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SMOOTHNESS IS NOT AN OBSTRUCTION TO EXACT REALIZABILITY

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Abstract

A sequence of non-negative integers $(\phi_n)_{n=1}^{\infty}$ is called *exactly realizable* if there is a map T of a set X such that $\phi_n = \#\{x : T^n x = x\}$. We prove that any exactly realizable sequence can be realized by a C^∞ diffeomorphism of $\mathbb{T}^2$.

1. *Introduction*

There is a natural class of sequences of non-negative integers, called the exactly realizable sequences, which arise as the sequence of the number of periodic points of period n for some dynamical system. This class of sequences was first introduced in the thesis of Puri [**4**], some of which appears as [**5**]. An intriguing consequence of this for the Fibonacci recurrence can be found in [**6**] while further number theoretic consequences appear in [**1**].

Definition. *Let $\phi = (\phi_n)_{n=1}^{\infty}$ be a sequence of non-negative integers. We say ϕ is exactly realizable if there is a set X and a map $T : X \to X$ such that*

$$\phi_n = \#\{x \in X : T^n x = x\}$$

for all $n \geq 1$.

Puri and Ward observe that the sole obstruction to being exactly realizable is the natural restriction that the number of periodic orbits of length n is a non-negative integer i.e.

$$\phi_n^o := \frac{1}{n} \sum_{d|n} \mu(n/d)\phi_d$$

is a non-negative integer for all $n \geq 1$.

Furthermore they observe that the class of exactly realizable sequences is unchanged if we require T to be a homeomorphism of a compact metric space X. It is natural to ask whether the class of exactly realizable sequences changes if we require that T be a C^∞ diffeomorphism of a smooth manifold M.

This question would seem to be harder because of the presence of non-trivial obstructions coming from both the topological and smooth structures. Hunt and Kaloshin [**2**] [**3**] prove that for a prevalent diffeomorphism the growth of periodic points is at most stretched exponential.

Depending on the manifold there may be restrictions arising from the Lefshetz

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Fixed point theorem. The choice of $(\phi_n)_{n=1}^{\infty}$ therefore restricts the choice of manifolds on which even a homeomorphism could realize $(\phi_n)_{n=1}^{\infty}$. For example every homeomorphism of the sphere must have a fixed point.

Given an exactly realizable sequence ϕ we construct a diffeomorphism T of $\mathbb{T}^2$ which realizes this sequence. We introduce a number of invariant circles on which the induced diffeomorphism has rational rotation number. The only complication occurs with the fact that the circles must accumulate. They accumulate on another invariant circle on which the induced diffeomorphism has an irrational rotation number. All periodic points have index 0. The resulting diffeomorphism is not expansive and has zero topological entropy.

Main Result. *If $(\phi_n)_{n=1}^{\infty}$ is an exactly realizable sequence of non-negative integers then there exists $T \in \mathrm{Diff}^{\infty}(\mathbb{T}^2)$ which realizes the sequence.*

2. *Construction*

The main result is a corollary of a slightly more general theorem about maps of an annulus. Let $I := [0,1]$.

Theorem. *Let $\epsilon > 0$ be arbitrary and (ϕ_n) be an exactly realizable sequence of non-negative integers. Define $m = \min\{n : \phi_n > 0\}$. Suppose $\varphi \in C^{\infty}(I, \mathbb{R}^+)$ satisfies*

(i) $\varphi(0), \varphi(1) \in \mathbb{R}\backslash\mathbb{Q}$.
(ii) $\operatorname{osc}\varphi := \max\varphi - \min\varphi > \frac{1}{m}$.

Define $T \in \mathrm{Diff}^{\infty}(\mathbb{T}\times I, \mathbb{T}\times I)$ by

$$T(x,y) := (x + \varphi(y), y).$$

Then there is $\widetilde{T} \in \mathrm{Diff}^{\infty}(\mathbb{T}\times I)$ such that

(i) $d_{\infty}(T, \widetilde{T}) < \epsilon$.
(ii) *the number of periodic points of period n is ϕ_n.*
(iii) $T(x,0) = \widetilde{T}(x,0)$ *and* $T(x,1) = \widetilde{T}(x,1)$.
(iv) *all periodic points are isolated and have index* 0.

This follows rather straight-forwardedly from a lemma about perturbations in a neighbourhood of an invariant circle. We show that a small perturbation can be made to make the induced rational rotation have any given number of periodic orbits with all other points asymptotic to a periodic orbit.

Lemma. *Let $\epsilon > 0$ be arbitrary, $\phi^o \in \mathbb{Z}^+$, and $\varphi \in C^{\infty}([-\delta,\delta], \mathbb{R}^+)$ satisfy $\varphi(0) = a/b$ with $(a,b) = 1$. Define $T \in \mathrm{Diff}^{\infty}(\mathbb{T}\times[-\delta,\delta])$ by*

$$T(x,y) := (x + \varphi(y), y)$$

Then there is a transformation $\widehat{T} \in \mathrm{Diff}^{\infty}(\mathbb{T}\times[-\delta,\delta])$ such that

(i) $d_{\infty}(\widehat{T}, T) < \epsilon$.
(ii) $\widehat{T} = T$ *in a neighbourhood of the boundary.*
(iii) $\widehat{T}(x,0)$ *has precisely ϕ^o orbits of prime period b and no other periodic points.*

Proof of the Lemma. Indeed the transformation $\widehat{T}$ can be chosen with the form

$$\widehat{T}(x,y) = (x + \varphi(y) + f(x,y), y) \quad \text{with} \quad f(x,y) = \xi(x)\zeta(y).$$

The function $\zeta \in C^\infty([-\delta,\delta],\mathbb{R})$ is a scaled bump function which is zero in a neighbourhood of the boundary and positive at 0. We choose $\xi \in C^\infty(\mathbb{T},\mathbb{R})$ to have the following properties:

(i) $\xi(x) \geq 0$.

(ii) $\xi(x) = 0$ if and only if $x = \frac{i}{\phi^o b}$ for some $i \in \mathbb{Z}$.

The properties we have required depend only on the zeros and the sign of the function and not on the magnitude. Hence we may multiply the function f by any positive constant and still get a function with all the required properties.

The map $\widehat{T}$ is invertible if we ensure that $f_x(x,y) > -1$. Then by the inverse function theorem we have that $\widehat{T}$ is a C^∞-diffeomorphism. By controlling the C^n norm of f we may ensure that $d_\infty(T,\widehat{T}) < \epsilon$ as required.

Thus $\widehat{T}$ is a circle diffeomorphism. Let $\widehat{x} = \frac{i}{\phi^o b}$ for some $i \in \mathbb{Z}$. Since $f(\widehat{x},0) = 0$ we have

$$\widehat{T}^j(\widehat{x},0) = (\widehat{x} + j\frac{a}{b}, 0) \mod 1 \tag{2.1}$$

and $\widehat{x}$ is periodic of prime period b. Therefore $\widehat{T}$ has rotation number a/b and any periodic point must have period b. Now

$$\widehat{T}^b(x,0) = (x + S_b f(x,0), 0) \mod 1$$

where

$$S_b(x,0) = \sum_{i=0}^{b-1} f\big(T^i(x,0)\big).$$

If $x \neq \frac{i}{\phi^o b}$ then

$$0 < S_b f(x,0) < \frac{1}{\phi^o} \leq 1$$

and thus x cannot be a periodic point. □

Now we apply the lemma to a countable collection of invariant circles and then perturb to ensure that only these periodic points persist.

Proof of the Theorem. Let ϕ^o_n be the corresponding sequence of the number of orbits of prime period n. If $\phi^o_n > 0$ then we choose a circle $C_n := \{(x,y) : y = y_n\}$ such that $\varphi(y_n) = a/n$ with $(a,n) = 1$. This is always possible because of the hypothesis on the oscillation of φ.

For each such circle C_n we can find an annular neighbourhood N_n of C_n such that the collection of neighbourhoods $\{N_n\}$ is pairwise disjoint and disjoint from the boundary. Applying the lemma to each neighbourhood we get a new diffeomorphism $\widehat{T}$ such that the required sequence of periodic points is realized by the restriction of $\widehat{T}$ to the family of circles $\{C_n\}$ and $d_\infty(T,\widehat{T}) < \epsilon/2$.

In order to ensure there are no further periodic points we introduce a second perturbative term. Define $\widetilde{T}$ by

$$\widetilde{T}(x,y) = (x + \varphi(y) + f(x,y), y + g(y)).$$

The function $g \in C^\infty(I,R)$ is chosen with the following properties:

(i) $g(y) \geq 0$.
(ii) $g(y) = 0$ if and only if $y = y_n$, where y_n defines one of our circles C_n, $y = 0$, or $y = 1$.

For any n the map g can be chosen to have arbitrarily small C^n norm. In addition, though it is not used here, we notice that we could take g to be C^∞ flat at $y = 0$ and $y = 1$.

This map is invertible provided $f_x(x, y) > -1$ and $g_y(y) > -1$. The estimate on f is automatic as f comes from $\widehat{T}$ which is a diffeomorphism and we can choose g accordingly. If we choose g with a sufficiently small C^n norm then we may ensure that $d_\infty(\widetilde{T}, \widehat{T}) < \epsilon/2$.

On each circle C_n, and on the two boundary circles, we have $\widetilde{T} = \widehat{T}$. On the two boundary circles we have $\widetilde{T} = T$. Since the rotation induced by T on the boundary is irrational these circles contain no periodic points. Since $g(y) \geq 0$ the y-coordinate is non-decreasing under the action of $\widetilde{T}$. If (x, y) is not on the boundary or one of the circles C_n then $g(y) > 0$ and thus (x, y) cannot be periodic. Thus the only periodic points occur on the specified circles C_n and these periodic points realize the sequence ϕ_n. □

Finally we give the proof of the corollary from the theorem.

Corollary. *If $(\phi_n)_{n=1}^\infty$ is an exactly realizable sequence of non-negative integers then there exists $T \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ which realizes this sequence.*

Proof Proof of the Corollary. Choose $\alpha \in \mathbb{R} \backslash \mathbb{Q}$. Define $m = \min\{n : \phi_n > 0\}$. Choose $\varphi \in C^\infty(I, \mathbb{R}^+)$ such that $\varphi(0) = \varphi(1) = \alpha$, φ is C^∞ flat at $y = 0$ and $y = 1$, and $\operatorname{osc} \varphi > 1/m$. Apply the main theorem to get a diffeomorphism $\widetilde{T}$ observing that the perturbation by $g(y)$ respects the C^∞ flatness on the boundary. Thus we may identify the boundaries to get a diffeomorphism of $\mathbb{T}^2$. □

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