TOPOLOGICALLY CROSSING HETEROCLINIC CONNECTIONS TO INVARIANT TORI

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Abstract. We consider transition tori of Arnold which have topologically crossing heteroclinic connections. We prove the existence of shadowing orbits to a bi-infinite sequence of tori, and of symbolic dynamics near a finite collection of tori. Topological crossing intersections of stable and unstable manifolds of tori can be found as non-trivial zeroes of certain Melnikov functions. Our treatment relies on an extension of Easton's method of correctly aligned windows due to Zgliczyński.

1. Introduction

When a completely integrable Hamiltonian system with a partially hyperbolic fixed point of general elliptic type is perturbed, the following scenario is typical: there is a Cantor family of invariant tori that survive the perturbation, and there is a subfamily of these tori which have transverse heteroclinic connections. A detailed model is described below.

Let M be a symplectic manifold of dimension $2n_c + 2n_h$, with $n_c, n_h > 0$, and let $f_\mu : M \to M$ be a family of symplectic diffeomorphisms that depends smoothly on μ with $|\mu - \mu_0| < a_0$. We make the following assumptions:

(A) We assume that each f_μ has a partially hyperbolic fixed point p_μ. More precisely, we assume that the derivative of f_μ at p_μ has n_h eigenvalues λ with $|\lambda| < 1$, n_h eigenvalues λ with $|\lambda| > 1$, and $2n_c$ eigenvalues λ with $|\lambda| = 1$. The eigenvalues are counted with multiplicity. Then p_μ has an unstable manifold $W^u(p_\mu)$ of dimension n_h, a stable manifold $W^s(p_\mu)$ of dimension n_h, and a center manifold $W^c(p_\mu)$ of dimension $2n_c$ (the center manifold is only locally invariant). The sizes of the center manifolds corresponding to different values of $\mu \in (\mu_0 - a_0, \mu_0 + a_0)$ are uniformly bounded below with respect to μ, provided that a_0 is chosen sufficiently small.

(B) We assume that f_{μ_0} satisfies a twist condition on $W^c(p_{\mu_0})$. We assume that the corresponding center manifold can be globally described in action-angle coordinates (I, θ) relative to which f_{μ_0} can be written as $f_{\mu_0}(I, \theta) = (I', \theta')$, and so the twist condition is

$$\det\left(\frac{\partial \theta'}{\partial I}\right) \neq 0.$$

Moreover, we assume that in a neighborhood of $W^c(p_{\mu_0})$ there exists a coordinate system (I, θ, x, y) with $W^c(p_{\mu_0})$ corresponding to $x = y = 0$,

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$W^u_{loc}(p_{\mu_0})$ corresponding to $I = \theta = 0$ and $y = 0$, and $W^s_{loc}(p_{\mu_0})$ corresponding to $I = \theta = 0$ and $x = 0$. By the KAM Theorem, there exists $0 < a_1 < a_0$ and a family of tori $\{T_{\mu,\alpha}\}_{\alpha\in\mathcal{I}}$ invariant under f_μ, for all $|\mu - \mu_0| < a_1$, indexed by a Cantor set $\mathcal{I}$. The restriction of f_μ to each such a torus $T_{\mu,\alpha}$ is quasi-periodic, and it is described by a rotation vector $\Omega = \Omega(\alpha)$, whose components are the rotation numbers in each direction. All of the components of these rotation vectors are irrational numbers, and they are linearly independent over the integers as they satisfy a Diophantine condition $|\Omega \cdot k| > \gamma\|k\|_1^{-\tau}$ for all $k \in \mathbb{Z}^{n_c} \setminus \{0\}$, with $\gamma > 0$ and $\tau > n_c - 1$. The index set $\mathcal{I}$ can be chosen as the set of the rotation vectors.

(C) We assume that there exists $0 < a_2 < a_1$, $b(\mu) > 0$ and $\alpha_0 \in \mathcal{I}$ such that, for every $\mu \in (\mu_0 - a_2, \mu_0 + a_2)$, and for every

$$\alpha, \beta \in \mathcal{I}_\mu := \{\alpha \in \mathcal{I} \,|\, |\alpha - \alpha_0| < b(\mu)\},$$

the unstable manifold $W^u(T_{\mu,\alpha})$ of the torus $T_{\mu,\alpha}$ has a topologically crossing intersection with the stable manifold $W^s(T_{\mu,\beta})$ of the torus $T_{\mu,\beta}$. We will refer to the tori $\{T_{\mu,\alpha}\}_{\alpha\in\mathcal{I}_\mu}$ as 'transition tori'.

Such a situation can be obtained from an initially hyperbolic Hamiltonian system and its corresponding Graff tori, as described in [2].

Let us discuss the topological crossing assumption. We assume that the heteroclinic connections to certain invariant tori have topologically crossing intersections, rather than transverse intersections. The usual way of obtaining transition tori is by perturbing a completely integrable Hamiltonian to measure the splitting of some stable and unstable manifolds. For this reason, one usually uses a 'splitting potential' or 'Melnikov potential', since the homoclinic orbits are given by critical points of such a potential. The Melnikov potential is sometimes difficult or impossible to compute explicitly, but the existence of critical points can be inferred by geometrical arguments. If the critical points are non-degenerate, the corresponding homoclinic orbits are transverse. In many instances, one can only ensure the existence of such critical points, which only guarantees the topological crossing of the homoclinic orbits (see [4]). There are examples of this type in twist maps or billiards, and in simple resonances of Hamiltonian systems.

The results below describe the existence of chaotic sets arbitrarily near the transition tori.

Theorem 1.1 (Shadowing lemma). *There exists $0 < a_3 < a_2$ such that, for any fixed value of μ with $|\mu-\mu_0| < a_3$, any bi-infinite collection $\{T_{\mu,\alpha_i}\}_{i\in\mathbb{Z}}$ of transition tori, and any bi-infinite collection $\{\epsilon_i\}_{i\in\mathbb{Z}}$ of positive real numbers, there exits a bi-infinite sequence $\{n_i\}_{i\in\mathbb{Z}}$ of positive integers and an orbit $\{z_i\}_{i\in\mathbb{Z}}$ in M with*

$$z_{i+1} = F^{n_i}(z_i),$$
$$d(z_i, T_{\mu,\alpha_i}) < \epsilon_i,$$

for all $i \in \mathbb{Z}$.

The above 'shadowing lemma' type of result has been proved in the case of differentiable transverse heteroclinic connections in [18], and in the case of topologically crossing heteroclinic connections, but only for a particular class of Hamiltonian systems, in [12].

Theorem 1.2 (Existence of symbolic dynamics)**.** *There exists* $0 < a_3 < a_2$ *such that, for any fixed value of* μ *with* $|\mu - \mu_0| < a_3$*, any finite collection of transition tori* $\{T_{\mu,\alpha_i}\}_{i=1,\dots,d}$*, any positive real number* ϵ*, and any positive integer* N*, there exist* $n > N$ *and a locally maximal compact invariant set* S_μ *relative to* f_μ^n*, satisfying the following properties:*

(i) *The set* S_μ *is contained in an* ϵ*-neighborhood of* $\bigcup_{i=1,\dots,d} T_{\mu,\alpha_i}$*.*

(ii) *There exists a surjective continuous map* $\rho_\mu : S_\mu \to \Sigma_d$ *with* $\rho_\mu \circ f_\mu^n = \sigma \circ \rho_\mu$*, such that the inverse image of each periodic orbit of* σ *contains a periodic orbit of* f_μ^n*. Moreover, there is a* $\nu > 0$ *such that the mapping* $\rho_{\mu'}$ *can be chosen to depend continuously on* μ' *with* $|\mu' - \mu| < \nu$*, in the compact-open topology.*

The above 'existence of symbolic dynamics' type of result has been proved in the more restrictive case of differentiably transverse heteroclinic connections in [11]. In the same case, a similar type of result has been obtained by J. Cresson in [3], but using a different method. His method uses standard differentiably crossing of windows in order to first get hyperbolic periodic points near the tori, and then uses standard hyperbolic theory to obtain symbolic dynamics.

The existence of symbolic dynamics for a certain power f_μ^n of f_μ, with n arbitrarily large, confirms the so called Holmes-Marsden conjecture on the existence of periodic motions of arbitrarily high period, close to two-way transition chains of tori [10]. This chaotic drift of an orbit is a weak form of 'Arnold diffusion'. However, the usual Arnold diffusion is characterized by orbits that drift along the tori for some positive distance in the action coordinate, independent of $\mu \to \mu_0$. In our case, the index set $\mathcal{I}_\mu$ on which topologically crossing heteroclinic intersection hold, may shrink down to zero, that is $b(\mu) \to 0$ as $\mu \to \mu_0$. By the above theorem, we can only guarantee the drift for parameters within some small neighborhood of a fixed value of μ.

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2. Normal forms

Let f_μ be a family of symplectic diffeomorphisms on M satisfying a twist condition for $\mu = \mu_0$, and let $T_{\mu,\alpha}$ be a transition torus whose frequency vector $\Omega_0 := \Omega(\alpha)$ satisfies a Diophantine condition $|\Omega_0 \cdot k| > \gamma \|k\|_1^{-\tau}$, for all $k \in \mathbb{Z}^{n_c} \setminus \{0\}$, with $\gamma > 0$ and $\tau > n_c - 1$. Then there exists $0 < a_3 < a_2$, and a smooth coordinate system (ρ, ϕ, x, y) in some neighborhood $V(T_{\mu,\alpha})$ of $T_{\mu,\alpha}$, relative to which the diffeomorphism f is given by

$$f(\rho, \phi, x, y) = (\rho, \phi + v(\rho), A_-(\rho, \phi)x, A_+(\rho, \phi)y) + r(\rho, \phi, x, y) \tag{2.1}$$

provided that $\|\mu - \mu_0\| < a_3$, where $0 < \lambda < 1$, $\|A_-(\rho, \phi)\| < \lambda$, $\|A_+(\rho, \phi)^{-1}\| \leq \lambda$, and $r(\rho, \phi, 0, 0)$ is of order 3 in ρ, and $\partial r/\partial x(\rho, \phi, 0, 0) = 0 = \partial r/\partial y(\rho, \phi, 0, 0)$. Moreover, we have $v(\rho) = \Omega_0 + \Omega_1(\rho) + \Omega_2(\rho, \rho)$, where $\Omega_1 : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ is a linear map with $\det(\Omega_1) \neq 0$, and $\Omega_2 : \mathbb{R}^{n_c} \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ is a bilinear mapping. With respect to such a coordinate system, the torus $T_{\mu,\alpha}$ is given by $\rho = I_\alpha$, $x = y = 0$. We will call the mapping $(\rho, \phi, x, y) \to (\rho, \phi + v(\rho), A_-(\rho, \phi)x, A_+(\rho, \phi)y)$ a *normal form* for f.

The normal form around the invariant torus we are using here follows from results of Fontich and Martin in [9]. We point out that their result does not need to assume a Diophantine condition on the torus, but only a non-resonance condition. Eliasson [8] and Niederman [16] construct similar normal forms in an analytical setting, and provide a careful estimates on the size of the maximal domain $V(T_{\mu,\alpha})$ where the normal form is defined; however, these results only work in the case when the hyperbolic dimension n_h is equal to one.

3. Oriented intersection number and local Brouwer degree

In this section we set up some notation and briefly recall on some basic notions from differential topology (see [13]).

Let M and N be oriented manifolds, P be a closed oriented submanifold of N with $\dim M + \dim P = \dim N$, $f : M \to N$ be a smooth map transverse to P, and D be an open set in M such that $f^{-1}(P) \cap D$ is finite. The intersection number of f with P is defined by

$$\#(f, P; D) = \sum_{p \in f^{-1}(P) \cap D} \#_p(f, P; D),$$

where $\#_p(f, P; D) = \pm 1$ depending on whether the composite map

$$T_pM \xrightarrow{df_p} T_f(p)N \longrightarrow T_{f(p)}N/T_{f(p)}P$$

preserves or reverses orientation. If W^1 and W^2 are two transverse, embedded submanifolds of complementary dimension in M, and D an open set in M such that $W^1 \cap W^2 \cap D$ is finite, then the *oriented intersection number* $\#(M, N; D)$ is given by $\#(i, W^2; D)$, where $i : W^1 \hookrightarrow M$ is the inclusion.

Now consider $f : M \to N$ a smooth map between two manifolds of same dimension with N connected, q a point in N, and D an open set in M such that $f^{-1}(q) \cap D$ is finite. The local Brouwer degree of f at q in the set D is defined by $\#(f, \{q\}; D)$. If q happens to be a regular point of f such that the set $f^{-1}(c) \cap D$ is compact, the *local Brouwer degree* of f at q in the set D is also given by

$$\deg(f, q; D) = \sum_{p \in f^{-1}(q) \cap D} \text{sign } df_p,$$

where sign df_p is ± 1 depending on whether df_p preserves or reverses orientation. Since the degree remains the same under small homotopy deformations, one can define the degree for a map f which is only continuous. If the set D is a disk, one can compute the degree by counting how many times the image of the boundary of D wraps around q. More precisely, for a continuous map $s : S^{n-1} \to S^{n-1}$, let $d(s)$ be the unique integer defined by $f_*(u) = d(s)u$, where $f_* : H_*(S^{n-1}) \to H_*(S^{n-1})$ is the homomorphism induced in homology, and u is any generator in $H_*(S^{n-1})$. Now let $f : \overline{B_n}(0, 1) \to \mathbb{R}^n$ be a continuous map with $0 \notin f(\partial B_u(0, 1))$. Define $s_f : S^{n-1} \to S^{n-1}$ by

$$s_f(p) = \frac{f(p)}{\|f(p)\|}.$$

Then,

$$\deg(f, 0; \overline{B_n}(0, 1)) = ds_f.$$

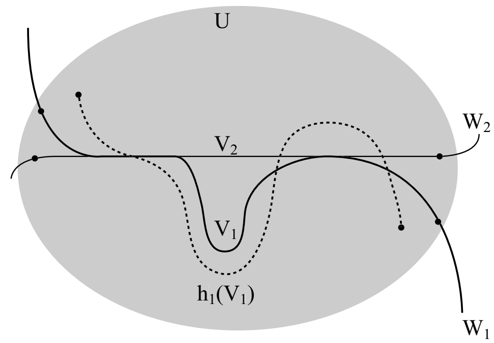

FIGURE 1. Topological crossing

4. Topological crossing

We define topological crossing following [1].

Definition 4.1 (Topological crossing)**.** Let W^1 and W^2 be two submanifolds of dimensions n_1 and respectively n_2 in M, such that $n_1 + n_2 = n$. We say that W^1 and W^2 have a *topologically crossing intersection* if there exist an orientable open set $U \subseteq M$, a compact orientable embedded n_1-dimensional submanifold with boundary $V^1 \subseteq W^1$, and a compact orientable embedded n_2-dimensional submanifold with boundary $V^2 \subseteq W^2$, such that

(i) $\mathrm{bd}V^1 \cap V^2 = \mathrm{bd}V^2 \cap V^1 = \emptyset$,
(ii) $V^1 \cap V^2 \subseteq U$,
(iii) For every $0 < \epsilon < \min\big(\mathrm{dist}(\mathrm{bd}V^1, V^2), \mathrm{dist}(\mathrm{bd}V^2, V^1)\big)$, there exits a homotopy $h : [0,1] \times M \to M$ such that
  (iii.a) $h_0 = \mathrm{id}$ and h_1 is an embedding,
  (iii.b) $d(h_t(p), p) < \epsilon$ for all $p \in \mathbb{R}^n$ and all $t \in [0,1]$,
  (iii.c) $h_1(V^1)$ and V^2 are transverse submanifolds, and their oriented intersection number $\iota := \#(h_1(V^1), V^2)$ is nonzero, for some choice of orientation on V^1, V^2 and U.

See Figure 1.

A direct consequence of this definition is that W^1 and W^2 have a non-empty intersection. It is important to notice that the intersection need not be transverse and could be of infinite order. The submanifolds V^1 and V^2 in the above definition will be referred as a *good pair* for W^1 and W^2.

The following construction provides an alternate description of topological crossing, which will be used later. Assume that there exists a local coordinate system (x, y) near $V^1 \cap V^2$ and a smooth parametrization ψ, such that, under this parametrization, we have the following identifications

$$\begin{aligned} U &= \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \\ V^1 &= \overline{B_{n_1}}(0, r_1) \times \{0\}, \\ V^2 &\text{ is the image of } \psi : \overline{B_{n_2}}(0, r_2) \to U, \\ V^2 \cap (\mathbb{R}^{n_1} \times \{0\}) &\subseteq B_{n_1}(0, r_1) \times \{0\}, \end{aligned}$$

where $r_1, r_2 > 0$. Define the map $A : \overline{B_{n_2}}(0, r_2) \subseteq \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ by

$$A = \pi_{\mathbb{R}^{n_2}} \circ \psi,$$

where $\pi_{\mathbb{R}^{n_2}}$ is the projection into the y-coordinate. The definition of the degree as an intersection number implies almost immediately that

$$0 \notin A(\partial(\overline{B_{n_1}}(0, r_2))),$$
$$\deg(A, 0; \overline{B_{n_2}}(0, r_2) = \iota.$$

As noticed earlier, we have $\deg(A, 0; \overline{B_{n_2}}(0, r_2) = ds_A$. Thus, W^1 and W^2 have a topological crossing intersection if and only if $ds_A \neq 0$.

In (C), we have assumed that the unstable manifold and the stable manifold of any pair of transition tori topologically cross one another. In general, the topologically crossing intersection of two manifolds can be a 'relatively large' compact set. Further, we will assume, for simplicity, that for each pair $W^u(T_{\mu,\alpha})$ and $W^s(T_{\mu,\beta})$ as in (C), there exists a good pair V^1, V^2 contained in some coordinate neighborhood U in M, with V^1 and V^2 given by parametric equations as described above. For the general case, one only has to patch together different parametrization.

In the case of Lagrangian submanifolds of a symplectic manifold there is a simple perturbation mechanism which produces topological crossing. If (M, ω) is a symplectic manifold, a submanifold $L \subseteq M$ is called isotropic if $\omega(\xi, \eta) = 0$ for any $x \in L$ and any $\xi, \eta \in T_xL$. The invariant tori described in condition (B) are isotropic. A submanifold $L \subseteq M$ is called Lagrangian if it is isotropic and $\dim L = (1/2) \dim M$. The stable and unstable manifolds of the invariant tori described in condition (B) are Lagrangian submanifolds. A symplectic diffeomorphism $f : M \to M$ is called exact symplectic if there exists a differential 1-form α on M such that $d\alpha = \omega$ and $f^*(\alpha) - \alpha = dS$ for some real valued function S on M.

The following result is due to Xia [19].

Proposition 4.2. *Assume that f_{μ_0} is an exact symplectic diffeomorphism on M. Assume that T_{μ_0,α_0} is an invariant torus in M as in condition (B), such that $W^u(T_{\mu_0,\alpha_0}) = W^s(T_{\mu_0,\alpha_0})$, and $W^u(T_{\mu_0,\alpha_0})$, $W^s(T_{\mu_0,\alpha_0})$ have a common (compact) fundamental domain. Then there exists $0 < a_2 < a_1$, such that, for every $|\mu - \mu_0| < a_2$ we have $W^u(T_{\mu,\alpha_0}) \cap W^s(T_{\mu,\alpha_0}) \neq \emptyset$. Moreover, if all connected components of $W^u(T_{\mu,\alpha_0}) \cap W^s(T_{\mu,\alpha_0})$ are contractible, then $W^u(T_{\mu,\alpha_0})$ and $W^s(T_{\mu,\alpha_0})$ have a topological crossing.*

In the context described in Section 1, the persistence of topological crossing under small perturbations yields to the following practical conclusion.

Corollary 4.3. *In the conditions of the previous theorem, there exists $b(\mu) > 0$ such that, for each*

$$\alpha, \beta \in \mathcal{I}_\mu := \{\alpha \in \mathcal{I} \,|\, |\alpha - \alpha_0| < b(\mu)\},$$

we have that $W^u(T_{\mu,\alpha})$ and $W^s(T_{\mu,\beta})$ have a topological crossing.

5. Correctly aligned windows

The method of correctly aligned windows has been introduced by Easton in a series of papers [5, 6, 7]. In short, correctly aligned windows are topological rectangles whose behavior under iteration mimics that of the rectangles of a Markov partition. Easton's method has been recently refined by Mischaikow and Mrozek

[15], Kennedy and Yorke [14], and several others. We use an extension of this method developed by Zgliczyński (see [12] and the references listed there). The main advantage of such a topological refinement is that neither hyperbolic structure nor transversal intersection of various sections are required. We will give a brief account of this method and refer to [12] for details and proofs (the reader should be aware of the different terminology used there).

Given a compact subset K in $\mathbb{R}^n$, a map $c : K \to c(K) \subseteq M$ is said to be a homeomorphism provided that the domain $\mathrm{dom}(c)$ of c is an open neighborhood of K in $\mathbb{R}^n$, the range $\mathrm{im}(c)$ of c is an open neighborhood of $c(K)$ in M, and $c : \mathrm{dom}(c) \to \mathrm{im}(c)$ is a homeomorphism.

Definition 5.1 (Window)**.** An (u, s)-window in M is a compact subset N of M together with a homeomorphism $c_N : [0,1]^u \times [0,1]^s \to N$, where $u + s = n$. The set $N^- = c_N(\partial([0,1]^u) \times [0,1]^s)$ is called the 'exit set' and the set $N^+ = c_N([0,1]^u \times \partial([0,1]^s))$ is called the 'entry set' of N.

Since c_N is merely a homeomorphism, in the above definition one can always replace the rectangle $[0,1]^u \times [0,1]^s$ by a a homeomorphic copy of it.

Definition 5.2 (Forward correctly aligned windows)**.** Assume that N_1 and N_2 are two (u, s)-windows in M and f is a continuous map on M such that $f(\mathrm{im}(c_{N_1})) \subseteq \mathrm{im}(c_{N_2})$. Let $f_c : \mathrm{dom}(c_{N_1}) \supseteq [0,1]^u \times [0,1]^s \to \mathbb{R}^n$ be defined by $f_c = c_{N_2}^{-1} \circ f \circ c_{N_1}$. We say that the windows N_1 forward correctly aligns with the window N_2 under f provided that

(i)

$$f_c(c_{N_1}^{-1}(N_1^-)) \cap c_{N_2}^{-1}(N_2) = \emptyset,$$
$$f_c(c_{N_1}^{-1}(N_1)) \cap c_{N_2}^{-1}(N_2^+) = \emptyset.$$

(ii) There exists a continuous homotopy $h : [0,1] \times [0,1]^u \times [0,1]^s \to \mathbb{R}^n$ such that the following conditions hold

(ii.a)

$$h_0 = f_c,$$
$$h_t(c_{N_1}^{-1}(N_1^-)) \cap c_{N_2}^{-1}(N_2) = \emptyset,$$
$$h_t(c_{N_1}^{-1}(N_1)) \cap c_{N_2}^{-1}(N_2^+) = \emptyset,$$

for all $t \in [0,1]$.

(ii.b) If $u = 0$, then $h_1 \equiv 1$. If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that

$$h_1(x, y) = (A(x), 0),$$
$$A(\partial([0,1]^u)) \subseteq \mathbb{R}^u \setminus [0,1]^u,$$
$$ds_A \neq 0,$$

where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$.

The number ds_A is called the 'degree of alignment'. In the case $u = 0$, the degree ds_A is set equal to 1 by default. It may happen that the homotopy map h can be chosen so that A is a linear map; in such a case $ds_A = \mathrm{sign}(\det(A))$. One could have used the degree of the map A in the above definition. It is however more convenient to use ds_A: in order to verify a correct alignment, one should only check what happens to the boundaries of the windows after iteration.

Sometimes windows can be correctly aligned under backward iterations in a sense made precise next.

Definition 5.3 (Transpose of a window)**.** If N is a (u, s)-window described by the coordinate mapping c_N, its transpose N^T is the (s, u)-window given by the homeomorphism $c_N^T : [0,1]^s \times [0,1]^u \to N^T$, where $c_N^T(y, x) = c_N(x, y)$. Its exist set $\left(N^T\right)^-$ of N^T is N^+ and its entry set $\left(N^T\right)^+$ is N^-.

Definition 5.4 (Backward correctly aligned windows)**.** Assume that N_1 and N_2 are (u, s)-windows, f is a continuous map on M such that f^{-1} is well defined and continuous on $\mathrm{im}(c_{N_2})$ and $f^{-1}(\mathrm{im}(c_{N_2})) \subseteq (\mathrm{im}(c_{N_1}))$. We say that the window N_1 backward correctly aligns with the window N_2 under f provided that N_2^T forward correctly aligns with N_1^T under f^{-1}.

Definition 5.5. Assume that N_1 and N_2 are (u, s)-windows, f is a continuous map on M. We say that N_1 correctly aligns with with N_2 under f provided that it correctly aligns either forward or backwards.

The following is a sufficient criterion for correct alignment.

Proposition 5.6 (Correct alignment criterion)**.** *Let N_1, N_2 be two (u, s)-windows in M, and f be a continuous map on M with $f(im(c_{N_1})) \subseteq im(c_{N_2})$. Assume that the following conditions are satisfied:*

(i)
$$f_c(c_{N_1}^{-1}(N_1^-)) \cap c_{N_2}^{-1}(N_2) = \emptyset,$$
$$f_c(c_{N_1}^{-1}(N_1)) \cap c_{N_2}^{-1}(N_2)^+ = \emptyset.$$

(ii) *there exists a point $y_0 \in [0,1]^s$ such that*

(ii.a) $f_c([0,1]^u \times \{y_0\}) \subseteq int[[0,1]^u \times [0,1]^s \cup (\mathbb{R}^u \setminus (0,1)^u) \times \mathbb{R}^s]$,

(ii.b) *If $u = 0$, then $f_c(c_{N_1}^{-1}((N_1)) \subseteq int c_{N_2}^{-1}(N_2)$. If $u > 0$, then the map $A_{y_0} : \mathbb{R}^u \to \mathbb{R}^u$ defined by $A_{y_0}(x) = \pi_u(f_c(x, y_0))$ satisfies*
$$A_{y_0}(\partial([0,1]^u)) \subseteq \mathbb{R}^u \setminus [0,1]^u,$$
$$ds_{A_{y_0}} \neq 0.$$

Then N_1 forward correctly aligns with N_2 under f.

Here π_u denotes the projection $(x, y) \in \mathbb{R}^u \times \mathbb{R}^s \to x \in \mathbb{R}^u$.

If $u = 0$, the above proposition gives a correct alignment of degree one; if $u > 0$, the degree is equal to $ds_{A_{y_0}}$.

The precise feature that makes correct alignment verifiable in concrete examples is its stability under sufficiently small perturbations.

Theorem 5.7 (Stability under small perturbations)**.** *Suppose that the (u, s)-window N_1 correctly aligns with the (u, s)-window N_2 under a continuous map f on M. There exists $\epsilon > 0$ such that for continuous map g on M for which g_c is ϵ-close to f_c in the compact-open topology, N_1 correctly aligns with N_2 under g.*

The main result regarding this construction can be stated as 'one can see through a sequence of correctly aligned windows' (compare [7]).

Theorem 5.8 (Existence of orbits with prescribed trajectories)**.** *Let N_i be a collection of (u, s)-windows in M, where $i \in \mathbb{Z}$ or $i \in \{0, \ldots, d-1\}$, with $d > 0$ (in the latter case, for convenience, we let $N_i := N_{(i mod\ d)}$ for all $i \in \mathbb{Z}$). Let f_i be a* *collection of continuous maps on M. If N_i correctly aligns with N_{i+1} under f_i, for all i, then there exists a point $p \in N_0$ such that*

$$f_i \circ \ldots \circ f_0(p) \in N_{i+1},$$

Moreover, if $N_{i+k} = N_i$ for some $k > 0$ and all i, then the point p can be chosen so that

$$f_{k-1} \circ \ldots \circ f_0(p) = p.$$

This result can be effectively used in detecting chaotic behavior.

Corollary 5.9 (Detection of chaos)**.** *Let $N_0, \ldots, N_{d-1}$ be a collection of mutually disjoint (u, s)-windows and f a continuous map on M. Assume that for every i and j in $\{0, \ldots, d-1\}$, the window N_i correctly aligns with the window N_j under f. There exist a maximal f-invariant set S in $\bigcup_{i=0,\ldots,d-1} \mathrm{int} N_i$, and a continuous surjective map $\rho : S \to \Sigma_d$ such that $\rho \circ f = \sigma \circ \rho$, and the inverse image of every periodic orbit of σ contains a periodic orbit of f.*

Here (Σ_d, σ) designates a full shift over d symbols. A surjective continuous map $\rho : S_1 \to S_2$ from a f_1-invariant set S_1 to a f_2-invariant set is called a semi-conjugacy provided that it maps f_1-orbits into f_2-orbits, i.e. $\rho \circ f_1 = f_2 \circ \rho$. The existence of a such a semi-conjugacy means that the dynamics on S_1 is at least as complicated as the dynamics on S_2. Using topological entropy h_{top} as a measure of complexity, we have $h_{top}(f_1) \geq h_{top}(f_2)$. If ρ is bijective, then it is called a conjugacy and we have $h_{top}(f_1) = h_{top}(f_2)$. See [17] for more background.

Remark 5.10. In Corollary 5.9, in practical situations, the map f may be the power g^n of some map g. The condition that each N_i is correctly aligned with each N_j under g^n may be relaxed to a condition that there exists a length n chain of correctly aligned windows under g. This means that for each i, j, there exists a sequence $W_0, W_1, \ldots, W_n$, such that N_i is correctly aligned with W_0 under the identity mapping, W_j is correctly aligned with W_{j+1} under g, $j = 0, \ldots, n-1$, and W_n is correctly aligned with N_j under the identity mapping. We can allow that $W_0 = N_i$ or $W_n = N_j$.

For the rest of the paper, we will omit the (u, s)-specification on a window whenever this data is clear from context.

6. Construction of windows

In this section, the value of μ is fixed and, to simplify the notation, the symbol μ will be dropped.

Let T_α and T_β be a pair of transition tori with a topological crossing intersection of $W^u(T_\alpha)$ and $W^s(T_\beta)$. Let $V^u(T_\alpha)$ and $V^s(T_\beta)$ be a good pair for $W^u(T_\alpha)$ and $W^s(T_\beta)$. Let (x_c, x_h, y_c, y_h) and (x'_c, x'_h, y'_c, y'_h) be local coordinate systems near $V^u(T_\alpha) \cap V^s(T_\beta)$, and U and U' coordinate neighborhoods, defined by the following properties:

(1) In the (x_c, x_h, y_c, y_h) coordinates we have the following
   (i) $U = \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}$,
   (ii) The submanifold with boundary $V^s(T_\beta)$ is a disk in the subspace $\{0\} \times \{0\} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}$,

$$V^s(T_\beta) = \{0\} \times \{0\} \times \overline{B_{n_c}}(0, r_c) \times \overline{B_{n_h}}(0, r_h), \text{ and}$$

$$W^s(q) \cap \mathrm{V}^s(T_\beta) = \{0\} \times \{0\} \times \{\text{const.}\} \times \overline{B_{n_h}}(0, r_h),$$

for any $q \in T_\beta$,

(iii) The submanifold with boundary $V^u(T_\alpha)$ is the image of an embedding

$$V^u(T_\alpha) = \psi'\left(\overline{B_{n_c}}(0, r'_c) \times \overline{B_{n_h}}(0, r'_h)\right), \text{ and}$$
$$W^u(p) \cap V^u(T_\alpha) = \psi'\left(\{\text{const.}\} \times \overline{B_{n_h}}(0, r'_h)\right),$$

for any $p \in T_\alpha$,

(iv) The submanifold $V^u(T_\alpha)$ only intersects the subspace containing $V^s(T_\beta)$ within $V^s(T_\beta)$,

$$V^u(T_\beta) \cap [\{0\} \times \{0\} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}] \subseteq \text{int} V^s(T_\beta),$$

(2) Dually, in the (x'_c, x'_h, y'_c, y'_h) coordinates we have

(i') $U' = \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}$,

(ii') The submanifold with boundary $V^u(T_\alpha)$ is a disk in the subspace $\mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \times \{0\} \times \{0\}$,

$$V^u(T_\alpha) = \overline{B_{n_c}}(0, r'_c) \times \overline{B_{n_h}}(0, r'_h) \times \{0\} \times \{0\}, \text{ and}$$
$$W^u(p) \cap V^u(T_\alpha) = \{\text{const.}\} \times \overline{B_{n_h}}(0, r'_h) \times \{0\} \times \{0\},$$

for any $p \in T_\alpha$,

(iii') The submanifold with boundary $W^s(T_\beta)$ is the image of an embedding

$$V^s(T_\beta) = \psi\left(\overline{B_{n_c}}(0, r_c) \times \overline{B_{n_h}}(0, r_h)\right), \text{ and },$$
$$W^s(q) \cap V^s(T_\beta) = \psi\left(\{\text{const.}\} \times \overline{B_{n_h}}(0, r_h)\right),$$

for any $q \in T_\beta$,

(iv') The submanifold $V^s(T_\beta)$ only intersects the subspace containing $V^u(T_\alpha)$ within $V^u(T_\alpha)$,

$$V^s(T_\beta) \cap [\mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \times \{0\} \times \{0\}] \subseteq \text{int} V^u(T_\alpha).$$

By these choices of coordinates, observe that $\text{int}V^u(T_\alpha)$, which corresponds to $y'_c = 0$ and $y'_h = 0$, is foliated by unstable manifolds of points in T_α (corresponding to $x'_c = \text{const.}$), and by leaves transverse to the unstable ones (corresponding to $x'_h = \text{const.}$). The images of the leaves $x'_h = \text{const.}$ under backward iterates of f C^1-approach T_α. Similarly, $\text{int}V^s(T_\beta)$, which corresponds to $x_c = 0$ and $x_h = 0$, is foliated by stable manifolds of points in T_β (corresponding to $y_c = \text{const.}$), and by leaves transverse to the stable ones (corresponding to $y_h = \text{const.}$). The images of the leaves $y_h = \text{const.}$ under forward iterates of f C^1-approach T_β.

The heteroclinic intersection $K_{\alpha\beta} = V^u(T_\alpha) \cap V^s(T_\beta) \subseteq U \cap U'$ can be, in general, a relatively large set. Let $0 < \kappa_c < r_c$, $0 < \kappa_h < r_h$, $0 < \kappa'_c < r'_c$, $0 < \kappa'_h < r'_h$ such that, in (x_c, x_h, y_c, y_h) coordinates we have

$$K_{\alpha\beta} \subseteq \{0\} \times \{0\} \times B_{n_c}(0, \kappa_c) \times B_{n_h}(0, \kappa_h),$$

and in (x'_c, x'_h, y'_c, y'_h) coordinates we have

$$K_{\alpha\beta} \subseteq B_{n_c}(0, \kappa'_c) \times B_{n_h}(0, \kappa'_h) \times \{0\} \times \{0\}.$$

We define two (n_c+n_h, n_c+n_h)-windows $N^u_{\alpha\beta}$ and $N^s_{\alpha\beta}$ as tubular neighborhoods of $V^u(T_\alpha)$ and $V^s(T_\beta)$, respectively, such that $N^u_{\alpha\beta}$ is correctly aligned with $N^s_{\alpha\beta}$ under the identity map. See Figure 2.

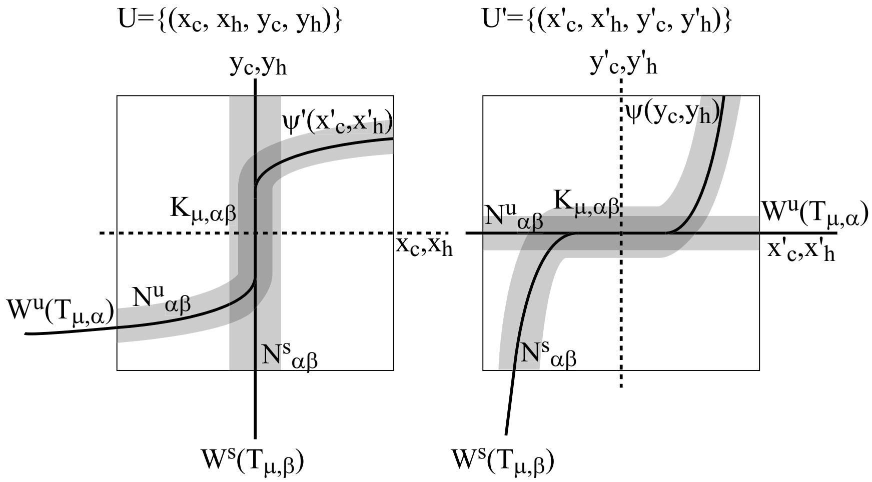

FIGURE 2. The construction of correctly aligned windows near a heteroclinic intersection $K_{\alpha\beta}$.

In (x'_c, x'_h, y'_c, y'_h) coordinates, the window $N^u_{\alpha\beta}$ is given by

$$N^u_{\alpha\beta} = \overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h) \times \overline{B_{n_c}}(0,\rho'_c) \times \overline{B_{n_h}}(0,\rho'_h), \tag{6.1}$$

$$N^u_{\alpha\beta}{}^- = \partial\left(\overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h)\right) \times \overline{B_{n_c}}(0,\rho'_c) \times \overline{B_{n_h}}(0,\rho'_h), \tag{6.2}$$

$$N^u_{\alpha\beta}{}^+ = \overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h) \times \partial\left(\overline{B_{n_c}}(0,\rho'_c) \times \overline{B_{n_h}}(0,\rho'_h)\right). \tag{6.3}$$

In (x_c, x_h, y_c, y_h) coordinates, the window $N^s_{\alpha\beta}$ is given by

$$N^s_{\alpha\beta} = \overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h) \times \overline{B_{n_c}}(0,\kappa_c) \times \overline{B_{n_h}}(0,\kappa_h), \tag{6.4}$$

$$N^s_{\alpha\beta}{}^- = \partial\left(\overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h)\right) \times \overline{B_{n_c}}(0,\kappa_c) \times \overline{B_{n_h}}(0,\kappa_h), \tag{6.5}$$

$$N^s_{\alpha\beta}{}^+ = \overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h) \times \partial\left(\overline{B_{n_c}}(0,\kappa_c) \times \overline{B_{n_h}}(0,\kappa_h)\right). \tag{6.6}$$

Lemma 6.1. *For every positive real numbers $\varepsilon_c, \varepsilon_h, \varepsilon'_c, \varepsilon'_h$, there exist $0 < \rho_c < \varepsilon_c$, $0 < \rho_h < \varepsilon_h$, $0 < \rho'_c < \varepsilon'_c$, $0 < \rho'_h < \varepsilon'_h$, such that the windows $N^u_{\alpha\beta}$ is correctly aligned with $N^s_{\alpha\beta}$ under the identity mapping.*

Proof. Choose $\delta > 0$ sufficiently small such that, relative to the (x_c, x_h, y_c, y_h) coordinate system we have

$$\delta < \text{dist}\left(\partial V^u(T_\alpha), \{0\} \times \{0\} \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}\right), \tag{6.7}$$

and, relative to the (x'_c, x'_h, y'_c, y'_h) coordinate system we have

$$\delta < \text{dist}\left(\partial V^s(T_\alpha), \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \times \{0\} \times \{0\}\right). \tag{6.8}$$

First, with respect to the (x_c, x_h, y_c, y_h) coordinates, we choose $0 < \rho_c < \varepsilon_c$ and $0 < \rho_h < \varepsilon_h$ such that $\rho_c < \delta/2$ and $\rho_h < \delta/2$. As a consequence of this choice, we have

$$\text{dist}\left(\left[\overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h) \times \overline{B_{n_c}}(0,\kappa_c) \times \overline{B_{n_h}}(0,\kappa_h)\right], \partial V^u(T_\alpha)\right) > \delta/2,$$
$$\text{dist}\left(\left[\overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h) \times \partial\left(\overline{B_{n_c}}(0,\kappa_c) \times \overline{B_{n_h}}(0,\kappa_h)\right)\right], V^u(T_\alpha)\right) > \delta/2.$$

At this point, $N^s_{\alpha\beta}$ has been completely described, and we have

$$\operatorname{dist}\left(N^s_{\alpha\beta}, \partial V^u(T_\alpha)\right) > \delta/2, \tag{6.9}$$

$$\operatorname{dist}\left({N^s_{\alpha\beta}}^+, V^u(T_\alpha)\right) > \delta/2. \tag{6.10}$$

Next, with respect to the (x'_c, x'_h, y'_c, y'_h) coordinates, we choose $0 < \rho'_c < \varepsilon'_c$ and $0 < \rho'_h < \varepsilon'_h$ such that $\rho'_c < \delta/2$ and $\rho'_h < \delta/2$. As a consequence of this choice, and of (6.9) and (6.10), we have

$$\operatorname{dist}\left(\left[\overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h) \times \overline{B_{n_c}}(0,\rho'_c) \times \overline{B_{n_h}}(0,\rho'_h)\right], {N^s_{\alpha\beta}}^+\right) > 0,$$
$$\operatorname{dist}\left(\left[\partial\left(\overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h)\right) \times \overline{B_{n_c}}(0,\rho'_c) \times \overline{B_{n_h}}(0,\rho'_h)\right], N^s_{\alpha\beta}\right) > 0.$$

At this point, $N^u_{\alpha\beta}$ has been also completely described, and we have

$$\operatorname{dist}\left(N^u_{\alpha\beta}, {N^s_{\alpha\beta}}^+\right) > 0, \tag{6.11}$$

$$\operatorname{dist}\left({N^u_{\alpha\beta}}^-, N^s_{\alpha\beta}\right) > 0. \tag{6.12}$$

In order to show that $N^u_{\alpha\beta}$ is forward correctly aligned with $N^s_{\alpha\beta}$ under the identity mapping, we check that the conditions in Proposition 5.6 are verified. By (6.11) and (6.12), we obviously have ${N^u_{\alpha\beta}}^- \cap N^s_{\alpha\beta} = \emptyset$ and $N^u_{\alpha\beta} \cap {N^s_{\alpha\beta}}^+ = \emptyset$, which show that (i) is satisfied. In order to check (ii), we go back to the (x_c, y_c, x_h, y_h) coordinates. By condition (6.7) on δ, we have that

$$\begin{aligned} V^u(T_\alpha) \subseteq \operatorname{int}\big[&\overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h) \times \overline{B_{n_c}}(0,\kappa_c) \times \overline{B_{n_h}}(0,\kappa_h) \cup \\ &\cup (\mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \setminus B_{n_c}(0,\rho_c) \times B_{n_h}(0,\rho_h)) \times \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}\big]. \end{aligned} \tag{6.13}$$

This shows that condition (ii.a) is satisfied. For (ii.b), let $\pi_{\mathbb{R}^{n_c}\times\mathbb{R}^{n_h}}$ be the projection into the (x_c, x_h) coordinates and $A : \overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h) \to \mathbb{R}^{n_c} \times \mathbb{R}^{n_h}$ be given by

$$A(x'_c, x'_h) = \pi_{\mathbb{R}^{n_c}\times\mathbb{R}^{n_h}} \circ \psi'(x'_c, x'_h).$$

By the choice of ρ_c and ρ_h, we have that

$$A\left(\partial\left(\overline{B_{n_c}}(0,\kappa'_c) \times \overline{B_{n_h}}(0,\kappa'_h)\right)\right) \subseteq \mathbb{R}^{n_c} \times \mathbb{R}^{n_h} \setminus \overline{B_{n_c}}(0,\rho_c) \times \overline{B_{n_h}}(0,\rho_h).$$

This makes the degree ds_A well defined. The alternate description of topological crossing in Section 4 leads to

$$ds_A \neq 0.$$

This shows that (ii.b) is satisfied. Applying Proposition 5.6 yields to the desired conclusion. □

If $p, q > 0$, we can transport the window $N^u_{\alpha\beta}$ to a window $f^{-p}\left(N^u_{\alpha\beta}\right)$ close to T_α, and the window $N^s_{\alpha\beta}$ to a window $f^q\left(N^s_{\alpha\beta}\right)$ close to T_β.

We now construct a pair of windows along each torus T_α. We rearrange the coordinates (I, θ, x, y) in the order (θ, x, I, y). Let $q_{\gamma\alpha}$, $q_{\alpha\beta}$ be a pair of arbitrary points on T_α, and let $(\theta_{\gamma\alpha}, 0, I_\alpha, 0)$, $(\theta_{\alpha\beta}, 0, I_\alpha, 0)$ be their coordinates. Relative to the (θ, x, I, y) coordinate system, we define two windows $M_{\gamma\alpha}$ and $P_{\alpha\beta}$ by

$$M_{\gamma\alpha} = \overline{B_{n_c}}(\theta_{\gamma\alpha},\kappa_{\gamma\alpha}) \times \overline{B_{n_h}}(0,\eta_\alpha) \times \overline{B_{n_c}}(I_\alpha,\eta_\alpha) \times \overline{B_{n_h}}(0,\eta_\alpha), \tag{6.14}$$

$$M^-_{\gamma\alpha} = \overline{B_{n_c}}(\theta_{\gamma\alpha},\kappa_{\gamma\alpha}) \times \partial\left(\overline{B_{n_h}}(0,\eta_\alpha) \times \overline{B_{n_c}}(I_\alpha,\eta_\alpha)\right) \times \overline{B_{n_h}}(0,\eta_\alpha), \tag{6.15}$$

$$\begin{aligned} M^+_{\gamma\alpha} = \ &\partial\overline{B_{n_c}}(\theta_{\gamma\alpha},\kappa_{\gamma\alpha}) \times \overline{B_{n_h}}(0,\eta_\alpha) \times \overline{B_{n_c}}(I_\alpha,\eta_\alpha) \times \overline{B_{n_h}}(0,\eta_\alpha) \cup \\ &\cup \overline{B_{n_c}}(\theta_{\gamma\alpha},\kappa_{\gamma\alpha}) \times \overline{B_{n_h}}(0,\eta_\alpha) \times \overline{B_{n_c}}(I_\alpha,\eta_\alpha) \times \partial\overline{B_{n_h}}(0,\eta_\alpha), \end{aligned} \tag{6.16}$$

and

$$P_{\alpha\beta} = \overline{B_{n_c}}(\theta_{\alpha\beta}, \kappa_{\alpha\beta}) \times \overline{B_{n_h}}(0, \eta_\alpha) \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha) \times \overline{B_{n_h}}(0, \eta_\alpha), \tag{6.17}$$

$$P^-_{\alpha\beta} = \partial\left(\overline{B_{n_c}}(\theta_{\alpha\beta}, \kappa_{\alpha\beta}) \times \overline{B_{n_h}}(0, \eta_\alpha)\right) \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha) \times \overline{B_{n_h}}(0, \eta_\alpha), \tag{6.18}$$

$$P^+_{\alpha\beta} = \overline{B_{n_c}}(\theta_{\alpha\beta}, \kappa_{\alpha\beta}) \times \overline{B_{n_h}}(0, \eta_\alpha) \times \partial\left(\overline{B_{n_c}}(I_\alpha, \eta_\alpha) \times \overline{B_{n_h}}(0, \eta_\alpha)\right), \tag{6.19}$$

where $\eta_\alpha > 0$.

We emphasize the interchange of the action and angle directions in the exit sets for these two windows. This interchange is responsible for the hyperbolic-like behavior of the system, and is related to the so called 'transversality-torsion phenomenon' (see [3]). We shall see that this mechanism survives in its essence when transversality is replaced with topological crossing.

By choosing η_α sufficiently small, one can ensure that $M_{\gamma\alpha}$, $P_{\alpha\beta}$ are contained in the neighborhood $V(T_\alpha)$ of T_α, where the normal form provided in Section 2 is defined.

We will be looking closely at the geometry of the windows along the center directions. For this reason, let

$$\tilde{M}_{\gamma\alpha} = \overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha}) \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha), \tag{6.20}$$

$$\tilde{M}^-_{\gamma\alpha} = \overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha}) \times \partial\overline{B_{n_c}}(I_\alpha, \eta_\alpha), \tag{6.21}$$

$$\tilde{M}^+_{\gamma\alpha} = \partial\overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha}) \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha), \tag{6.22}$$

and

$$\tilde{P}_{\alpha\beta} = \overline{B_{n_c}}(\theta_{\alpha\beta}, \kappa_{\alpha\beta}) \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha), \tag{6.23}$$

$$\tilde{P}^-_{\alpha\beta} = \partial\overline{B_{n_c}}(\theta_{\alpha\beta}, \kappa_{\alpha\beta}) \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha), \tag{6.24}$$

$$\tilde{P}^+_{\alpha\beta} = \overline{B_{n_c}}(\theta_{\alpha\beta}, \kappa_{\alpha\beta}) \times \partial\overline{B_{n_c}}(I_\alpha, \eta_\alpha). \tag{6.25}$$

Lemma 6.2. *Given $\gamma, \alpha, \beta \in \mathcal{I}_\mu$, $\epsilon > 0$ and the integers $P, Q > 0$, there exist $p > P$, $q > Q$, $\eta_\alpha < \epsilon$, $\kappa_{\gamma\alpha} > 0$, $\kappa_{\alpha\beta} > 0$, and sufficiently small $\rho_c, \rho_h, \rho'_c, \rho'_h$, such that $N^s_{\gamma\alpha}$ is correctly aligned with $M_{\gamma\alpha}$ under f^q, and $P_{\alpha\beta}$ is correctly aligned with $N^u_{\alpha\beta}$ under f^p.*

Proof. By the Stable Manifold Theorem (see [17]), there exist $0 < C$, $0 < \lambda < 1$ such that $d(f^n(p_0), f^n(p_1)) < C\lambda^n d(p_0, p_1)$ for for each $p_0 \in W^c(p_\mu)$ and each $p_1 \in W^s_{loc}(p_0)$, and $d(f^{-n}(p_0), f^{-n}(p_1)) < C\lambda^n d(p_0, p_1)$ for each $p_0 \in W^c(p_\mu)$ and for each $p_1 \in W^u_{loc}(p_0)$. The image of the cross section $x_c = 0$, $x_h = 0$, $y_c =$ const. through the window $N^s_{\gamma\alpha}$ under f^q is a shrinking topological disk contained in $W^s(T_\alpha)$, and so its y-projection is contained in $\overline{B_{n_h}}(0, \eta_\alpha)$ provided that q is chosen sufficiently large. By the continuity of the foliation, the same is true for each cross section $x_c =$ const., $x_h =$ const., $y_c =$ const. through the window $N^s_{\gamma\alpha}$, provided ρ_h and ρ_c are sufficiently small. Due to the Lambda Lemma (see [17]), if q is sufficiently large, the image of each cross section $x_c = 0$, $x_h = 0$, $y_h =$ const. thorough $N^s_{\gamma\alpha}$ under f^q approaches a n_c-disk contained in $W^c(p)$, in the C^1-topology, so its x-projection contains $\overline{B_{n_h}}(0, \eta_\alpha)$ within its interior. By the continuity of the foliation the same is true for each cross section $x_c =$ const., $x_h =$ const., $y_h =$ const. thorough $N^s_{\gamma\alpha}$, provided ρ_h and ρ_c are sufficiently small. Hence the hyperbolic directions of an appropriately high order iteration of $N^s_{\gamma\alpha}$ align with the hyperbolic directions of N_α. For the remaining directions we also use the Lambda Lemma and the continuity of the foliations of $N^s_{\gamma\alpha}$. The sections corresponding to $y_c =$

const. and y_h = const. through $N^s_{\gamma\alpha}$ are transverse to $W^s(T_\beta)$. The I-projection of the image of each cross section x_h = const., y_h = const., y_c = const. under f^q contains $\overline{B_{n_c}}(I_\alpha, \eta_\alpha)$, for sufficiently large q and sufficiently small ρ_c. By the ergodicity of the quasi-periodic motion on T_α, the θ-projection of the image of each cross section x_h = const., x_c = const., y_h = const. under f^q is contained in $\overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha})$, for sufficiently large q, sufficiently large $\kappa_{\gamma\alpha}$, and sufficiently small ρ_h and ρ_c. Intuitively, we have that the (x_h, x_c)-directions of $N^s_{\gamma\alpha}$ stretch across the (x, I)-directions of $M_{\gamma\alpha}$. Thus, the conditions of Proposition 5.6 are satisfied and so $N^s_{\gamma\alpha}$ is correctly aligned with $M_{\gamma\alpha}$ under f^q with degree ± 1, where q can be chosen arbitrarily large. The statement regarding $P_{\alpha\beta}$ and $N^u_{\alpha\beta}$ follows similarly. □

Let
$$\tilde{f}(\phi, \rho) = (\tilde{f}_\phi, \tilde{f}_\rho)(\phi, \rho) := (\phi + v(\rho), \rho) + r(\phi, 0, \rho, 0),$$
where $(\phi, \rho) \in V(T_\alpha) \cap W^c(p_\mu)$, $v(\rho) = \Omega_0 + \Omega_1(\rho) + \Omega_2(\rho, \rho)$, and $r(\phi, 0, \rho, 0)$ is of order 3 in ρ.

In the sequel, both f and $\tilde{f}$ will be considered as defined on the corresponding covering spaces. We emphasize that the theory of correctly aligned windows as exposed above works only in Euclidean spaces (or in homeomorphic copies of them).

We will proceed with constructing a chain of correctly aligned windows that links $M_{\gamma\alpha}$ to $P_{\alpha\beta}$. Before we do so, we need the following technical little lemma to be used for estimating the growth in size of these windows under iteration, in various directions.

Lemma 6.3. *Let x_n be the sequence of positive real numbers given by $x_{n+1} = x_n - b x_n^{a+1}$, with $a, b > 0$, $0 < x_0 < 1$ and $0 < b x_0^a < 1$. Then $x_n = O(n^{-1/a})$, $x_1^a + x_2^a + \ldots + x_n^a = O(\ln(n))$, $x_1^p + x_2^p + \ldots + x_n^p = O(n^{1-p/a})$ for $1 < a < p$, so $x_n \to 0$, $n x_n \to \infty$, and $n x_n / (x_1^p + x_2^p + \ldots + x_n^p) \to \infty$ provided $1 < a \leq p$, as $n \to \infty$.*

Proof. The asymptotic behavior of the successive iterates of the map $g(x) = x - b x^{a+1}$ is the same as of the solutions of the differential equation $dx/dt = -b x^{a+1}$, as $t \to \infty$. The general solution of this equation is given by

$$x(t) = [x_0^{-a} + bat]^{-1/a}. \tag{6.26}$$

To see that $x_n := g^n(x_0)$ has the same asymptotic behavior as of $x(n) = [x_0^{-a} + ban]^{-1/a}$, note that

$$\begin{aligned} g(x(n)) &= x(n)\left(1 - b\,(x(n))^a\right) = x(n)\left(1 - \frac{b}{x_0^{-a} + ban}\right) \\ &= x(n+1)\left(1 - \frac{ba}{x_0^{-a} + ba(n+1)}\right)^{-1/a}\left(1 - \frac{b}{x_0^{-a} + ban}\right) \\ &= x(n+1)\left[1 + O\left(\frac{1}{n^2}\right)\right]. \end{aligned}$$

From the form of the solution (6.26) of the differential equation, we have $t^{1/a} x(t) \to (ba)^{-1/a}$ as $t \to \infty$, thus $n^{1/a} x_n = n^{1/a} f^n(x_0) \to (ba)^{-1/a}$ as $n \to \infty$. The rest of the statement is a matter of basic calculus. □

Lemma 6.4. *Given $\epsilon > 0$, there exist $0 < \eta_\alpha < \epsilon$, $n \geq 1$ and a chain of windows*
$$M_{\gamma\alpha} = W_0, W_1, \ldots, W_n, W_{n+1} = P_{\alpha\beta},$$

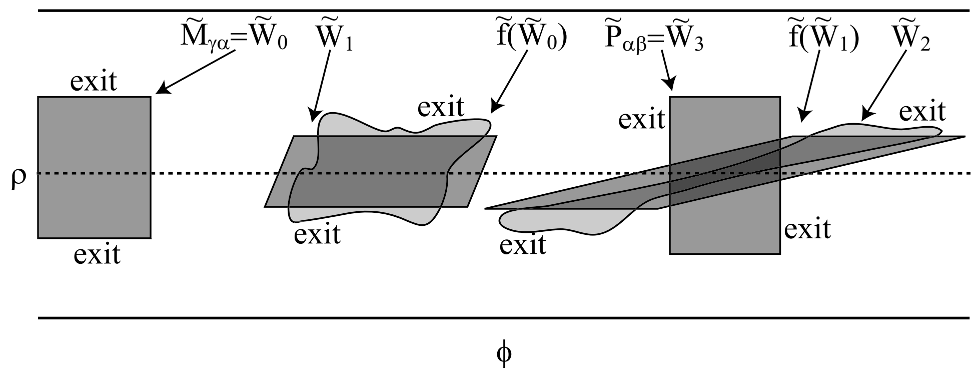

FIGURE 3. A chain of correctly aligned windows.

such that W_i correctly aligns with W_{i+1} under f, for all $i = 0, \ldots, n-1$, and W_n correctly aligns with W_{n+1} under the identity map.

Proof. Using the continuity of the foliations of $M_{\gamma\alpha}$ and of $P_{\alpha\beta}$, and the fact that their hyperbolic directions align correctly under iteration, what we really need to prove is that correct alignment along the center directions can be achieved.

We claim that there exist $0 < \eta_\alpha < \epsilon$ and a chain of windows in $W^c(p_\mu)$,

$$\tilde{M}_{\gamma\alpha} = \tilde{W}_0, \tilde{W}_1, \ldots, \tilde{W}_n, \tilde{W}_{n+1} = \tilde{P}_{\alpha\beta},$$

such that $\tilde{W}_i$ correctly aligns with $\tilde{W}_{i+1}$ under $\tilde{f}$, for all $i = 0, \ldots, n-1$, and $\tilde{W}_n$ correctly aligns with $\tilde{W}_{n+1}$ under the identity map. We construct the chain $\{\tilde{W}_i\}$ inductively.

There exist $C > 0$ such that

$$\|\tilde{f}(\phi, \rho) - (\phi + \Omega_0 + \Omega_1(\rho) + \Omega_2(\rho, \rho), \rho)\| < C\|\rho\|^3, \tag{6.27}$$

within some compact neighborhood of the torus relative to $W^c(p_\mu)$.

Let $r = \min_{\|\rho\|=1} \|\Omega_1(\rho)\| > 0$. We have that $\|\Omega_1(\rho)\| \geq r\|\rho\|$ for every ρ and that the image through Ω_1 of a ball of radius η contains a ball of radius $r\eta$, for every $\eta > 0$. Let $\|\Omega_2\| = \sup_{\|\rho\|=1} \|\Omega_2(\rho, \rho)\|$. Choose $\eta_\alpha > 0$ sufficiently small such that the following two conditions hold

$$\|\Omega_2\| \|\eta_\alpha\| < r/8, \tag{6.28}$$

$$C\|\eta_\alpha\|^2 < r/8. \tag{6.29}$$

The image of each 'vertical' disk $\{\phi\} \times \overline{B_{n_c}}(I_\alpha, \eta_\alpha)$ through the normal form of $\tilde{f}$ is a 'tilted' topological disk $\{(\phi + v(\rho), \rho) \,|\, \rho \in \overline{B_{n_c}}(I_\alpha, \eta_\alpha)\}$, with its 'center' (the image of the center of the original disk through the normal form) contained in $\{\rho = I_\alpha\}$. The projection of this tilted disk into the $\{\rho = I_\alpha\}$-parameter space is the topological disk $\{\phi + v(\rho) \,|\, \rho \in \overline{B_{n_c}}(I_\alpha, \eta_\alpha)\}$. The image of each 'horizontal' disk $\overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha}) \times \{\rho\}$ through the normal form is a horizonal disk of radius $\kappa_{\gamma\alpha}$.

By (6.27), the 'true' image of a vertical disk $\{\phi\} \times \overline{B_{n_c}}(0, \eta_\alpha)$ through $\tilde{f}$ is a topological disk whose center and boundary are each displaced by at most $C\|\eta_\alpha\|^3$

from the ones corresponding to the topological disk generated by the normal form. The projection of this topological disk into the $\{\rho = I_\alpha\}$-parameter space contains a disk of radius

$$\frac{r}{2}\|\eta_\alpha\| < \|\Omega_1(\eta_\alpha)\| - 2\|\Omega_2\|\|\eta_\alpha\|^2 - 2C\|\eta_\alpha\|^3.$$

The projection of the 'true' image of a horizontal disk $\overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha}) \times \{\rho\}$ through $\tilde{f}$ in the $\{\rho = I_\alpha\}$-parameter space is contained in a disk centered at $\theta_{\gamma\alpha} + \Omega_0 + \Omega_1(\rho)$, of radius $\kappa_{\gamma\alpha} + 2\|\Omega_2\|\|\eta_\alpha\|^2 + 2C\|\eta_\alpha\|^3$.

Set $x_0 = \|\eta_\alpha\|$ and $x_1 = \|\eta_\alpha\| - 2C\|\eta_\alpha\|^3$. We define the window $\tilde{W}_1$ by

$$\tilde{W}_1 = \bigcup_{\phi \in \overline{B_{n_c}}(\theta_{\gamma\alpha}+\Omega_0, \kappa_{\gamma\alpha}+2\|\Omega_2\|x_0^2+2Cx_0^3)} \{(\phi + \Omega_1(\rho), \rho) \mid \rho \in \overline{B_{n_c}}(I_\alpha, x_1)\}, \tag{6.30}$$

$$\tilde{W}_1^- = \bigcup_{\phi \in \overline{B_{n_c}}(\theta_{\gamma\alpha}+\Omega_0, \kappa_{\gamma\alpha}+2\|\Omega_2\|x_0^2+2Cx_0^3)} \{(\phi + \Omega_1(\rho), \rho) \mid \rho \in \partial\overline{B_{n_c}}(I_\alpha, x_1)\}, \tag{6.31}$$

$$\tilde{W}_1^+ = \bigcup_{\phi \in \partial\overline{B_{n_c}}(\theta_{\gamma\alpha}+\Omega_0, \kappa_{\gamma\alpha}+2\|\Omega_2\|x_0^2+2Cx_0^3)} \{(\phi + \Omega_1(\rho), \rho) \mid \rho \in \overline{B_{n_c}}(I_\alpha, x_1)\}. \tag{6.32}$$

Loosely speaking, $\tilde{W}_1$ is a 'parallelogram' of 'width'

$$\overline{B_{n_c}}(\theta_{\gamma\alpha} + \Omega_0 + \Omega_1(I_\alpha), \kappa_{\gamma\alpha} + 2\|\Omega_2\|x_0^2 + 2Cx_0^3),$$

and of 'height'

$$\overline{B_{n_c}}(I_\alpha, x_1).$$

Compared to $\tilde{W}_0$, the base of $\tilde{W}_1$ has been enlarged by a quantity of $2\|\Omega_2\|x_0^2 + 2Cx_0^3$ in all directions, while the height of $\tilde{W}_1$ has been shortened by a quantity of $x_0 - 2Cx_0^3$ in all directions. With these choices, we have that $\tilde{f}(\tilde{W}_0^-) \cap \tilde{W}_1 = \emptyset$ and $\tilde{f}(\tilde{W}_0) \cap \tilde{W}_1^+ = \emptyset$. Since $\tilde{f}$ is a diffeomorphism, the projection A into the $\{\rho = I_\alpha\}$-parameter space maps the topological disk $\tilde{f}\left(\overline{B_{n_c}}(\theta_{\gamma\alpha}, \kappa_{\gamma\alpha}) \times \{I_\alpha\}\right)$ within the disk $\overline{B_{n_c}}(\theta_{\gamma\alpha} + \Omega_0 + \Omega_1(I_\alpha), \kappa_{\gamma\alpha} + 2\|\Omega_2\|x_0^2 + 2Cx_0^3)$, with a degree

$$ds_A = \pm 1,$$

where $A = \pi_{\{\rho = I_\alpha\}} \circ \tilde{f}_c$. This and Proposition 5.6 show that $\tilde{W}_0$ and $\tilde{W}_1$ are forward correctly aligned under $\tilde{f}$.

Now we define the sequence $\tilde{W}_i$ inductively. Define the sequence x_i by the recurrent relation $x_{i+1} = x_i - 2Cx_i^3$. Define the sequence κ_i by $\kappa_0 = \kappa_{\gamma\alpha}$ and $\kappa_{i+1} = \kappa_i + 2\|\Omega_2\|x_i^2 + 2Cx_i^3$, that is $\kappa_{i+1} = \kappa_{\gamma\alpha} + 2\|\Omega_2\|(x_0^2 + x_1^2 + \ldots + x_i^2) + 2C(x_0^3 + x_1^3 + \ldots + x_i^3)$.

Assume that $\tilde{W}_i$, has already been constructed by

$$\tilde{W}_i = \bigcup_{\phi \in \overline{B_{n_c}}(\theta_{\gamma\alpha}+i\Omega_0, \kappa_i)} \{(\phi + i\Omega_1(\rho), \rho) \mid \rho \in \overline{B_{n_c}}(I_\alpha, x_i)\}, \tag{6.33}$$

$$\tilde{W}_i^- = \bigcup_{\phi \in \overline{B_{n_c}}(\theta_{\gamma\alpha}+i\Omega_0, \kappa_i)} \{(\phi + i\Omega_1(\rho), \rho) \mid \rho \in \partial\overline{B_{n_c}}(I_\alpha, x_i)\}, \tag{6.34}$$

$$\tilde{W}_i^+ = \bigcup_{\phi \in \partial\overline{B_{n_c}}(\theta_{\gamma\alpha}+i\Omega_0, \kappa_i)} \{(\phi + i\Omega_1(\rho), \rho) \mid \rho \in \overline{B_{n_c}}(I_\alpha, x_i)\}. \tag{6.35}$$

Thus $\tilde{W}_i$ is foliated by horizontal disks $\overline{B_{n_c}}(\theta_{\gamma\alpha} + i\Omega_0 + i\Omega_1(\rho), \kappa_i) \times \{\rho\}$, with $\rho \in \overline{B_{n_c}}(I_\alpha, x_i)$. The image of a disk $\overline{B_{n_c}}(\theta_{\gamma\alpha} + i\Omega_0 + i\Omega_1(\rho), \kappa_i) \times \{\rho\}$, under the normal form of $\tilde{f}$ is a disk $\overline{B_{n_c}}(\theta_{\gamma\alpha}+(i+1)\Omega_0+(i+1)\Omega_1(\rho)+\Omega_2(\rho,\rho),\kappa_i)\times\{\rho\}$, which is contained in the disk $\overline{B_{n_c}}(\theta_{\gamma\alpha}+(i+1)\Omega_0+(i+1)\Omega_1(\rho),\kappa_i+2\|\Omega_2\|x_i^2)\times\{\rho\}$. So the projection into the $\{\rho=I_\alpha\}$-parameter space maps the image of the original horizontal disk under $\tilde{f}$ is within the disk

$$\overline{B_{n_c}}(\theta_{\gamma\alpha}+(i+1)\Omega_0+(i+1)\Omega_1(\rho),\kappa_i+2\|\Omega_2\|x_i^2+2Cx_i^3)= \\ =\overline{B_{n_c}}(\theta_{\gamma\alpha}+(i+1)\Omega_0+(i+1)\Omega_1(\rho),\kappa_{i+1}).$$

The projection into the $\{\phi=0\}$-parameter space maps the image of $\tilde{W}_i$ under $\tilde{f}$ onto a topological disk that contains $\overline{B_{n_c}}(I_\alpha,x_{i+1})$ inside it. These facts and Proposition 5.6 show that $\tilde{W}_i$ correctly aligns with $\tilde{W}_{i+1}$ under $\tilde{f}$, with degree $ds_A=\pm1$, where $A=\pi_{\{\rho=I_\alpha\}}\circ\tilde{f}_c$.

Now we want to show that for some large enough n, $\tilde{W}_n$ correctly aligns with $\tilde{P}_{\alpha\beta}$ under the identity mapping. That is, we need to show that $\tilde{W}_n$ stretches across $\tilde{W}_{\alpha\beta}$, in a manner that is correctly aligned with respect to the exit sets of the two windows. Notice that during the inductive construction, the windows $\tilde{W}_i$ become 'shorter' and 'more and more sheared'. Using the estimates from Lemma 6.3, we will show that the 'shearing effect' eventually overcomes the 'shortening effect'.

The projection into the $\{\phi=0\}$-parameter space maps the window $\tilde{W}_n$ onto $\overline{B_{n_c}}(I_\alpha,x_n)$. Applying Lemma 6.3 for $a=3$ and $b=2C$, we have $x_n\to0$, so, for large enough n, we have $\overline{B_{n_c}}(I_\alpha,x_n)\subseteq\overline{B_{n_c}}(I_\alpha,\eta_\alpha)$, where the latter disk represents the projection of $\tilde{P}_{\alpha\beta}$ into the $\{\phi=0\}$-parameter space.

The window $\tilde{W}_{n+1}$ is foliated by 'slanted' disks $\{(\phi+n\Omega_1(\rho),\rho)\,|\,\rho\in\overline{B_{n_c}}(I_\alpha,x_n)\}$, where $\phi\in\overline{B_{n_c}}(\theta_{\gamma\alpha}+n\Omega_0,\kappa_n)$. The projection into the $\{\rho=I_\alpha\}$-parameter space maps such a 'slanted' disk onto the set $\{\theta_{\gamma\alpha}+\phi+n\Omega_0+n\Omega_1(\rho)\,|\,\rho\in\overline{B_{n_c}}(I_\alpha,x_n)\}$, which contains the disk $\overline{B_{n_c}}(\theta_{\gamma\alpha}+n\Omega_0,rnx_n-\kappa_n)$, provided that $rnx_n-\kappa_n>0$. This fact follows from our earlier choice of r and the fact that the range of the parameter ϕ is a disk of radius κ_n. For any given $\kappa>0$, there exist sufficiently large n, such that, according to Lemma 6.3 applied for $a=2$, $b=2C$ and $p=2,3$, we have

$$rnx_n-2\|\Omega_2\|(x_0^2+x_1^2+\ldots+x_{n-1}^2)-2C(x_0^3+x_1^3+\ldots+x_{n-1}^3)>\kappa. \tag{6.36}$$

Choose and fix $\kappa=\kappa_{\gamma\alpha}+\kappa_{\alpha\beta}$. For all sufficiently large n, we then have $rnx_n-\kappa_n>\kappa_{\alpha\beta}$. Using the ergodicity of the quasi-periodic motion on T_α, there exists such a large n such that the disk $\overline{B_{n_c}}(\theta_{\gamma\alpha}+n\Omega_0,rnx_n-\kappa_n)$ contains the disk $\overline{B_{n_c}}(\theta_{\alpha\beta},\kappa_{\alpha\beta})$. This shows that $\tilde{W}_n^-\cap\tilde{P}_{\alpha\beta}=\emptyset$ and $\tilde{W}_n\cap\tilde{P}_{\alpha\beta}^+=\emptyset$. For any choice of such a 'slanted' disk, the degree of the mapping $A=\pi_{\{\rho=I_\alpha\}}\circ\mathrm{id}_c$, defined by the identity mapping as in Proposition 5.6, is well defined and equal to ±1. Thus $\tilde{W}_n$ correctly aligns with $\tilde{W}_{n+1}$ under the identity mapping. □

Remark 6.5. The above proof outlines a geometrical method of controlling the error in approximating f (or $\tilde{f}$ rather) by a normal form during an iterative process.

The following lemma has been proved in [11].

Lemma 6.6. *Let $T_1,T_2,\ldots,T_s$ be a family of n-dimensional tori. For each $i=1,\ldots,s$, let $\tau_i:T_i\to T_i$, $i=1,\ldots,s$ be a translation by irrational angles ω_i^j ($j=1,\ldots,n$) in each dimension, with all angular frequencies $\omega_i^1,\ldots,\omega_i^n$ independent over the integers. Assume that the angular frequency vectors $\Omega_i:=(\omega_i^1,\ldots,\omega_i^n)$, $i=1,\ldots,s$, are linearly independent over the integers. Let p_i, p_i' be a fixed pair of* *points on T_i, for each $i = 1, \ldots, s$. Then, for every $\epsilon > 0$ and every integer $h_0 > 0$, there exists an integer $h > h_0$ such that $d(\tau_i^h p_i, p_i') < \epsilon$ for all $i = 1, \ldots, s$.*

Remark 6.7. The constructions and the lemmas in this section refer to forward correctly aligned windows. Similar constructions and results are valid when we consider backward correct alignment.

7. Proof of Theorem 1.1.

Let μ be fixed, $\{T_{\mu,\alpha_i}\}_{i\in\mathbb{Z}}$ be a bi-infinite sequence sequence of transition tori, and ϵ_i be a bi-infinite sequence of positive reals. We start with the topological crossing intersection of $W^u(T_{\mu,\alpha_0})$ and $W^s(T_{\mu,\alpha_1})$ and we first construct $N^u_{\alpha_0\alpha_1}$ is correctly aligned with $N^s_{\alpha_0\alpha_1}$ under the identity map, as in Lemma 6.1. Hence the windows $f_\mu^{-p}(N^u_{\alpha_0\alpha_1})$ and $f_\mu^q(N^s_{\alpha_0\alpha_1})$ are near T_{μ,α_0} and T_{μ,α_1}, respectively. At this point we have some initial choices for the positive reals $\rho_h(N^s_{\alpha_0\alpha_1})$, $\rho_c(N^s_{\alpha_0\alpha_1})$, $\kappa_h(N^s_{\alpha_0\alpha_1})$, $\kappa_c(N^s_{\alpha_0\alpha_1})$, $\rho'_h(N^u_{\alpha_0\alpha_1})$, $\rho'_c(N^u_{\alpha_0\alpha_1})$, $\kappa'_h(N^u_{\alpha_0\alpha_1})$, $\kappa'_c(N^u_{\alpha_0\alpha_1})$.

We continue with constructing correctly aligned windows about the topological crossing intersection of $W^u(T_{\mu,\alpha_1})$ and $W^s(T_{\mu,\alpha_2})$. Similarly, we have that $f_\mu^{-p'}(N^u_{\alpha_1\alpha_2})$ and $f^{q'}(N^u\alpha_1\alpha_2)$ are near T_{μ,α_1} and T_{μ,α_2}, respectively. We have also some initial choices for the positive reals $\rho_h(N^s_{\alpha_1\alpha_2})$, $\rho_c(N^s_{\alpha_1\alpha_2})$, $\kappa_h(N^s_{\alpha_1\alpha_2})$, $\kappa_c(N^s_{\alpha_1\alpha_1})$, $\rho'_h(N^u_{\alpha_1\alpha_2})$, $\rho'_c(N^u_{\alpha_1\alpha_2})$, $\kappa'_h(N^u_{\alpha_1\alpha_2})$, $\kappa'_c(N^u_{\alpha_1\alpha_2})$.

The windows $f_\mu^q(N^s_{\alpha_0\alpha_1})$ and $f_\mu^{-p'}(N^u_{\alpha_1\alpha_2})$ are both near the torus T_{μ,α_1}. Now we construct $M_{\alpha_0\alpha_1}$ and $P_{\alpha_1\alpha_2}$ near T_{μ,α_1} as follows

- Choose and fix $\eta_1 < \epsilon_1$ sufficiently small such that there exists a chain of correctly aligned widows under f_μ linking $M_{\alpha_0\alpha_1}$ and $P_{\alpha_1\alpha_2}$, as in Lemma 6.4.
- Make $\rho_c(N^s_{\alpha_0\alpha_1})$, $\rho_h(N^s_{\alpha_0\alpha_1})$ smaller, if necessary, so that $N^s_{\alpha_0\alpha_1}$ is correctly aligned with $M_{\alpha_0\alpha_1}$ under f_μ^q, as in Lemma 6.2. We emphasize that making these quantities smaller does not affect the correct alignment of $f_\mu^{-p}(N^u_{\alpha_0\alpha_1})$ and $f_\mu^q(N^s_{\alpha_0\alpha_1})$ as established above.
- Make $\rho'_c(N^u_{\alpha_1\alpha_2})$, $\rho'_h(N^u_{\alpha_1\alpha_2})$ smaller, if necessary, so that $P_{\alpha_1\alpha_2}$ is correctly aligned with $N^s_{\alpha_1\alpha_2}$ under $f_\mu^{p'}$, as in Lemma 6.2. We emphasize that making these quantities smaller does not affect the correct alignment of $f_\mu^{q'}(N^s_{\alpha_1\alpha_2})$ and $f_\mu^{-p'}(N^u_{\alpha_1\alpha_2})$ as established above.

Then we focus on the topological crossing intersection of the invariant manifolds $W^u(T_{\mu,\alpha_{-1}})$ and $W^s(T_{\mu,\alpha_0})$. As pointed out in Remark 6.7, analogue constructions can be made with respect to backward correct alignment. So one obtains $N^s_{\alpha_{-1}\alpha_0}$ correctly aligned with $M_{\alpha_{-1}\alpha_0}$ under $f^{q''}$, $M_{\alpha_{-1}\alpha_0}$ linked by a chain of correctly aligned windows under f with $P_{\alpha_0\alpha_1}$, and $P_{\alpha_0\alpha_1}$ correctly aligned with $N^u_{\alpha_0\alpha 1}$ under $f^{p''}$. For this to happen, we may need to make $\rho'_c(N^u_{\alpha_0\alpha 1})$ and $\rho'_h(N^u_{\alpha_0\alpha 1})$ even smaller, which does not affect the correct alignment of $f_\mu^{-p}(N^u_{\alpha_0\alpha_1})$ and $f_\mu^q(N^s_{\alpha_0\alpha_1})$ as established above.

We continue this construction inductively. All ρ_h's, ρ_c's, ρ'_h's, ρ'_c's chosen at previous steps may need to be made simultaneously even smaller in order to pass to the next step. We end up with a bi-infinite sequence of widows which are correctly aligned under various powers of f. This sequence contains windows like $M_{\alpha_{i-1}\alpha_i}$ or $P_{\alpha_i\alpha_{i+1}}$, with all their points within an ϵ_i-distance from T_{μ,α_i}. Thus, by Theorem 5.8, there is an orbit z_i with $d(z_i, T_{\mu,\alpha_i}) < \epsilon_i$ and $z_{i+1} = f_\mu^{n_i}(z_i)$, for some $n_i > 0$. This ends the proof. □

Remark 7.1. In Theorem 1.1 there is no restriction on the sequence $\{\epsilon_i\}_i$, so ϵ_i may tend to zero at any speed. This will result in a construction of a sequence of windows which shrink about the tori with the speed of the convergence of ϵ_i. At this point, we do not claim any stability result. In order to ensure the stability of the shadowing orbit under small perturbations, one needs to require that the sequence of ratios $\epsilon_i/\epsilon_{i+1}$ is bounded away from zero and above. Justification for this restriction is provided in [7].

8. Proof of Theorem 1.2.

Let us fix μ. Consider a finite collection of tori $T_{\mu,\alpha_1},\ldots,T_{\mu,\alpha_d}$ and let $N>0$ and $\epsilon>0$. At each topological crossing intersection of $W^u(T_{\mu,\alpha_k})$ with $W^s(T_{\mu,\alpha_i})$, we construct windows $N^u_{\alpha_k\alpha_i}$ correctly aligned with $N^s_{\alpha_k\alpha_i}$ under the identity mapping, as in Lemma 6.1. There exist positive integers $p=q>N/2$ such that $f^q_\mu(N^s_{\alpha_k\alpha_i})$ and $f^{-p}_\mu(N^u_{\alpha_i\alpha_j})$ are contained in an $\epsilon/2$-neighborhood of $W^c(p_\mu)$, for each i and all $k,j\in\{1,\ldots,d\}$. There exist $p=q>N/2$ and a family of KAM tori T_{μ,α'_i}, $i=1,\ldots,d$, that is $d(I_{\alpha_i},I_{\alpha'_i})<\epsilon/2$ for all i, satisfying the non-resonance condition from Lemma 6.6, such that the following conditions hold

- in each cross section through $f^q_\mu(N^s_{\alpha_k\alpha_i})$ parallel to the center manifold $W^c(p_\mu)$, all points with $\rho=I_{\alpha'_i}$ are either in the interior of the cross section, or on the portion of the cross section that is part of the entry set of $f^q_\mu(N^s_{\alpha_k\alpha_i})$,
- in each cross section through $f^{-p}_\mu(N^u_{\alpha_i\alpha_j})$ parallel to the center manifold $W^c(p_\mu)$, all points with $\rho=I_{\alpha'_i}$ are either in the interior of the cross section, or on the portion of the cross section which is part of the exit set of $f^{-p}_\mu(N^u_{\alpha_i\alpha_j})$,

for all i. Such choices are possible since the KAM tori form a perfect set and since $f^q(N^s_{\alpha_k\alpha_i})$ and $f^{-p}(N^u_{\alpha_i\alpha_j})$ approach T_{μ,α_i} in a manner as described by Lemma 6.2. Then we construct the windows $M_{\alpha'_i}$, $P_{\alpha'_i}$ about the torus T_{μ,α'_i}, with

$$M_{\alpha'_i}= \overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\times\overline{B_{n_h}}(0,\eta), \tag{8.1}$$

$$M^-_{\alpha'_i}= \overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\partial\left(\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\right)\times\overline{B_{n_h}}(0,\eta), \tag{8.2}$$

$$\begin{aligned} M^+_{\alpha'_i}= {}& \partial\overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\times\overline{B_{n_h}}(0,\eta)\cup\\ &\cup\overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\times\partial\overline{B_{n_h}}(0,\eta), \end{aligned} \tag{8.3}$$

and

$$P_{\alpha'_i}= \overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\times\overline{B_{n_h}}(0,\eta), \tag{8.4}$$

$$\begin{aligned} P^-_{\alpha'_i}= {}& \partial\overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\times\overline{B_{n_h}}(0,\eta)\cup\\ &\cup\overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\times\partial\overline{B_{n_h}}(0,\eta), \end{aligned} \tag{8.5}$$

$$P^+_{\alpha'_i}= \overline{B_{n_c}}(\theta_{\alpha'_i},\kappa_{\alpha'_i})\times\partial\left(\overline{B_{n_h}}(0,\eta)\times\overline{B_{n_c}}(I_{\alpha'_i},\eta)\right)\times\overline{B_{n_h}}(0,\eta), \tag{8.6}$$

with $0<\eta<\epsilon$, $0<\kappa_{\alpha'_i}$, such that

(i) Each $f^q_\mu(N^s_{\alpha_k\alpha_i})$ is correctly aligned with $M_{\alpha'_i}$ under the identity map, for all k,
(ii) $P_{\alpha'_i}$ is correctly aligned with each $f^{-p}_\mu(N^u_{\alpha_i\alpha_j})$ under the identity map, for all j.

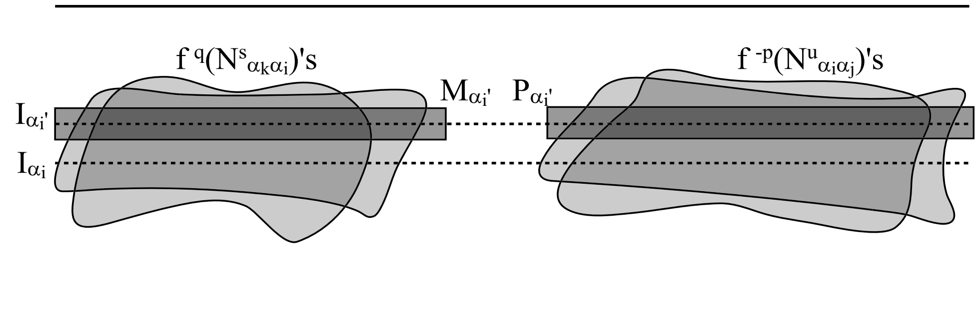

Figure 4. Correctly aligned windows along the non-resonant tori (the hyperbolic directions are ignored in this figure).

See Figure 4. Of course that the numbers $\kappa_{\alpha_i'}$ may be quite large, which means that $M_{\alpha_i'}$ and $P_{\alpha_i'}$ may wrap around the torus multiple times, but this is alright, since, again, we check the correct alignment of windows in the corresponding covering space. By Lemma 6.4 each $M_{\alpha_i'}$ is linked with $P_{\alpha_i'}$ by a chain of correctly aligned windows under f. By Lemma 6.6, the length of this chain can be uniformly chosen equal to some $r > 0$.

Using Corollary 5.9 and Remark 5.10 applied to the windows $\{M_{\alpha_i'}\}_{i=1,\dots,d}$ and the mapping f_μ^{p+q+r}, there exists a compact set S_μ invariant to $f_\mu^n := f_\mu^{p+q+r}$, which is semi-conjugate to the full shift on d symbols.

Since we deal with a finite number of windows, we can apply Theorem 5.7 to conclude that the symbolic dynamics is stable under small perturbations. Since the maximal invariant set $S_{\mu'}$ varies continuously with respect to μ', the semi-conjugacy $\rho_{\mu'}$ can be chosen to depend continuously on μ' with $|\mu' - \mu| < \nu$, for some small $\nu > 0$. □

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