Real and Complex Turbulence for the Stochastic Burgers Equation. *

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Abstract

The inviscid limit of Burgers equation, with body forces white noise in time, is discussed in terms of the level surfaces of the minimising Hamilton-Jacobi function and the classical mechanical caustic. Presurfaces and precaustics are introduced by using the classical mechanical flow map. When the prelevel surface touches the precaustic, the geometry (number of cusps) on the level surface changes infinitely rapidly causing 'real turbulence' (Davies, Truman and Zhao – DTZ) [4, 5]. Using an idea of Felix Klein [9], it is shown that the geometry (number of swallowtails) on the caustic also changes infinitely rapidly when the real part of the precaustic touches its complex counterpart, which we call 'complex turbulence'. These two new kinds of turbulence are both inherently stochastic in nature.

A complete analysis of this problem is given in terms of a reduced (one dimensional) action function. This characterises which parts of the original caustic are singular – an old problem in applied mathematics relevant for our 'elementary formula' with Elworthy and Zhao. It also determines when this turbulence is intermittent in terms of the recurrent behaviour of two processes.

1 Introduction

Burgers equations have been used in studying turbulence and in modelling the large scale structure of the universe [8], [15]. In the deterministic case they have also played a part in Arnol'd's pioneering work on caustics and Maslov's seminal works in semiclassical quantum mechanics which inspired much of the early work in this subject [1], [2], [12], [13].

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We consider the stochastic, viscous Burgers equation for velocity field $v^\mu(x,t)$, $x \in \mathbb{R}^d$, $t > 0$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla)\, v^\mu = \frac{\mu^2}{2}\Delta v^\mu - \nabla c(x) - \epsilon \nabla k_t(x) \dot{W}_t,$$

$$v^\mu(x,0) = \nabla S_0(x),$$

$\dot{W}_t$ being white noise, and μ^2 the coefficient of viscosity.

We are interested in the advent of discontinuities in,

$$v^0(x,t) = \lim_{\mu \searrow 0} v^\mu(x,t).$$

The corresponding *Stratonovich heat equation* reads:

$$\frac{\partial u^\mu}{\partial t} = \frac{\mu^2}{2}\Delta u^\mu + \mu^{-2} c(x) u^\mu + \epsilon \mu^{-2} k_t(x) u^\mu \circ \dot{W}_t,$$

$$u^\mu(x,0) = \exp\left(-\frac{S_0(x)}{\mu^2}\right) T_0(x),$$

where the convergence factor T_0 is related to the initial Burgers fluid density. Here the connection is the *Hopf-Cole transformation*

$$v^\mu(x,t) = -\mu^2 \nabla \ln u^\mu(x,t).$$

Following Donsker, Freidlin et al [7], as $\mu \to 0$

$$-\mu^2 \ln u^\mu(x,t) \to \inf_{X(0)} \left[S_0(X(0)) + A(X(0),x,t)\right] = \mathcal{S}(x,t)$$

where

$$A(X(0),x,t) = \inf_{\substack{X(s) \\ X(t)=x}} A[X],$$

$$A[X] = \frac{1}{2}\int_0^t \dot{X}^2(s)\,\mathrm{d}s - \int_0^t c(X(s))\,\mathrm{d}s - \epsilon \int_0^t k_s(X(s))\,\mathrm{d}W_s.$$

This gives the *minimal entropy solution* of Burgers equation [3]. Necessary conditions for X to be an extremiser of $\mathcal{A}[X] = A[X] + S_0(X_0)$ are

$$\mathrm{d}\dot{X}(s) + \nabla c(X(s))\,\mathrm{d}s + \epsilon \nabla k_s(X(s))\,\mathrm{d}W_s = 0,$$

$$\dot{X}(0) = \nabla S_0(X(0)).$$

Minimising $\mathcal{A}[X]$ over $X(0)$ gives $\mathcal{S}(x,t)$ the minimal solution of the *Hamilton-Jacobi equation*

$$\mathrm{d}S_t + \left(\frac{1}{2}|\nabla S_t|^2 + c(x)\right)\,\mathrm{d}t + \epsilon k_t(x)\,\mathrm{d}W_t = 0,$$

$$S_{t=0}(x) = S_0(x).$$

Definition 1.1 *Stochastic wavefront $\mathcal{W}_t$ in x space has equation $\mathcal{S}(x,t) = 0$ at time t.*

For small μ, the heat equation solution $u^\mu(x,t)$ *switches continuously* from being exponentially large to small as we cross the wavefront $\mathcal{W}_t$. However, u^μ and v^μ can also switch *discontinuously* as we now explain.

Define the classical flow map $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ by,

$$\mathrm{d}\dot{\Phi}_s + \nabla c(\Phi_s)\,\mathrm{d}s + \epsilon \nabla k_s(\Phi_s)\,\mathrm{d}W_s = 0,$$

$$\Phi_0 = \mathrm{id}, \quad \dot{\Phi}_0 = \nabla S_0.$$

Since $X(t) = x$ by definition,

$$X(s) = \Phi_s \Phi_t^{-1} x,$$

where we accept that $x_0(x,t) = \Phi_t^{-1}x$ is not necessarily unique. Given some regularity and boundedness, the global inverse function theorem gives a caustic time $T(\omega) > 0$ such that for $0 < s < T(\omega)$, Φ_s is a random diffeomorphism. For $t < T(\omega)$,

$$v^0(x,t) = \dot{\Phi}_t\left(\Phi_t^{-1}x\right),$$

is a classical solution of Burgers equation with probability one.

The method of characteristics suggests that discontinuities in $v^0(x,t)$ are associated with non-uniqueness of $x_0(x,t)$. This is related to when an infinitesimal volume of points $\mathrm{d}x_0$ focus into zero volume $\mathrm{d}X(t)$ under the classical flow map Φ_t.

Definition 1.2

$$\det\left(\frac{\partial X(t)}{\partial x_0}\right) = 0 \qquad \left(\begin{array}{l} \textit{Precaustic } \Phi_t^{-1}C_t \textit{ in } x_0, \\ \textit{Caustic } C_t \textit{ in } x = \Phi_t x_0. \end{array}\right)$$

When $\Phi_t^{-1}\{x\} = \{x_0^1(x,t), x_0^2(x,t), \ldots, x_0^n(x,t)\}$, the Feynman-Kac formula and Laplace's method in infinite dimensions give for a non-degenerate critical point:

$$u^\mu(x,t) = \sum_{i=1}^{n} \theta_i \exp\left(-\frac{S_0^i(x,t)}{\mu^2}\right),$$

where

$$S_0^i(x,t) = S_0\left(x_0^i(x,t)\right) + A\left(x_0^i(x,t), x, t\right),$$

θ_i being an asymptotic series in μ^2.

When $x_0(x,t)$ is unique, $t < T(\omega)$, Hamilton Jacobi theory (Truman and Zhao) [16] gives for each integer m:

$$\begin{aligned} v^\mu(x,t) \;=\; & \sum_{j=0}^{m} \mu^{2j} v_j(x,t) \\ & -\mu^2 \nabla \ln \mathbb{E}\Bigg\{ \exp \frac{-\mu^{2m}}{2} \int_0^t \nabla \cdot v_m(y_s^\mu, t-s)\,\mathrm{d}s \\ & +\frac{1}{2} \sum_{j=m+1}^{2m} \mu^{2(j-1)} \sum_{\substack{0 \le i_1 \le i_2 \le m \\ i_1+i_2=j}} \int_0^t v_{i_1} \cdot v_{i_2}(y_s^\mu, t-s)\,\mathrm{d}s \Bigg\}, \end{aligned}$$

where

$$v_i(x,t) = \nabla S_i(x,t),$$

and S_i satisfies the H-J continuity equations

$$\frac{\partial S_j}{\partial t} + \frac{1}{2}\sum_{i_1,i_2\geq 0} \nabla S_{i_1}\cdot\nabla S_{i_2} = \frac{1}{2}\Delta S_{j-1}$$

for $j = 0, 1, 2, \ldots$, with convention $\frac{1}{2}\Delta S_{-1} = -c - \epsilon k_t \dot{W}_t$. Here each S_i can be found explicitly [16].

Moreover,the Nelson diffusion process

$$\mathrm{d}y_s^\mu = \mu\,\mathrm{d}B_s - \nabla\sum_{j=0}^{m}\mu^{2j}S_j(y_s^\mu, t-s),$$

$$y_0^\mu = x,$$

so that the asymptotic series θ is known explicitly and as $\mu \sim 0$

$$v^\mu(x,t) \sim \nabla S_0(x,t) + O(\mu^2),$$

S_0 being the *minimising action* as expected.

When $\Phi_t^{-1}\{x\} = \{x_0^1(x,t), x_0^2(x,t), \ldots, x_0^n(x,t)\}$ there is a similar asymptotic series θ_i to above corresponding to $x_0^i(x,t)$. Observe that

$$\mathcal{S}(x,t) = \min_{i=1,2,\ldots,n} S_0^i(x,t).$$

Definition 1.3 (Zero level surface)

$$H_t^0 = \left\{x : S_0^i(x,t) = 0 \text{ for some } i\right\},$$

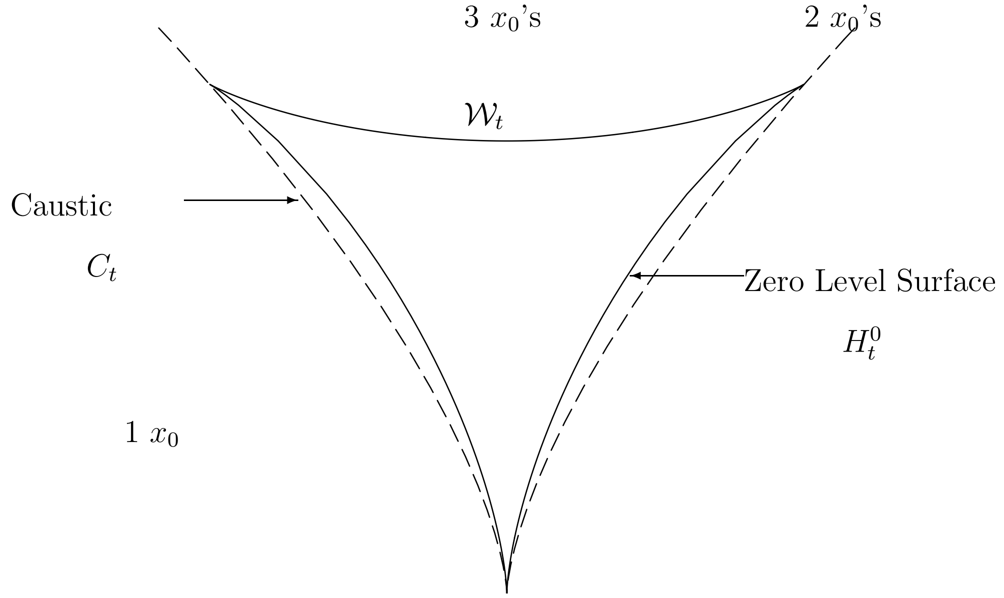
where H_t^0 includes the wavefront $\mathcal{W}_t$.

The dominant term for $v^0(x,t)$ comes from the minimising $x_0^i(x,t) = \tilde{x}_0(x,t)$ (assumed unique) and we obtain the corresponding inviscid limit of the Burgers fluid velocity

$$v^0(x,t) = \dot{\Phi}_t\tilde{x}_0(x,t).$$

Two $x_0^i(x,t)$'s can coalesce and disappear as we cross the caustic surface C_t. When this corresponds to the minimising x_0^i jumping, $u^{\mu=0}$ and $v^{\mu=0}$ have jump discontinuities.

Example 1.4 *(Cusp and Tricorn in 2 dimensions)* $c \equiv 0$, $k_t \equiv 0$, $S_0(x,y) = \frac{x^2 y}{2}$.



$$C_t \ : \ \begin{array}{l} x(x_0,t) = t^2 x_0^3 \\ y(x_0,t) = \frac{3}{2} t x_0^2 - \frac{1}{t} \end{array}$$

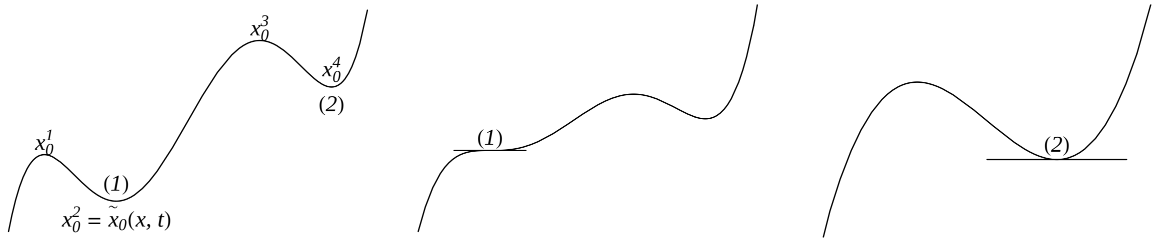
$$H_t^0 \ : \ \begin{array}{l} x(x_0,t) = \frac{x_0}{2}\left(1 \pm \sqrt{1 - t^2 x_0^2}\right) \\ y(x_0,t) = \frac{1}{2t}\left(t^2 x_0^2 - 1 \pm x_0 \sqrt{1 - t^2 x_0^2}\right) \end{array}$$

Evidently n, the multiplicity of $x_0(x,t)$, depends upon x and t. This multiplicity changes by multiples of 2 as we cross the caustic surface. It turns out that this is associated with level surfaces of Hamilton's characteristic function having cusps on the caustic. We illustrate this in 1 dim by considering:

$$I(x,t) = \int_{\mathbb{R}} G(x_0) \exp\left(i \frac{F(x_0,x,t)}{\mu^2}\right) \mathrm{d}x_0,$$

where $G \in C_0^\infty(\mathbb{R})$, $i = \sqrt{-1}$. Consider the graph of the phase function $F_{(x,t)}(x_0) = F(x_0,x,t)$ as x crosses the caustic surface C_t

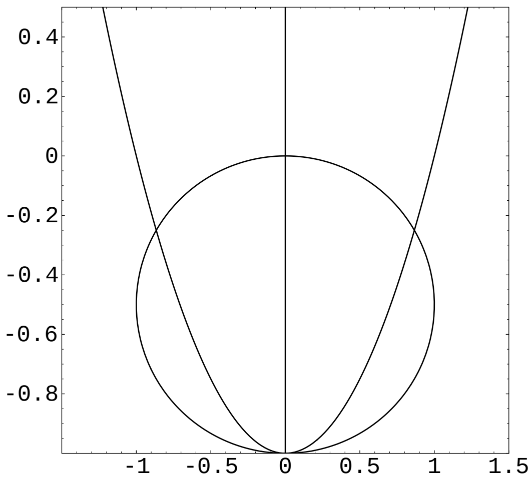
Cusped Side | **On Cool part of Caustic** | **Beyond Caustic**



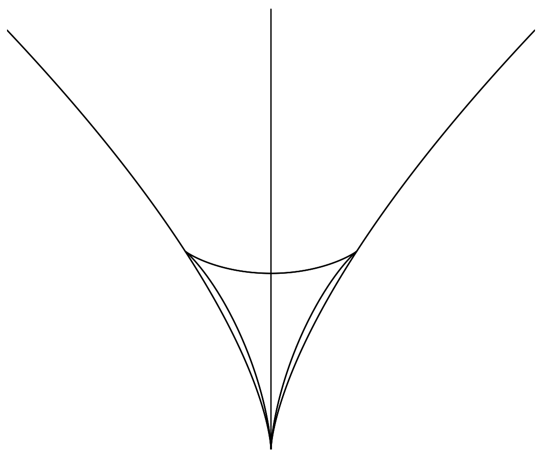
Minimiser $= x_0^2 = \tilde{x}_0$ | *Two* x_0*'s coalescing form point of inflexion* | *Minimiser jumps*

Here $\tilde{x}_0(x,t)$ jumps from (1) to (2) causing u^μ and v^μ for small μ to have jump discontinuities. This only happens when the point of inflexion is the global minimiser of F. When two x_0's coalesce at a minimum of F, we are on a "cool part of caustic" giving jump discontinuities in u^μ and v^μ for small μ. Two x_0's coalescing corresponds to the level surface H_t^c having cusps.

When does H_t have cusps?



Precaustic and Zero Prelevel Surfaces

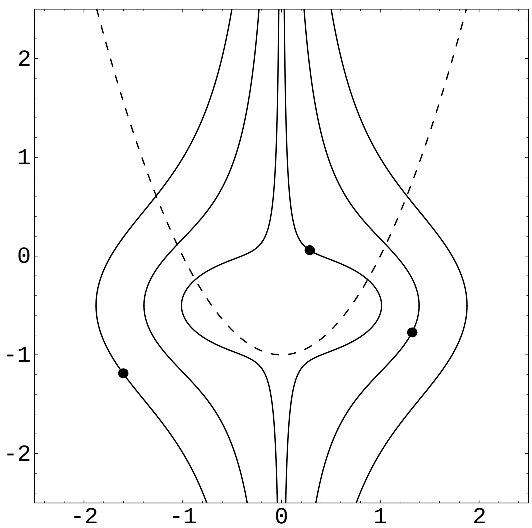


Caustic and Zero Level Surfaces

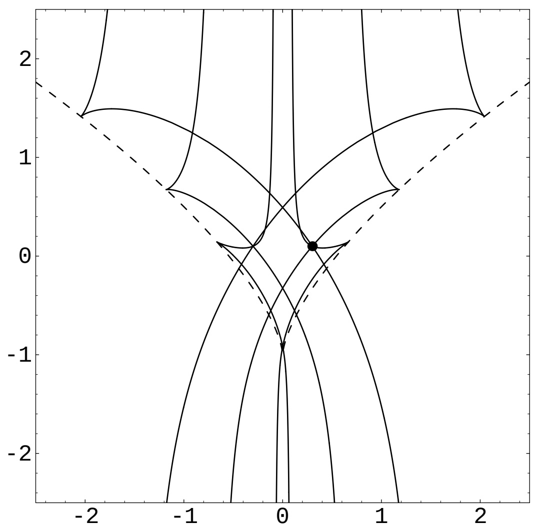
$$\mathrm{Cusp}\,(H_t) = \Phi_t\left(\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t\right)$$

Note $\Phi_t^{-1}H_t \neq \Phi_t^{-1}(H_t)$ the former being determined algebraically allowing for the non-uniqueness of $x_0(x,t)$.

Considering H_t^c for $c > 0$ makes this clearer.



Precaustic and Prelevel Surfaces



Caustic and Level Surfaces

If you want to find cusps on level surfaces of Hamilton's characteristic function you look for intersections of corresponding prelevel surfaces with precaustic. Let $n_c(t)$ be the number of cusped curves in $C_t \cap H_t$ and ask when does $n_c(t)$ change? The simplest way for this to happen is for the corresponding presurface to touch the precaustic. The times t when $n_c(t)$ changes are the zeros of a stochastic process ζ (zeta). i.e. times t when $\zeta(t) = 0$. Such times t form a perfect set i.e. an infinite set with no isolated points. Thus, at such times the geometry of the level surface of the H-J function changes infinitely rapidly as it would in turbulent behaviour. We call this new phenomenon *Real Stochastic Turbulence*. Where does this occur and when is it intermittent?

It turns out that turbulent behaviour can only occur at points in the cool part of the caustic where its tangent has a scalar product zero with the Burgers fluid velocity. So we have to find the boundary between hot and cool parts of caustic and find the ζ (zeta) process at appropriate points of the caustic. The ζ process is just the action functional evaluated at these points on the caustic. The recurrence of ζ is equivalent to the turbulence being intermittent.

There is another kind of *Complex Stochastic Turbulence* associated with the infinitely fast creation and destruction of tiny swallowtails on the caustic determined by the zeros of an η (eta) process as we shall show. This occurs when the real precaustic touches its complex counterpart.

2 Some Results

2.1 Geometrical results of DTZ in 2 dims (or more)

We investigate the geometrical relationship between curves which are level surfaces of the Hamilton Jacobi function and shockwaves or caustics for Burgers equation [5, 4].

Definition 2.1 *A curve $x = x(\gamma)$, $\gamma \in N(\gamma_0, \delta)$, is said to have a generalised cusp at $\gamma = \gamma_0$, γ being an intrinsic variable such as arc length, if*

$$\left.\frac{\mathrm{d}x}{\mathrm{d}\gamma}\right|_{\gamma=\gamma_0} = 0.$$

We begin by considering the deterministic case.

$$\mathcal{A} = \mathcal{A}(x_0, x, t) = S_0(x_0) + A(x_0, x, t),$$

where

$$A(x_0, x, t) = \inf_{\substack{X(0)=x_0\\ X(t)=x}} \left[\int_0^t \left\{\frac{1}{2}\dot{X}(s)^2 \text{-} c(X(s))\right\} \mathrm{d}s\right].$$

The corresponding Euler Lagrange equation reads

$$\ddot{X}(s) = -\nabla c(X(s)), \qquad s \in [0, t],$$

$$X(t) = x, \quad X(0) = x_0.$$

The free case $c \equiv 0$ gives

$$\mathcal{A}(x_0, x, t) = \frac{(x - x_0)^2}{2t} + S_0(x_0).$$

The key identity here is

$$\frac{\partial \mathcal{A}}{\partial x_0^\alpha} = 0, \quad \alpha = 1, 2, \ldots, d \Leftrightarrow x = \Phi_t x_0 = x_0 + t\nabla S_0(x_0).$$

This result is true in enormous generality as we shall see.

Consider the level surface H_t^0 obtained by eliminating x_0 between

$$\mathcal{A}(x_0, x, t) = 0 \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0, \quad \alpha = 1, 2, \ldots, d.$$

Eliminating x alternatively gives the prelevel surface $\Phi_t^{-1} H_t^0$. Similarly the precaustic $\Phi_t^{-1} C_t$ (and caustic C_t) are obtained by eliminating x (or x_0) between

$$\det \left(\frac{\partial^2 \mathcal{A}}{\partial x_0^\alpha \partial x_0^\beta}(x_0, x, t) \right)_{\alpha,\beta=1,2,\ldots,d} = 0 \quad \text{and} \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$

In the free case the equation for the zero prelevel surface is the *eikonal equation*

$$\frac{t}{2} |\nabla S_0(x_0)|^2 + S_0(x_0) = 0,$$

and the derivative map $D\Phi_t(x_0)$ is given by the Hessian

$$D\Phi_t(x_0) = 1 + t\nabla^2 S_0(x_0).$$

The following elementary identity is vital in the free case

$$\nabla_{x_0} \left\{ \frac{t}{2} |\nabla S_0(x_0)|^2 + S_0(x_0) \right\} = \left(1 + t\nabla^2 S_0\right) \nabla S_0(x_0).$$

Lemma 2.2 (Free case in 2 dims) *Assume the prelevel surface meets the precaustic at x_0 where*

$$\left| \left(1 + t\nabla^2 S_0(x_0)\right) \nabla S_0(x_0) \right| \neq 0$$

and

$$\mathit{Dim}\left(\mathit{Ker}\left(1 + t\nabla^2 S_0(x_0)\right)\right) = 1.$$

Then tangent plane to the prelevel surface T_{x_0} is spanned by

$$\mathit{Ker}\left(1 + t\nabla^2 S_0(x_0)\right).$$

Proof: In the free case, the proof is obvious from the last 2 identities. □

Proposition 2.3 (Free case in 2 dims) *Assume that*

$$\left|\left(1+t\nabla^2 S_0(x_0)\right)\nabla S_0(x_0)\right| \neq 0$$

so that x_0 is not a singular point of $\Phi_t^{-1}H_t$. Then $\Phi_t x_0$ can only be a generalised cusp if $\Phi_t(x_0)\in C_t$, the caustic. Moreover, if

$$x=\Phi_t x_0 \in \Phi_t\left(\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t\right),$$

x will indeed be a generalised cusp of the level surface.

Proof: We have normal $n(x_0)\neq 0$ and $\frac{\mathrm{d}x_0}{\mathrm{d}\gamma}(\gamma)\Big|_{\gamma=\gamma_0} \neq 0$ and

$$\frac{\mathrm{d}x}{\mathrm{d}\gamma}(\gamma)=\left(1+t\nabla^2 S_0(x_0)\right)\frac{\mathrm{d}x_0}{\mathrm{d}\gamma}.$$

For this to be zero, it is necessary that

$$\det\left(1+t\nabla^2 S_0(x_0)\right)=0.$$

Trivially from above

$$\frac{\mathrm{d}x}{\mathrm{d}\gamma}(\gamma)\Big|_{\gamma=\gamma_0}=0,$$

since

$$\frac{\mathrm{d}x}{\mathrm{d}\gamma}\| e_0 \in \ker\left(1+t\nabla^2 S_0(x_0)\right). \qquad \square$$

The above generalises to d dimensions, to very general deterministic systems and to systems with noise. Let the stochastic action be defined by

$$A(x_0,p_0,t)=\frac{1}{2}\int_0^t \dot{X}(s)^2\,\mathrm{d}s-\int_0^t\Big[c(X(s))\,\mathrm{d}s+\epsilon k_s(X(s))\,\mathrm{d}W_s\Big],$$

where $X_s=X(s)=X(s,x_0,p_0)\in\mathbb{R}^d$ and

$$\mathrm{d}\dot{X}(s)=-\nabla c(X(s))\,\mathrm{d}s-\epsilon\nabla k_s(X(s))\,\mathrm{d}W_s,\quad s\in[0,t],$$

with $X(0)=x_0$, $\dot{X}(0)=p_0$, and $x_0,p_0\in\mathbb{R}^d$. We assume X_s is $\mathcal{F}_s$ measurable and unique. If $\mathrm{d}u_s\,\mathrm{d}\dot{X}(s)=0$, we have

$$\int_0^t u(s)\,\mathrm{d}\dot{X}(s)=u(t)\dot{X}(t)-u(0)\dot{X}(0)-\int_0^t \dot{u}(s)\dot{X}(s)\,\mathrm{d}s.$$

In particular this is true when $u_s=\frac{\partial X_s}{\partial x_0^\alpha}$ where $\alpha=1,2\ldots,d$. Using Kunita [11], mild regularity gives

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial X_s}{\partial x_0^\alpha}\right)=\frac{\partial \dot{X}_s}{\partial x_0^\alpha}\quad \alpha=1,2,\ldots,d,$$

almost surely.

Lemma 2.4 (n dims) *Assume $S_0, c \in C^2$ and $k \in C^{2,0}$, $\nabla c, \nabla k$ Lipschitz with Hessians $\nabla^2 c, \nabla^2 k$ and all second derivatives w.r.t. space variables of c and k bounded. then for p_0 possibly x_0 dependent*

$$\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \dot{X}(t) \cdot \frac{\partial X(t)}{\partial x_0^\alpha} - \dot{X}_\alpha(0), \qquad \alpha = 1, 2, \ldots, d.$$

Now let

$$\mathcal{A}(x_0, x, t) = A(x_0, p_0, t)|_{p_0 = p_0(x_0, x, t)}$$

where $p_0 = p_0(x_0, x, t)$ is the random minimiser (assumed unique) of $A(x_0, p_0, t)$ when $X(t, x_0, p_0) = x$. (Here we need the map $p_0 \mapsto X(t, x_0, p_0) \in \mathbb{R}^d$ to be onto for all x_0. Methods of Kolokoltsov et al [10] guarantee this for small t.)

Theorem 2.5 (n dims) *The stochastic flow map Φ_t is defined by*

$$\frac{\partial}{\partial x_0^\alpha}\left[S_0(x_0) + \mathcal{A}(x_0, x, t)\right] = 0, \quad \alpha = 1, 2, \ldots, d,$$

so that $x = \Phi_t x_0$.

Assume that $\mathcal{A}(x_0, x, t)$ is C^4 in space variables and

1. $\det\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^\alpha \partial x^\beta}\right) \neq 0.$

2. $\mathrm{Ker}(D\Phi_t)$ is 1 dimensional

Then we have:-

Proposition 2.6 (n dims) *The random classical flow map has Frechet derivative a.s.*

$$\left(D\Phi_t\right)(x_0) = \left(-\frac{\partial^2 \mathcal{A}}{\partial x \partial x_0}\right)^{-1}\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x_0}(x_0, x, t)\right)$$

and the normal to the prelevel surface is

$$n(x_0) = -\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x_0}\right)\left(\frac{\partial^2 \mathcal{A}}{\partial x_0 \partial x}\right)^{-1}\dot{X}\left(t, x_0, \nabla S_0(x_0)\right).$$

(These are analogues of

$$D\Phi_t(x_0) = \left(1 + t\nabla^2 S_0(x_0)\right)$$

and

$$\nabla_{x_0}\left\{\frac{t}{2}\left|\nabla S_0(x_0)\right|^2 + S_0(x_0)\right\} = \left(1 + t\nabla^2 S_0(x_0)\right)\nabla S_0(x_0)$$

in the free case.)

We content ourselves with quoting:

Theorem 2.7 (3 dims) *Let*

$$\begin{aligned} x &\in Cusp\,(H_t) \\ &= \left\{x \in \Phi_t\left(\Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t\right), x = \Phi_t(x_0), n(x_0) \neq 0\right\}. \end{aligned}$$

Then in 3 dims in the stochastic case, with probability one, T_x the tangent space to the level surface at x is at most one dimensional.

Proof: $\det\left(\frac{\partial^2 \mathcal{A}}{\partial x_0^\alpha \partial x_0^\beta}\right) = 0$ so there exists $e_0 \in \ker\left(D\Phi_t(x_0)\right)$. From above $e_0 \cdot n = 0$ so $e_0 \in T_{x_0}$ tangent plane to prelevel surface at x_0. Similarly $(n \wedge e_0) \in T_{x_0}$ and $D\Phi_t(x_0)e_0 = 0$ so T_{x_0} is spanned by $D\Phi_t(x_0)(n \wedge e_0)$. □

2.2 A one dimensional analysis

2.2.1 Global reducibility

We now explain how a one dimensional analysis first described by Reynolds, Truman and Williams (RTW) [17] can reveal information on the casutic.

Definition 2.8 *Classical flow map Φ_t is globally reducible if*

$$y = \Phi_t y_0, \quad y = (y_1, y_2, \ldots, y_d), \quad y_0 = (y_0^1, y_0^2, \ldots, y_0^d)$$

$$\Rightarrow y_0^r = y_0^r(y, y_0^1, y_0^2, \ldots y_0^{r-1}, t), \quad r = d, d-1, \ldots, 2. \quad \star$$

We want C^2 functions $y_0^d, y_0^{d-1}, \ldots$ such that

$$\begin{aligned} & y_0^d = y_0^d(y, y_0^1, y_0^2, \ldots y_0^{d-1}, t) \\ & \quad \Leftrightarrow \frac{\partial \mathcal{A}}{\partial y_0^d}(y_0, y, t) = 0, \\ & y_0^{d-1} = y_0^{d-1}(y, y_0^1, y_0^2, \ldots y_0^{d-2}, t) \\ & \quad \Leftrightarrow \frac{\partial \mathcal{A}}{\partial y_0^{d-1}}(y_0^1, y_0^2, \ldots, y_0^d(\ldots), y, t) = 0, \\ & \quad \vdots \\ & y_0^2 = y_0^2(y, y_0^1, t) \\ & \quad \Leftrightarrow \frac{\partial \mathcal{A}}{\partial y_0^2}(y_0^1, y_0^2, y_0^3(y, y_0^1, y_0^2, t), \ldots, y_0^d(\ldots), y, t) = 0, \end{aligned}$$

where $y_0^d(\ldots) = y_0^d(y, y_0^1, y_0^2, \ldots, y_0^{d-1}, t)$. No root is repeated so second derivatives of $\mathcal{A}$ do not vanish. Evidently we are assuming a favoured ordering of coordinates and a corresponding decomposition of Φ_t so that non-uniqueness is reduced to the level of y_0^1 coordinate. We begin with a result of RTW.

Proposition 2.9 *Assume Φ_t is globally reducible. Define the reduced action:*

$$f(y_0^1, y, t) = \mathcal{A}(y_0^1, y_0^2(y, y_0^1, t), y_0^3(\ldots), \ldots, y, t).$$

Then:

1. $\frac{\partial f}{\partial y_0^1}(y_0^1, y, t) = 0$ *and previous equations* $\star \Leftrightarrow y = \Phi_t y_0$,

2. $\frac{\partial f}{\partial y_0^1}(y_0^1, y, t) = \frac{\partial^2 f}{(\partial y_0^1)^2}(y_0^1, y, t) = 0$ *and the previous equations* $\star$ $\Leftrightarrow y = \Phi_t y_0$ *is such that the number of solutions* y_0 *of this equation changes.*

Proof: Key is:-

Lemma 2.10

$$\left| Det \left(\frac{\partial^2 \mathcal{A}}{\partial x_0^2}(\boldsymbol{x}_0, \boldsymbol{x}, t) \right) \Big|_{\boldsymbol{x} = \Phi_t \boldsymbol{x}_0} \right| = \prod_{i=0}^{n-1} \left| \frac{\partial^2 \mathcal{A}}{\partial x_{0_{n-i}}^2}(x_{0_1}, x_{0_2}(\boldsymbol{x}, x_{0_1}, t), \ldots, x_{0_n}(\ldots), \boldsymbol{x}, t) \right|$$

the last term is $f''_{(\boldsymbol{x},t)}(x_{0_1})$ *and the first* $d-1$ *terms are non zero.*

Proof: Principle of stationary phase applied to the evaluation of

$$I = \int_{\mathbb{R}^d} G(y_0) \exp\left(-\frac{i}{\mu^2} \mathcal{A}(y_0, y, t) \right) \mathrm{d}y_0,$$

by repeated integration shows that, if we assume

$$\frac{\partial f}{\partial y_0^1}(y_0^1, y, t) = 0,$$

has n roots and y such that

$$\frac{\partial^2 f}{(\partial y_0^1)^2}(y_0^1, y, t) \neq 0,$$

then the first equation will have n simple roots, critical points of f corresponding to $\{x_0^i(x, t)\}_{i=1,2,\ldots,n}$ and varying y now so that

$$\frac{\partial^2 f}{(\partial y_0^1)^2}(y_0^1, y, t) = 0,$$

typically 2 of these n critical points will coalesce. □

The linchpin is that the phase function F in the introduction is

$$F(x_0^1, x, t) = f(x_0^1, x, t)$$

where $x \in \mathbb{R}^d$.

2.2.2 Caustic parameterisation

Corollary 2.11 *Let $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^{d-1}) \in \mathbb{R}^{d-1}$ where $\lambda \mapsto (\lambda, x_0^d(\lambda))$ is a parameterisation of the precaustic, so that $x_t(\lambda) = \Phi_t(\lambda, x_0^d(\lambda))$ is a parameterisation of the caustic. Then*

$$f'_{(x_t(\lambda),t)}(\lambda^1) = f''_{(x_t(\lambda),t)}(\lambda^1) = 0.$$

i.e. There is a critical point of inflexion at $x_0^1 = \lambda^1$ for this parameterisation.

The geometry of the caustic is determined by the higher derivatives of f. In d dimensions we call a part of the caustic with $(d-2)$ dimensional tangent space associated with the above parameterisation, the *principal subcaustic*.

$$x_t^{sc} = \Phi_t\left(\lambda, x_0^d(\lambda)\right)\big|_{\lambda^{d-1}=\lambda^{d-1}(\lambda^1,\ldots,\lambda^{d-2})}$$

where $\lambda^{d-1}(\ldots)$ is determined by

$$\det_{i,j=1,\ldots,d-1}\left(\frac{\partial x_t}{\partial \lambda^i}\cdot\frac{\partial x_t}{\partial \lambda^j}\right) = 0.$$

In 2 dimensions this corresponds to cusps and in 3 dimensions to creasing the caustic along a curve. Here we must assume that $\left\{\frac{\partial \Phi_t(x_0)}{\partial x_0^i}\right\}_{i=2,3,\ldots,d}$ are always linearly independent.

Proposition 2.12 *If $x_t(\lambda)$ is on the subcaustic, then*

$$f'_{(x_t(\lambda),t)}(\lambda^1) = f''_{(x_t(\lambda),t)}(\lambda^1) = f'''_{(x_t(\lambda),t)}(\lambda^1) = 0.$$

More generally:

Proposition 2.13 *If we assume $f_{(x_t(\lambda),t)}(x_0^1) \in C^{p+1}$ then in n dimensions if the tangent to the caustic is at most $(n-p+1)$ dimensional at $\Phi_t(\lambda, x_0^d(\lambda))$ then*

$$f'_{(x_t(\lambda),t)}(\lambda^1) = f''_{(x_t(\lambda),t)}(\lambda^1) = \ldots f^{(p)}_{(x_t(\lambda),t)}(\lambda^1) = 0,$$

and

$$f^{(p+1)}_{(x_t(\lambda),t)}(\lambda^1) \neq 0.$$

2.2.3 Hot and cool caustics

We define the renormalised reduced action

$$F(x_0^1) = f_{(x_t(\lambda),t)}(x_0^1) - f_{(x_t(\lambda),t)}(\lambda^1).$$

Then by inspection

$$F(\lambda^1) = F'(\lambda^1) = F''(\lambda^1) = 0.$$

Assume that $F(x_0^1)$ is a real analytic function in a neighbourhood of $\lambda_1 \in \mathbb{R}$. Then

$$F(x_0^1) = (x_0^1 - \lambda^1)^3 \tilde{F}(x_0^1)$$

so that $F(x_0^1)$ has a critical point of inflexion at $x_0^1 = \lambda^1$. The caustic will be cool when this inflexion is the minimising stationary point of F. Therefore, we will be on a hot/cool boundary if this inflexion is about to become or cease being the minimiser.

Proposition 2.14 *A* necessary condition *for $x_t(\lambda) \in C_t$ to be on a hot/cool boundary is that either*

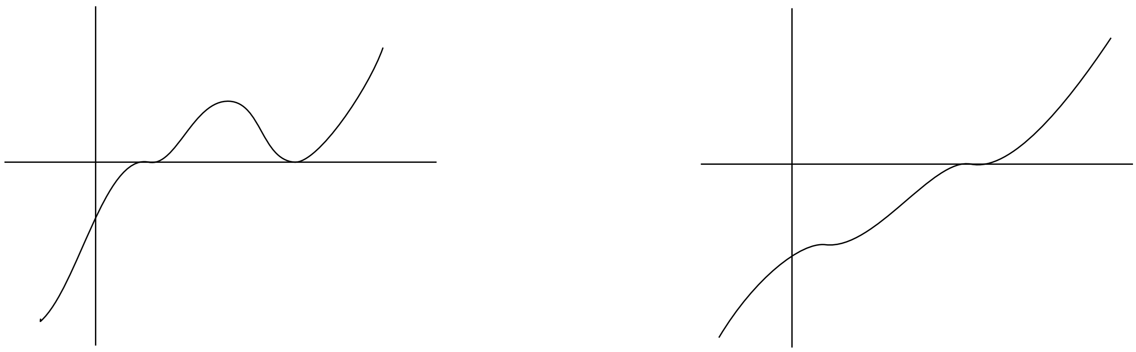
1. *$\tilde{F}(x_0^1)$ or,*
2. *$\tilde{G}(x_0^1) = 3\tilde{F}(x_0^1) + (x_0^1 - \lambda^1)\tilde{F}'(x_0^1)$,*

has a repeated root at $x_0^1 = \alpha^1 \neq \lambda^1$.

This follows since there are two ways in which the minimiser could change:-

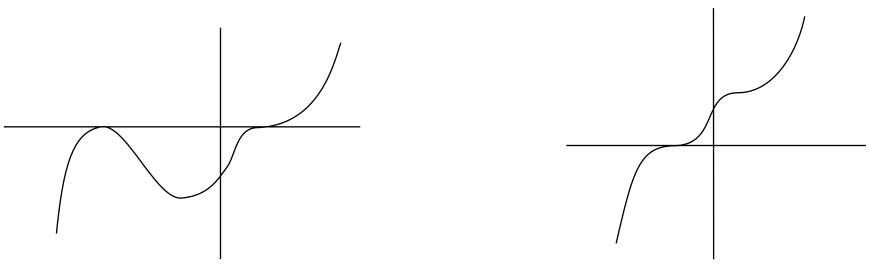
1. $\tilde{F}$ has a repeated root which is also the minimiser.
2. There is another inflexion at a lower value which is the minimiser (point of self-intersection of caustic).

Graphs of F:-



The condition is not sufficient as it includes cases where the minimiser is not about to change, for instance:-

Graphs of F:-



Minimiser not changing

Minimiser unchanged if second (higher) inflexion disappears

Example 2.15 *Let* $c \equiv 0$, $k_t(x, y) \equiv x$,

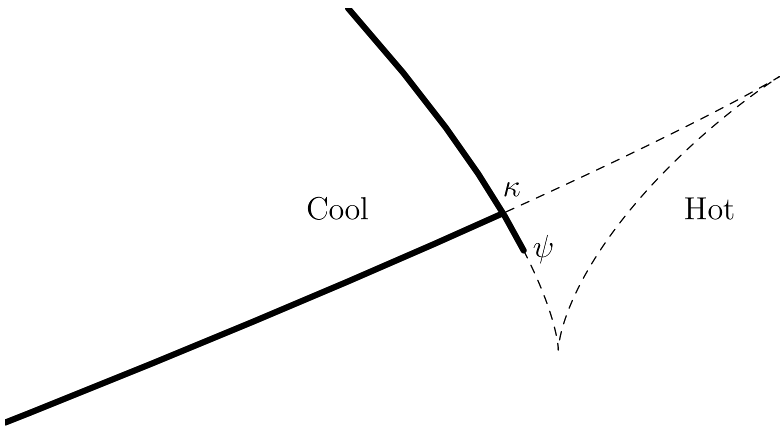
$$S_0(x_0, y_0) = x_0^5 + x_0^2 y_0.$$

This gives global reducibility and $k_t(x, y) \equiv x$ *means that the effect of the noise is to translate* $\epsilon = 0$ *picture through* $\left(-\epsilon \int_0^t W_s \, \mathrm{d}s, 0\right)$. *A simple calculation gives*

$$\tilde{F}(x_0) = 12\lambda^2 - 3\lambda t + 6\lambda x_0 - t x_0 + 2x_0^2$$

$$\tilde{G}(x_0) = 15\lambda^2 - 4\lambda t + 10\lambda x_0 - 2t x_0 + 5x_0^2,$$

giving cool part of polynomial swallowtail.



$$\kappa = \left(-\frac{t^5}{500} - \epsilon \int_0^t W_s \, \mathrm{d}s, -\frac{1}{2t} + \frac{t^3}{50}\right)$$

$$\psi = \left(-\frac{t^5(3 + 8\sqrt{6})}{18000} - \epsilon \int_0^t W_s \, \mathrm{d}s, -\frac{1}{2t} + \frac{t^3(9 - \sqrt{6})}{450}\right)$$

3 Some Applications in 2 Dims

3.1 Real turbulence and the ζ process

Definition 3.1 *The turbulent times t are times when the prelevel surface of minimising Hamilton Jacobi function* touches *the precaustic. Such times t are zeros of a stochastic process $\zeta^c(.)$.*

We begin with some minor generalisations of results in RTW [17].

Proposition 3.2 *Assume Φ_t is globally reducible, with associated parameterisation of the precaustic $\lambda \mapsto (\lambda, x_0^2(\lambda))$ in 2 dimensions. Then turbulence process at λ*

$$\zeta^c(t) = f_{(x_t(\lambda_0),t)}(\lambda_0) - c,$$

where $f_{(x,t)}(x_0^1)$ is the reduced action *evaluated at points $x = x_t(\lambda_0) = \Phi_t(\lambda_0, x_0^2(\lambda_0)) \in C_t$, $\lambda = \lambda_0$ satisfying*

$$\dot{X}_t(\lambda) \cdot \frac{\mathrm{d}x_t}{\mathrm{d}\lambda}(\lambda) = 0,$$

where $\dot{X}(\lambda) = \dot{\Phi}_t(\lambda, x_0^2(\lambda))$ and $\Phi_t(\lambda_0, x_0^2(\lambda_0)) \in C_t^c$, the cool part of the caustic.

Hence, there are three kinds of real stochastic turbulence:-

1. *Cusped*, where there is a cusp on the caustic,
2. *Zero speed*, where the Burgers fluid velocity is zero,
3. *Orthogonal*, where the Burgers fluid velocity is orthogonal to the caustic.

Proof: The number of cusps on relevant prelevel surface is

$$n_c(t) = \#\left\{\lambda \in \mathbb{R} : f_{(x_t(\lambda),t)}(\lambda) = c\right\},$$

where the roots $\lambda = \lambda_0$ correspond to points in *cool part* of caustic. The presurfaces touch when $n_c(t)$ changes, which occurs when

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f_{(x_t(\lambda),t)}(\lambda) = 0.$$

□

For stochastic turbulence to be intermittent we require that the process $\zeta^c(t)$ be recurrent.

Proposition 3.3 *Let $k_t(x,y) \equiv x$ and*

$$S_0(x_0, y_0) = f(x_0) + g(x_0)y_0,$$

where f, g, f' and g' are zero at $x_0 = a$ but $g''(a) \neq 0$. Then for orthogonal turbulence at a

$$\zeta^c(t) = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s \,\mathrm{d}s - \frac{\epsilon^2}{2} \int_0^t W_s^2 \,\mathrm{d}s - c.$$

In this connection we note the result of Reynolds, Truman and Williams [17].

Lemma 3.4 *Let W be a $BM(\mathbb{R})$ process starting at 0, c any real constant and*

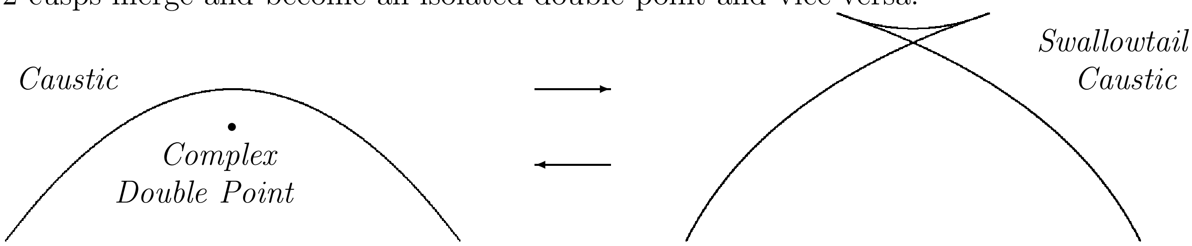
$$Y_t = -a\epsilon W_t + \epsilon^2 W_t \int_0^t W_s \, \mathrm{d}s - \frac{\epsilon^2}{2} \int_0^t W_s^2 \, \mathrm{d}s - c.$$

Then, with probability one, there exists a sequence of times $t_n \nearrow \infty$ such that

$$Y_{t_n} = 0 \quad \textit{for every } n.$$

3.2 The complex caustic

In his study of Riemann surfaces, Felix Klein first investigated the formation of swallowtails on algebraic curves [9]. In his work he describes how a swallowtail will disappear if its 2 cusps merge and become an isolated double point and vice versa.



We do not have any double isolated points for $x_t(\lambda)$, $\lambda \in \mathbb{R}$ so we allow $\lambda \in \mathbb{C}$ and consider the *complex caustic*. We look for complex values $\lambda = a + i\eta$, $a, \eta \in \mathbb{R}$ being mapped into *real* points $x_t(a + i\eta)$. If classical flow map is *real analytic*

$$\Phi_t(\overline{z_0}) = \overline{\Phi_t(z_0)}$$

then

$$x_t(a + i\eta) = x_t(a - i\eta),$$

hence when $x_t(a + i\eta)$ is real, we call these points *complex double points* of the caustic. Here swallowtails appear and disappear on the caustic where the real precaustic touches the complex precaustic, at a point mapping under Φ_t to a real image.

Lemma 3.5 *Let $x_t(\lambda) = (x_t^1(\lambda), x_t^2(\lambda))$ be a real analytic parameterisation of the caustic. In the (a, η) plane the curves*

$$0 = \frac{1}{\eta} Im \left\{ x_t^1(a + i\eta) \right\}, \quad 0 = \frac{1}{\eta} Im \left\{ x_t^2(a + i\eta) \right\}$$

intersect at a point $(a, 0)$ if and only if there is a generalised cusp on caustic at $x_t(a)$.

Example 3.6

$$S_0(x_0) = x_0^5 + x_0^6 y_0, \quad k_t(x,y) = x.$$

This is a case of bodily translation of caustic so we need only consider $\epsilon = 0$. *Equation of caustic is* $\lambda = a \in \mathbb{R}$

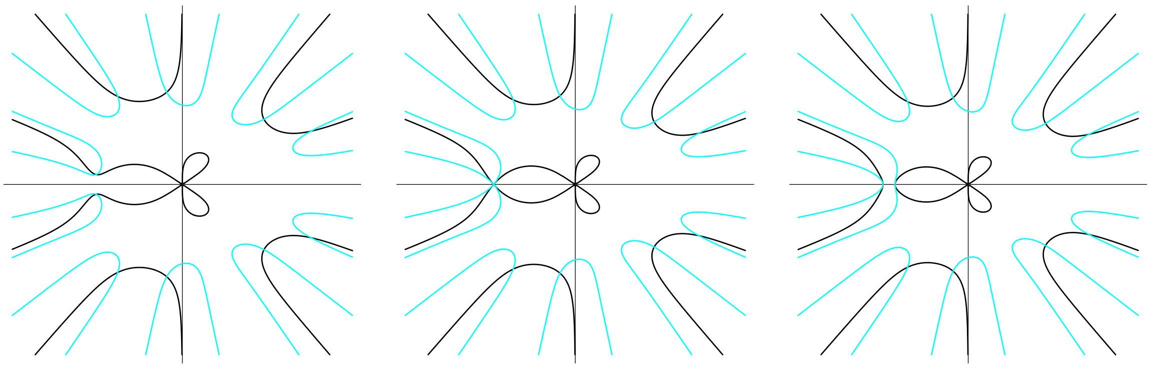
$$x_t(a) = \left(\frac{a}{5} \left(4 + 5a^3 t + 36a^{10}t^2\right), \frac{1}{30a^4 t}\left(-1 - 20a^3 t + 66a^{10}t^2\right) \right).$$

Caustic has no cusps for times $t < t_c$ *and* 2 *cusps for times* $t > t_c$, *where*

$$t_c = \frac{4}{7}\sqrt{2}\left(\frac{33}{7}\right)^{\frac{3}{4}} = 2.5854\ldots$$

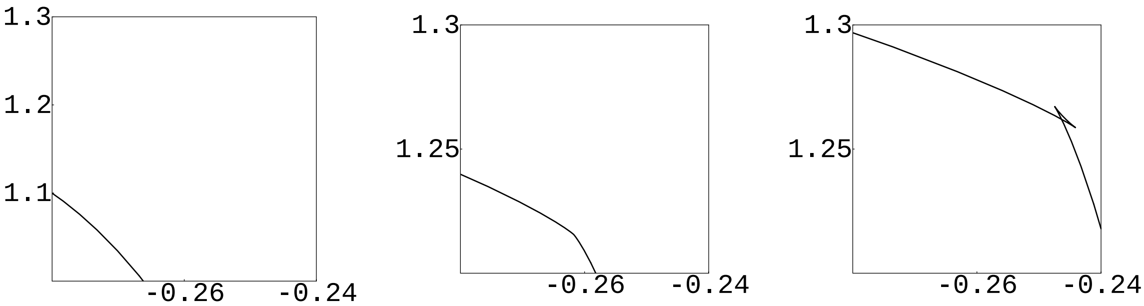
This is an example of the above mechanism for forming a swallowtail as shown in the pictures below.

Curves $\mathrm{Im}\{x_t(a + i\eta)\} = 0$ **in** (a, η) **plane.**



The grey curves are $\mathrm{Im}\{x_t^1(a + i\eta)\} = 0$ and the black curves are $\mathrm{Im}\{x_t^2(a + i\eta)\} = 0$. Intersections of these curves are the complex double points which give caustic swallowtails when $\eta \to 0$ and remains zero.

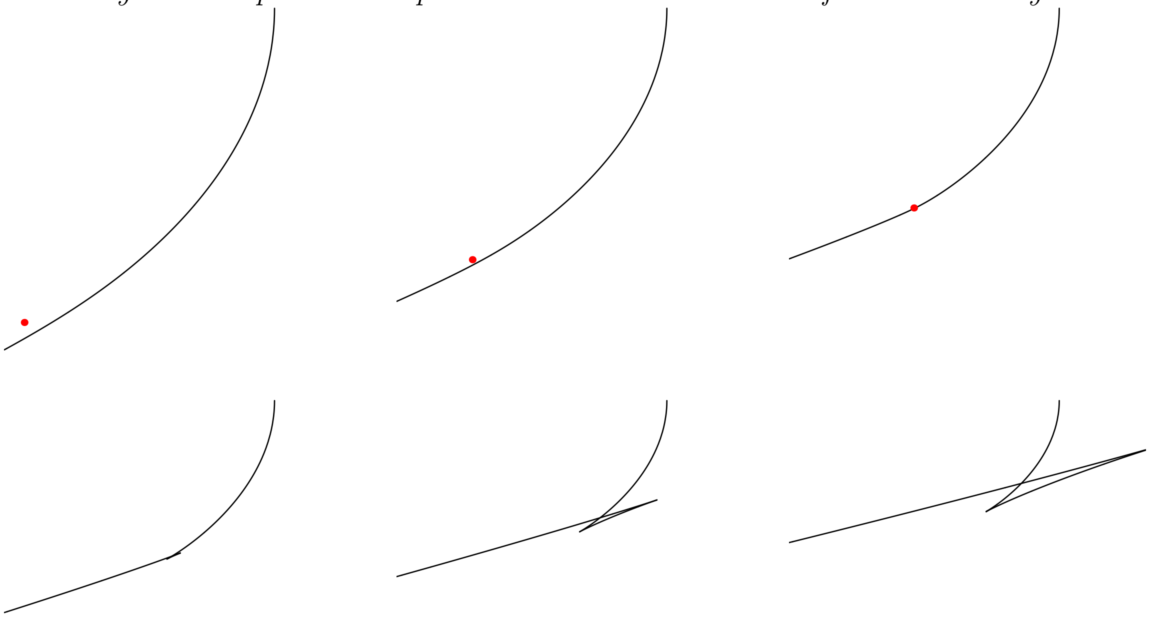
Caustic plotted at corresponding times.



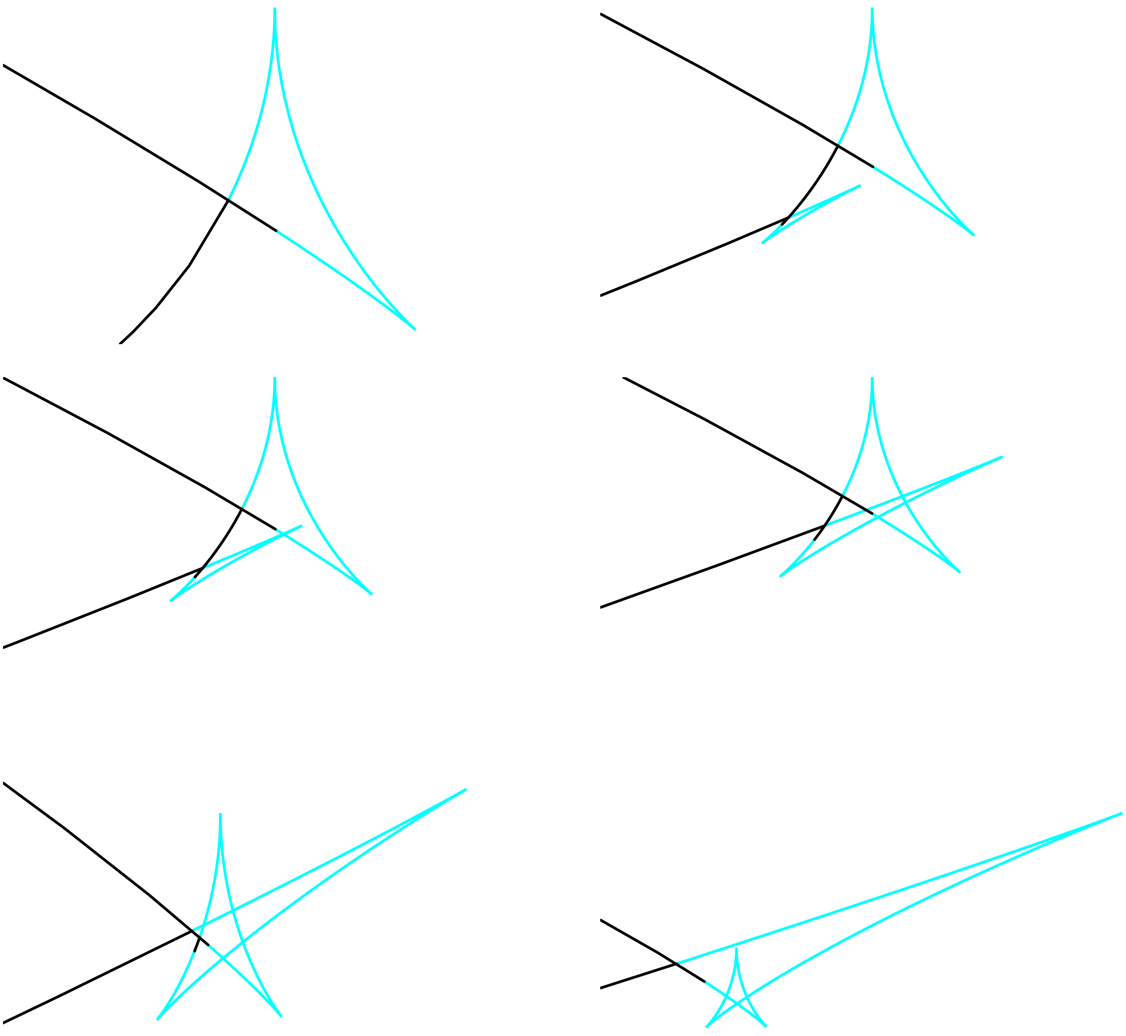
Example 3.7 *For the non generic swallowtail,*

$$S_0(x_0, y_0) = x_0^5 + |x_0|^{\frac{3}{2}}\, y_0,$$

there is only one complex double point which mathematica can find numerically.



With this additional swallowtail the caustic develops into into a five pointed star. The following diagram illustrates this development and also indicates the hot and cool parts as calculated numerically using our method. Cool parts are drawn in black and hot parts in grey.



As you would expect, the reduced action function has a role to play here.

Proposition 3.8 *For a 2 dimensional setup assume that $x_t(\lambda)$ is real analytic. If a* complex double point *joins the caustic at $\lambda = a$ at time t_c, then there is a real solution x to:*

$$0 = f'_{(x,t_c)}(a) = f''_{(x,t_c)}(a) = f'''_{(x,t_c)}(a) = f^{(4)}_{(x,t_c)}(a).$$

This follows from:

Proposition 3.9 *A complex double point joins the caustic at a time t_c if and only if there is a solution $a \in \mathbb{R}$ to the equations*

$$\frac{\partial x_t}{\partial a}(a) = \frac{\partial^2 x_t}{\partial a^2}(a) = 0,$$

$x_t(a)$ being precaustic parameterisation of caustic which is real analytic.

We can bring these results together in the following theorem.

Theorem 3.10 *Let $\lambda \in \mathbb{C}$ be a* complex critical point of inflexion *of the reduced action in the sense that*

$$0 = f'_{(\Phi_t(\lambda, x_0^2(\lambda)),t)}(x_0^1) = f''_{(\Phi_t(\lambda, x_0^2(\lambda)),t)}(x_0^1),$$

when $x_0^1 = \lambda$. Let $\lambda(t) = a(t) + i\eta(t)$ where $\eta(t) > 0$ for $t < t_c$, $a(t), \eta(t) \in \mathbb{R}$ and $\lambda(t)$ satisfies

$$Im\left\{\Phi_t(\lambda(t), x_0^2(\lambda(t)))\right\} = 0$$

with $\eta(t) \to 0$ as $t \nearrow t_c$ and $\eta(t) = 0$ for $t > t_c$.

Then, assuming that $\Phi_t(z)$ is real analytic in z, $a(t) - i\eta(t)$ is also a solution, and

$$0 = f'_{(x,t_c)}(a(t_c)) = f''_{(x,t_c)}(a(t_c)) = f'''_{(x,t_c)}(a(t_c)) = f^{(4)}_{(x,t_c)}(a(t_c))$$

where $x = \Phi_{t_c}(a(t_c), x_0^2(a(t_c))) \in C_t$.

Moreover

$$f'_{(\Phi_t((\lambda), x_0^2(\lambda)),t)}(x_0^1) = (x_0^1 - \lambda)^2 Q_{(\lambda,t)}(x_0^1) H_{(\lambda,t)}(x_0^1)$$

where Q is a quadratic in x_0^1.

We assume that for $t \in (t_c - \epsilon, t_c)$, the quadratic $Q_{(\lambda,t)}(x_0^1)$ has two complex roots for all $\lambda \in \mathbb{R}$, and that as $t \to t_c$, these roots converge to the real point a_{t_c}. Further we assume that for $t \in (t_c, t_c + \epsilon)$ there exists an interval of λ for which $Q_{(\lambda,t)}(x_0^1)$ has two real distinct roots, and for λ immediatly outside this interval $Q_{(\lambda,t)}(x_0^1)$ has two complex roots. Then a swallowtail must form on the caustic at time t_c.

3.3 Complex turbulence and the η process.

When $\text{Im}\{\Phi_t(a + i\eta, x_0^2(a + i\eta))\}$ is random, the above $\eta(t)$ will be a stochastic process which is just the discriminant of the quadratic factor Q. When the zeros of η form a perfect set, swallowtails will spontaneously appear and disappear on the caustic infinitely rapidly. We call this complex turbulence occurring at the turbulent times which are the zeros of the eta process.

This is in fact related to real turbulence. When the swallowtail forms on the caustic, swallowtails also form on level surfaces in that region. For this to occur the presurfaces must touch, and so we have a form of cusped turbulence

4 Some Applications in 3 Dims

4.1 Hot and cool caustics in 3 dims

Using our new technique for identifying hot and cool parts we can, for the first time, obtain an explicit expression for the hot and cool boundary on the 3 dimensional swallowtail caustic. Previously only a numerical approximation was possible due to the complexity of earlier techniques. Let

$$S_0(x_0, y_0, z_0) = x_0^7 + x_0^3 y_0 + x_0^2 z_0, \quad c \equiv 0, \quad k_t \equiv 0.$$

The parametric form of the precaustic is

$$
\begin{aligned}
x_0 &= \lambda_1 \\
y_0 &= \lambda_2 \\
z_0(\lambda_1,\lambda_2) &= -\frac{1}{2t}\left(42t\lambda_1^5 - 9t^2\lambda_1^4 - 4t^2\lambda_1^2 + 6t\lambda_1\lambda_2 + 1\right).
\end{aligned}
$$

The caustic is

$$
\begin{aligned}
x_t(\lambda_1,\lambda_2) &= \lambda_1^2 t\left(-35\lambda_1^4 + 9t\lambda_1^3 + 4t\lambda_1 - 3\lambda_2\right) \\
y_t(\lambda_1,\lambda_2) &= \lambda_2 + t\lambda_1^3 \\
z_t(\lambda_1,\lambda_2) &= -21\lambda_1^5 + 4.5t\lambda_1^4 + 3t\lambda_1^2 - 3\lambda_1\lambda_2 - \frac{1}{2t}.
\end{aligned}
$$

Then following the earlier definitions the boundary is given by repeated roots of:

$$
\begin{aligned}
\tilde{F}(x_0) &= -30\lambda_1^4 - 2\lambda_2 + 3\lambda_1 t + 8\lambda_1^3 t - 20\lambda_1^3 x_0 + tx_0 + 6\lambda_1^2 t x_0 - 12\lambda_1^2 x_0^2 + 3\lambda_1 t x_0^2 \\
&\quad -6\lambda_1 x_0^3 + tx_0^3 - 2x_0^4 \\
\tilde{G}(x_0) &= -35\lambda_1^4 - 3\lambda_2 + 4\lambda_1 t + 9\lambda_1^3 t - 28\lambda_1^3 x_0 + 2tx_0 + 9\lambda_1^2 t x_0 - 21\lambda_1^2 x_0^2 + 6\lambda_1 t x_0^2 \\
&\quad -14\lambda_1 x_0^3 + 3tx_0^3 - 7x_0^4.
\end{aligned}
$$

We can then find the boundary (on the precaustic) explicitly as

$$
\begin{aligned}
\lambda_2 &= \lambda_2(\lambda_1) \\
&= \begin{cases}
\frac{1}{4096}\Big(-35952\lambda_1^4 + 4608\lambda_1 t + 6176\lambda_1^3 t + 256t^2 + 408\lambda_1^2 t^2 + 72\lambda_1 t^3 \\
\quad +9t^4 + \{P_1(\lambda_1) - Q_1(\lambda_1)\}^{\frac{1}{3}} + \{P_1(\lambda_1) + Q_1(\lambda_1)\}^{\frac{1}{3}} \Big) & : \ \lambda_1 < \lambda_c \\
\frac{1}{87808}\Big(-653072\lambda_1^4 + 87808\lambda_1 t + 98784\lambda^3 t + 6272t^2 + 7448\lambda_1^2 t^2 \\
\quad +1512\lambda_1 t^3 + 243t^4 + \{P_2(\lambda_1) - Q_2(\lambda_1)\}^{\frac{1}{3}} + \{P_2(\lambda_1) + Q_2(\lambda_1)\}^{\frac{1}{3}} \Big) & : \ \lambda_1 > \lambda_c,
\end{cases}
\end{aligned}
$$

where P_1 and P_2 are polynomials of degree 12 in λ_1 and t, and Q_1 and Q_2 are the square roots of polynomials of degree 24 in λ_1 and t. Also λ_c is the unique real solution for λ to the equation

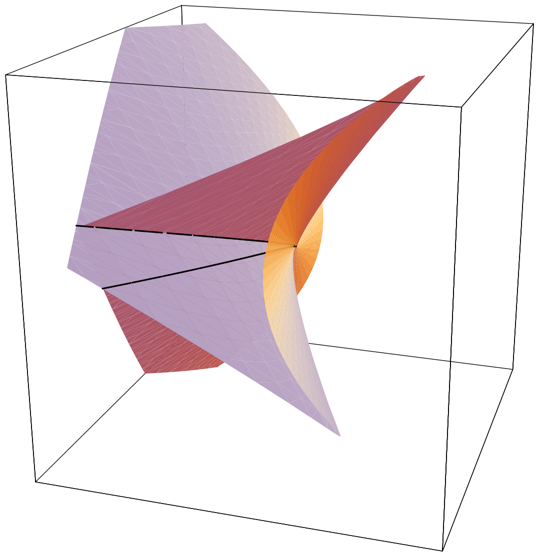
$$70\lambda^3 - 15\lambda^2 t - t = 0.$$

It can then be easily demonstrated that the point on the caustic $x_t(\lambda_1,\lambda_2)$ is

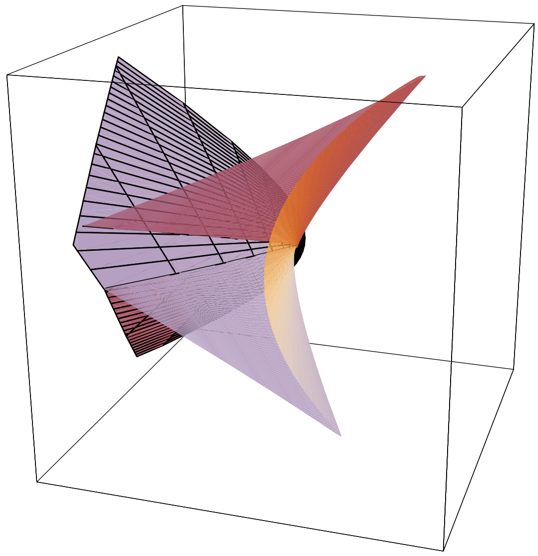
1. HOT if $\lambda_2 < \lambda_2(\lambda_1)$,
2. COOL if $\lambda_2 \geq \lambda_2(\lambda_1)$.

This is illustrated below where the hatched areas represent the cool parts and the plain areas the hot parts.

3D Swallowtail caustic



Boundary on caustic



Hot and cool parts

4.2 The complex caustic in 3 dims

We consider how to extend our work on the complex caustic to a three dimensional setting. There is no immediate analogue of Klein's work for three dimensions and so we instead consider what can be gained from the derivatives of the reduced action functional. We have already found a geometrical interpretation for zeros of each of the first four derivatives in terms of the subcaustic, and so our attention turns to the fifth derivative. The natural way to extend our work is to reduce the three dimensional case to a two dimensional setting so that we can again apply Klein's ideas. We achieve this by considering the subcaustic and its projection onto each of the planes $x = 0$, $y = 0$ and $z = 0$. Setting the first three derivatives of f to zero forces us onto the subcaustic, and so we want to consider when complex double points of the caustic join the subcaustic. That is we want to solve

$$\frac{1}{\eta}\text{Im}\left\{\Phi_t\left(a+i\eta, \lambda_2(a+i\eta), z_t(a+i\eta, \lambda_2(a+i\eta))\right)\right\} = 0,$$

where $\lambda_2 = \lambda_2(\lambda_1)$ denotes the equation of the presubcaustic. However this gives us three equations in the two unknowns a and η. Thus we are forced to consider the projections of the subcaustic onto three orthogonal planes. Let P_x, P_y, and P_z denote projections onto the planes $x = 0$, $y = 0$, and $z = 0$ respectively. For brevity we only give results for P_x.

Lemma 4.1 *Let $x_t(\lambda_1, \lambda_2)$ be a real analytic parameterisation of the caustic and let $\lambda_2 = \lambda_2(\lambda_1)$ denote the presubcaustic. Then the projected subcaustic $P_x x_t(\lambda_1, \lambda_2(\lambda_1))$ has a cusp if and only if the curves*

$$\frac{1}{\eta}\text{Im}\left\{P_x x_t(a+i\eta, \lambda_2(a+i\eta))\right\} = 0,$$

intersect at a point $(a,0)$ in the (a,η) plane.

Proposition 4.2 *A complex double point joins the projected subcaustic at a time t_c if and only if there is a solution $a \in \mathbb{R}$ to the equations*

$$\frac{\partial P_x x_t^{sc}}{\partial \lambda_1} = \frac{\partial^2 P_x x_t^{sc}}{\partial \lambda_1^2} = 0.$$

As in the two dimensional case a swallowtail will form on the projected subcaustic at such times subject to suitable conditions. Therefore if we find a time t_c and a parameter a at which a complex double point joins the projected subcaustic for each projection P_x, P_y and P_z, then a swallowtail should form simultaneously on each of the three projections. Moreover at such a time and position

$$\frac{\partial x_t^{sc}}{\partial \lambda_1} = \frac{\partial^2 x_t^{sc}}{\partial \lambda_1^2} = 0.$$

Which leads us to the following proposition regarding the reduced action functional.

Proposition 4.3 *If a complex double point joins each of the projected subcaustics at a time t_c at a point a then there is a real solution $\boldsymbol{x}$ to*

$$f'_{(x,t_c)}(a) = f''_{(x,t_c)}(a) = f'''_{(x,t_c)}(a) = f^{(4)}_{(x,t_c)}(a) = f^{(5)}_{(x,t_c)}(a) = 0.$$

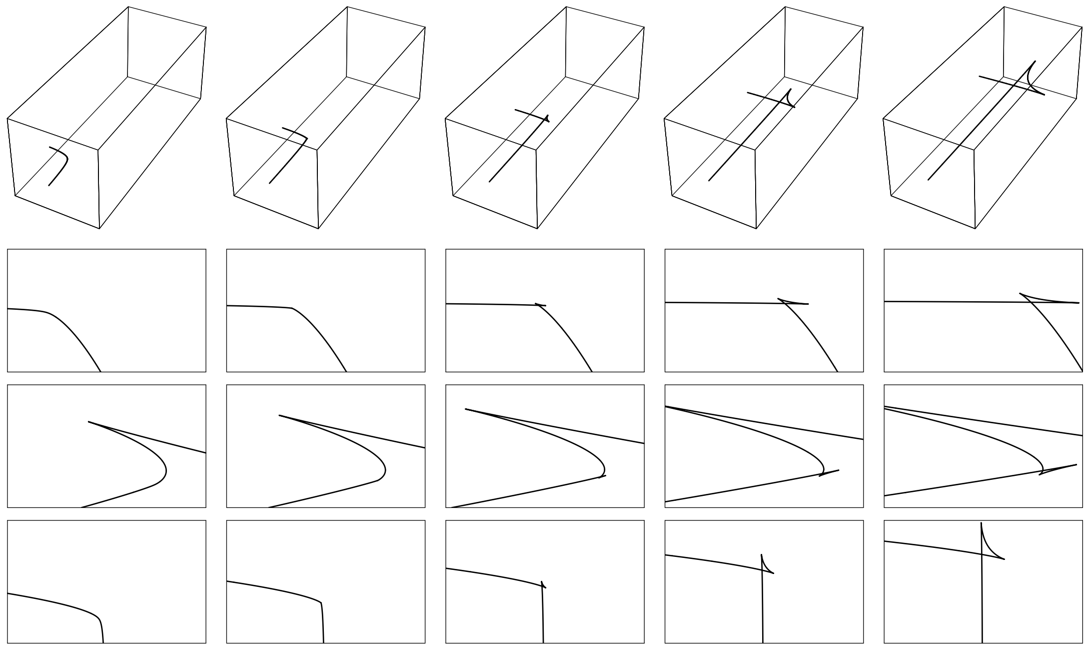
The formation of a swallowtail on each of the projections of the subcaustic produces an interesting shape on the caustic.

Example 4.4 *Let us consider the initial condition*

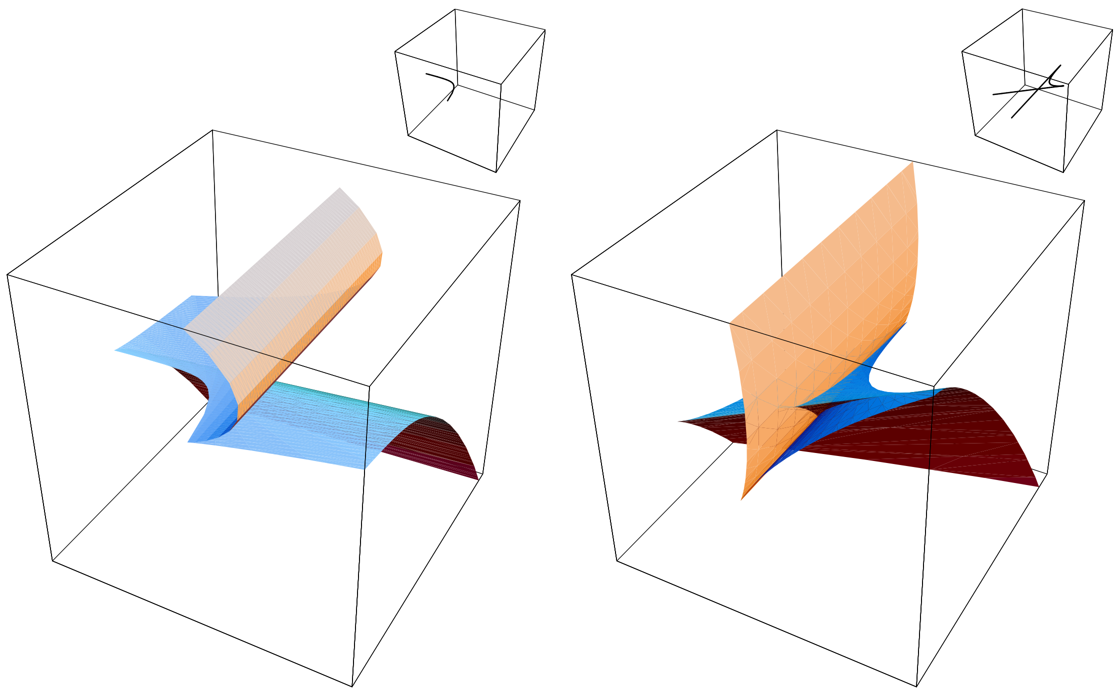
$$S_0(x_0, y_0, z_0) = x_0^4 + x_0^3 + x_0^2 + x_0^5 y_0 + x_0^2 z_0$$

Then (to 5 d.p.) at time $t = 5.98056$ when $x = -0.199789$, $y = 1.62976$, $z = -1.34006$ and $x_0 = 0.23091$ the first five derivatives of the reduced action functional will be zero. Therefore from the preceding propositions we would expect swallowtails to form on each of the 3 projections of the subcaustic. Moreover if we consider the caustic, two dimensional swallowtails form on slices taken parallel to any of the axes. In fact a pyramid like structure forms on the caustic.

Subcaustic with projections at times t=5, 6, 7, 8, 9.



The caustic (with subcaustic inset) at times t=5, and 9.



References

[1] V.I. Arnol'd, *Mathematical methods of classical mechanics,* (Springer-Verlag, New York 1989).

[2] V.I. Arnol'd, *Singularities of caustics and wave fronts,* Mathematics and its applications (Soviet Series) 62, (Kluwer Academic Publishers Group, Dordrecht, 1990).

[3] C. Dafermos, *Hyperbolic conservation laws in continuum physics,* Grundlehren der Mathematischen Wissenschaten 325, (Springer-Verlag, Berlin, 2000).

[4] I.M. Davies, A. Truman, H. Zhao, *Stochastic heat and Burgers equations and their singularities - geometrical and analytic properties (the fish the butterfly, and why.),* UWS MRRS preprint,
http://www.ma.texas.edu/mp_ arc-bin/mpa?yn=01-45, (2001).

[5] I.M. Davies, A. Truman, H. Zhao, *Stochastic heat and Burgers equations and their singularities I - geometric properties,* J. Math. Phys. **43**, 3293-3328 (2002).

[6] K.D. Elworthy, A. Truman and H. Zhao, *Stocahstic elementary formulae on caustics 1: One dimensional linear heat equations,*UWS MRRS preprint.

[7] M.I. Freidlin and A.D. Wentzell, *Random perturbations of dynamical systems*, (Springer-Verlag, NewYork, 1998).

[8] W.E.K Khanin, A. Mazel and Y. Sinai, *Invariant measures for Burgers equations with stochastic forcing*, Ann. Math. **151**, 877-960 (2000).

[9] F. Klein, *Über den Verlauf der Abelschen Integrale bei den Kurven vierten Grades*, in *Gesammelte Mathematische Abhandlungen II*, Edited by R. Fricke , H. Vermeil. (Springer, Berlin, 1922)

[10] V.N. Kolokoltsov, R.L. Schilling, and A.E. Tyokov, *Estimates for multiple stochastic integrals and stochastic Hamilton-Jacobi equations,* Submitted.

[11] H. Kunita, *Stochastic differential equations and stochastic flows of homeomorphisms*, in *Stochastic Analysis and Applications*, edited by M.A. Pinsky, Advances in Probability and Related Topics (Marcel Dekker, New York, 1984), Vol. 7, pp 269-291.

[12] V.P. Maslov, *Perturbation theory and asymptotic methods*, (Dunod, Paris 1972).

[13] V.P. Maslov and M.V. Fedoriuk, *Semi-classical approximation in quantum mechanics* Mathematical physics and applied mathematics vol. 7, (D. Riedel Publishing Company, 1981)

[14] C. Reynolds *On the polynomial swallowtail and cusp singularities of stochastic Burgers equations*, PhD thesis, UWS, (2002).

[15] S.F. Shandarin and Y.B. Zeldovich , *The large scale structure of the universe: turbulence, intermittency , structures in a self gravitating medium,* Rev. Mod. Phys. **6**, 185-220 (1989).

[16] A. Truman and H. Zhao, *Stochastic Burgers equations and their semi classical expansions,* Comm. Math. Phys. **194**, 231-248 (1998).

[17] A. Truman, C.N. Reynolds and D. Williams, *Stochastic Burgers equation in d-dimensions - a one dimensional analysis: Hot and cool caustics and intermittence of stochatsic turbulence,* in *Probabilistic methods in fluids*, edited by I.M. Davies, O. Hassan, N. Jacob, K. Morgan, A. Truman, N.P. Weatherill. (World Scientific, 2003).