{X}^0(t) = \dot{N}(t) = \epsilon\int_0^t \cos\left[\Omega(t-u)\right] a(u)\,dW_u$. Then, working in a coordinate system in which Ω^2 is diagonal, $\dot{N}(t)$ has mean zero* *and is gaussian with covariance*

$$\begin{aligned} A_{ij}^{-1}(t) &= \mathbb{E}\left[\dot N_i(t)\dot N_j(t)\right] \\ &= \frac{\epsilon^2}{4}\sum_{\pm}\Big\{ \cos\left((\Omega_i \pm \Omega_j)t\right)\int_0^t \cos\left((\Omega_i \pm \Omega_j)u\right) a_i(u)a_j(u)\,du \\ &\qquad + \sin\left((\Omega_i \pm \Omega_j)t\right)\int_0^t \sin\left((\Omega_i \pm \Omega_j)u\right) a_i(u)a_j(u)\,du\Big\}, \quad t>0. \end{aligned}$$

Proof. The point is that, if $0 = s_0 < s_1 < \cdots < s_{n+1} = t$ is a partition of $[0,t]$,

$$\dot N_i(t) = \lim \sum_j \cos\left[\Omega_i(t-s_j)\right] a_i(s_j)\left[W(s_{j+1}) - W(s_j)\right]$$

exhibits $\dot N(t)$ as the limit of a sum of independent gaussians. So for each t $\dot N(t)$ is gaussian with mean and covariance as above. □

A strikingly simple result emerges if we assume that $a(\cdot)$ is almost periodic in the sense that in our coordinate system $a(s) = (a_1(s), a_2(s), \ldots, a_d(s))$, where each a_j is almost periodic. (The class of almost periodic functions is a natural generalisation of periodic functions in that they can be realised as uniform limits of trigonometric polynomials and if f is almost periodic $\lim_{t\uparrow\infty} t^{-1}\int_{t_0}^{t_0+t} f(s)\,ds = M(f)$ exists, the class being closed under addition and multiplication.) When $a(\cdot)$ is almost periodic in our set-up we refer to almost periodic white noise forces. The possibility of resonance between the almost periodic noise and the restraining harmonic oscillator force affects the long time behaviour of the Burgers fluid.

Theorem 3.5. *For almost periodic white noise forces, for underlying harmonic oscillator potentials, the inviscid limit of Burgers fluid velocity $v^0(x,t)$ satisfies*

$$\mathbb{E}(v^0(x,t)) = \dot X^0(x,t) = \cos(\Omega t)\tilde x_0(x,t) + \Omega^{-1}\sin(\Omega t)\nabla S_0(\tilde x_0(x,t))$$

where $\tilde x_0(x,t)$ is the minimiser (assumed to be unique) of the deterministic $\mathcal{A}(x_0,x,t)$. Moreover,

$$\mathbb{P}\left(\left(v^0(x,t) - \dot X^0(x,t)\right) \in dv\right) = \exp\left(-\frac{v^T A(t) v}{2}\right)(2\pi)^{-d/2}(\det A)^{-1/2}\,dv,$$

where the matrix $A^{-1}(t)$ is specified above. In particular as $t \sim \infty$ we obtain

$$t^{-1}A_{ij}^{-1}(t) = t^{-1}\mathbb{E}\left[\left(v_i^0(x,t) - \dot X_i^0(x,t)\right)\left(v_j^0(x,t) - \dot X_j^0(x,t)\right)\right]$$

$$\sim \frac{\epsilon^2}{4} \sum_{\pm} \Big\{ \cos\left((\Omega_i \pm \Omega_j)t\right) M \left[\cos\left((\Omega_i \pm \Omega_j)\cdot\right) a_i(\cdot) a_j(\cdot)\right] + \sin\left((\Omega_i \pm \Omega_j)t\right) M \left[\sin\left((\Omega_i \pm \Omega_j)\cdot\right) a_i(\cdot) a_j(\cdot)\right] \Big\},$$

where M denotes the infinite time average.

Proof. The proof is a simple consequence of the properties of almost periodic functions Corduneanu [6]. □

Remark 3.4. (i) The ensemble average of the Burgers fluid velocity $\mathbb{E}\left(v^0(x,t)\right)$ inherits the singularity structure of $\dot{X}^0(x,t)$ with caustics depending on the initial velocity field ∇S_0.
(ii) The distribution of Burgers fluid velocity depends only on the forces in this harmonic oscillator case.
(iii) Evidently, if there is resonance between the noise and the harmonic oscillator forces, there is no invariant measure for this problem as expected.

We emphasize here that the above results are true for any value of $\epsilon > 0$. We see in the next section to what extent the above results generalise to the non-linear setting for small noise. Our main results will require new detailed estimates for this problem.

4 General Potentials with Small Noise Perturbation

Having considered a special case in the previous section we now demonstrate for small noise the closeness of the stochastic X_t, C_t and H_t to their deterministic counterparts for more general potentials. Let X^ϵ, with ϵ highlighted, satisfy

$$d\dot{X}^\epsilon(x_0, s) = -\nabla c(X^\epsilon(x_0, s))ds - \epsilon \nabla k(X^\epsilon(x_0, s))dW_s, \tag{4.1}$$

with $X^\epsilon(x_0, 0) = x_0$ and $\dot{X}^\epsilon(x_0, 0) = \nabla S_0(x_0)$, for $0 < s < t$. Abusing notation, let $X^0(x_0, s) = \Phi_s x_0$ be the deterministic version, where $\epsilon = 0$, and let $\mathcal{G}$ be given by $\mathcal{G}_{ij}(x_0, s, u) = \left\{X_i^0(u), X_j^0(s)\right\}\theta(s-u)$, the first term being the Poisson bracket, the second a Heaviside Function. The following elementary lemma is key.

Lemma 4.1. *With X^0 defined as above, $\mathcal{G}$ satisfies the matrix Jacobi equation*

$$\left(\frac{d^2}{ds^2} + \nabla^2 c(X^0(x_0, s))\right)\mathcal{G} = 0, \mathcal{G}(x_0, s_+, s) = 0, \left.\frac{d\mathcal{G}}{ds}(x_0, s, u)\right|_{s=u_+} = I.$$

Furthermore, $\frac{\partial}{\partial x_0^i}\mathcal{G}(x_0, s, u)$ *satisfies*

$$\frac{d^2}{ds^2}\frac{\partial}{\partial x_0^i}\mathcal{G}(x_0, s, u) + \sum_{j=1}^{n}\frac{\partial}{\partial X_j^0}\nabla^2 c(X^0(x_0, s))(\frac{\partial}{\partial x_0^i}X_j^0(x_0, s))\mathcal{G}(x_0, s, u)$$
$$+\nabla^2 c(X^0(x_0, s))\frac{\partial}{\partial x_0^i}\mathcal{G}(x_0, s, u) = 0, \tag{4.2}$$

and

$$\nabla_{x_0^i}\mathcal{G}(x_0, s_+, s) = 0, \qquad \left.\frac{d}{ds}\nabla_{x_0^i}\mathcal{G}(x_0, s, u)\right|_{s=u_+} = 0. \tag{4.3}$$

Proof. A trivial computation using the properties of Poisson brackets. □

We now come to one of our main results.

Theorem 4.2. *Subject to certain conditions on the continuity and boundedness of c and k and their derivatives, define*

$$\tilde{X}^\epsilon(x_0, s) = \Phi_s x_0 - \epsilon\int_0^s \mathcal{G}(x_0, s, u)\nabla k(\Phi_u x_0)\, dW_u,$$

for $s \in [0,t]$. Then there exists a constant $M > 0$ such that for any $\delta > 0$ and sufficiently small $\epsilon > 0$,

$$P\{\epsilon^{-\frac{3}{2}} \sup_{x_0 \in \mathbb{R}^d} |X^\epsilon(x_0,s) - \tilde{X}^\epsilon(x_0,s)| > \delta, \text{ some } s \in [0,t]\} < \frac{M\epsilon^2}{\delta^4}, \qquad (4.4)$$

and

$$P\{\epsilon^{-\frac{3}{2}} \sup_{x_0 \in \mathbb{R}^d} |\nabla X^\epsilon(x_0,s) - \nabla\tilde{X}^\epsilon(x_0,s)| > \delta, \text{ some } s \in [0,t]\} < \frac{M\epsilon}{\delta^2}. \quad (4.5)$$

We have $X^\epsilon(x_0,s) - \tilde{X}^\epsilon(x_0,s) = \circ(\epsilon^{\frac{3}{2}})$, $\nabla X^\epsilon(x_0,s) - \nabla\tilde{X}^\epsilon(x_0,s) = \circ(\epsilon^{\frac{3}{2}})$ as $\epsilon \to 0$ in probability.

Proof. (i) We first prove (4.4). From the definition of $\tilde{X}^\epsilon(x_0,s)$ and Lemma 4.1, it is easy to see

$$\dot{\tilde{X}}^\epsilon(x_0,s) = \dot{\Phi}_s x_0 - \epsilon\int_0^s \frac{\partial}{\partial s}\mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u, \qquad s \in [0,t].$$

Differentiating with respect to s again we have

$$\begin{aligned} d\dot{\tilde{X}}^\epsilon(x_0,s) &= -\nabla c(\Phi_s x_0)ds - \epsilon\nabla k(\Phi_s x_0)\, dW_s \\ &\quad - \epsilon\int_0^s \frac{\partial^2}{\partial s^2}\mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u ds \\ &= -\nabla c(\Phi_s x_0)ds + (\nabla^2 c(\Phi_s x_0)\epsilon\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u)ds \\ &\quad - \epsilon\nabla k(\Phi_s x_0)\, dW_s. \end{aligned}$$

It turns out that by Taylor's Theorem, there exist ξ_1 and ξ_2 such that for $0 \le s \le t$,

$$\begin{aligned} d\dot{\tilde{X}}^\epsilon(x_0,s) &= -\nabla c(\Phi_s x_0 - \epsilon\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u)ds \\ &\quad - \epsilon\nabla k(\Phi_s x_0 - \epsilon\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u)\, dW_s \\ &\quad + \epsilon^2\sum_{i,j=1}^d \frac{\partial^2}{\partial\xi_1^i\xi_1^j}\nabla c(\xi_1)(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u)_i \\ &\qquad \left(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u\right)_j ds \\ &\quad + \epsilon^2\nabla^2 k(\xi_2)(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u)dW_s \\ &= -\nabla c(\tilde{X}^\epsilon(x_0,s))ds - \epsilon\nabla_{\tilde{X}^0} k(\tilde{X}^\epsilon(x_0,s))dW_s \end{aligned}$$

$$+\epsilon^2 N_1(s)ds + \epsilon^2 N_2(s)dW_s.$$

Here N_1 and N_2 are given by

$$\begin{aligned}
N_1(s) &= \sum_{i,j=1}^{d} \frac{\partial^2}{\partial \xi_1^i \xi_1^j} \nabla c(\xi_1) \Big(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u\Big)_i \\
&\qquad \Big(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u\Big)_j, \\
N_2(s) &= \nabla^2 k(\xi_2) \int_0^s \mathcal{G}(x_0,s,u)\nabla_{X^0} k(\Phi_u x_0)\, dW_u. \qquad (4.6)
\end{aligned}$$

Now it is easy to see that

$$\begin{aligned}
\tilde{X}^\epsilon(x_0,s) &= x_0 - s\nabla S_0(x_0) - \int_0^s \int_0^r \nabla c(\tilde{X}^\epsilon(x_0,u))dudr \\
&\quad - \epsilon \int_0^s \int_0^r \nabla k(\tilde{X}^\epsilon(x_0,u))dW_u dr \\
&\quad + \epsilon^2 \int_0^s \int_0^r N_1(u)dudr + \epsilon^2 \int_0^s \int_0^r N_2(u)dW_u dr \\
&= x_0 - s\nabla S_0(x_0) - \int_0^s (s-r)\nabla c(\tilde{X}^\epsilon(x_0,r))dr \\
&\quad - \epsilon \int_0^s (s-r)\nabla k(\tilde{X}^\epsilon(x_0,r))dW_r \\
&\quad + \epsilon^2 \int_0^s (s-r)N_1(r)dr + \epsilon^2 \int_0^s (s-r)N_2(r)dW_r. \qquad (4.7)
\end{aligned}$$

On the other hand, the solution of (4.1) can be represented by

$$\begin{aligned}
X^\epsilon(x_0,s) &= x_0 - s\nabla S_0(x_0) - \int_0^s (s-r)\nabla c(X^\epsilon(x_0,r))dr \\
&\quad - \epsilon \int_0^s (s-r)\nabla k(X^\epsilon(x_0,r))dW_r. \qquad (4.8)
\end{aligned}$$

Therefore it turns out that there exists $M_1 > 0$ such that

$$\begin{aligned}
&|\tilde{X}^\epsilon(x_0,s) - X^\epsilon(x_0,s)|^4 \\
&\quad = \Big| - \int_0^s (s-r)(\nabla c(\tilde{X}^\epsilon(x_0,r)) - \nabla c(X^\epsilon(x_0,r)))dr \\
&\qquad - \epsilon \int_0^s (s-r)(\nabla k(\tilde{X}^\epsilon(x_0,r)) - \nabla k(X^\epsilon(r)))dW_r \\
&\qquad + \epsilon^2 \int_0^s (s-r)N_1(r)dr + \epsilon^2 \int_0^s (s-r)N_2(r)dW_r\Big|^4 \qquad (4.9) \\
&\quad \le M_1 \Big|\int_0^s (s-r)(\nabla c(\tilde{X}^\epsilon(x_0,r)) - \nabla c(X^\epsilon(x_0,r)))dr\Big|^4 \\
&\qquad + M_1\epsilon^4 \Big|\int_0^s (s-r)(\nabla k(\tilde{X}^\epsilon(x_0,r)) - \nabla k(X^\epsilon(x_0,r)))dW_r\Big|^4
\end{aligned}$$

$$+ M_1\epsilon^8|\int_0^s (s-r)N_1(r)dr|^4 + M_1\epsilon^8|\int_0^s (s-r)N_2(r)dW_r|^4.$$

Now using Hölder's inequality, the inequality $E(\int_0^s f(r)dW_r)^4 \leq M_2 s\int_0^s Ef^4(r)dr$ for a constant $M_2 > 0$, and the Lipschitz continuity we have

$$\begin{aligned}
&E\sup_{x_0\in\mathbb{R}^d}|\tilde{X}^\epsilon(x_0,s) - X^\epsilon(x_0,s)|^4 \\
&\quad\leq M_1(\int_0^s (s-r)^{\frac{4}{3}}dr)^3 E\int_0^s \sup_{x_0\in\mathbb{R}^d}|\nabla c(\tilde{X}^\epsilon(x_0,r)) - \nabla c(X^\epsilon(x_0,r))|^4 dr \\
&\qquad + M_1M_2\epsilon^4 s\int_0^s (s-r)^4 E\sup_{x_0\in\mathbb{R}^d}|\nabla k(\tilde{X}^\epsilon(x_0,r)) - \nabla k(X^\epsilon(r))|^4 dr \\
&\qquad + M_1\epsilon^8(\int_0^s (s-r)^{\frac{4}{3}}dr)^3\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_1(r)|^4 dr \\
&\qquad + M_1M_2\epsilon^8 s\int_0^s (s-r)^4 E\sup_{x_0\in\mathbb{R}^d}|N_2(r)|^4 dr \qquad (4.10)\\
&\quad\leq M_3 s^7 L^4\int_0^s E\sup_{x_0\in\mathbb{R}^d}|\tilde{X}^\epsilon(x_0,r) - X^\epsilon(x_0,r)|^4 dr \\
&\qquad + M_3 s^4\epsilon^4 L^4\int_0^s E\sup_{x_0\in\mathbb{R}^d}|\tilde{X}^\epsilon(x_0,r) - X^\epsilon(r)|^4 dr \\
&\qquad + M_3\epsilon^8 s^7\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_1(r)|^4 dr + M_3 s^4\epsilon^8\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_2(r)|^4 dr.
\end{aligned}$$

Here L is a Lipschitz constant of ∇c and ∇k and M_3 is a constant. But using inequality $E(\int_0^s f(r)dW_r)^8 \leq M_4 s^{\frac{4}{3}}\int_0^s E(f(r))^8 dr$ for a $M_4 > 0$, and taking $M_5 = n\max_{l,\xi_1,\xi_2}\{|\text{eigenvalue of the Hessian}(\nabla^2(\nabla_l c(\xi_1)))|, |\text{eigenvalue of the Hessian}(\nabla^2(k(\xi_2)))|\}$, then

$$\begin{aligned}
&E\sup_{x_0\in\mathbb{R}^d}|N_1(r)|^4 \\
&\quad\leq M_4M_5 r^{\frac{4}{3}}\int_0^r \sup_{x_0\in\mathbb{R}^d} < \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0), \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0) >^4 du \\
&\quad\leq M_6 r^{\frac{4}{3}}, \qquad (4.11)
\end{aligned}$$

and

$$\begin{aligned}
&E\sup_{x_0\in\mathbb{R}^d}(N_2(r))^4 \\
&\quad= M_4M_2 r\int_0^r \sup_{x_0\in\mathbb{R}^d} < \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0), \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0) >^2\ du \\
&\quad\leq M_6 r. \qquad (4.12)
\end{aligned}$$

Here $M_6 > 0$ is a constant.
It follows from (4.10) that there exists $M_7 > 0$ such that

$$
\begin{aligned}
E & \sup_{x_0\in\mathbb{R}^d} (\tilde{X}^\epsilon(x_0,s) - X^\epsilon(x_0,s))^4 \\
& \qquad \le M_3 s^7 L^4 \int_0^s E \sup_{x_0\in\mathbb{R}^d} (\tilde{X}^\epsilon(x_0,r) - X^\epsilon(x_0,r))^4 dr \\
& \qquad\quad + M_3 s^4 \epsilon^4 L^4 \int_0^s E \sup_{x_0\in\mathbb{R}^d} (\tilde{X}^\epsilon(x_0,r) - X^\epsilon(x_0,r))^4 dr \\
& \qquad\quad + M_7 s^{\frac{28}{3}} \epsilon^8 + M_7 s^6 \epsilon^8. \qquad (4.13)
\end{aligned}
$$

By using the Gronwall inequality we know that for $0 \le s \le t$,

$$E \sup_{x_0\in\mathbb{R}^d} (\tilde{X}^\epsilon(x_0,s) - X^\epsilon(x_0,s))^4 \le M\epsilon^8, \qquad (4.14)$$

for a constant $M > 0$. Then (4.4) follows from the Chebyshev inequality.
Following a similar method, one can prove that $0 \le s \le t$,

$$E \sup_{x_0\in\mathbb{R}^d} (X(x_0,s))^4 \le M, \qquad (4.15)$$

for a constant $M > 0$.
(ii) Now we prove (4.5). Denote $\nabla_i = \frac{\partial}{\partial x_0^i}$. From the definition of $\tilde{X}^\epsilon(x_0,s)$ we know

$$
\begin{aligned}
\nabla_i \tilde{X}^\epsilon(x_0,s) = \nabla_i \Phi_s x_0 - \epsilon \int_0^s & \mathcal{G}(x_0,s,u) \nabla^2 k(\Phi_u x_0) \nabla_i \Phi_u x_0 \, dW_u \\
- \epsilon \int_0^s & \nabla_i \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u.
\end{aligned}
$$

Thus, applying Lemma 4.1 again we have

$$
\begin{aligned}
\frac{d}{ds} \nabla_i \tilde{X}^\epsilon(x_0,s) = \nabla_i \dot{\Phi}_s x_0 - \epsilon \int_0^s & \frac{\partial}{\partial s} \mathcal{G}(x_0,s,u) \nabla^2 k(\Phi_u x_0) \nabla_i \Phi_u x_0 \, dW_u \\
- \epsilon \int_0^s & \frac{\partial}{\partial s} \nabla_i \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u.
\end{aligned}
$$

Then differentiating again and using Lemma 4.1 we have

$$
\begin{aligned}
d & \frac{d}{ds} \nabla_i \tilde{X}^\epsilon(x_0,s) \\
& = -\nabla^2 c(\Phi_s x_0)(\nabla_i \Phi_s x_0) ds - \epsilon \nabla^2 k(\Phi_s x_0)(\nabla_i \Phi_s x_0) \, dW_s \\
& \quad - \epsilon \int_0^s \frac{\partial^2}{\partial s^2} \mathcal{G}(x_0,s,u) \nabla^2 k(\Phi_u x_0) \nabla_i \Phi_u x_0 \, dW_u ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \epsilon \int_0^s (\frac{\partial^2}{\partial s^2} \nabla_i \mathcal{G}(x_0,s,u)) \nabla k(\Phi_u x_0)\, dW_u ds \\
&= -\nabla^2 c(\Phi_s x_0)(\nabla_i \Phi_s x_0) ds \\
&\quad + \nabla^2 c(\Phi_s x_0) \epsilon \int_0^s \mathcal{G}(x_0,s,u) \nabla^2 k(\Phi_u x_0) \nabla_i \Phi_u x_0 \, dW_u ds \\
&\quad + \sum_{j=1}^d \frac{\partial}{\partial X_j} \nabla^2 c(\Phi_s x_0)(\nabla_i \Phi_s^j x_0) \epsilon \int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u ds \\
&\quad + \nabla^2 c(\Phi_s x_0) \epsilon \int_0^s (\nabla_i \mathcal{G}(x_0,s,u)) \nabla k(\Phi_u x_0)\, dW_u ds \\
&\quad - \epsilon \nabla^2 k(\Phi_s x_0)(\nabla_i \Phi_s x_0)\, dW_s \\
&= -\nabla^2 c(\Phi_s x_0) \nabla_i \tilde{X}^\epsilon(x_0,s) ds \\
&\quad + \sum_{j=1}^d \frac{\partial}{\partial X_j} \nabla^2 c(\Phi_s x_0)(\nabla_i \Phi_s^j x_0) \epsilon \int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u ds \\
&\quad - \epsilon \nabla^2 k(\Phi_s x_0)(\nabla_i \Phi_s x_0)\, dW_s.
\end{aligned}
$$

Now using Taylor's theorem we know that there exist ξ_3 and ξ_4 such that

$$
\begin{aligned}
& d\frac{d}{ds} \nabla_i \tilde{X}^\epsilon(x_0,s) \\
&\quad = -\nabla^2 c(\Phi_s x_0 - \epsilon \int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u) \nabla_i \tilde{X}^\epsilon(x_0,s) ds \\
&\qquad - \epsilon \nabla^2 k(\Phi_s x_0 - \epsilon \int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u) \nabla_i \tilde{X}^\epsilon(x_0,s)\, dW_s \\
&\qquad - \epsilon^2 \sum_{j=1}^d \frac{\partial}{\partial X_j} \nabla^2 c(\Phi_s x_0) \left[\int_0^s \mathcal{G}(x_0,s,u) \nabla^2 k(\Phi_u x_0) \nabla_i X^0(x_0,u)\, dW_u \right. \\
&\qquad \left. + \int_0^s \nabla_i \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u \right] \\
&\qquad\qquad \times \left(\int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u \right)_j ds \\
&\qquad + \epsilon^2 \sum_j \sum_l (\nabla_{\xi_3^j} \nabla_{\xi_3^l} \nabla^2 c(\xi_3)) (\int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u)_j \\
&\qquad\qquad \times (\int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u)_l \nabla_i \tilde{X}^\epsilon(x_0,s) ds \\
&\qquad - \epsilon^2 \sum_j \nabla_{\xi_4^j} \nabla^2 k(\xi_4) (\int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0)\, dW_u))_j \nabla_i \tilde{X}^\epsilon(x_0,s)) dW_s \\
&\quad = -\nabla^2 c(\tilde{X}^\epsilon(x_0,s)) \nabla_i \tilde{X}^\epsilon(x_0,s) ds - \epsilon \nabla^2 k(\tilde{X}^\epsilon(x_0,s)) \nabla_i \tilde{X}^\epsilon(x_0,s) dW_s \\
&\qquad + \epsilon^2 N_3(s) ds + \epsilon^2 N_4(s) dW_s, \quad s \in [0,t].
\end{aligned}
$$

Here N_3 and N_4 are given by

$$\begin{aligned}
N_3(s) = &-\sum_{j=1}^{d}\frac{\partial}{\partial X_j}\nabla^2 c(\Phi_s x_0)\Big[\int_0^s \mathcal{G}(x_0,s,u)\nabla^2 k(\Phi_u x_0)\nabla_i X^0(x_0,u)\,dW_u \\
&+\int_0^s \nabla_i\mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\,dW_u\Big] \\
&\quad\times\Big(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\,dW_u\Big)_j \\
&+\sum_j\sum_l(\nabla_{\xi_3^j}\nabla_{\xi_3^l}c''(\xi_3))(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\,dW_u)_j \qquad (4.16)\\
&\quad\times(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\,dW_u)_l\nabla_i\tilde{X}^\epsilon(x_0,s) \\
N_4(s) = &-\sum_j\nabla_{\xi_4^j}\nabla^2 k(\xi_4)(\int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\,dW_u))_j\nabla_i\tilde{X}^\epsilon(x_0,s)).
\end{aligned}$$

Using the Fubini theorem, it is easy to see that

$$\begin{aligned}
\nabla_i\tilde{X}^\epsilon(x_0,s) &= \mathrm{e}_i - s\nabla_i\nabla S_0(x_0) - \int_0^s\int_0^r \nabla^2 c(\tilde{X}^\epsilon(x_0,u))du\nabla_i\tilde{X}^\epsilon(x_0,r))dr \\
&\quad-\epsilon\int_0^s\int_0^r \nabla^2 k(\tilde{X}^\epsilon(x_0,u))dW_u\nabla_i\tilde{X}^\epsilon(x_0,r))dr \\
&\quad+\epsilon^2\int_0^s\int_0^r N_3(u)dudr + \epsilon^2\int_0^s\int_0^r N_4(u)dW_u dr \\
&= \mathrm{e}_i - s\nabla_i\nabla S_0(x_0) - \int_0^s (s-r)\nabla^2 c(\tilde{X}^\epsilon(x_0,r))\nabla_i\tilde{X}^\epsilon(x_0,r))dr \\
&\quad-\epsilon\int_0^s (s-r)\nabla^2 k(\tilde{X}^\epsilon(x_0,r))\nabla_i\tilde{X}^\epsilon(x_0,r))dW_r \\
&\quad+\epsilon^2\int_0^s (s-r)N_3(r)dr + \epsilon^2\int_0^s (s-r)N_4(r)dW_r. \qquad (4.17)
\end{aligned}$$

Here $\mathrm{e}_i = (0,0,\cdots,0,1,0,\cdots,0)$.
On the other hand, differentiating (4.8) we have

$$\begin{aligned}
\nabla_i X^\epsilon(x_0,s) &= \mathrm{e}_i - s\nabla_i\nabla S_0(x_0) - \int_0^s (s-r)\nabla^2 c(X^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r))dr \\
&\quad-\epsilon\int_0^s (s-r)\nabla^2 k(X^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r))dW_r. \qquad (4.18)
\end{aligned}$$

Note here $\nabla_i\nabla S_0, \nabla^2 c, \nabla^2 k$ are bounded. So it is easy to prove there exists $M_1 > 0$ such that

$$E\sup_{x_0\in\mathbb{R}^d}|\nabla_i X^\epsilon(x_0,s)|^4 \le M_1. \qquad (4.19)$$

Therefore it turns out that

$$
\begin{aligned}
&|\nabla_i\tilde{X}^\epsilon(x_0,s)-\nabla_i X^\epsilon(x_0,s)|^2\\
&\qquad=\Big|-\int_0^s(s-r)(\nabla^2 c(X^\epsilon(x_0,r))-\nabla^2 c(\tilde{X}^\epsilon(x_0,r)))\nabla_i X^\epsilon(x_0,r)dr\\
&\qquad\quad-\int_0^s(s-r)(\nabla^2 c(\tilde{X}^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))dr\\
&\qquad\quad-\epsilon\int_0^s(s-r)(\nabla^2 k(X^\epsilon(x_0,r))-\nabla^2 k(\tilde{X}^\epsilon(x_0,r)))\nabla_i X^\epsilon(x_0,r)dW_r\\
&\qquad\quad-\epsilon\int_0^s(s-r)\nabla^2 k(\tilde{X}^\epsilon(x_0,r)(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))dW_r\\
&\qquad\quad+\epsilon^2\int_0^s(s-r)N_3(r)dr+\epsilon^2\int_0^s(s-r)N_4(r)dW_r\Big|^2. \qquad (4.20)
\end{aligned}
$$

This leads to for any $0\le s\le t$,

$$
\begin{aligned}
&E\sup_{x_0\in\mathbb{R}^d}|\nabla_i\tilde{X}^\epsilon(x_0,s)-\nabla_i X^\epsilon(x_0,s)|^2\\
&\quad\le 6E\sup_{x_0\in\mathbb{R}^d}|\int_0^s(s-r)(\nabla^2 c(\tilde{X}^\epsilon(x_0,r))-\nabla^2 c(X^\epsilon(x_0,r)))\nabla_i X^\epsilon(x_0,r)dr|^2\\
&\qquad+6E\sup_{x_0\in\mathbb{R}^d}|\int_0^s(s-r)(\nabla^2 c(\tilde{X}^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))dr|^2\\
&\qquad+6\epsilon^2E\sup_{x_0\in\mathbb{R}^d}|\int_0^s(s-r)(\nabla^2 k(\tilde{X}^\epsilon(x_0,r))-\nabla^2 k(X^\epsilon(r)))\nabla_i X^\epsilon(x_0,r)dW_r|^2\\
&\qquad+6\epsilon^2E\sup_{x_0\in\mathbb{R}^d}|\int_0^s(s-r)\nabla^2 k(\tilde{X}^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))dW_r|^2\\
&\qquad+6\epsilon^4E\sup_{x_0\in\mathbb{R}^d}|\int_0^s(s-r)N_3(r)dr|^2+6\epsilon^4E|\int_0^s(s-r)N_4(r)dW_r|^2. \qquad (4.21)
\end{aligned}
$$

Denote by $M_2=\sup\limits_{x\in\mathbb{R}^d}\{|\text{eigenvalue of }\nabla^2 c(x))|,|\text{eigenvalue of }\nabla^2 k(x))|\}$ and by L the Lipschitz constant of $\nabla^2 c$ and $\nabla^2 k$. So by the Hölder inequality and the Lipschitz continuity we have

$$
\begin{aligned}
&E\sup_{x_0\in\mathbb{R}^d}|\nabla_i\tilde{X}^\epsilon(x_0,s)-\nabla_i X^\epsilon(x_0,s)|^2\\
&\quad\le 6E\sup_{x_0\in\mathbb{R}^d}\int_0^s(s-r)^2|\nabla^2 c(\tilde{X}^\epsilon(x_0,r))-\nabla^2 c(X^\epsilon(x_0,r))|^2dr\int_0^s|\nabla_i X^\epsilon(x_0,r)|^2dr\\
&\qquad+6E\sup_{x_0\in\mathbb{R}^d}\int_0^s(s-r)^2dr\int_0^s|\nabla^2 c(\tilde{X}^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))|^2dr\\
&\qquad+6\epsilon^2E\sup_{x_0\in\mathbb{R}^d}\int_0^s(s-r)^2|\nabla^2 k(\tilde{X}^\epsilon(x_0,r))-\nabla^2 k(X^\epsilon(r)))\nabla_i X^\epsilon(x_0,r)|^2dr
\end{aligned}
$$

$$
\begin{aligned}
&+6\epsilon^2 E\sup_{x_0\in\mathbb{R}^d}\int_0^s (s-r)^2|\nabla^2 k(\tilde{X}^\epsilon(x_0,r))(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))|^2 dr\\
&+6\epsilon^4 E\sup_{x_0\in\mathbb{R}^d}\int_0^s (s-r)^2 dr\int_0^s |N_3(r)|^2 dr+6\epsilon^4 E\sup_{x_0\in\mathbb{R}^d}\int_0^s (s-r)^2|N_4(r)|^2 dr\\
\le{}& 6\{E\sup_{x_0\in\mathbb{R}^d}(\int_0^s (s-r)^2|\nabla^2 c(\tilde{X}^\epsilon(x_0,r))-\nabla^2 c(X^\epsilon(x_0,r))|^2 dr)^2\}^{\frac{1}{2}}\\
&\times\{E\sup_{x_0\in\mathbb{R}^d}(\int_0^s (\nabla_i X^\epsilon(x_0,r))^2 dr)^2\}^{\frac{1}{2}}\\
&+2M_2 s^3\int_0^s E\sup_{x_0\in\mathbb{R}^d}|\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r)|^2 dr\\
&+6\epsilon^2\int_0^s (s-r)^2\{E\sup_{x_0\in\mathbb{R}^d}(\nabla^2 k(\tilde{X}^\epsilon(x_0,r))-\nabla^2 k(X^\epsilon(r)))^4\}^{\frac{1}{2}}\\
&\{E\sup_{x_0\in\mathbb{R}^d}(\nabla_i X^\epsilon(x_0,r))^4\}^{\frac{1}{2}}dr\\
&+6\epsilon^2 M_2 s^2 E\int_0^s E\sup_{x_0\in\mathbb{R}^d}(\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r)))^2 dr\\
&+2\epsilon^4 s^3\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_3(r)|^2 dr+6s^2\epsilon^4\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_4(r)|^2 dr.
\end{aligned}
\tag{4.22}
$$

This leads to

$$
\begin{aligned}
&E\sup_{x_0\in\mathbb{R}^d}|\nabla_i\tilde{X}^\epsilon(x_0,s)-\nabla_i X^\epsilon(x_0,s)|^2\\
&\le 6L^2\{\int_0^s (s-r)^4 dr\int_0^s E\sup_{x_0\in\mathbb{R}^d}|\tilde{X}^\epsilon(x_0,r)-X^\epsilon(x_0,r)|^4 dr\}^{\frac{1}{2}}\\
&\quad\times\{\int_0^s E\sup_{x_0\in\mathbb{R}^d}|\nabla_i X^\epsilon(x_0,r)|^4 dr\}^{\frac{1}{2}}\\
&\quad+6L^2\epsilon^2\int_0^s (s-r)^2\{E\sup_{x_0\in\mathbb{R}^d}|\tilde{X}^\epsilon(x_0,r))-X^\epsilon(r)|^4\}^{\frac{1}{2}}\\
&\quad\times\{E\sup_{x_0\in\mathbb{R}^d}|\nabla_i X^\epsilon(x_0,r)|^4\}^{\frac{1}{2}}dr\\
&\quad+2M_2 s^3\int_0^s E|\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r)|^2 dr\\
&\quad+6\epsilon^2 M_2 s^2\int_0^s E\sup_{x_0\in\mathbb{R}^d}|\nabla_i X^\epsilon(x_0,r)-\nabla_i\tilde{X}^\epsilon(x_0,r))|^2 dr\\
&\quad+2\epsilon^4 s^3\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_3(r)|^2 dr+6s^2\epsilon^4\int_0^s E\sup_{x_0\in\mathbb{R}^d}|N_4(r)|^2 dr.
\end{aligned}
\tag{4.23}
$$

Note (4.19). Similar to the estimate of N_1 and N_2, we can prove that there exists $M_3 > 0$ such that

$$E \sup_{x_0 \in \mathbb{R}^d} |N_3(r)|^2 \le M_3 < \infty, \tag{4.24}$$

and

$$E \sup_{x_0 \in \mathbb{R}^d} |N_4(r)|^2 \le M_3 < \infty. \tag{4.25}$$

It follows from (4.14) and (4.23-4.25) that there exists $M_4 > 0$ such that

$$\begin{aligned} E &\sup_{x_0 \in \mathbb{R}^d} |\nabla_i \tilde{X}^\epsilon(x_0, s) - \nabla_i X^\epsilon(x_0, s)|^2 \\ &\le M_4 s^3 \int_0^s E \sup_{x_0 \in \mathbb{R}^d} |\nabla_i \tilde{X}^\epsilon(x_0, r) - \nabla_i X^\epsilon(x_0, r)|^2 dr \\ &\quad + M_4 \epsilon^2 \int_0^s E \sup_{x_0 \in \mathbb{R}^d} |\nabla_i \tilde{X}^\epsilon(x_0, r) - \nabla_i X^\epsilon(x_0, r)|^2 dr \\ &\quad + M_4 \epsilon^4. \end{aligned} \tag{4.26}$$

By using the Gronwall inequality we know that

$$E \sup_{x_0 \in \mathbb{R}^d} |\nabla_i \tilde{X}^\epsilon(x_0, s) - \nabla_i X^\epsilon(x_0, s)|^2 \le M \epsilon^4, \tag{4.27}$$

for a constant $M > 0$. Then (4.5) follows from Chebyshev inequality. □

4.1 Behaviour of the Caustic

Define for any given $0 \le T < \infty$

$$\Omega_0 = \{\omega \in \Omega : \epsilon^{-\frac{3}{2}} \sup_{x_0 \in \mathbb{R}^d} |\nabla X^\epsilon(x_0, s) - \nabla \tilde{X}^\epsilon(x_0, s)| < \delta \text{ for all } s \in [0, T]\}.$$

Then from Theorem 4.2

$$P\{\Omega_0\} > 1 - \frac{M\epsilon}{\delta^2}.$$

Now we define for each $\omega \in \Omega_0$,
In the case of the caustic C_t we define

$$\begin{aligned} \mathcal{D}^\epsilon_{pre} &= \{(t, x_0) : \text{Det}\, \nabla_{x_0} X^\epsilon_t(x_0) \ne 0, 0 \le t \le T\}, \\ \mathcal{D}_{pre} &= \{(t, x_0) : \text{Det}\, \nabla_{x_0} \Phi_t x_0 \ne 0, 0 \le t \le T\}. \end{aligned}$$

Lemma 4.3. *As $\epsilon \to 0$, $[0, T] \times \mathbb{R}^d - \mathcal{D}^\epsilon_{pre} \to [0, T] \times \mathbb{R}^d - \mathcal{D}_{pre}$ in probability for any given $T > 0$. That is to say the precaustic surface of the stochastic dynamics converges to the precaustic surface of the classical mechanics as $\epsilon \to 0$ in probability.*

Proof. From Lemma 4.1, it is easy to see that for each $\omega \in \Omega_0$

$$\nabla_{x_0} X^\epsilon(x_0, s) = \nabla\Phi_s x_0 - \epsilon \int_0^s \mathcal{G}(x_0, s, u)\nabla^2 k(\Phi_u x_0)\nabla_{x_0}\Phi_u x_0 dW_u + \circ(\epsilon).$$

So if $\text{Det}(\nabla\Phi_s x_0) \neq 0$, then for sufficiently small ϵ, $\text{Det}(\nabla X^\epsilon(x_0, s)) \neq 0$. That is to say, $\mathcal{D}^\epsilon_{pre} \to \mathcal{D}_{pre}$ as $\epsilon \to 0$ in probability. So as $\epsilon \to 0$, $[0, T] \times \mathbb{R}^d - \mathcal{D}^\epsilon_{pre} \to [0, T] \times \mathbb{R}^d - \mathcal{D}_{pre}$ in probability. □

Theorem 4.4. *Denote for any given* $T > 0$

$$\begin{aligned} \mathcal{D}^\epsilon &= \{(t, x) : \text{Det}(\nabla(X^\epsilon)_t^{-1} x) \neq 0, 0 \leq t \leq T\}, \\ \mathcal{D} &= \{(t, x) : \text{Det}(\nabla\Phi_t^{-1} x) \neq 0, 0 \leq t \leq T\}. \end{aligned}$$

Then $[0, T] \times \mathbb{R}^d - \mathcal{D}^\epsilon \to [0, T] \times \mathbb{R}^d - \mathcal{D}$*, as* $\epsilon \to 0$*. That is to say the caustic surface of the stochastic dynamics with noise converges in probability to the caustic surface of the classical mechanics without noise as* $\epsilon \to 0$.

Proof. Denote by $\mathcal{D}^\epsilon_{pre}(s)$ a projection of $\mathcal{D}^\epsilon_{pre}$ on $\mathbb{R}^d$ such that $(s, \mathcal{D}^\epsilon_{pre}(s)) \subset \mathcal{D}^\epsilon_{pre}$. So

$$\mathcal{D}^\epsilon = \{(s, X^\epsilon_s(\mathcal{D}^\epsilon_{pre}(s))),\ \text{all } s\}. \tag{4.28}$$

But

$$X^\epsilon_s(\mathcal{D}^\epsilon_{pre}(s)) = X_s(\mathcal{D}^\epsilon_{pre}(s)) + (X^\epsilon_s(\mathcal{D}^\epsilon_{pre}(s)) - X_s(\mathcal{D}^\epsilon_{pre}(s))). \tag{4.29}$$

But from Lemma 4.1 and Lemma 4.3, as $\epsilon \to 0$ in probability,

$$X_s(\mathcal{D}^\epsilon_{pre}(s)) \to X_s(\mathcal{D}_{pre}(s)) \tag{4.30}$$

and

$$X^\epsilon_s(\mathcal{D}^\epsilon_{pre}(s)) - X_s(\mathcal{D}^\epsilon_{pre}(s)) \to 0. \tag{4.31}$$

Therefore $X^\epsilon_s(\mathcal{D}^\epsilon_{pre}(s)) \to X_s(\mathcal{D}_{pre}(s))$, i.e. $\mathcal{D}^\epsilon(s) \to \mathcal{D}(s)$. So $\mathcal{D}^\epsilon \to \mathcal{D}$. □

4.2 Behaviour of the Wavefront

To characterize the wavefront H_t we must consider the Hamilton principal function $S^\epsilon(x, t)$ where we have, once again, emphasized the ϵ dependence.

Theorem 4.5. *Let* ϕ_s *be the minimizer of*

$$\frac{1}{2}\int_0^t |\dot{\phi}_s|^2 ds + S_0(\phi_t) - \int_0^t c(\phi_s) ds,$$

satisfying $\phi_t = x$ *and* ϕ^ϵ_s *be the minimizer of* $\frac{1}{2}\int_0^t |\dot{\phi}^\epsilon_s|^2 ds + S_0(\phi^\epsilon_t) - \int_0^t c(\phi^\epsilon_s)ds - \epsilon\int_0^t k(\phi^\epsilon_s)dW_s$*, satisfying* $\phi^\epsilon_t = x$ *for almost all* $\omega \in \Omega$*. Then we have for almost all* $\omega \in \Omega$

$$S(x,t) - \epsilon\int_0^t k(\phi^\epsilon_s)dW_s \le S^\epsilon(x,t) \le S(x,t) - \epsilon\int_0^t k(\phi_s)dW_s. \qquad (4.32)$$

In particular, as $\epsilon \to 0$*,* $S^\epsilon(x,t) \to S(x,t)$ *a.s..*
Furthermore, for the classical mechanics, assume there exists a unique x_0 *for fixed* t *and* x *such that* $\Phi_t x_0 = x$*. The random wavefront for the heat equation has* (x,t) *equation*

$$S^0(x,t) = \epsilon\int_0^t k(\Phi_s x_0)\,dW_s + \circ(\epsilon),$$

where $S^0(x,t)$ *is Hamilton's principal function for the path* $X^0(x,t)(s)$*.*

Proof. By the definition of ϕ_s and ϕ^ϵ_s, it is easy to see for almost all ω,

$$\frac{1}{2}\int_0^t |\dot{\phi}^\epsilon_s|^2 ds + S_0(\phi^\epsilon_t) - \int_0^t c(\phi^\epsilon_s)ds \ge \frac{1}{2}\int_0^t |\dot{\phi}_s|^2 ds + S_0(\phi_t) - \int_0^t c(\phi_s)ds, \quad (4.33)$$

and

$$\begin{aligned}&\frac{1}{2}\int_0^t |\dot{\phi}_s|^2 ds + S_0(\phi_t) - \int_0^t c(\phi_s)ds - \epsilon\int_0^t k(\phi_s)dW_s \\ &\quad \ge \frac{1}{2}\int_0^t |\dot{\phi}^\epsilon_s|^2 ds + S_0(\phi^\epsilon_t) - \int_0^t c(\phi^\epsilon_s)ds - \epsilon\int_0^t k(\phi^\epsilon_s)dW_s. \qquad (4.34)\end{aligned}$$

It turns out that

$$\begin{aligned}&\frac{1}{2}\int_0^t |\dot{\phi}_s|^2 ds + S_0(\phi_t) - \int_0^t c(\phi_s)ds - \epsilon\int_0^t k(\phi_s)dW_s \\ &\ge \frac{1}{2}\int_0^t |\dot{\phi}^\epsilon_s|^2 ds + S_0(\phi^\epsilon_t) - \int_0^t c(\phi^\epsilon_s)ds - \epsilon\int_0^t k(\phi^\epsilon_s)dW_s \\ &\ge \frac{1}{2}\int_0^t |\dot{\phi}_s|^2 ds + S_0(\phi_t) - \int_0^t c(\phi_s)ds - \epsilon\int_0^t k(\phi^\epsilon_s)dW_s. \qquad (4.35)\end{aligned}$$

That is (4.32). Denote

$$\Omega_1 = \{\omega \in \Omega : \epsilon \sup_{\substack{x_o\in\mathbb{R}^d \\ 0\le t\le T}} |\int_0^t k(\phi_s)dW_s| \le \delta\}, \qquad (4.36)$$

and

$$\Omega_2 = \{\omega \in \Omega : \epsilon \sup_{\substack{x_o\in\mathbb{R}^d \\ 0\le t\le T}} |\int_0^t k(\phi^\epsilon_s)dW_s| \le \delta\}. \qquad (4.37)$$

But it is easy to see that

$$P\{\Omega_1\} \geq P\{\epsilon \int_0^t k(\phi_s)dW_s - \frac{\delta\epsilon^2}{2\epsilon^2 \max\limits_x k^2(x)T} \int_0^t k^2(\phi_s)ds \leq \frac{1}{2}\delta\}$$
$$\geq 1 - \mathrm{e}^{-\frac{\delta^2}{2\epsilon^2 \max\limits_x k^2(x)T}}. \tag{4.38}$$

Similarly

$$P\{\Omega_2\} \geq 1 - \mathrm{e}^{-\frac{\delta^2}{2\epsilon^2 \max\limits_x k^2(x)T}}. \tag{4.39}$$

So for $\omega \in \Omega_1 \cap \Omega_2$,

$$S(x,t) - \delta \leq S^\epsilon(x,t) \leq S(x,t) + \delta. \tag{4.40}$$

From this, it is easy to conclude as $\epsilon \to 0$, $S^\epsilon(x,t) \to S(x,t)$ almost surely. Now we assume that there exists a unique minimiser $x_0 = x_0(x,t)$ for fixed t and x such that $\Phi_t x_0 = x$. Consider the solution of the Hamilton-Jacobi equation for $\mathcal{L}(q,\dot{q}) = \dot{q}^2/2 + c(q)$,

$$S(x,t) = S_0(x_0(x,t)) + \int_0^t \mathcal{L}(q(s),\dot{q}(s))ds|_{q=q_0},$$

where q_0 is $X_0(s) = \Phi_s x_0(x,t)$. Here S satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \frac{1}{2}|\nabla S|^2 + c(x) = 0 \text{ and } S(x,0) = S_0(x). \tag{4.41}$$

Consider

$$\delta q(s) = \Phi_s x_0(x + \epsilon \int_0^t \mathcal{G}(x_0,t,u)\nabla k(\Phi_u x_0)\, dW_u, t)$$
$$- \epsilon \int_0^s \mathcal{G}(x_0,s,u)\nabla k(\Phi_u x_0)\, dW_u - \Phi_s x_0(x,t).$$

By Taylor's theorem, there exists ξ such that

$$\delta q(s) = (\frac{\partial}{\partial x_0}\Phi_s x_0)\frac{\partial}{\partial x}x_0(x,t)(\epsilon \int_0^t \mathcal{G}(x_0,t,u)\nabla k(\Phi_u x_0)\, dW_u)$$
$$+ \sum_{i,j,l=1}^d \Bigg[\sum_{k=1}^d (\nabla_k \nabla_l \Phi_s x_0(\xi,t))\frac{\partial}{\partial \xi_i}x_0^k(\xi,t)\frac{\partial}{\partial \xi_j}x_0^l(\xi,t)$$
$$+ (\nabla_l \Phi_s x_0(\xi,t))\frac{\partial^2}{\partial \xi_i \xi_j}x_0^l(\xi,t)\Bigg]$$
$$\times (\epsilon \int_0^t \mathcal{G}(x_0,t,u)\nabla k(\Phi_u x_0)\, dW_u))_i (\epsilon \int_0^t \mathcal{G}(x_0,t,u)\nabla k(\Phi_u x_0)\, dW_u))_j$$

$$- \epsilon \int_0^s \mathcal{G}(x_0, s, u) \nabla k(\Phi_u x_0) \, dW_u,$$

and computing $\dot{q}(s)$ gives

$$\begin{aligned}
\delta \dot{q}(s) = &(\frac{\partial}{\partial x_0}) \dot{\Phi}_s x_0 \frac{\partial}{\partial x} x_0(x,t)(\epsilon \int_0^t \mathcal{G}(t,u) \nabla k(\Phi_u x_0) \, dW_u) \\
&+ \sum_{i,j,l=1}^{d} \Bigg[\sum_{k=1}^{d} (\nabla_k \nabla_l \dot{\Phi}_s x_0(\xi,t)) \frac{\partial}{\partial \xi_i} x_0^k(\xi,t) \frac{\partial}{\partial \xi_j} x_0^l(\xi,t) \\
&\quad + (\nabla_l \dot{\Phi}_s x_0(\xi,t)) \frac{\partial^2}{\partial \xi_i \xi_j} x_0^l(\xi,t) \Bigg] \\
&\times (\epsilon \int_0^t \mathcal{G}(x_0,t,u) \nabla k(\Phi_u x_0) \, dW_u))_i (\epsilon \int_0^t \mathcal{G}(x_0,t,u) \nabla k(\Phi_u x_0) \, dW_u))_j \\
&- \epsilon \int_0^s \frac{\partial}{\partial s} \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u.
\end{aligned}$$

Therefore, as required,

$$\delta \dot{q}(s) = \frac{d}{ds} \delta q(s). \tag{4.42}$$

We see by definition that

$$\delta q(t) = 0, \tag{4.43}$$

$$\delta q(0) = x_0(x + \epsilon \int_0^t \mathcal{G}(x_0,t,u) \nabla k(\Phi_u x_0) \, dW_u, t) - x_0(x,t). \tag{4.44}$$

Define

$$\Omega_3 = \{ \sup_{\substack{x_0 \in \mathbb{R}^d \\ 0 \le s \le t}} \epsilon^{\frac{1}{2}} | \int_0^s \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) dW_u | \le \delta \}. \tag{4.45}$$

It is easy to see that

$$P\{\Omega_3\} > 1 - \mathrm{e}^{-\frac{\delta^2}{2\epsilon \sup_{x_0 \in \mathbb{R}^d} \int_0^T (\mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0))^2 du}}. \tag{4.46}$$

Then for each $\omega \in \Omega_3$ there exists ξ_1 such that,

$$\begin{aligned}
\delta q(0) &= \frac{\partial}{\partial x} x_0(x,t)(\epsilon \int_0^t \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u) \\
&\quad + \sum_{i,j=1}^{d} \frac{\partial^2}{\partial \xi_i \partial \xi_j} x_0(\xi_1,t)(\epsilon \int_0^t \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u)_i \\
&\qquad (\epsilon \int_0^t \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u)_j \\
&= \frac{\partial}{\partial x} x_0(x,t)(\epsilon \int_0^t \mathcal{G}(x_0,s,u) \nabla k(\Phi_u x_0) \, dW_u) + \circ(\epsilon).
\end{aligned} \tag{4.47}$$

Now we have

$$\begin{aligned} \delta S & \qquad \delta S_0 + \delta A \\ &=< (\frac{\partial}{x_0} S_0), (\frac{\partial x_0}{\partial x})(\epsilon \int_0^t \mathcal{G}(x_0, s, u) \nabla k(\Phi_u x_0)\, dW_u) > + \delta A. \end{aligned} \tag{4.48}$$

And,

$$\begin{aligned} \delta A \epsilon & \int_0^t k(\Phi_s x_0) dW_s \\ & + \int_0^t \mathcal{L}(q_0 + \delta q_0, \dot{q}_0 + \delta \dot{q}_0) ds - \int_0^t \mathcal{L}(q_0, \dot{q}_0) ds \\ & = \epsilon \int_0^t k(\Phi_s x_0) dW_s + \int_0^t (\frac{\partial \mathcal{L}}{\partial q} \delta q_0 + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q}_0) ds \\ & = \epsilon \int_0^t k(\Phi_s x_0) dW_s + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q_0 \right]_{s=0}^{s=t} \\ & + \int_0^t (\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{ds} (\frac{\partial \mathcal{L}}{\partial \dot{q}})) \delta q_0 ds \\ & = \epsilon \int_0^t k(\Phi_s x_0) dW_s - < \dot{q}(0), \delta q(0) > . \end{aligned} \tag{4.49}$$

It turns out that for each $\omega \in \Omega_3$,

$$\begin{aligned} \delta A = \epsilon & \int_0^t k(\Phi_s x_0) dW_s \\ & - < \nabla S_0(x_0), \frac{\partial}{\partial x} x_0(x,t) (\epsilon \int_0^t \mathcal{G}(x_0, s, u) \nabla k(\Phi_u x_0)\, dW_u) > + \circ(\epsilon). \end{aligned} \tag{4.50}$$

Therefore for each $\omega \in \Omega_3$,

$$\delta S = \epsilon \int_0^t k(\Phi_s x_0) dW_s + \circ(\epsilon). \tag{4.51}$$

The wave front, for each $\omega \in \Omega_3$, satisfies

$$S(x,t) = \epsilon \int_0^t k(\Phi_s x_0) dW_s + \circ(\epsilon). \tag{4.52}$$

□

4.3 Large time behaviour for almost periodic small noise

Our Theorem (4.2) shows that for small noise $\dot{\tilde{X}}^\epsilon(x,t)$ is close to the actual solution of our problem in the inviscid limit. Also, we have an exact expression for $\tilde{X}^\epsilon$ in terms of the Jacobi fields for $\tilde{X}^0$, If we now assume that

$\nabla k(x,s) = -a(s)$ is independent of x, we can define $\dot{\tilde{X}}^\epsilon(x,t)$ and $\tilde{X}^\epsilon(x,t)$ in the same way as we did for the harmonic oscillator and, since we are working only to first order in ϵ, the $x^0(x,t)$'s are just the deterministic ones. It is therefore not surprising that there is an analogue of Theorem (3.5).

Theorem 4.6. *Assume that $\tilde{X}^0(x,t)$ is periodic together with its Jacobi fields $\frac{\partial \tilde{X}^0}{\partial \gamma_0^\alpha}$, $\alpha = 1,2,\ldots,2d$ where $\gamma_0 = (x_0,p_0)^T$. Then, if $\int_0^t |a(s)|^2\,ds < \infty$, dropping $\tilde{}$, the ensemble average is given by*

$$\mathbb{E}\left(\dot{\tilde{X}}^\epsilon(x,t)\right) = \dot{\tilde{X}}^0(x,t) = \dot{X}^0(x,t).$$

Moreover,

$$\mathbb{P}\left(\left(\dot{\tilde{X}}^\epsilon(x,t) - \dot{X}^0(x,t)\right) \in dv\right) = \exp\left(-\frac{v^T A(t) v}{2}\right)(2\pi)^{-d/2}(\det A)^{-1/2}\,dv,$$

where the $d \times d$ matrix $A^{-1}(t)$ is given by

$$A^{-1}_{ii'}(t) = \epsilon^2 \int_0^t \frac{\partial X_i^0(u)}{\partial \gamma_0^\alpha} a_j(u) \frac{\partial X_{i'}^0(u)}{\partial \gamma_0^{\alpha'}} a_{j'}(u)\,du J_{\alpha\beta} J_{\alpha'\beta'} \frac{\partial \dot{X}_j^0(t)}{\partial \gamma_0^\beta} \frac{\partial \dot{X}_{j'}^0(t)}{\partial \gamma_0^{\beta'}},$$

$i,i' = 1,2,\ldots,d$, where J is the complex structure and the summation convention is enforced. In particular, if $a(\cdot)$ is almost periodic, with the above product almost periodic uniformly in (x_0,p_0) in a compact set K,

$$t^{-1} A^{-1}_{ii'}(t) \sim \epsilon^2 M\left(\frac{\partial X_i^0}{\partial \gamma_0^\alpha} a_j \frac{\partial X_{i'}^0}{\partial \gamma_0^{\alpha'}} a_{j'}\right) J_{\alpha\beta} J_{\alpha'\beta'} \frac{\partial \dot{X}_j^0(t)}{\partial \gamma_0^\beta} \frac{\partial \dot{X}_{j'}^0(t)}{\partial \gamma_0^{\beta'}},$$

where M denotes the infinite time average.

Proof. This is a simple rewrite of the proof of Theorem (3.5). We need uniform almost periodicity because at the last step we have to put $p_0 = \nabla S_0(x_0)$ and $x_0 = \tilde{x}_0(x,t)$ the minimiser (assumed unique) of $\mathcal{A}(x_0,x,t)$. We tacitly assume that $(\tilde{x}_0(x,t), \nabla S_0(\tilde{x}_0(x,t))) \in K$, our compact set. □

Remark 4.1. The remarks after Theorem (3.5) are all relevant here save that the distribution of the Burgers fluid velocity now depends on the initial conditions. This distribution is in fact discontinuous across the cool part of the caustic because $\tilde{x}_0(x,t)$ jumps as we cross this part of the caustic, making the Jacobi fields jump here as well. The periodic behaviour is in $\frac{\partial \dot{X}_j^0(t)}{\partial \gamma_0^\beta}$ and the resonance which is relevant is that between the zero noise Jacobi fields $\frac{\partial X_i^0}{\partial \gamma_0^\alpha}$ and the almost periodic noise term a. As expected the resonance destroys any possibility of an invariant measure existing. When the infinite time averages above are all zero the invariant measure should exist for a suitable restricted class of initial conditions.

4.4 Archetypal Example with some Illustrations

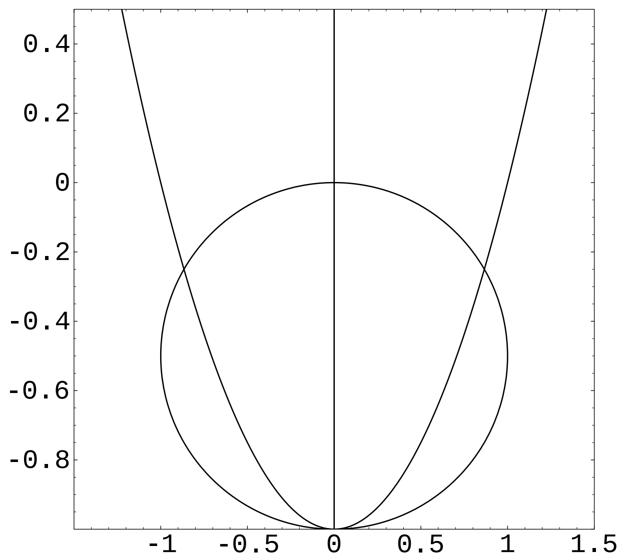
Having discussed the properties of the surfaces C_t and H_t we now illustrate what is perhaps the archetypal case with $S_0(x_0, y_0) = x_0^2 y_0/2$ and zero potentials.
Figure 2 illustrates the critical case $\mathcal{A} = 0$ where the precaustic is a parabola and the prelevel surface consists of an ellipse and a line pair. The zero prelevel surface has equation

$$x_0^2 \left(\frac{(y_0 + \frac{1}{2t})^2}{(\frac{1}{2t})^2} + \frac{x_0^2}{(\frac{1}{t})^2} - 1 \right) = 0,$$

and the precaustic has equation

$$1 + ty_0 = t^2 x_0^2.$$

Figure 2. Precaustic and Prelevel Surface



We now map $(x, y) = \Phi_t(x_0, y_0)$ to obtain the hypocycloid tricorn

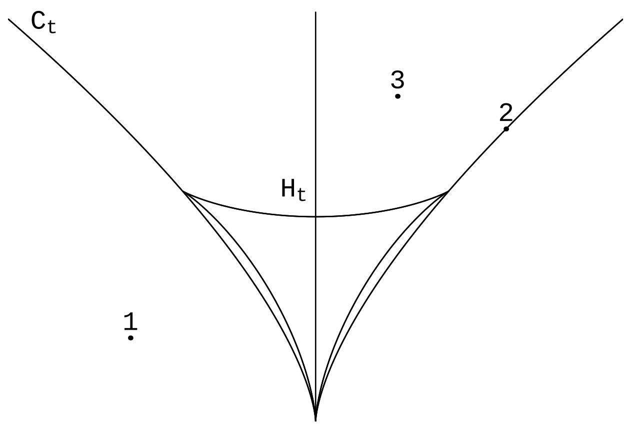
$$x = \frac{\cos\theta(1 + \sin\theta)}{2t}, \qquad y = \frac{\sin\theta(1 - \sin\theta)}{2t}, \tag{4.53}$$

for $0 \leq \theta < 2\pi$, as the zero level surface at time t for the heat equation and semi-cubical parabolic cusp

$$8\,(yt+1)^3 = 27t^2x^2, \tag{4.54}$$

as the caustic at time t for the corresponding Burgers velocity field. Figure 3 illustrates the level surface H_t and the caustic C_t. The number of solutions $x_0(x,t)$ is also indicated for the three different regions (within, on and outwith the semicubical parabola).

Figure 3. Cusp and Tricorn



In this case $x_0(x,t)$ is the point $(a, y - ta^2/2)$ where $a \in \mathbb{R}$ satisfies the cubic $t^2a^3/2 - a(1+ty) + x = 0$ and $\tilde{x}_0(x,t)$ corresponds to the value of a minimising $a^2(t^3a^4/8 - ta^3(1+4ty)/8 + y(1+ty)/2)$. We note that the vanishing of the discriminant of the cubic for a defines the Cusp caustic and that it can be shown that $\tilde{x}_0(x,t)$ jumps on crossing this caustic and nowhere else.

We now consider the stochastic case at $t = 1$ where $\epsilon = \frac{1}{10}$ and $k(x, y, t) = x$. Figures 4 and 5 illustrate the case where we have $\mathcal{A} = 0$.

Figure 4. Stochastic Precaustic and Prelevel Surface

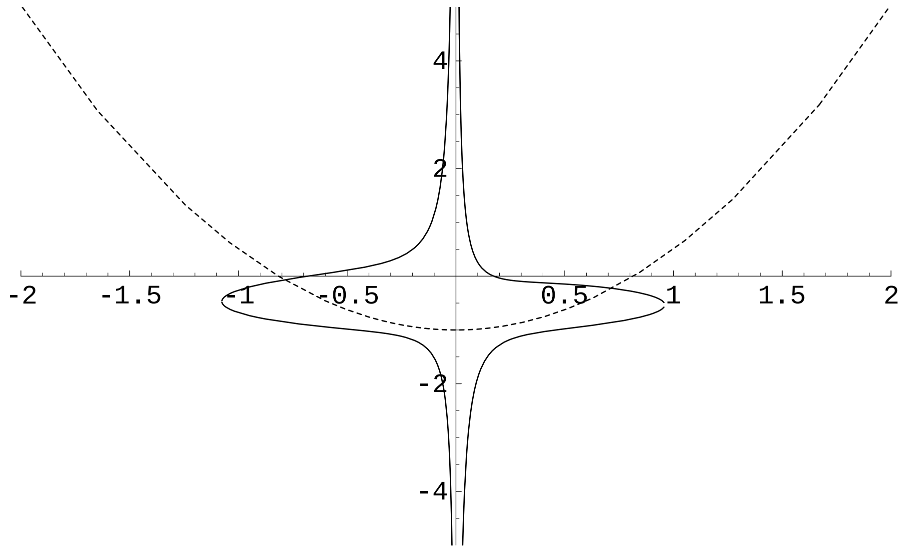
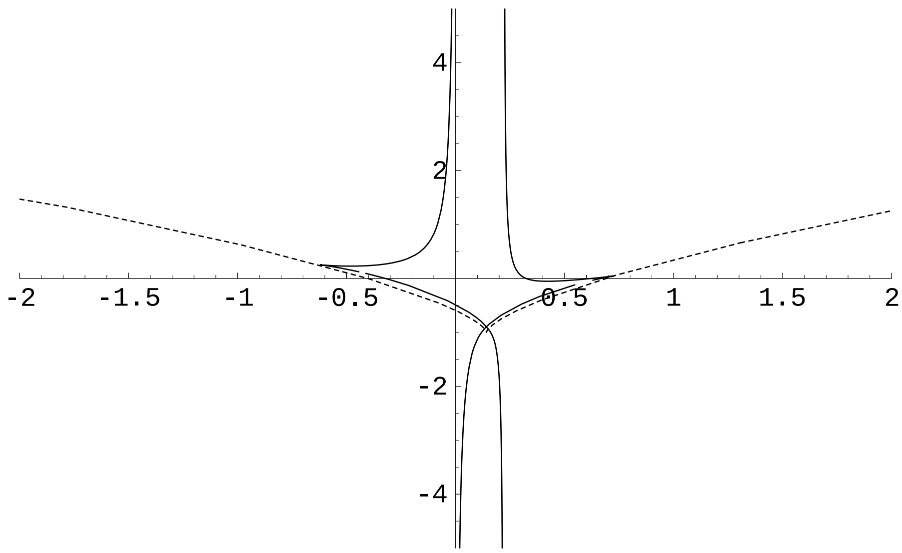


Figure 5. Stochastic Caustic and Level Surface



Figures 6 and 7 illustrate the case of $\mathcal{A} = -\frac{1}{64}$.

Figure 6. Stochastic Precaustic and Prelevel Surface

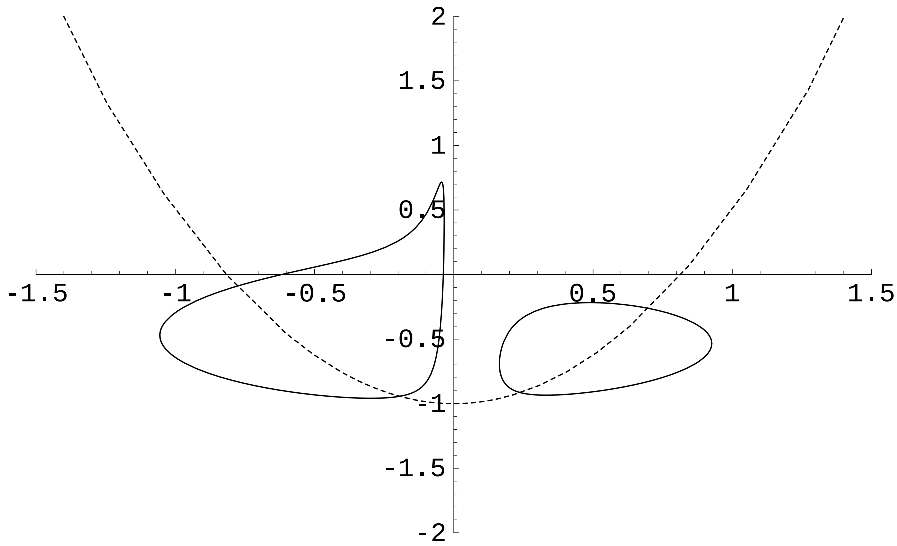
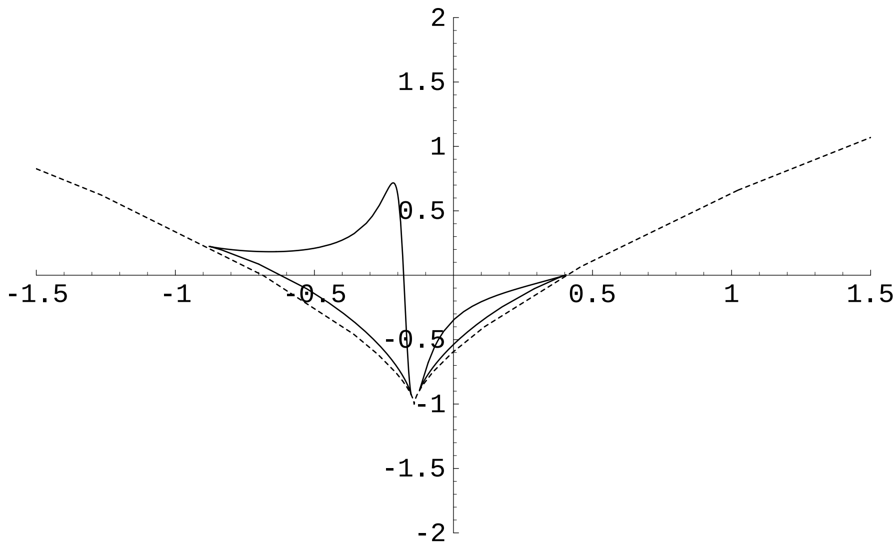


Figure 7. Stochastic Caustic and Level Surface



We refrain from presenting a gallery of two and three dimensional images herein but such can be found in the recent doctoral theses of B. T. Reynolds and C.N. Reynolds [31, 32].

5 Singularities and Intermittence of Turbulence

We have analysed the stochastic Burgers equation for small noise and shown how for our initial conditions resonance between the almost periodic noise and the underlying deterministic problem destroys any possibility of the existence of an invariant measure. We have also seen how for small noise the ensemble average and the distribution of the Burgers velocity inherit the singularity structure of the underlying Hamiltonian system, in particular the way that the caustics and shockwaves are still relevant. This is the main object lesson of the present work. These results rely heavily on our exact analysis of the harmonic oscillator and free cases.

Previously we detailed the caustic wavefront interaction in both the classical and stochastic case by appealing to geometrical properties of the Hamiltonian system. Now we have detailed information about these stochastic caustic wavefront structures in terms of the classical structures, at least in the case of small ϵ.

We now emphasise that new features emerge if one tries to use the above ideas to analyse the "intermittence" of stochastic turbulence as opposed to deterministic turbulence. The reason for the sharp contrast between stochastic and deterministic turbulence can already be seen here if we associate turbulent behaviour with (an infinitely rapid) change in the number of cusped curves on the minimising level surface of the Hamilton-Jacobi function. The times t when this occurs are just the times when the prelevel surfaces *touch* the precaustic. The times t when this number of curves changes in the deterministic case are simply the zeros of a deterministic function ζ, usually isolated zeros.

In the stochastic case ζ is a stochastic process whose zeros usually form a perfect set i.e. an infinite set containing no isolated points. At these times the number of cusped curves changes with infinite frequency because of the infinitely rapid oscillation of the stochastic process ζ. This is in line with what one would expect for turbulent behaviour. When the stochastic process ζ is *recurrent* this turbulent behaviour is "intermittent" so that the scale of turbulent fluctuations varies in a random periodic way.
For example, in the two dimensional case of $c \equiv 0$, $k_t(x, y) \equiv x$ and $S_0(x_0, y_0) = f(x_0) + g(x_0)y_0$, where f, g, f' and g' are zero at $x_0 = a$, $g''(a) \neq 0$, the turbulent times t at which $n_c(t)$, the number of cusps on the zero pre-level surface of the Hamilton-Jacobi function changes are the zeros of the stochastic tur-

bulence process ζ,

$$\zeta(t) = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s \, ds - \frac{\epsilon^2}{2} \int_0^t W_s^2 \, ds \ .$$

The turbulence appears at the point $\Phi_t(a, -\frac{1+tf''(a)}{tg''(a)})$ where $(a, -\frac{1+tf''(a)}{tg''(a)})$ is a point on the pre-caustic with equation

$$y_0 = \frac{-1 + t^2(g'(x_0))^2 - tf''(x_0)}{tg''(x_0)}.$$

Now $\{t : \zeta(t) = 0\}$ is a perfect set and $\zeta(t)$ is recurrent to 0, whilst $\zeta_c(t) = \zeta(t) - c$ has exactly the same properties. Zeros of $\zeta_c(t)$ are times at which the number of cusps on the c pre-level surface of the Hamilton-Jacobi function changes (see Reynolds, Truman and Williams [33]).

We may also introduce a singularity periodic in the (x_0, y_0) coordinates.

Lemma 5.1. *Consider the initial function* $S_0(x_0, y_0) = f(x_0) + g(x_0)\gamma(y_0)$. *If* f, f', g, g' *are zero at* α_i *for* $i = 1, 2, \ldots, n$ *and* $g''(\alpha_i) \neq 0$ *then the zero pre-curves will touch at*

$$\left(\alpha_i, \gamma^{-1}\left(\frac{-1 - tf''(\alpha_i)}{tg''(\alpha_i)}\right)\right) \ ,$$

for $i = 1, 2, \ldots, n$ *if* γ *is invertible.*

Proof. The pre-curves meet at solutions x_0 of $F_0(x_0, t) = 0$ where

$$F_0(x_0, t) = \frac{t}{2}\left(f' + g'\gamma(y_0(x_0))\right)^2 + \frac{t}{2} g^2 \gamma'(y_0(x_0))^2 + f + g\gamma(y_0(x_0)) \ ,$$

and $y_0(x_0)$ is the pre-caustic. Now

$$\begin{aligned} \frac{\partial F_0}{\partial x_0}(x_0, t) &= t(f' + g'\gamma(y_0(x_0))\frac{\partial}{\partial x_0}\left(f' + g'\gamma(y_0(x_0))\right) + tgg'\gamma'(y_0(x_0))^2 \\ &+ tg^2\gamma'(y_0(x_0))\gamma''(y_0(x_0))y_0'(x_0) + f' + g'\gamma(y_0(x_0)) + g\gamma'(y_0(x_0))y_0'(x_0) \ , \end{aligned}$$

so that clearly $F_0(\alpha_i, t) = 0$ and $\left.\frac{\partial F_0}{\partial x_0}\right|_{x_0=\alpha_i} = 0$. Hence the pre-curves will touch at $(\alpha_i, y_0(\alpha_i))$ where $y_0(x_0)$ is the pre-caustic. However in this case the pre-caustic is given by

$$\left(1 + t(f'' + g''\gamma(y_0)\right)\left(1 + tg\gamma''(y_0)\right) - t^2 g'^2 \gamma'(y_0)^2 = 0 \ ,$$

so that at $x_0 = \alpha_i$

$$\gamma(y_0) = \frac{-1 - tf''(\alpha_i)}{tg''(\alpha_i)} \ .$$

Hence

$$y_0(\alpha_i) = \gamma^{-1}\left(\frac{-1 - tf''(\alpha_i)}{tg''(\alpha_i)}\right) \ .$$

□

Corollary 5.2. *If $S_0(x_0, y_0) = f(x_0) + g(x_0)\gamma(y_0)$ where γ is a periodic function with period b and f, g, f', g' are zero at α_i for $i = 1, 2, \ldots, n$ whilst $g''(\alpha_i) \neq 0$ then the pre-curves will touch at*

$$(x_0, y_0) = \left(\alpha_i, \gamma^{-1}\left(\frac{-1 - tf''(\alpha_i)}{tg''(\alpha_i)}\right) + lb\right) \ ,$$

for $l \in \mathbb{Z}$ and $i = 1, 2, \ldots, n$.

In order to obtain a periodic cusp singularity we set $\gamma(\cdot) := \frac{b}{2\pi}\sin\left(\frac{2\pi}{b}\ \cdot\right)$, where $b > 0$, so that γ has period b and $\gamma(y) \sim y$ for $y \sim 0$. Clearly if we take $f \equiv 0$ and $g(x_0) := \frac{a^2}{2\pi^2}\sin^2\left(\frac{\pi x_0}{a}\right)$ then f, g, f' and g' are zero at $x_0 = ka$ ($k \in \mathbb{Z}$) and $g''(ka) = 1$. Moreover

$$\begin{aligned}\gamma^{-1}\left(\frac{-1 - tf''(ka)}{tg''(ka)}\right) &= \gamma^{-1}\left(-\frac{1}{t}\right) \\ &= \frac{b}{2\pi}\arcsin\left(-\frac{2\pi}{bt}\right) \ ,\end{aligned}$$

which exists if $b \geq \frac{2\pi}{t}$. Since the conditions of Corollary 5.2 are satisfied we obtain the following proposition.

Proposition 5.3. *Consider the initial function $S_0(x_0, y_0) = \frac{a^2 b}{4\pi^3}\sin^2\left(\frac{\pi x_0}{a}\right)\sin\left(\frac{2\pi y_0}{b}\right)$. If $b \geq \frac{2\pi}{t}$ then the pre-curves will touch at*

$$(x_0, y_0) = \left(ka, \frac{b}{2\pi}\arcsin\left(-\frac{2\pi}{bt}\right) + lb\right) \ ,$$

for $k, l \in \mathbb{Z}$.

In Figures 8 and 9 we have shown the pre-curves and image curves respectively where $a = 3$, $b = 4$ and $t = 2$ so that $b \geq \frac{2\pi}{t}$.

Figure 8. Periodic zero pre-level surface and pre-caustic

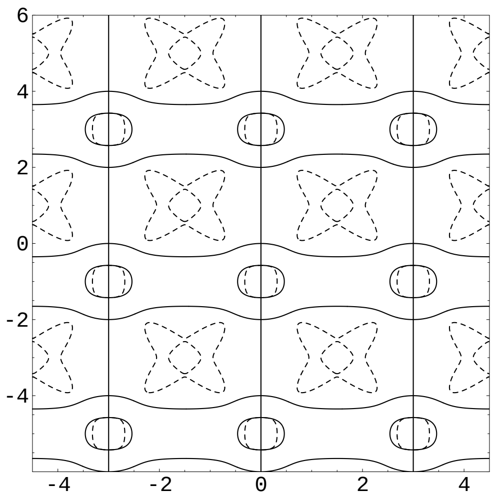
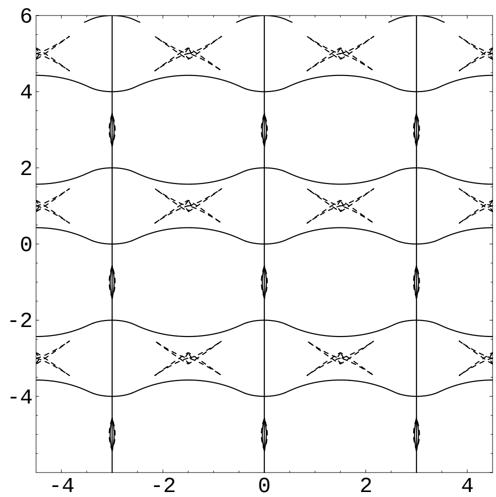


Figure 9. Periodic zero level surface and caustic



We shall not pursue the notion of periodic singularities any further, but suffice to say many examples may be easily created.

Hence, the second object lesson of the present study is that the number of cusped curves on the wavefront will change infinitely rapidly in the stochastic case when the presurfaces touch and that this behaviour will recur in a random periodic way if the stochastic process ζ is recurrent. This is the "intermittence" of stochastic turbulence in our model. There is no analogue of this for the deterministic Burgers equation.

We hope to investigate these phenomena in more detail in a future paper.

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References

[1] S. Albeverio, A. Hilbert and E. Zehnder, *Hamiltonian systems with stochastic forces: nonlinear versus linear and a Girsanov formula*, Stochastics and Stochastics Reports **39** (1992), pp. 159-188.

[2] S. Albeverio, S. Molchanov and D. Surgailis, *Stratified structure of the universe and Burgers equation - a probability approach*, Probab. Theory Relat. Fields **100** (1994), pp. 457-484.

[3] V. I. Arnol'd, *Singularities of caustics and wave fronts, Mathematics and its Applications (Soviet Series), 62*, (Kluwer Academic Publishers Group, Dordrecht, 1990)

[4] L. Bertini, N. Cancrini and G. Jona-Lasinio, *The stochastic Burgers equation,* Comm. Math. Phys. **165** (1994), pp. 211-232.

[5] Z. Brzeźniak and F. Flandoli, *Regularity of solutions and random evolution operator for stochastic parabolic equations*, Pitman Research Notes in Mathematics Series **268** (1992), pp. 54.

[6] C. Corduneanu *Almost Periodic Functions*, (Interscience, New York, 1968)

[7] G. Da Prato, A. Debusche and R. Temam, *Stochastic Burgers equation,* Nonlinear Differential Equations Appl **1** (1994), pp. 389–402.

[8] I. M. Davies and A. Truman, *On the Laplace asymptotic expansion of conditional Wiener Integrals and the Bender-Wu formula for x^{2n}-anharmonic oscillators*, J. Math. Phys. **24** (1983), pp. 255–266.

[9] I. M. Davies and A. Truman, *Laplace asymptotic expansions of conditional Wiener Integrals and generalized Mehler kernel formulæ for Hamiltonians on $L^2(R^n)$*, J. Phys. **A17** (1984), pp. 2773.

[10] I. M. Davies, A. Truman and H. Zhao, *Stochastic heat and Burgers equations and their singularities I – Geometrical properties*, J. Math. Phys. **43** (2002), pp. 3293–3328.

[11] W. E, K. Khanin, A. Mazel and Ya Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. Math. **151** (2000), pp. 877–960.

[12] W. E and E. Vanden Eijnden, *Statistical theory for the stochastic Burgers equation in the inviscid limit*, Comm. Pure Appl. Math. **53** (2000), pp. 852–901.

[13] K. D. Elworthy, A. Truman and H. Zhao, *Stochastic Elementary Formulæ on Caustics I: One-Dimensional Linear Heat Equations*, MRRS preprint.

[14] R. S. Ellis and J. S. Rosen, *Asymptotic analysis of Gaussian integrals I*, Trans. Amer. Math. Soc. **273** (1982), pp. 447.

[15] R. S. Ellis and J. S. Rosen, *Asymptotic analysis of Gaussian integrals II*, Comm. Math. Phys. **82** (1981), pp. 153.

[16] R. S. Ellis and J. S. Rosen, *Laplace's Method for Gaussian integrals with an application to statistical mechanics*, Ann. Probability **10** (1982), pp. 47.

[17] M. I. Freidlin and A. D. Wentzell, *Random Perturbations of Dynamical systems*, (Springer-Verlag, New York, 1998)

[18] M. I. Freidlin, *Probabilistic approach to the small viscosity asymptotics for the Navier-Stokes equations*, Nonlinear Analysis **30** (1997), pp. 4069.

[19] M. I. Freidlin, *Functional Integration and Partial Differential Equations*, (Princeton University Press, Princeton, 1985)

[20] I. Gyöngy and D. Nualart, *On the stochastic Burgers' equation in the real line*, Ann. Prob. **27** (1999), pp. 782–802.

[21] H. Holden, T. Lindstrom, B. Øksendal, J. Uboe and T. S. Zhang, *The Burgers' equation with a noise force and the stochastic heat equations*, Comm. PDE **19** (1994), pp. 119–141.

[22] H. Holden, T. Lindstrom, B. Øksendal, J. Uboe and T. S. Zhang, *The stochastic Wick-type Burgers equation*, in: *Stochastic Partial Differential Equations* edited by A.M. Etheridge, London Mathematical Society Lecture Note Series 216, Cambridge University Press, (1995), pp. 141–161.

[23] M. Kardar, G. Parisi and Y.C. Zhang, *Dynamical scaling of growing interfaces*, Phys, Rev. Letter **56** (1986), pp. 889–892.

[24] Y. Kifer, *The Burgers equations with a random force and a general model for directed polymers in random environments*, Prob. Theory Relat. Fields **108** (1997), pp. 29–65.

[25] V. N. Kolokoltsov and A. E. Tyukov, *Small time and semiclassical asymptotics for stochastic heat equations driven by Lévy noise*, Stoch. Stoch. Rep. **75** (2003), pp. 1–38.

[26] V. N. Kolokoltsov, R. L. Schilling and A. E. Tyukov, *Transience and non-explosion of certain stochastic Newtonian systems*, Electron. J. Probab. **7**, no. 19 (2002), pp. 1–19.

[27] H. Kunita, *Stochastic Differential Equations and Stochastic Flows of Homeomorphisms*, 269–291, in: *Stochastic Analysis and Applications*, ed. M. A. Pinsky, Advances in Probability and Related Topics 7, Marcel Dekker, New York, 1984.

[28] L. Markus and A. Weerasinghe, *Stochastic oscillators*, J. Diff. Eqn. **21** (1988), pp. 288-314.

[29] H. P. McKean, *Stochastic Integrals*, (Academic Press, New York, 1969)

[30] R. Mikulevicius and B. L. Rozovskii, *On equations of stochastic fluid mechanics*, *Stochastics in finite and infinite dimensions*, (Birkhäuser, Boston, 2001), pp 285–302.

[31] B. T. Reynolds *Some Exact Singularities of Burgers and Heat Equations*, PhD thesis, Swansea 2002.

[32] C. N. Reynolds, *On the Polynomial Swallowtail and Cusp Singularities of Stochastic Burgers Equation*, PhD thesis, Swansea 2002.

[33] C.N. Reynolds, A. Truman and D. Williams, *Stochastic Burgers equation in d-dimensions — A one dimensional analysis: Hot and Cool caustics and intermittence of stochastic turbulence*, pp 239–262, *Probabilistic Methods in Fluids*, (World Scientific, Singapore, 2003)

[34] S. F. Shandarin and Ya. B Zeldovich, *The large-scale structure of the universe: turbulence, intermittency, structures in a self gravitating medium*, Rev. Mod. Phys. **6** (1989), pp. 185–220.

[35] R. Tribe and O. Zaboronski, *On large time asymptotics of decaying Burgers turbulence*, Comm. Math. Phys. **212** (2000), pp. 415-436.

[36] A. Truman and T. Zastawniak, *Stochastic PDE's of Schrödinger type and stochastic Mehler kernels – a path integral approach*, Progr. Probab. **45** (1999), pp. 275.

[37] A. Truman and H.Z. Zhao, *Stochastic Burgers' equations and their semi-classical expansions*, Comm. Math. Phys. **194** (1998), pp. 231–248.

[38] A. Truman and H.Z. Zhao, *The stochastic Hamilton-Jacobi equation, stochastic heat equation and Schrödinger equations*, pp 441–464, *Stochastic Analysis and Applications*, (World Scientific, River Edge, 1996)

[39] A. Truman and H.Z. Zhao, *Stochastic Hamilton-Jacobi equations and related topics*, LMS Lecture Note Series **216** (1995), pp. 287–303.

[40] A. Truman and H.Z. Zhao, *On stochastic diffusion equations and stochastic Burgers' equations*, J. Math. Phys. **37** (1996), pp. 283–307.

[41] A. Truman and H.Z. Zhao, *Quantum Mechanics of charged particles in random electromagnetic fields*, J. Math. Phys. **37** (1996), pp. 3180–3197.