HOLOMORPHIC SOBOLEV SPACES AND THE GENERALIZED SEGAL–BARGMANN TRANSFORM

BRIAN C. HALL AND WICHARN LEWKEERATIYUTKUL

ABSTRACT. We consider the generalized Segal–Bargmann transform C_t for a compact group K, introduced in B. C. Hall, *J. Funct. Anal.* **122** (1994), 103-151. Let $K_{\mathbb{C}}$ denote the complexification of K. We give a necessary-and-sufficient pointwise growth condition for a holomorphic function on $K_{\mathbb{C}}$ to be in the image under C_t of $C^{\infty}(K)$. We also characterize the image under C_t of Sobolev spaces on K. The proofs make use of a holomorphic version of the Sobolev embedding theorem.

CONTENTS

1. INTRODUCTION AND STATEMENT OF RESULTS

The Segal–Bargmann transform, in a form convenient for the purposes of this paper, is the map $C_t : L^2(\mathbb{R}^d) \to \mathcal{H}(\mathbb{C}^d)$ given by

$$(1) \qquad C_t f(z) = \int_{\mathbb{R}^d} (2\pi t)^{-d/2} e^{-(z-x)^2/2t} f(x)\, dx, \quad z \in \mathbb{C}^d.$$

Here $(z-x)^2 = (z_1 - x_1)^2 + \cdots + (z_d - x_d)^2$ and $\mathcal{H}(\mathbb{C}^d)$ denotes the space of (entire) holomorphic functions on $\mathbb{C}^d$. It is easily verified that the integral in (1) is absolutely convergent for all $z \in \mathbb{C}^d$ and that the result is a holomorphic function of z. If we restrict attention to $z \in \mathbb{R}^d$, then we may recognize the function

$$(2) \qquad (2\pi t)^{-d/2} e^{-(z-x)^2/2t}$$

as the *heat kernel* for $\mathbb{R}^d$, that is, the integral kernel for the time-t heat operator. This means that $C_t f$ may alternatively be described as

$$(3) \qquad C_t f = \text{ analytic continuation of } e^{t\Delta/2}f.$$

Here the analytic continuation is from $\mathbb{R}^d$ to $\mathbb{C}^d$ with t fixed, and $e^{t\Delta/2}$ is the time-t (forward) heat operator. (We take the Laplacian to be a negative operator and follow the probabilists' normalization of the heat operator.)

Theorem 1 (Segal–Bargmann). *For each $t > 0$, the map C_t is a unitary map of $L^2(\mathbb{R}^d)$ onto $\mathcal{H}L^2(\mathbb{C}^d, \nu_t)$. Here $\mathcal{H}L^2(\mathbb{C}^d, \nu_t)$ denotes the space of holomorphic functions that are square-integrable with respect to the measure $\nu_t(z)\, dz$, where dz denotes $2d$-dimensional Lebesgue measure on $\mathbb{C}^d$ and where ν_t is the density given by*

$$\nu_t(x + iy) = (\pi t)^{-d/2} e^{-y^2/t}, \quad x, y \in \mathbb{R}^d.$$

Our normalization of the Segal–Bargmann transform is different from that of Segal [Se] and Bargmann [B1]; see [H5], [H8], or [H1] for a comparison of normalizations. Note that the function ν_t is simply the heat kernel at the origin in the y variable, evaluated at time $t/2$. (That is, to get ν_t, put $z = 0$ in (2), replace x with y, and replace t by $t/2$.)

One of the distinctive features of L^2 spaces of holomorphic functions is that "pointwise evaluation is continuous." This means, in the present setting, that for each $z \in \mathbb{C}^d$, the map $F \to F(z)$ is a continuous linear functional on $\mathcal{H}L^2(\mathbb{C}^d, \nu_t)$. One can shown (adapting a result of Bargmann [B1] to our normalization) that the norm of the "evaluation at z" functional is precisely $(4\pi t)^{-d/4} e^{y^2/2t}$. This means that elements F of $\mathcal{H}L^2(\mathbb{C}^d, \nu_t)$ satisfy the pointwise bounds

$$(4) \qquad |F(x + iy)|^2 \leq A e^{y^2/t},$$

where the optimal value of A is $(4\pi t)^{-d/2} \|F\|^2_{L^2(\mathbb{C}^d, \nu_t)}$. Conversely, if a holomorphic function F satisfies a polynomially better bound, say,

$$(5) \qquad |F(x + iy)|^2 \leq A \frac{e^{y^2/t}}{1 + (x^2 + y^2)^{d+\varepsilon}}, \quad \varepsilon > 0,$$

then, by direct calculation, F will be square-integrable with respect to the measure $\nu_t(z)\, dz$ and thus will be in $\mathcal{H}L^2(\mathbb{C}^d, \nu_t)$.

Theorem 1 characterizes the image under C_t of $L^2(\mathbb{R}^d)$ exactly as a holomorphic L^2 space over $\mathbb{C}^d$. The image of $L^2(\mathbb{R}^d)$ can also be characterized by the necessary pointwise bounds (4) and the slightly stronger sufficient pointwise bounds (5). It is natural to ask in addition for a characterization of other spaces of functions, for example, the Schwarz space. The "polynomial closeness" between the necessary bounds (4) and the sufficient bounds (5) is a key ingredient in the following result of Bargmann [B2, Theorem 1.7] (adapted, as always, to our normalization of the transform).

Theorem 2 (Bargmann). *Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwarz space. If F is a holomorphic function on $\mathbb{C}^d$ then there exists $f \in \mathcal{S}(\mathbb{R}^d)$ with $C_t f = F$ if and only if F satisfies*

$$|F(x + iy)|^2 \leq A_n \frac{e^{y^2/t}}{[1 + (x^2 + y^2)]^{2n}}$$

for some sequence of constants A_n, $n = 1, 2, 3, \ldots$.

See also [H8] for related results. Roughly speaking, smoothness of f gives a polynomial improvement in the behavior of F (compared to (4)) in the imaginary (y) directions, while decay at infinity of f gives polynomial improvement of F in the real (x) directions.

The purpose of this paper is to obtain similar results for the generalized Segal–Bargmann transform introduced in [H1]. (The paper [H1] was motivated by results of Gross [Gr]. For more information about the generalized Segal–Bargmann transform and its connections to the work of Gross, see [H7] and [H9].) Let K be an arbitrary connected compact Lie group. Fix once and for all a bi-invariant Riemannian metric on K and let Δ_K denote the (negative) Laplacian operator with respect to this metric. Let $K_{\mathbb{C}}$ denote the complexification of K, which is a certain complex Lie group containing K as a maximal compact subgroup. (For example, if $K = \mathsf{U}(n)$ then $K_{\mathbb{C}} = \mathsf{GL}(n; \mathbb{C})$.) Let dx denote the Haar measure on K, normalized to coincide with the Riemannian volume measure. Then, by analogy to the $\mathbb{R}^d$ case, we define a map $C_t : L^2(K, dx) \to \mathcal{H}(K_{\mathbb{C}})$ by

$$C_t f = \text{analytic continuation of } e^{t\Delta_K/2} f.$$

It can be shown [H1, Sect. 4] that for any $f \in L^2(K, dx)$ and any fixed $t > 0$, $e^{t\Delta_K/2} f$ admits a unique analytic continuation from K to $K_{\mathbb{C}}$. One of the main results of [H1] is the following.

Theorem 3. *For each $t > 0$ there exists a smooth positive function ν_t on $K_{\mathbb{C}}$ such that C_t is a unitary isomorphism of $L^2(K, dx)$ onto $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. Here $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ denotes the space of holomorphic functions on $K_{\mathbb{C}}$ that are square-integrable with respect to the measure $\nu_t(g)\,dg$, where dg is the Haar measure on $K_{\mathbb{C}}$.*

We will use a convenient normalization of the Haar measure on $K_{\mathbb{C}}$, given in (9) below. As in the $\mathbb{R}^d$ case, the function ν_t is the "heat kernel at the origin in the imaginary variables." This means, more precisely, that ν_t is the heat kernel at the identity coset for the noncompact symmetric space $K_{\mathbb{C}}/K$, viewed as a bi-K-invariant function on $K_{\mathbb{C}}$ (and evaluated at time $t/2$). There is an explicit formula for ν_t, due to Gangolli, which we will make use of repeatedly in what follows. (See (10).)

In this paper, we consider the image under C_t of spaces other than $L^2(K)$. Since K is compact, there is no behavior at infinity to worry about, and therefore the natural function spaces to consider are ones with various degrees of smoothness. We will consider Sobolev spaces on K and also $C^\infty(K)$. In particular we will give (Theorem 5) a single necessary-and-sufficient pointwise condition that a holomorphic function must satisfy in order to be in the image under C_t of $C^\infty(K)$. This result is the analog for a compact group of Bargmann's result (Theorem 2) for $\mathbb{R}^d$.

To describe our results we introduce *polar coordinates* on $K_{\mathbb{C}}$, which are analogous to the coordinates $z = x + iy$ on $\mathbb{C}^d$. If $\mathfrak{k}$ denotes the Lie algebra of K, then the Lie algebra of $K_{\mathbb{C}}$ is $\mathfrak{k}_{\mathbb{C}} := \mathfrak{k} + i\mathfrak{k}$, and so we may consider the exponential mapping from $\mathfrak{k} + i\mathfrak{k}$ into $K_{\mathbb{C}}$.

Proposition 4 (Polar Coordinates)**.** *For each g in $K_{\mathbb{C}}$ there exists a unique x in K and Y in $\mathfrak{k}$ such that*

(6) $$g = x e^{iY}, \quad x \in K, Y \in \mathfrak{k}.$$

Furthermore, the map $(x, Y) \to x e^{iY}$ is a diffeomorphism of $K \times \mathfrak{k}$ with $K_{\mathbb{C}}$.

This is a standard result in the case in which K is semisimple [Kn, Theorem 6.31] and is easily extended to the general case, as discussed in Section 11 of [H1]. Consider, for example, the case $K = \mathsf{U}(n)$. Then $K_\mathbb{C} = \mathsf{GL}(n;\mathbb{C})$ and the elements of $\mathfrak{k} = \mathfrak{u}(n)$ are skew-self-adjoint matrices. Thus for $Y \in \mathfrak{k}$, iY will be self-adjoint and e^{iY} will be self-adjoint and positive. Thus the decomposition $g = xe^{iY}$ for a matrix $g \in \mathsf{GL}(n;\mathbb{C})$ is the ordinary polar decomposition into the product of a unitary matrix x and a positive matrix e^{iY}.

Now let Φ be the unique Ad-K-invariant function on $\mathfrak{k}$ whose restriction to any maximal commutative subspace $\mathfrak{t}$ of $\mathfrak{k}$ is given by

$$(7) \qquad \Phi(H) = \prod_{\alpha \in R^+} \frac{\alpha(H)}{\sinh \alpha(H)}, \quad H \in \mathfrak{t}.$$

Here $R \subset \mathfrak{t}^*$ denotes the set of real roots of $\mathfrak{k}$ relative to $\mathfrak{t}$ and R^+ denotes a set of positive roots for this root system. We are now ready to state the main result of this paper.

Theorem 5. *Suppose F is a holomorphic function on $K_\mathbb{C}$ and t is a fixed positive number. Then there exists $f \in C^\infty(K)$ with $F = C_t f$ if and only if F satisfies*

$$(8) \qquad \left|F(xe^{iY})\right|^2 \le A_n \Phi(Y) \frac{e^{|Y|^2/t}}{(1 + |Y|^2)^{2n}}$$

for some sequence of constants A_n, $n = 1, 2, 3, \dots$.

In the right-hand side of (8) we think of $\mathfrak{k}$ as the tangent space to K at the identity. Then $|Y|$ is computed with respect to the restriction of the bi-invariant metric on K to $\mathfrak{k} = T_e(K)$.

Since the function Φ plays a critical role in this paper, it is worth taking a moment to consider its behavior. If K is commutative then there are no roots and so we have

$$\Phi(Y) \equiv 1 \quad \text{(commutative case)}.$$

If K is semisimple then the roots span $\mathfrak{t}^*$ and as a result the function Φ has exponential decay at infinity. For example, consider the rank-one case $K = \mathsf{SU}(2)$ and equip $\mathsf{SU}(2)$ with the bi-invariant Riemannian metric whose restriction to $\mathsf{su}(2) = T_e(\mathsf{SU}(2))$ is given by $|Y|^2 = 2\,\mathrm{trace}(Y^*Y)$. Then we have

$$\Phi(Y) = \frac{|Y|}{\sinh |Y|} \quad \text{($\mathsf{SU}(2)$ case)}$$

for all Y in $\mathsf{su}(2)$. The function Φ is related to the exponential growth (in the noncommutative case) of Haar measure on $K_\mathbb{C}$. Specifically, the Haar measure on $K_\mathbb{C}$ can be written in polar coordinates as follows:

$$(9) \qquad dg = \frac{1}{\Phi(Y)^2}\, dx\, dY,$$

where dx is the Haar measure on K and dY is the Lebesgue measure on $\mathfrak{k}$ (normalized by the inner product). (See [H3, Lem. 5].)

To understand the significance of the bounds (8), we need to look at the expression for the measure $\nu_t(g)\, dg$. The *function ν_t* has the following expression, due to Gangolli [Ga, Prop. 3.2]:

$$(10) \qquad \nu_t(xe^{iY}) = c_t \Phi(Y) e^{-|Y|^2/t}, \quad x \in K,\, Y \in \mathfrak{k},$$

where

$$(11) \qquad c_t = (\pi t)^{-d/2} e^{-|\delta|^2 t}.$$

In (11), δ is half the sum of the positive roots for K and $d = \dim K$. (See also [H3, Eq. (11)].) By combining (10) and (9) we obtain an expression for the *measure* $\nu_t(g)\, dg$, namely,

$$(12) \qquad \nu_t(g)\, dg = c_t \frac{e^{-|Y|^2/t}}{\Phi(Y)}\, dx\, dY, \quad g = x e^{iY}.$$

Let us compare to the $\mathbb{R}^d$ case, where $\nu_t(x + iy) = (\pi t)^{-d/2} e^{-|y|^2/t}$. If K is commutative then Φ is identically equal to one and things behave as in the $\mathbb{R}^d$ case. If K is semisimple, then $\Phi(Y)$ decays exponentially. This means that although the function ν_t has *faster* decay at infinity than $e^{-|Y|^2/t}$, the measure $\nu_t(g)\, dg$ (which is the quantity that really matters) has *slower* decay at infinity than the measure $e^{-|Y|^2/2}\, dx\, dY$. The slower decay at infinity of the heat kernel measure reflects that (if K is semisimple) $K_\mathbb{C}/K$ has a lot of negative curvature. The negative curvature causes the heat to flow out to infinity faster than in the Euclidean case, which makes the heat kernel measure larger near infinity than in the Euclidean case.

The paper [H3] establishes the following pointwise bounds for elements F of $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$

$$(13) \qquad \left|F(x e^{iY})\right|^2 \leq A\Phi(Y) e^{|Y|^2/t}.$$

In the semisimple case, this bound is better (as a function of Y) than in the $\mathbb{R}^d$ case, because of the exponentially decaying factor $\Phi(Y)$. Intuitively, the reason for this is the slower decay at infinity of the measure (12): If F is to be square-integrable with respect to the slower-decaying measure, then F must have correspondingly better behavior at infinity. (Nevertheless, actually proving the bounds in (13) is not especially easy; see Section 5.) As in the $\mathbb{R}^d$ case, we can see by direct calculation (using (12)) that if a holomorphic function F on $K_\mathbb{C}$ satisfies polynomially stronger bounds than (13), say,

$$(14) \qquad \left|F(x e^{iY})\right|^2 \leq A\Phi(Y) \frac{e^{|Y|^2/t}}{(1 + |Y|^2)^{d/2+\varepsilon}}, \quad \varepsilon > 0,$$

then F is square-integrable with respect to the measure in (12) and is therefore in $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$.

It is the polynomial closeness of the necessary bounds (13) and the sufficient bounds (14) that is the key to the proof of Theorem 5. Specifically, instead of considering $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$, which is the image under C_t of $L^2(K)$, we will consider the image under C_t of the Sobolev space $H^{2n}(K)$, consisting of functions on K having all derivatives up to order $2n$ in L^2. We will give necessary pointwise bounds and sufficient pointwise bounds for the image of $H^{2n}(K)$. These bounds are the same as (13) and (14), except with an extra factor of $(1 + |Y|^2)^{-2n}$ on the right-hand side. (Compare (13) and (14) to (15) and (17).) The polynomial closeness of the two sets of bounds means that the necessary bounds for one value of n become sufficient for some slightly smaller value of n. Thus, after intersecting over all n we obtain the single necessary-and-sufficient condition on the image of $C^\infty(K)$ given in Theorem 5.

In the remainder of this section, we state the results concerning Sobolev spaces and describe the strategy for proving Theorem 5. The Sobolev space $H^{2n}(K)$ can also be described as the set of f in $L^2(K)$ such that $\Delta_K^n f$ (computed in the distributional sense) is again in $L^2(K)$. We then think of $C^\infty(K)$ as the intersection of $H^{2n}(K)$ over $n = 1, 2, 3, \ldots$. In Section 2, we will give the following (easy) characterization of the image of $H^{2n}(K)$ under C_t.

Theorem 6. *For $f \in L^2(K)$, let $F = C_t f$. Then f is in $H^{2n}(K)$ if and only if $\Delta_K^n F \in \mathcal{H}L^2(K_\mathbb{C}, \nu_t)$.*

In computing $\Delta_K^n F$, we regard Δ_K as a left-invariant differential operator on $K_\mathbb{C}$; see Section 2 for details. We wish to think of the set of holomorphic functions F with $F \in \mathcal{H}L^2(K_\mathbb{C}, \nu_t)$ and $\Delta_K^n F \in \mathcal{H}L^2(K_\mathbb{C}, \nu_t)$ as a sort of "holomorphic Sobolev space." Now, it may seem odd at first to speak of Sobolev spaces in the holomorphic context. After all, every element F of $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$ is automatically infinitely differentiable, and $\Delta_K^n F$ is automatically holomorphic again. Nevertheless, given $F \in \mathcal{H}L^2(K_\mathbb{C}, \nu_t)$, there is no reason that $\Delta_K^n F$ must be again square-integrable with respect to $\nu_t(g)\,dg$. Thus having $\Delta_K^n F$ be in $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$ is a nontrivial "regularity" condition on F.

Definition 7. *The $2n^{th}$ **holomorphic Sobolev space** on $K_\mathbb{C}$, denoted $\mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$, is the space of holomorphic functions F on $K_\mathbb{C}$ such that $F \in L^2(K_\mathbb{C}, \nu_t)$ and $\Delta_K^n F \in L^2(K_\mathbb{C}, \nu_t)$.*

Note that on K, membership in the Sobolev space $H^{2n}(K)$ is a local condition: Since K is compact, if f is in $H^{2n}(K)$ locally then it is in $H^{2n}(K)$ globally. By contrast, membership in the holomorphic Sobolev space is a condition on the behavior of the function at infinity: If F is holomorphic then $\Delta_K^n F$ is automatically square-integrable locally, and it is only the behavior at infinity that one needs to worry about. Using holomorphic Fourier series (see [H1, Sect. 8]) it is easy to show that if F is *any* holomorphic function on $K_\mathbb{C}$ such that $\Delta_K^n F \in \mathcal{H}L^2(K_\mathbb{C}, \nu_t)$, F itself will automatically be in $L^2(K_\mathbb{C}, \nu_t)$ and therefore F will be in $\mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$. The same sort of reasoning shows that if $F \in \mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$, then $\Delta_K^m F \in \mathcal{H}L^2(K_\mathbb{C}, \nu_t)$ for $1 \le m \le n$ and, more generally, all left-invariant derivatives of F up to order $2n$ are in $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$.

We wish to relate membership of a function F in the holomorphic Sobolev space to behavior of F at infinity. Such a relationship can be thought of as a holomorphic version of the Sobolev embedding theorem: existence of derivatives in L^2 translates into improved pointwise behavior of the function itself. Such a result is obtained in Section 2 by estimating the reproducing kernel for the $2n^{\text{th}}$ holomorphic Sobolev space, leading to the following.

Theorem 8 (Holomorphic Sobolev embedding theorem)**.** *If F belongs to the $2n^{th}$ holomorphic Sobolev space $\mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$, then for some constant A (depending on F and n) we have*

$$(15) \qquad \left|F(xe^{iY})\right|^2 \le A\frac{\Phi(Y)e^{|Y|^2/t}}{(1+|Y|^2)^{2n}},$$

where Φ is as in (7).

This bound is the same as in (13) except for the extra factor of $(1+|Y|^2)^{2n}$ in the denominator.

In this holomorphic setting we can also reason in the opposite direction, in a way that is impossible in ordinary Sobolev spaces. That is, good pointwise bounds imply membership in holomorphic Sobolev spaces. In Section 3, we will prove the following result of this sort.

Theorem 9. *If F is a holomorphic function on $K_{\mathbb{C}}$ then F belongs to the $2n^{th}$ holomorphic Sobolev space $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ if and only if*

$$(16) \qquad \int_{K_{\mathbb{C}}} |F(g)|^2 \, (1 + |Y|^2)^{2n} \nu_t(g) \, dg < \infty, \quad g = x e^{iY}.$$

That is, $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t) = \mathcal{H}L^2(K_{\mathbb{C}}, (1 + |Y|^2)^{2n} \nu_t(g) \, dg)$.

This result is proved by means of integration by parts. Specifically, the function $(1 + |Y|^2)^{2n}$ has the same behavior as a certain logarithmic-type derivative of ν_t, and it is this logarithmic-type derivative that comes out of the integration by parts.

Suppose, now, that F is a holomorphic function satisfying polynomially better bounds than those in Theorem 8, say,

$$(17) \qquad \left|F(x e^{iY})\right|^2 \le A\Phi(Y)\frac{e^{|Y|^2/t}}{(1 + |Y|^2)^{2n+d/2+\varepsilon}} \quad (d = \dim K).$$

Then Theorem 9, together with the expression (12) for the measure $\nu_t(g) \, dg$, shows that F belongs to $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$.

By combining Theorems 8 and 9 we can immediately obtain Theorem 5. Consider $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ and let $f = C_t^{-1} F$. In one direction, if F satisfies the bounds in (8) for a given n, then (17) shows that the integral in (16) is finite for all n' with $n' < n - d/4$. Thus if (8) holds for all n, then so does (16). In such cases, $F \in \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ and $f \in H^{2n}(K)$ for all n, which implies that $f \in C^\infty(K)$.

In the other direction, if $f \in C^\infty(K)$, then certainly $f \in H^{2n}(K)$ for all n and, therefore, $F \in \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ for all n. This, by Theorem 8, implies that F satisfies the bounds in Theorem 5.

2. The holomorphic Sobolev embedding theorem

The goal of this section is to estimate the reproducing kernel for the holomorphic Sobolev spaces introduced in Definition 7, leading to a proof of the holomorphic Sobolev embedding theorem (Theorem 8). We begin with the proof of Theorem 6, which asserts that the image under C_t of the Sobolev space $H^{2n}(K)$ is the holomorphic Sobolev space $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ described in Definition 7. This result holds essentially because both the heat operator and analytic continuation commute with Δ_K.

Before turning to the proof of the theorem, we explain how Δ_K is to be viewed as a differential operator on $K_{\mathbb{C}}$. On K, we have $\Delta_K = \sum_k X_k^2$, where $X_1, \ldots, X_d$ is an orthonormal basis for $\mathfrak{k}$ and each X_k is viewed as a left-invariant differential operator on K. Since $\mathfrak{k} \subset \mathfrak{k}_{\mathbb{C}}$, we may also regard each X_k as a left-invariant differential operator on $K_{\mathbb{C}}$. Then on $K_{\mathbb{C}}$, we define $\Delta_K = \sum_k X_k^2$. This means that, for any C^∞ function ϕ on $K_{\mathbb{C}}$ we have

$$(18) \qquad (\Delta_K \phi)(g) = \sum_{k=1}^d \frac{d^2}{dt^2} \phi\left(g e^{t X_k}\right)\Big|_{t=0}.$$

Note that although Δ_K is a bi-K-invariant operator on K, on $K_{\mathbb{C}}$, Δ_K is only left-$K_{\mathbb{C}}$-invariant and not bi-$K_{\mathbb{C}}$-invariant. (When applied to holomorphic functions, however, Δ_K coincides with the analogously defined right-$K_{\mathbb{C}}$-invariant operator.) The operator Δ_K preserves the space of holomorphic functions on $K_{\mathbb{C}}$. Furthermore, if a function f on K admits an analytic continuation to a holomorphic function on $K_{\mathbb{C}}$, then $\Delta_K f$ also has an analytic continuation to $K_{\mathbb{C}}$, given by $(\Delta_K f)_{\mathbb{C}} = \Delta_K(f_{\mathbb{C}})$, where $(\cdot)_{\mathbb{C}}$ denotes analytic continuation. That is, Δ_K commutes with analytic continuation. We now proceed with the proof of Theorem 6.

Proof of Theorem 6. On K, we consider Δ_K, defined at first on the space of finite linear combinations of matrix entries. (A matrix entry is a function on K of the form $f(x) = \text{trace}(\pi(x)A)$, where π is an irreducible representation of K acting on a finite-dimensional space V and where A is a linear operator on V.) Then Δ_K^n is essentially self-adjoint on this space. This holds because $L^2(K)$ is the orthogonal direct sum of the spaces of matrix entries (as the representation varies over equivalence classes of irreducible representations of K) and the restriction of Δ_K^n to each space of matrix entries (for a fixed representation) is a real multiple of the identity. Then we define $H^{2n}(K)$ as the domain of the closure of the operator "Δ_K^n on finite linear combinations of matrix entries." It is not hard to show that $H^{2n}(K)$ coincides with the space of all $f \in L^2(K)$ such that $\Delta_K^n f$ (computed in the distribution sense) is in $L^2(K)$. Furthermore, if $f \in H^{2n}(K)$, then f is in the domain of Δ_K^m for all m between 1 and n.

To compute the image under C_t of $H^{2n}(K)$, suppose, first, that $f \in H^{2n}(K)$. Since Δ_K commutes with the heat operator $e^{t\Delta_K/2}$ and with analytic continuation, we have $C_t(\Delta_K^n f) = \Delta_K^n C_t f$, which shows that $C_t f \in \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$.

In the reverse direction, it suffices to show that Δ_K^n is a symmetric operator on the domain $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t) \subset \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. After all, since C_t is unitary, $C_t \Delta_K^n C_t^{-1}$ is self-adjoint on $C_t(H^{2n}(K))$. But we have just shown that $C_t(H^{2n}(K)) \subset \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ and that $C_t \Delta_K^n C_t^{-1} F = \Delta_K^n F$ for all $F \in C_t(H^{2n}(K))$. This means that Δ_K^n on the domain $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ is an extension of a self-adjoint operator, and one cannot have a nontrivial symmetric extension of a self-adjoint operator.

Now, to prove that Δ_K^n is symmetric on $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$, we write out the inner product in polar coordinates. It is convenient in this calculation to have the K-factor on the right, so we write $g = xe^{iY} = e^{iY'}x$, where $Y' = \text{Ad}_x(Y)$. Then we have (by (9))

$$(19) \qquad \langle \Delta_K^n F_1, F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} = \int_{\mathfrak{k}} \int_K [\Delta_K^n \bar{F}_1](e^{iY'}x) F_2(e^{iY'}x) \, dx \, \frac{\nu_t(e^{iY'})}{\Phi(Y')^2} dY',$$

since ν_t is bi-K-invariant. In the expression $[\Delta_K^n \bar{F}_1](e^{iY'}x)$ we are thinking of Δ_K^n as a left-invariant differential operator on $K_{\mathbb{C}}$, applied to the function $\bar{F}_1$ and evaluated at the point $e^{iY'}x$. However, from (18) we see that this is the same as applying Δ_K in the x-variable with Y' fixed. Then, since F_1 and F_2 are smooth and since Δ_K^n is symmetric on $C^\infty(K)$, we can integrate by parts in the inner integral to get

$$\langle \Delta_K^n F_1, F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} = \int_{\mathfrak{k}} \int_K \bar{F}_1(e^{iY'}x)[\Delta_K^n F_2](e^{iY'}x) \, dx \, \frac{\nu_t(e^{iY'}x)}{\Phi(Y')^2} dY'$$
$$= \langle F_1, \Delta_K^n F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} .$$

$\square$

We now wish to compute the reproducing kernel for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$. (See [H5] for generalities on reproducing kernels.) Let us first recall the situation concerning the reproducing kernel for $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, since the reproducing kernel for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ is computed by relating it to the reproducing kernel for $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. For each $g \in K_{\mathbb{C}}$, the pointwise evaluation map $F \to F(g)$ is a continuous linear functional on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. Thus, by the Riesz representation theorem, there exists a unique vector $\chi_g \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ such that

$$(20) \qquad F(g) = \langle \chi_g, F \rangle_{L^2(K_{\mathbb{C}}, \nu_t)}$$

for all $F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. (We adopt the convention that the inner product be linear in the second factor.) The vector χ_g is called the **coherent state** for $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ at the point g. This state also depends on t, but we have suppressed this dependence in the notation.

In [H1] it is shown that

$$(21) \qquad \chi_g(h) = \rho_{2t}(g^* h).$$

Here ρ_t is the heat kernel at the identity for K, analytically continued from K to $K_{\mathbb{C}}$ [Section 4 of [H1]], and the map $g \to g^*$ is the unique antiholomorphic anti-involution of $K_{\mathbb{C}}$ such that $x^* = x^{-1}$ for $x \in K$. If $K = \mathsf{U}(n)$ and $K_{\mathbb{C}} = \mathsf{GL}(n; \mathbb{C})$ then g^* is simply the usual matrix adjoint. In polar coordinates we have $(xe^{iY})^* = e^{iY} x^{-1}$. It can be shown that $\overline{\rho_{2t}(g^* h)} = \rho_{2t}(gh^*)$ for all $g \in K_{\mathbb{C}}$. Thus (20) and (21) become

$$F(g) = \int_{K_{\mathbb{C}}} \rho_{2t}(gh^*) F(h) \nu_t(h) \, dh.$$

The function

$$k_t(g, h) := \overline{\chi_g(h)} = \rho_{2t}(gh^*)$$

is called the **reproducing kernel** for $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$.

The norm of the pointwise evaluation functional is equal to the norm of the corresponding coherent state, which can be computed as

$$\|\chi_g\|^2 = \langle \chi_g, \chi_g \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} = k_t(g, g) = \rho_{2t}(gg^*).$$

The pointwise bounds (13) from [H3] are obtained by estimating the behavior of the quantity $\rho_{2t}(gg^*)$.

We now turn to the case of the Sobolev spaces. We consider on $H^{2n}(K)$ the inner product given by

$$(22) \qquad \langle f_1, f_2 \rangle_{H^{2n}(K)} = \langle (cI - \Delta_K)^n f_1, (cI - \Delta_K)^n f_2 \rangle_{L^2(K)},$$

where c is a positive constant whose value will be chosen later. (Recall that our Laplacian is negative.) Different positive values of c give equivalent inner products on $H^{2n}(K)$, and for any c, the inner product (22) is equivalent to the inner product $\langle f_1, f_2 \rangle_{L^2(K)} + \langle \Delta_K^n f_1, \Delta_K^n f_2 \rangle_{L^2(K)}$. The Sobolev space $H^{2n}(K)$ is complete in the inner product (22), because Δ_K^n is closed on $H^{2n}(K)$.

We consider the image under C_t of this space, which is denoted $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ and is characterized in Theorem 6. The map $C_t : H^{2n}(K) \to \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ will be isometric if we use on $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ the inner product

$$(23) \qquad \langle F_1, F_2 \rangle_{\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)} = \langle (cI - \Delta_K)^n F_1, (cI - \Delta_K)^n F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)}.$$

The holomorphic Sobolev space $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ is then complete with respect to the inner product (23).

We now define the **coherent states** for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ to be the elements χ_g^{2n} of $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ such that

$$F(g) = \left\langle \chi_g^{2n}, F \right\rangle_{\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)}.$$

The **reproducing kernel** for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ is then defined as the function $k_t^{2n}(g, h) = \overline{\chi_g^{2n}(h)}$.

Proposition 10. *The reproducing kernel for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ is the function $k_t^{2n}(g, h)$ given by*

$$k_t^{2n}(g, h) = [(cI - \Delta_K)^{-2n} \rho_{2t}](gh^*)$$

which may be computed as

$$(24) \qquad k_t^{2n}(g, h) = \frac{1}{(2n-1)!} \int_0^\infty s^{2n-1} e^{-cs} \rho_{2(t+s)}(gh^*) \, ds.$$

Thus, for all $g \in K_{\mathbb{C}}$ and all $F \in \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ we have

$$(25) \qquad |F(g)|^2 \le \|F\|^2_{\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)} \, k_t^{2n}(g, g).$$

Note that since Δ_K is a non-positive self-adjoint operator on $L^2(K)$ and also on $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$, $(cI - \Delta_K)^{-2n}$ is a bounded self-adjoint operator on both $L^2(K)$ and $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$. The expression $(cI - \Delta_K)^{-2n} \rho_{2t}$ may be interpreted in one of two equivalent ways. We may think of ρ_{2t} as a function on K, apply $(cI - \Delta_K)^{-2n}$, and then analytically to $K_{\mathbb{C}}$. Alternatively, we may first analytically continue ρ_{2t} to $K_{\mathbb{C}}$, think of it as an element of $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$, and then apply $(cI - \Delta_K)^{-2n}$. Since Δ_K (and so also $(cI - \Delta_K)^{-2n}$) commutes with analytic continuation, these two views are equivalent.

Proof. For $F \in \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t) \subset \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ we have

$$\begin{aligned} F(g) &= \langle \chi_g, F \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} \\ &= \left\langle (cI - \Delta_K)^n (cI - \Delta_K)^{-2n} \chi_g, (cI - \Delta_K)^n F \right\rangle_{L^2(K_{\mathbb{C}}, \nu_t)} \\ &= \left\langle (cI - \Delta_K)^{-2n} \chi_g, F \right\rangle_{\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)}, \end{aligned}$$

because $(cI - \Delta_K)^{-n}$ is a self-adjoint operator on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. This means that the coherent state for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ is $(cI - \Delta_K)^{-2n} \chi_g$.

To compute $(cI - \Delta_K)^{-2n}$ we use the elementary calculus identity

$$(26) \qquad \frac{1}{a^{2n}} = \frac{1}{(2n-1)!} \int_0^\infty s^{2n-1} e^{-as} \, ds, \quad a > 0.$$

Applying this formally with $a = (cI - \Delta_K)$ we have

$$(27) \qquad (cI - \Delta_K)^{-2n} = \frac{1}{(2n-1)!} \int_0^\infty s^{2n-1} e^{-cs} e^{s\Delta_K} \, ds.$$

It is not hard to show that this formal argument is correct. Note that since $\Delta_K \le 0$, the integral on the right-hand side of (27) is absolutely convergent in the operator norm topology.

Now applying this to the function ρ_{2t} and noting that $e^{s\Delta_K} \rho_{2t} = \rho_{2t+2s}$ (with our normalization of the heat equation) we obtain

$$(28) \qquad (cI - \Delta_K)^{-2n} \rho_{2t} = \frac{1}{(2n-1)!} \int_0^\infty s^{2n-1} e^{-cs} \rho_{2(t+s)} \, ds.$$

Here we may initially think of the integral in (28) as taking values in the Hilbert space $\mathcal{H}L^2(K_\mathbb{C}, \nu_t)$. However, the estimates below will show that the integral is convergent pointwise for all h in $K_\mathbb{C}$. (More precisely, the estimates will show convergence for points of the form $h = g^*g = e^{2iY}$. For general $h \in K_\mathbb{C}$ we use the inequality, deduced from the matrix-entry expansion of ρ_t, $\left| \rho_t(xe^{iY}) \right| \le \left| \rho_t(e^{iY}) \right|$.)
□

Lemma 11. *For all $t > 0$ there exists a constant α_t such that for all $\tau > t$ and all $g \in K_\mathbb{C}$ we have*

$$\rho_{2\tau}(gg^*) \le \alpha_t \tau^{(r-d)/2} e^{|\delta|^2 \tau} e^{|Y|^2/\tau} \Phi(Y).$$

Here d is the dimension of K, $r = \dim \mathfrak{t}$ is the rank of K, δ is half the sum of the positive roots, and Φ is as given in (7).

This result is a sharpening of Theorem 2 of [H3], obtained by estimating the behavior of the constants a_t in that theorem as t tends to infinity. Assuming this result for the moment, let us complete the proof of the holomorphic Sobolev embedding theorem (Theorem 8).

Proof of Theorem 8. We use Proposition 10 and apply Lemma 11 with $\tau = t + s$. Since $d = \dim K \ge r = \dim \mathfrak{t}$, we have $\tau^{(r-d)/2} \le t^{(r-d)/2}$ for $\tau \ge t$ and we obtain

$$k_t^{2n}(g, g) \le \frac{\alpha_t t^{(r-d)/2} e^{t|\delta|^2}}{(2n-1)!} \Phi(Y) \int_0^\infty s^{2n-1} e^{-(c-|\delta|^2)s} e^{|Y|^2/(s+t)} \, ds.$$

We now choose c so that $c > |\delta|^2$. Multiplying and dividing by $e^{|Y|^2/t}$ and doing some algebra gives

$$(29) \qquad k_t^{2n}(g, g) \le \frac{\beta_t}{(2n-1)!} \Phi(Y) e^{|Y|^2/t} \int_0^\infty s^{2n-1} e^{-Bs} e^{-s|Y|^2/t(s+t)} \, ds,$$

where $B = c - |\delta|^2$ and β_t is independent of g.

We now divide the integral (29) into the region where $s \le t$ and the region where $s > t$. When $s \le t$, $s/t(s+t) \ge s/2t^2$ and we get

$$\int_0^t s^{2n-1} e^{-Bs} e^{-s|Y|^2/t(s+t)} ds \le \int_0^\infty s^{2n-1} \exp\left\{ -s \left[B + \frac{|Y|^2}{2t^2} \right] \right\} ds$$

$$= \frac{(2n-1)!}{(B + |Y|^2 / 2t^2)^{2n}}$$

by (26). When $s > t$, $s/t(s+t) > s/t(2s) = 1/2t$ and we get

$$\int_t^\infty s^{2n-1} e^{-Bs} e^{-s|Y|^2/t(s+t)} ds \le e^{-|Y|^2/2t} \int_0^\infty s^{2n-1} e^{-Bs} ds$$

$$= \frac{(2n-1)!}{B^{2n}} e^{-|Y|^2/2t}.$$

Plugging these estimates into (29) gives

$$k_t^{2n}(g, g) \le \beta_t \Phi(Y) e^{|Y|^2/t} \left[\frac{1}{(B + |Y|^2 / 2t^2)^{2n}} + \frac{e^{-|Y|^2/2t}}{B^{2n}} \right]$$

$$\le \gamma_t \Phi(Y) e^{|Y|^2/t} \frac{1}{(1 + |Y|^2)^{2n}}.$$

This estimate, together with (25), implies the holomorphic Sobolev embedding theorem, Theorem 8. □

It now remains only to prove Lemma 11.

Proof of Lemma 11. The paper [H3] establishes the bound

$$(30) \qquad \rho_{2\tau}(gg^*) \leq a_\tau e^{|\delta|^2 \tau} (4\pi\tau)^{-d/2} \Phi(Y) e^{|Y|^2/\tau}, \quad g = xe^{iY},$$

where a_τ is a quantity independent of g. To establish Lemma 11 we must show that the optimal constants a_τ can be bounded by a constant times $\tau^{r/2}$ as τ tends to infinity. According to Proposition 3 of [H3] we have

$$a_\tau \leq \sum_{\gamma \in \bar{C} \cap \Gamma} P\left(\frac{|\gamma|}{\sqrt{\tau}}\right) e^{-|\gamma|^2/\tau}$$

where $\bar{C}$ is the closed fundamental Weyl chamber, P is a polynomial, and $\Gamma \subset \mathfrak{t}$ is the kernel of the exponential mapping, which is a lattice in $\mathfrak{t}$. We rewrite this as

$$a_\tau \leq \sum_{\eta \in \bar{C} \cap (\Gamma/\sqrt{\tau})} P(|\eta|) e^{-|\eta|^2}.$$

If we let $r = \dim \mathfrak{t}$ then it is straightforward to show, using dominated convergence, that

$$(31) \qquad \lim_{\tau \to \infty} \frac{1}{\tau^{r/2}} \sum_{\eta \in \bar{C} \cap (\Gamma/\sqrt{\tau})} P(|\eta|) e^{-|\eta|^2} = \frac{1}{A} \int_{\bar{C}} P(|x|) e^{-|x|^2} dx,$$

where A is the volume of a fundamental domain in Γ. (Approximate the integrand on the right-hand side of (31) by a function that is constant on each cell of the lattice Γ.) Thus, the left-hand side of (31) is bounded as τ tends to infinity. This means that on each interval of the form $[t, \infty)$ we will have a_τ bounded by a constant (depending on t) times $\tau^{r/2}$. This (together with (30)) gives the estimate in Lemma 11. □

3. Holomorphic Sobolev spaces and Toeplitz operators

Our goal in this section is to show that the holomorphic Sobolev space $\mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$ (Definition 7) can described as a holomorphic L^2 space in which the measure is the heat kernel measure $\nu_t(g)\, dg$ multiplied by the additional factor $(1+|Y|^2)^{2n}$. As explained at the end of Section 1, this result and the holomorphic Sobolev embedding theorem (proved in the previous section) together imply Theorem 5, characterizing the image under C_t of $C^\infty(K)$.

Our strategy is as follows. By a fairly simple integration-by-parts argument, we will obtain a positive function ϕ_{2n} with the property that for sufficiently nice holomorphic functions F_1 and F_2 we have

$$\langle (cI - \Delta_K)^n F, (cI - \Delta_K)^n F \rangle_{L^2(K_\mathbb{C}, \nu_t)} = \int_{K_\mathbb{C}} |F(g)|^2 \, \phi_{2n}(g) \nu_t(g)\, dg.$$

This means that (for sufficiently nice functions) the inner product on $\mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$ coincides with the inner product on $\mathcal{H}L^2(K_\mathbb{C}, \phi_{2n}\nu_t)$. It is then not difficult to show that the Hilbert space $\mathcal{H}^{2n}(K_\mathbb{C}, \nu_t)$ coincides with the Hilbert space $\mathcal{H}L^2(K_\mathbb{C}, \phi_{2n}\nu_t)$. The proof will then be completed by showing that the function $\phi_{2n}(g)$ has the same behavior at infinity as the function $(1 + |Y|^2)^{2n}$.

Definition 12. *Let ϕ be a complex-valued measurable function on $K_{\mathbb{C}}$ (not necessarily holomorphic). Consider the subspace $\mathcal{D}_{\phi}$ of $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ given by*

$$\mathcal{D}_{\phi} = \{F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t) | \phi F \in L^2(K_{\mathbb{C}}, \nu_t)\}.$$

*Then define the **Toeplitz operator** T_{ϕ} to be the (possibly unbounded) operator on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ with domain $\mathcal{D}_{\phi}$ given by*

$$T_{\phi}(F) = P_t(\phi F), \quad F \in \mathcal{D}_{\phi}.$$

*Here P_t is the orthogonal projection operator from $L^2(K_{\mathbb{C}}, \nu_t)$ onto the closed subspace $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$. The function ϕ is called the **Toeplitz symbol** of the Toeplitz operator T_{ϕ}.*

This means that on $\mathcal{D}_{\phi}$, T_{ϕ} is equal to $P_t M_{\phi}$, where M_{ϕ} denotes multiplication by ϕ. If ϕ is bounded then $\mathcal{D}_{\phi} = \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ and T_{ϕ} is a bounded operator. In general, T_{ϕ} may not be densely defined in $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$, though it will be densely defined for the examples we will consider. It is possible that two different symbols could give rise to the same Toeplitz operator.

Proposition 13. *For any $F_1 \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ and $F_2 \in \mathcal{D}_{\phi}$ we have*

$$\langle F_1, T_{\phi}F_2\rangle_{L^2(K_{\mathbb{C}},\nu_t)} = \int_{K_{\mathbb{C}}} \bar{F}_1(g)\phi(g)F_2(g)\nu_t(g)\,dg.$$

Proof. Since P_t is self-adjoint on $L^2(K_{\mathbb{C}}, \nu_t)$ and since $P_tF_1 = F_1$, we have $\langle F_1, T_{\phi}F_2\rangle = \langle F_1, P_tM_{\phi}F_2\rangle = \langle F_1, M_{\phi}F_2\rangle$. $\square$

Our goal is to express each left-invariant differential operator A acting on $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ as a Toeplitz operator with some symbol ϕ_A. The function ϕ_{2n} in the second paragraph of this section will then be the Toeplitz symbol of the operator $(cI - \Delta_K)^{2n}$.

We consider the universal enveloping algebra $U(\mathfrak{k})$ of $\mathfrak{k}$ (with complex coefficients). Then $U(\mathfrak{k})$ is isomorphic to the algebra of left-invariant differential operators on K (with complex coefficients). Each element of $U(\mathfrak{k})$ can also be regarded as a left-invariant differential operator on $K_{\mathbb{C}}$ (as in the case of Δ_K).

We then consider the universal enveloping algebra $U(\mathfrak{k}_{\mathbb{C}})$ of $\mathfrak{k}_{\mathbb{C}}$. Here we regard $\mathfrak{k}_{\mathbb{C}}$ as a *real* Lie algebra, but we use complex coefficients in constructing $U(\mathfrak{k}_{\mathbb{C}})$. Thus $U(\mathfrak{k}_{\mathbb{C}})$ is isomorphic to the algebra of left-invariant differential operators on $K_{\mathbb{C}}$ (with complex coefficients). So we now introduce the notation $J : \mathfrak{k}_{\mathbb{C}} \to \mathfrak{k}_{\mathbb{C}}$ for the "multiplication by i" map on $\mathfrak{k}_{\mathbb{C}}$. So for $X \in \mathfrak{k}$, we have two different objects, JX and iX. Viewed as differential operators, these satisfy

$$JX\phi(g) = \left.\frac{d}{dt}\phi(ge^{iX})\right|_{t=0}$$

$$iX\phi(g) = \left.i\frac{d}{dt}\phi(ge^{X})\right|_{t=0},$$

for $\phi \in C^{\infty}(K_{\mathbb{C}})$. If ϕ happens to be holomorphic, JX and iX will coincide. (In the same way, the operators $\partial/\partial y$ and $i\partial/\partial x$ on $\mathbb{C}$ are not equal, but they do agree on holomorphic functions.)

Proposition 14. *There exists a unique homomorphism $\Psi : U(\mathfrak{k}) \to U(\mathfrak{k}_{\mathbb{C}})$ such that $\Psi(1) = 1$ and such that*

$$\Psi(X) = \frac{1}{2}(X + iJX)$$

for all $X \in \mathfrak{k}$.

Proof. In light of standard properties of universal enveloping algebras, it suffices to compute that

$$\left[\frac{1}{2}(X + iJX), \frac{1}{2}(Y + iJY) \right] = \frac{1}{4}([X,Y] + i[JX,Y] + i[X,JY] - [JX,JY])$$

$$= \frac{1}{4}([X,Y] + iJ[X,Y] + iJ[X,Y] - J^2[X,Y])$$

$$= \frac{1}{2}([X,Y] + iJ[X,Y]).$$

That is to say, the map $X \to \frac{1}{2}(X + iJX)$ is a Lie algebra homomorphism. Here we use that $J^2 = -I$ and that the bracket on $\mathfrak{k}_{\mathbb{C}}$ is J-linear. $\qquad\square$

Suppose that F is holomorphic, so that $\bar{F}$ is antiholomorphic. Then $JX\bar{F} = -iX\bar{F}$. From this it follows that

$$\frac{1}{2}(X + iJX)\bar{F} = X\bar{F}$$

for all $X \in \mathfrak{k}$ and therefore

(32) $$A\bar{F} = \Psi(A)\bar{F}, \quad F \in \mathcal{H}(K_{\mathbb{C}})$$

for all $A \in U(\mathfrak{k})$. We will make use of this below.

Definition 15. *Let $\mathcal{F} \subset \mathcal{H}(K_{\mathbb{C}})$ denote the space of finite linear combinations of holomorphic matrix entries, that is, the space of finite linear combinations of functions of the form $F(g) = \mathrm{trace}(\pi(g)B)$, where π is a finite-dimensional irreducible holomorphic representation of $K_{\mathbb{C}}$ acting on some vector space V and where B is a linear operator on V.*

For each representation π, the space of matrix entries is finite-dimensional and invariant under all left-invariant differential operators. In particular, if F is a matrix entry for π, then $\Delta_K F = -\lambda_\pi F$, where λ_π is a non-negative constant depending on π but not on B. It is shown in [H1] that $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ is the orthogonal direct sum of the spaces of matrix entries, as π ranges over the equivalence classes of irreducible representations of $K_{\mathbb{C}}$. From these observations it follows that $\mathcal{F}$ is a core for $(cI - \Delta_K)^n$, for each n.

Theorem 16. *Fix $A \in U(\mathfrak{k})$. Let ϕ_A be the function on $K_{\mathbb{C}}$ given by*

$$\phi_A = \frac{\Psi(A)\nu_t}{\nu_t},$$

Then $\mathcal{F} \subset \mathcal{D}_{\phi_A}$ and for all $F \in \mathcal{F}$ we have

(33) $$AF = T_{\phi_A}F,$$

where on the left-hand side of (33), A is regarded as a left-invariant differential operator on $K_{\mathbb{C}}$.

Note that we are *not* asserting that $A = T_{\phi_A}$ (with equality of domains), but only $A = T_{\phi_A}$ on the subspace $\mathcal{F}$ of the domain of T_ϕ. Equality of domains probably does not hold in general, although we will see eventually that it is true if $A = (cI - \Delta_K)^n$ (Remark 19). Some cases of this result were announced in [H4].

Proof. Let us assume for the moment that $\mathcal{F} \subset \mathcal{D}_{\phi_A}$. Once this is established, it suffices (by Proposition 13) to show that

$$\langle F_1, AF_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)} = \langle F_1, M_{\phi_A} F_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)}$$

for all $F_1, F_2 \in \mathcal{F}$. It suffices to consider A of the form $A = X_1 \cdots X_n$ with $X_k \in \mathfrak{k}$, since every element of $U(\mathfrak{k})$ is a linear combination of elements of this form. We use integration by parts on $K_\mathbb{C}$ in the form

$$(34) \qquad \int_{K_\mathbb{C}} \phi(g)(Z\psi)(g)\, dg = - \int_{K_\mathbb{C}} (Z\phi)(g)\psi(g)\, dg$$

for any $Z \in \mathfrak{k}_\mathbb{C}$. This holds for all sufficiently regular functions ϕ and ψ on $K_\mathbb{C}$ (not necessarily holomorphic). (More on the conditions on ϕ and ψ below.) Since we have written this without any complex conjugates, we can extend this by linearity to complex linear combinations of elements of $\mathfrak{k}_\mathbb{C}$. In particular, for any $X \in \mathfrak{k}$ we have

$$(35) \qquad \int_{K_\mathbb{C}} \phi(g)((X + iJX)\psi)(g)\, dg = - \int_{K_\mathbb{C}} ((X + iJX)\phi)(g)\psi(g)\, dg.$$

Note that on the right-hand side we have (still) $X + iJX$ and *not* $X - iJX$.

Let us now proceed assuming that all necessary integrations by parts are valid, addressing this issue at the end. For $X_1, \ldots, X_n \in \mathfrak{k}$ we have

$$(36) \qquad \langle F_1, X_1 \cdots X_n F_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)} = \int_{K_\mathbb{C}} \bar{F}_1(g)(X_1 \cdots X_n F_2)(g)\nu_t(g)\, dg.$$

Since $\nu_t(g)$ is bi-K-invariant, it is annihilated by each X_k. Thus when we integrate by parts, the terms with X_k hitting on ν_t are zero and we get

$$\langle F_1, X_1 \cdots X_n F_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)}$$

$$(37) \qquad = (-1)^n \int_{K_\mathbb{C}} (X_n \cdots X_1 \bar{F}_1(g)) F_2(g)\nu_t(g)\, dg$$

$$(38) \qquad = (-1)^n \frac{1}{2^n} \int_{K_\mathbb{C}} [(X_n + iJX_n) \cdots (X_1 + iJX_1)\bar{F}_1](g)F_2(g)\nu_t(g)\, dg,$$

by (32). We now integrate by parts a second time. When we do so, the terms where $X_k + iJX_k$ hit F_2 are zero, since F_2 is holomorphic. Thus we get
$$(39)$$
$$\langle F_1, X_1 \cdots X_k F_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)} = \frac{1}{2^n} \int_{K_\mathbb{C}} \bar{F}_1(g)F_2(g)[(X_1 + iJX_1) \cdots (X_n + iJX_n)\nu_t](g)\, dg.$$

Multiplying and dividing by $\nu_t(g)$ we get

$$\langle F_1, X_1 \cdots X_n F_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)} = \left\langle F_1, \frac{\Psi(A)\nu_t}{\nu_t} F_2 \right\rangle_{L^2(K_\mathbb{C}, \nu_t)} = \langle F_1, \phi_A F_2 \rangle_{L^2(K_\mathbb{C}, \nu_t)},$$

which is what we wanted to show.

The heart of the proof of Theorem 16 is the integration by parts in the previous paragraph. It remains only to address two technical issues: showing that $\mathcal{F} \subset \mathcal{D}_{\phi_A}$ and showing that the boundary terms in the integration by parts vanish. We sketch the arguments here and provide more details in Section 4. By writing out what the left-invariant differential operator $\Psi(A)$ looks like in polar coordinates and by using the explicit formula for ν_t, it is not hard to show that $\Psi(A)\nu_t$ behaves at worst like $e^{-|Y|^2/t}$ times a function with exponential growth in Y. Thus $\phi_A = \Psi(A)\nu_t/\nu_t$

will have at most exponential growth in Y. Since the holomorphic matrix entries also have at most exponential growth in Y, this (together with the formula (12)) shows that $\int_{K_{\mathbb{C}}} |F(g)\phi_A(g)|^2 \nu_t(g)\, dg < \infty$ for any matrix entry F. That is to say, $\mathcal{F} \subset \mathcal{D}_{\phi_A}$.

We must also justify two integrations by parts, one in passing from (36) to (37) and one in passing from (38) to (39). The first of these involves only differentiation in the K-directions. If we write out the integral in reverse polar coordinates as in (19), the integration by parts will be only in the K-integral, where there are no boundary terms to worry about. So we need only worry about the passage from (38) to (39). We use the following criterion for applying (35): in polar coordinates, ψ, along with its partial derivatives in the Y variable, should have at most exponential growth in Y (with estimates uniform in x), while ϕ, along with its partial derivatives in the Y variable, should have faster than exponential decay. (See Section 4 for a justification of this condition.) As we do successive integrations by parts to pass from (38) to (39) we will apply this criterion with

$$(40) \qquad \psi(g) = [(X_{k-1} + iJX_{k-1}) \cdots (X_1 + iJX_1)\bar{F}_1](g)$$

and

$$(41) \qquad \phi(g) = F_2(g)[(X_{k+1} + iJX_{k+1}) \cdots (X_n + iJX_n)\nu_t](g).$$

Calculation in polar coordinates will show (Section 4) that these functions indeed satisfy the above criterion. $\qquad \square$

Proposition 17. *For any positive integer n, let ϕ_n denote the Toeplitz symbol of the operator $(c - \Delta_K)^n$, namely,*

$$(42) \qquad \phi_n = \frac{\Psi((c - \Delta_K)^n)\nu_t}{\nu_t}.$$

Then

$$\phi_n(xe^{iY}) = p_{n,c,t}(|Y|^2),$$

where $p_{n,c,t}$ is a polynomial of degree n. Furthermore, for each n and t, ϕ_n is a positive function for all sufficiently large values of c.

Proof. Since X_k commutes with JX_k, we have

$$\Psi(X_k^2) = \frac{1}{4}(X_k + iJX_k)^2 = \frac{1}{4}(X_k^2 + 2i(JX_k)X_k - (JX_k)^2).$$

Now, ν_t satisfies the differential equation $d\nu_t/dt = \frac{1}{4}\sum_{k=1}^d (JX_k)^2 \nu_t$. Furthermore, ν_t is bi-K-invariant and therefore annihilated by each X_k. Thus

$$\Psi(\Delta_K)\nu_t = -\frac{1}{4}\sum_{k=1}^d (JX_k)^2 \nu_t = -\frac{d\nu_t}{dt}.$$

Since t derivatives commute with spatial derivatives we then have

$$(43) \qquad \Psi((c - \Delta_K)^n)\nu_t = \left(c + \frac{d}{dt}\right)^n \nu_t = \sum_{k=0}^n \binom{n}{k} c^{n-k} \left(\frac{d}{dt}\right)^k \nu_t.$$

Looking at the formula for ν_t in (10) and (11), we see that repeated applications of the operator d/dt to ν_t will give back ν_t itself multiplied by a sum of terms of the form $t^{-a}|Y|^{2b}$, with coefficients involving $|\delta|^2$ and d. (Here a and b are non-negative integers and the result may be proved by induction on the number of derivatives.)

The highest power of $|Y|^2$ that will arise in computing (43) is $|Y|^{2n}$. This establishes that $\phi_n = [(c + d/dt)^n \nu_t]/\nu_t$ is a polynomial in $|Y|^2$ of degree n.

To establish the positivity of ϕ_n, let us think about the coefficient of $|Y|^{2l}$ in the computation of ϕ_n. From the $k = l$ term in (43) we get

$$(44) \qquad c^{n-l} \left[\frac{|Y|^{2l}}{t^{2l}} + \text{lower powers of } |Y|^2 \right].$$

From all terms in (43) with $k < l$, we get powers of $|Y|^2$ lower than $|Y|^{2l}$. From terms in (43) with $k > l$, we may get terms involving $|Y|^{2l}$, but they will be multiplied by a lower power of c than in (44). We see, then, that for each fixed value of n and t, the coefficient of $|Y|^{2l}$ in ϕ_n will be a polynomial in c of degree $n - l$ with positive leading term. Thus for all sufficiently large values of c, every power of $|Y|^2$ in the expression for ϕ_n will have a positive coefficient and ϕ_n will therefore be positive. $\qquad \square$

Proposition 18. *Choose c large enough that the function ϕ_{2n} in Proposition 17 is positive. Then the holomorphic Sobolev space $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ coincides with the Hilbert space $\mathcal{H}L^2(K_{\mathbb{C}}, \phi_{2n}(g)\nu_t(g)\,dg)$.*

Since, by Proposition 17, $\phi_{2n}(g)$ has the same behavior at infinity as $(1+|Y|^2)^{2n}$, Proposition 18 implies Theorem 9.

Proof. For $F_1, F_2 \in \mathcal{F}$ we have

$$\begin{aligned}
\langle F_1, F_2 \rangle_{\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)} &= \langle (c - \Delta_K)^n F_1, (c - \Delta_K)^n F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} \\
&= \langle F_1, (c - \Delta_K)^{2n} F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} \\
&= \langle F_1, T_{\phi_{2n}} F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)} \\
&= \langle F_1, \phi_{2n} F_2 \rangle_{L^2(K_{\mathbb{C}}, \nu_t)}.
\end{aligned}$$

(We have used Proposition 13 in the last equality.) The last expression is nothing but the inner product of F_1 and F_2 in $L^2(K_{\mathbb{C}}, \phi_{2n}\nu_t)$. Thus the inner product for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ and for $\mathcal{H}L^2(K_{\mathbb{C}}, \phi_{2n}\nu_t)$ coincide on $\mathcal{F}$. But (as in the proof of Theorem 6), $\mathcal{F}$ is dense in $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$. Furthermore, by the same argument as in the proof of Lemma 10 in [H1], $\mathcal{F}$ is dense in $\mathcal{H}L^2(K_{\mathbb{C}}, \phi_{2n}\nu_t)$. It then follows that the two Hilbert spaces $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ and $\mathcal{H}L^2(K_{\mathbb{C}}, \phi_{2n}\nu_t)$ must coincide. $\qquad \square$

Remark 19. *Proposition 17 tells us that $\phi_{2n}(g)$ has the same behavior at infinity as $\phi_n(g)^2$. (That is, the Toeplitz symbol of $(c - \Delta_K)^{2n}$ has the same behavior at infinity as the square of the Toeplitz symbol of $(c-\Delta_K)^n$.) From this we see that $\mathcal{D}_{\phi_n}$ is the same space as $\mathcal{H}L^2(K_{\mathbb{C}}, \phi_{2n}\nu_t)$. Thus $\mathcal{D}_{\phi_n} = \mathcal{H}L^2(K_{\mathbb{C}}, \phi_{2n}\nu_t) = \mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$. Thus $(c - \Delta_K)^n = T_{\phi_n}$, with equality of domains. For more general left-invariant operators A, there is no obvious reason that the symbol of A^2 should have the same behavior at infinity as the square of the symbol of A. Thus in general the domain of A may not coincide with $\mathcal{D}_{\phi_A}$.*

4. INTEGRATION BY PARTS AND GROWTH OF LOGARITHMIC DERIVATIVES

In this section we give more details concerning the technical issues in the proof of Theorem 16, namely, justifying integration by parts and bounding functions of the form $\Psi(A)\nu_t/\nu_t$. Our strategy is to write out left-invariant differential operators

on $K_{\mathbb{C}}$ in polar coordinates. This means that we think of $K_{\mathbb{C}}$ as $K \times \mathfrak{k}$ by means of polar coordinates and we express everything in terms of left-invariant vector fields on K and constant coefficient differential operators on $\mathfrak{k}$. So we introduce vector fields $\tilde{X}_k$ and $\partial/\partial y_k$ on $K_{\mathbb{C}}$, where $\tilde{X}_k$ is given by

$$(\tilde{X}_k\phi)(xe^{iY}) = \frac{d}{dt}\phi(xe^{tX_k}e^{iY})\Big|_{t=0}$$

and where $\partial/\partial y_k$ means partial differentiation in the Y variable with x fixed. Note that both $\tilde{X}_k$ and $\partial/\partial y_k$ are left-K-invariant operators, but neither is left-$K_{\mathbb{C}}$-invariant. Note that since they act in separate variables, the $\tilde{X}_k$'s commutes with the $\partial/\partial y_l$'s.

Since the vector fields $\tilde{X}_k$ and $\partial/\partial y_k$ span the tangent space at each point, the left-$K_{\mathbb{C}}$-invariant vector fields X_k and JX_k can be expressed as linear combinations of these vector fields with coefficients that are smooth functions on $K_{\mathbb{C}}$. Since all the vector fields involved are left-K-invariant, the coefficient functions (in polar coordinates) will depend only on Y and not on x. So we will have

$$(45) \qquad X_k = \sum_{l=1}^{d}\left(a_{kl}(Y)\tilde{X}_l + b_{kl}(Y)\frac{\partial}{\partial y_l}\right)$$

$$(46) \qquad JX_k = \sum_{l=1}^{d}\left(c_{kl}(Y)\tilde{X}_l + d_{kl}(Y)\frac{\partial}{\partial y_l}\right).$$

The coefficient functions a_{kl}, etc., can be computed explicitly by differentiating the polar coordinates map $(x, Y) \to xe^{iY}$. This calculation is done in [H3] with the result that

$$(47) \qquad \left(\begin{array}{cc} a(Y) & c(Y) \\ b(Y) & d(Y) \end{array}\right) = \left(\frac{\sin \mathrm{ad}Y}{\mathrm{ad}Y}\right)^{-1}\left(\begin{array}{cc} \frac{\sin \mathrm{ad}Y}{\mathrm{ad}Y} & \frac{\cos \mathrm{ad}Y - 1}{\mathrm{ad}Y} \\ \sin \mathrm{ad}Y & \cos \mathrm{ad}Y \end{array}\right).$$

Here $\sin \mathrm{ad}Y/\mathrm{ad}Y$ is to be computed using the power series for the function $\sin z/z$, which has infinite radius of convergence, and similarly for $(\cos \mathrm{ad}Y - 1)/\mathrm{ad}Y$. Note that the eigenvalues of $\mathrm{ad}Y$ are pure imaginary, so the eigenvalues of $\sin \mathrm{ad}Y/\mathrm{ad}Y$ are of the form $\sin(ia)/(ia) = \sinh a/a$, $a \in \mathbb{R}$. This means that $\sin \mathrm{ad}Y/\mathrm{ad}Y$ is invertible. In (47), each $d \times d$ block of the $(2d) \times (2d)$ matrix on the right-hand side is to be multiplied by the $d \times d$ matrix $(\sin \mathrm{ad}Y/\mathrm{ad}Y)^{-1}$. Note that the functions a, b, c, and d have at most linear growth as a function of Y.

Integration by parts. We start by verifying the criterion for integration by parts described in the previous section: ψ and its Y-derivatives should have at most exponential growth; ϕ and its Y-derivatives should have faster-than-exponential decay. In justifying the integration by parts, the X_k term is no problem, since then we write out things in reverse polar coordinates (as in (19)) and the integration by parts will be purely in the K-variable, where there is no boundary to worry about.

For the JX_k term, we write out the integration in polar coordinates, using (46). This gives

$$\int_{K_{\mathbb{C}}} (JX_k\phi)(g)\psi(g)\,dg = \sum_{k=1}^{d} \int_{\mathfrak{k}} \int_{K} c_{kl}(Y)(\tilde{X}_l\phi)(xe^{iY})\psi(xe^{iY})\,dx\,\Phi(Y)^{-2}\,dY$$

$$(48) \qquad\qquad + \sum_{k=1}^{d} \int_{K} \int_{\mathfrak{k}} d_{kl}(Y)\frac{\partial\phi}{\partial y_l}(xe^{iY})\psi(xe^{iY})\Phi(Y)^{-2}\,dY\,dx.$$

Under our assumptions on ϕ and ψ, the integrals are all convergent. In the first term, we use that $\tilde{X}_l$ is skew-symmetric on $C^{\infty}(K)$ to move $\tilde{X}_l$ from ϕ onto ψ (with a minus sign in front). In the second term, we compute the inner integral by integrating over a cube in $\mathfrak{k}$ and then letting the size of the cube tend to infinity. In the second term, we apply ordinary Euclidean integration by parts. This will give three integral terms (from the functions d_{kl}, ψ, and Φ^{-2} in the integrand) plus a boundary term. Two of the integral terms are "divergence" terms, namely,

$$\int_{K} \int_{\text{cube}} \left[\sum_{k=1}^{d} \left(\frac{\partial d_{kl}(Y)}{\partial y_l}\Phi^{-2}(Y) + d_{kl}(Y)\frac{\partial \Phi^{-2}(Y)}{\partial y_l} \right) \right] \phi(xe^{iY})\psi(xe^{iY})\,dY\,dx.$$

Now, the quantity in square brackets must be identically zero, or else JX_k would not be skew-symmetric on $C_c^{\infty}(K_{\mathbb{C}})$. (The skew-symmetry of JX_k is a consequence of the invariance of Haar measure under right translations on the unimodular group $K_{\mathbb{C}}$.)

We are left, then, with the term we want, namely,

$$-\sum_{k=1}^{d} \int_{K} \int_{\text{cube}} \phi(xe^{iY})d_{kl}(Y)\frac{\partial\psi}{\partial y_l}(xe^{iY})\Phi(Y)^{-2}\,dY\,dx$$

together with a boundary term, namely,

$$\sum_{k=1}^{d} \int_{K} \int_{\text{boundary}} d_{kl}(Y)\phi(xe^{iY})\psi(xe^{iY})\Phi(Y)^{-2}\,dY\,dx,$$

where "boundary" refers to integration over two opposite faces of the cube (with opposite signs). Since we assume that ψ and its Y-derivatives have at most exponential growth and that ϕ has faster-than-exponential decay, we can now let the size of the cube tend to infinity. The boundary term will drop out in the limit and the remaining term becomes

$$(49) \qquad -\sum_{k=1}^{d} \int_{K} \int_{\mathfrak{k}} \phi(xe^{iY})d_{kl}(Y)\frac{\partial\psi}{\partial y_l}(xe^{iY})\Phi(Y)^{-2}\,dY\,dx.$$

Recall that (49) is the second term from the right-hand side of (48). After we integrate by parts in the first term (in the x-variable only) we get

$$\int_{K_{\mathbb{C}}} (JX_k\phi)(g)\psi(g)\,dg = -\int_{K_{\mathbb{C}}} \phi(g)(JX_k\psi)(g)\,dg$$

and our criterion for integration by parts is justified.

It remains to check that the functions ϕ and ψ to which we want to apply integration by parts satisfy the just-obtained criterion. According to (40), ψ is obtained by applying left-invariant derivatives to the complex conjugate of a matrix entry. Since left-invariant derivatives of matrix entries are again matrix entries, ψ

will have at most exponential growth. Meanwhile, according to (41), ϕ is the heat kernel ν_t, with several left-invariant derivatives applied and multiplied by a matrix entry. As we will show below, left-invariant derivatives of ν_t give back ν_t itself, multiplied by a function with at most exponential growth. Thus ϕ has faster-than-exponential decay.

Logarithmic-type derivatives. The entries of (47) grow only linearly in Y. However, due to the noncommutative nature of differentiating matrix-valued functions, it is not altogether evident how the derivatives of these functions behave. Nevertheless, it is not hard to show that all the derivatives have at most exponential growth. For example, to differentiate the expression for $c(Y)$, namely,

$$c(Y) = \left(\frac{\sin \mathrm{ad}Y}{\mathrm{ad}Y} \right)^{-1} \frac{\cos \mathrm{ad}Y - 1}{\mathrm{ad}Y}$$

we use the rules for differentiating products and inverses of matrix-valued functions, giving, for any smooth path $Y(t)$,

$$\begin{aligned}
\frac{d}{dt}c(Y(t)) = &-\left(\frac{\sin \mathrm{ad}Y(t)}{\mathrm{ad}Y(t)} \right)^{-1} \left[\frac{d}{dt} \frac{\sin \mathrm{ad}Y(t)}{\mathrm{ad}Y(t)} \right] \left(\frac{\sin \mathrm{ad}Y(t)}{\mathrm{ad}Y(t)} \right)^{-1} \\
&+ \left(\frac{\sin \mathrm{ad}Y(t)}{\mathrm{ad}Y(t)} \right)^{-1} \left[\frac{d}{dt} \frac{\cos \mathrm{ad}Y(t) - 1}{\mathrm{ad}Y(t)} \right].
\end{aligned}$$

Term-by-term differentiation gives exponential estimates on the two terms in square brackets.

Since the derivatives of the coefficient functions a, b, c, and d grow at most exponentially, any left-invariant differential operator on $K_{\mathbb{C}}$ (built up from products of left-invariant vector fields) will be expressible in terms of the operators $\tilde{X}_k$ and $\partial/\partial Y_k$ with coefficients that have at most exponential growth. When we apply such an operator to ν_t, any term with $\tilde{X}_k$ in it is zero and we get only Y-derivatives of ν_t, multiplied by at most exponential coefficient functions. It is easily seen that the Y derivatives of ν_t can be expressed as ν_t itself, multiplied by functions of at most exponential growth. Thus functions of the form $\Psi(A)\nu_t/\nu_t$ will have at most exponential growth.

5. Concluding remarks

5.1. **Pointwise bounds.** The pointwise bounds (13) for $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$—or, equivalently, bounds for the on-diagonal reproducing kernel $k_t(g, g)$—play an essential role in the proof of Theorem 5. After all, the holomorphic Sobolev embedding theorem (which provides one direction of Theorem 5) is proved by relating the reproducing kernel $k_t^{2n}(g, g)$ for $\mathcal{H}^{2n}(K_{\mathbb{C}}, \nu_t)$ to the reproducing kernel $k_t(g, g)$ for $\mathcal{H}L^2(K_{\mathbb{C}}, \nu_t)$ and then using (a slight refinement of) estimates from [H3] for $k_t(g, g)$. (See the expression (24) in Proposition 10 and Lemma 11.)

In [H3], the bounds on $k_t(g, g)$ are obtained by using the expression $k_t(g, g) = \rho_{2t}(gg^*)$ from [H1]. This expression, in turn, depends on the special form of the measure $\nu_t(g)\, dg$, namely, that it is the heat kernel for $K_{\mathbb{C}}/K$. In light of the importance of the bounds (13), it is worthwhile to compare them to bounds obtainable by more general means. The elementary argument of [DG] gives the bound (for

$F \in \mathcal{H}L^2(K_{\mathbb{C}}, \nu(g) \; dg))$

(50) $$|F(g)|^2 \leq \|F\|^2_{L^2(K_{\mathbb{C}},\nu)} \, a_\varepsilon \sup_{h \in B_\varepsilon(g)} \frac{1}{\nu(h)},$$

where $B_\varepsilon(g)$ is the ball of radius ε around g, computed using a left-invariant Riemannian metric on $K_{\mathbb{C}}$. Here, $\nu(g)$ is an arbitrary positive, continuous density on $K_{\mathbb{C}}$. If we specialize to the case $\nu = \nu_t$, it is possible that by using a "local gauge transformation" as in [CL], one could allow ε to go to zero in the above estimate, and thereby obtain an estimate of the form

(51) $$|F(g)|^2 \leq \|F\|^2_{L^2(K_{\mathbb{C}},\nu_t)} \, a \frac{1}{\nu_t(g)} = A_t \frac{e^{|Y|^2/t}}{\Phi(Y)}.$$

This estimate would be "off" from the actual optimal estimate by a factor of $\Phi(Y)^2$ (since (13) has $\Phi(Y)$ in the numerator rather than the denominator).

Meanwhile, the estimate of Driver [D] together with the averaging lemma of [H1] imply the bound

(52) $$|F(g)|^2 \leq \|F\|^2_{L^2(K_{\mathbb{C}},\nu_t)} \, A_t e^{|g|^2/t},$$

where $|g|^2$ is the distance from the identity to g with respect to a certain left-invariant Riemannian metric on $K_{\mathbb{C}}$. This estimate holds for heat kernel measures on arbitrary complex Lie groups [DG] and is therefore less dependent on the special structure of ν_t than the argument in [H3]. On the other hand, it can be shown that $|Y|^2 \leq |g|^2 \leq |Y|^2 + C$, and therefore (52) is equivalent to

$$|F(g)|^2 \leq \|F\|^2_{L^2(K_{\mathbb{C}},\nu_t)} \, B_t e^{|Y|^2/t}.$$

This bound is better by a factor of $\Phi(Y)$ than (51), but still one factor of $\Phi(Y)$ off from the bounds in (13).

We see, then, that more general methods leave us exponentially short of the estimates from [H3] (at least when K is semisimple). To make the proofs in this paper work, we need bounds that are within a polynomial factor of the ones in (13). If there is a way to get bounds similar to (13) without using special properties of the density ν_t, it would presumably be by working in the right sort of holomorphic local coordinates about each point g. The coordinate neighborhood about g should have fixed volume with respect to the "phase volume measure" $dx\,dY$. If such coordinates can be constructed, then one might hope to get estimates involving the reciprocal of the density of $\nu_t(g)\,dg$ with respect to $dx\,dY$, which is precisely what we have in (13). (One would still need some sort of local gauge transformation to make this work.) By contrast, (50) and (51) involve the reciprocal of the density of $\nu_t(g)\,dg$ with respect to the Haar measure dg. In the noncommutative case, when the Haar measure has exponential volume growth, such bounds are not adequate.

5.2. **Connections with the inversion formula.** It is illuminating to think of the results of this paper in comparison to the inversion formula for the generalized Segal–Bargmann transform given in [H2]. Consider some f in $L^2(K)$ and let $F = C_t f$. Then according to [H2] we have the inversion formula

(53) $$f(x) = (2\pi t)^{-d/2} e^{-|\delta|^2 t/2} \int_{\mathfrak{k}} F(xe^{iY}) e^{-|Y|^2/2t} \frac{1}{\Phi(Y/2)} \, dY,$$

provided that the integral is absolutely convergent for all x. (This formula is obtained by writing out the integral in [H2, Eq. (2)] explicitly with $p = e^{iY}$ and then making the change of variables $Y' = 2Y$. Compare also Theorem 2.6 in [H6].) For any $f \in L^2(K)$, one can recover f from F by restricting the integral in (53) to a ball of radius R in $\mathfrak{k}$ and then taking the limit (in the $L^2(K)$ topology) as R tends to infinity.

The question of when the integral in (53) converges is an important one. The integral cannot be convergent for all f and x, since f is an arbitrary L^2 function on K and can equal infinity at some points. On the other hand, if f is sufficiently smooth (with estimates depending on the dimension and rank of K), Theorem 3 of [H2] shows that the integral is indeed convergent for all $x \in K$.

The pointwise bounds in [H3] and in this paper reflect this state of affairs. For general f in $L^2(K)$ we have (taking the square root of both sides of (13))

$$\left| F(xe^{iY}) \right| \le B e^{|Y|^2/2t} \Phi(Y)^{1/2}.$$

Since $\Phi(Y/2)$ has the same asymptotic behavior (up to a constant) as $\Phi(Y)^{1/2}$, these bounds are polynomially short of what is needed to guarantee convergence of (53). On the other hand, if we assume a sufficient degree of differentiability for f, then by the holomorphic Sobolev embedding theorem (Theorem 8), we get polynomially better bounds and thus convergence of (53). Therefore, the holomorphic Sobolev embedding theorem gives an alternative way of proving the convergence results in Theorem 3 of [H2]. It is interesting to note, however, that the convergence results of [H2] can be proved by comparatively soft methods which do not require detailed estimates of the reproducing kernel. (Compare also the results in [St] extending [H2] to arbitrary compact symmetric spaces.)

5.3. **Distributions.** In this paper we have considered functions smoother than those in $L^2(K)$, either Sobolev spaces or $C^\infty(K)$. One could also consider distributions, which are "functions" less smooth than those in $L^2(K)$. It is easily shown that the transform C_t, defined initially on $L^2(K)$, can be extended to a map of the space of distributions on K into the space of holomorphic functions on $K_{\mathbb{C}}$. (Compare [FMN1, FMN2].) We make the following conjecture concerning the image under C_t of the space of distributions.

Conjecture 20. *Suppose F is a holomorphic function on $K_{\mathbb{C}}$. Then there exists a distribution f on K with $F = C_t f$ if and only if F satisfies*

(54)
$$\left| F(xe^{iY}) \right|^2 \le A\Phi(Y)e^{|Y|^2/t}(1 + |Y|^2)^{2n}$$

for some positive integer n and some constant A.

This is the same bound as in Theorem 5 except that the factor of $(1 + |Y|^2)^{2n}$ is in the numerator instead of the denominator and the bound is required to hold only for *some* n rather than for all n. The most obvious approach to proving such a theorem would be to consider negative Sobolev spaces $H^{-2n}(K)$ and then to think of the space of distributions as the union of all the $H^{-2n}(K)$'s. One would then hope to "dualize" all the arguments in this paper. However, this approach requires some additional effort at each stage, so we do not attempt to carry it out here. We hope to return to this problem in a future paper.

5.4. **More general settings.** Finally, let us mention some additional settings in which problems similar to those in this paper could be considered. The most natural extension would be to consider the generalized Segal–Bargmann transform for compact symmetric spaces, as considered in Section 11 of [H1] and (in a better form) in [St]. (See also [HM].) The necessary estimates are more complicated in such cases; [HS] is a first attempt to provide the necessary estimates. In another direction, one could consider various Segal–Bargmann-type spaces over $\mathbb{C}^n$, with various measures. Here there would be no Segal–Bargmann transform, but one could still define holomorphic Sobolev spaces and try to derive pointwise bounds for elements of these spaces. The paper [Si] give the first results in this direction. See also [H8] for related results.

6. Acknowledgments

References

[CL]   K. Chailuek and W. Lewkeeratiyutkul, A pointwise bound for a holomorphic function which is square-integrable with respect to an exponential density function, *Soochow J. Math.*, **30** (2004), 75–83.

[B1]   V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, *Comm. Pure Appl. Math.* **14** (1961), 187–214.

[B2]   V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform. Part II. A family of related function spaces. Application to distribution theory, *Comm. Pure Appl. Math.* **20** (1967), 1–101.

[FMN1] C. A. Florentino, J. M. Mourão, and J. P. Nunes, Coherent state transforms and abelian varieties, *J. Funct. Anal.* **192** (2002), 410–424.

[FMN2] C. A. Florentino, J. M. Mourão, and J. P. Nunes, Coherent state transforms and vector bundles on elliptic curves, *J. Funct. Anal.*, **204** (2003), 355–398.

[Ga]   R. Gangolli, Asymptotic behavior of spectra of compact quotients of certain symmetric spaces, *Acta Math.* **121** (1968), 151–192

[Gr]   L. Gross, Uniqueness of ground states for Schrödinger operators over loop groups, *J. Funct. Anal.* **112** (1993), 373–441.

[D]   B. K. Driver, On the Kakutani-Itô-Segal-Gross and Segal-Bargmann-Hall isomorphisms, *J. Funct. Anal.* **133** (1995), 69–128.

[DG]   B. K. Driver and L. Gross, Hilbert spaces of holomorphic functions on complex Lie groups. In: New trends in stochastic analysis (K. Elworthy, S. Kusuoka, and I. Shigekawa, Eds.), 76–106, World Sci. Publishing, River Edge, NJ, 1997.

[H1]   B. C. Hall, The Segal-Bargmann "coherent state" transform for compact Lie groups, *J. Funct. Anal.* **122** (1994), 103–151.

[H2]   B. C. Hall, The inverse Segal-Bargmann transform for compact Lie groups, *J. Funct. Anal.* **143** (1997), 98–116.

[H3]   B. C. Hall, Phase space bounds for quantum mechanics on a compact Lie group, *Comm. Math. Phys.* **184** (1997), 233–250.

[H4]   B. C. Hall, Quantum mechanics in phase space. In: Perspectives on quantization (L. Coburn and M. Rieffel, Eds), 47–62, Contemp. Math., 214, Amer. Math. Soc., Providence, RI, 1998.

[H5]   B. C. Hall, Holomorphic methods in analysis and mathematical physics. In: First Summer School in Analysis and Mathematical Physics (S. Pérez-Esteva and C. Villegas-Blas, Eds.), 1–59, Contemp. Math., 260, Amer. Math. Soc., Providence, RI, 2000.

[H6]   B. C. Hall, Geometric quantization and the generalized Segal–Bargmann transform for Lie groups of compact type, *Comm. Math. Phys.* **226** (2002), 233-268.

[H7]   B. C. Hall, Harmonic analysis with respect to heat kernel measure, *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), 43–78.

[H8]   B. C. Hall, Bounds on the Segal-Bargmann transform of L^p functions, *J. Fourier Anal. Appl.* **7** (2001), 553–569.

[H9]   B. C. Hall, The Segal-Bargmann transform and the Gross ergodicity theorem. In: Finite and infinite dimensional analysis in honor of Leonard Gross (H.-H. Kuo and A. N. Sengupta, Eds.), 99–116, Contemp. Math., 317, Amer. Math. Soc., Providence, RI, 2003.

[HM]   B. C. Hall and J. J. Mitchell, Coherent states on spheres, *J. Math. Phys.* **43** (2002), 1211–1236.

[HS]   B. C. Hall and M. B. Stenzel, Sharp bounds for the heat kernel on certain symmetric spaces of non-compact type. In: Finite and infinite dimensional analysis in honor of Leonard Gross (H.-H. Kuo and A. N. Sengupta), 117–135, Contemp. Math., 317, Amer. Math. Soc., Providence, RI, 2003.

[Kn]   A. W. Knapp, "Lie groups beyond an introduction." Progress in Mathematics, 140. Birkhäuser Boston, Inc., Boston, MA, 1996.

[Se]   I. E. Segal, Mathematical characterization of the physical vacuum for a linear Bose-Einstein field. (Foundations of the dynamics of infinite systems. III) *Illinois J. Math.* **6** (1962), 500–523.

[Si]   B. Singphu, Certain properties of the domains of multiplication and differentiation operators on a generalized Segal–Bargmann space, Master's Thesis, Chulalongkorn University, Thailand, 2000.

[St]   M. B. Stenzel, The Segal-Bargmann transform on a symmetric space of compact type, *J. Funct. Anal.* **165** (1999), 44–58.

(Brian C. Hall) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, INDIANA 46556-4618, USA

E-mail address: bhall@nd.edu

(Wicharn Lewkeeratiyutkul) DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, CHULALONGKORN UNIVERSITY, BANGKOK 10330, THAILAND

E-mail address: Wicharn.L@chula.ac.th