Brownian Directed Polymers in Random Environment[1]

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Abstract

We study the thermodynamics of a continuous model of directed polymers in random environment. The environment is given by a space-time Poisson point process, whereas the polymer is defined in terms of the Brownian motion. We mainly discuss: (i) The normalized partition function, its positivity in the limit which characterizes the phase diagram of the model. (ii) The existence of quenched Lyapunov exponent, its positivity, and its agreement with the annealed Lyapunov exponent; (iii) The longitudinal fluctuation of the free energy, some of its relations with the overlap between replicas and with the transversal fluctuation of the path.

The model considered here, enables us to use stochastic calculus, with respect to both Brownian motion and Poisson process, leading to handy formulas for fluctuations analysis and qualitative properties of the phase diagram. We also relate our model to some formulation of the Kardar-Parisi-Zhang equation, more precisely, the stochastic heat equation. Our fluctuation results are interpreted as bounds on various exponents and provide a circumstantial evidence of super-diffusivity in dimension one. We also obtain an almost sure large deviation principle for the polymer measure.

Short Title: Brownian Polymers
Key words and phrases: Directed polymers, Kardar-Parisi-Zhang equation, random environment, stochastic calculus
AMS 1991 subject classifications: Primary 60K37; secondary 60Hxx, 82A51, 82D30

Contents

[1] draft March 4, 2003
[2] Partially supported by CNRS (UMR 7599 Probabilités et Modèles Aléatoires)
[3] Partially supported by JSPS Grant-in-Aid for Scientific Research, Wakatekenkyuu (B) 14740071

1 Introduction

Directed polymers in random environment can be thought of as paths of stochastic processes interacting with a quenched disorder (impurities), depending on both time and space. Roughly, individual paths are not only weighted according to their length, but also according to random impurities that they meet along their route, with a larger influence as the temperature is decreased. A physical example is the interface in the 2-dimensional Ising model with random bonds [12], within the Solid-On-Solid approximation – where the interface can be parametrized by one coordinate –. The heuristic picture at low temperature is that, typical paths are pinned down to clouds of favourable impurities. With the relevant clouds located at a large distance, the polymer behaves superdiffusively. Similarly, the free energy essentially depends on the characteristics of the relevant clouds, exhibits large fluctuations, as well as other thermodynamic quantities. Directed polymers in random environment, at positive or zero temperature, relate – even better, can sometimes be exactly mapped – to a number of interesting models of growing random surfaces (directed invasion percolation, ballistic deposition, polynuclear growth, low temperature Ising models), and non equilibrium dynamics (totally asymmetric simple exclusion, population dynamics in random environment); We refer to the survey paper [17] by Krug and Spohn for detailed account on these models and their relations.

On the other hand, stochastic calculus provides a number of natural tools for studying random processes and their fluctuations. In this paper, we introduce and study a model which allows using such tools as Doob-Meyer's martingale decomposition and Itô's formula.

1.1 The Brownian directed polymers in random environment

The model we consider in this paper is defined in terms of Brownian motion and of a Poisson random measure. Before introducing the polymer measure, we first fix some notations. In what follows, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$, d denotes a positive integer and $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ the class of Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$.

• *The Brownian motion:* Let $(\{\omega_t\}_{t\geq 0}, \{P^x\}_{x\in\mathbb{R}^d})$ denote a d-dimensional standard Brownian motion. Specifically, we let the measurable space $(\Omega, \mathcal{F})$ be the path space $C(\mathbb{R}_+ \to \mathbb{R}^d)$ with the cylindrical σ-field, and P^x be the Wiener measure on $(\Omega, \mathcal{F})$ such that $P^x\{\omega_0 = x\} = 1$.

• *The space-time Poisson random measure:* Let η denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with the unit intensity, defined on a probability space $(\mathcal{M}, \mathcal{G}, Q)$. Then, η is an integer valued random measure characterized by the following property: If $A_1, ..., A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ are disjoint and bounded, then

$$Q\left(\bigcap_{j=1}^{n}\{\eta(A_j) = k_j\}\right) = \prod_{j=1}^{n} \exp(-|A_j|)\frac{|A_j|^{k_j}}{k_j!} \quad \text{for } k_1, ..., k_n \in \mathbb{N}. \tag{1.1}$$

Here, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^{1+d}$. For $t > 0$, it is natural and convenient to introduce

$$\eta_t(A) = \eta(A \cap ((0, t] \times \mathbb{R}^d)) \,, \quad A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d) \tag{1.2}$$

and the sub σ-field

$$\mathcal{G}_t = \sigma[\eta_t(A) \,;\, A \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)] \,. \tag{1.3}$$

• *The polymer measure:* We let V_t denote a "tube"around the graph $\{(s, \omega_s)\}_{0<s\leq t}$ of the Brownian path,

$$V_t = V_t(\omega) = \{(s, x) \,;\, s \in (0, t],\ x \in U(\omega_s)\}, \tag{1.4}$$

where $U(x) \subset \mathbb{R}^d$ is the closed ball with the unit volume, centered at $x \in \mathbb{R}^d$. For any $t > 0$ and $x \in \mathbb{R}^d$, define a probability measure μ_t^x on the path space $(\Omega, \mathcal{F})$

$$\mu_t^x(d\omega) = \frac{\exp(\beta\eta(V_t))}{Z_t^x} P^x(d\omega), \tag{1.5}$$

where $\beta \in \mathbb{R}$ is a parameter and

$$Z_t^x = P^x[\exp(\beta\eta(V_t))] \,. \tag{1.6}$$

Under the measure μ_t^x, the graph $\{(s, \omega_s)\}_{0\leq s\leq t}$ may be interpreted as a polymer chain living in the $(1 + d)$-dimensional space, constrained to stretch in the direction of the first coordinate (t-axis). At the heuristic level, the polymer measure is governed by the formal Hamiltonian

$$\beta\mathbf{H}_t^\eta(\omega) = \frac{1}{2}\int_0^t |\dot{\omega}_s|^2 ds - \beta\,\#\Big\{\text{points } (s, x) \text{ in } \eta : s \leq t, x \in U(\omega_s)\Big\} \tag{1.7}$$

on the path space. Since this Hamiltonian is parametrized by η, the polymer measure μ_t^x is random. The path ω is attracted to Poisson points when $\beta > 0$, and repelled by them when $\beta < 0$. The sets $\{s\} \times U(x)$ with (s, x) a point of the Poisson field η, appear as "rewards" in the first case, and "soft obstacles" in the second one. Note that the obstacles stretches in the transverse direction (x-hyperplane): This is a key technical point, allowing a simple use of stochastic calculus with respect to the Poisson field.

In this paper, we address the question of understanding the large time behavior of the transversal motion $(\omega_t)_{t\geq 0}$ under the polymer measures $(\mu_t^x)_{t\geq 0}$, in particular, how its fluctuation in large time scale is affected by the random environment η. As is the general rule in statistical mechanics, much information will be obtained by investigating the asymptotic behavior of the partition function Z_t^x.

Let us finish the definition of the model with some remarks on the notation we use. An important parameter is

$$\lambda = \lambda(\beta) = e^\beta - 1 \in (-1, \infty) , \tag{1.8}$$

which is in fact the logarithmic moment generating function of a mean-one Poisson distribution. When we want to stress the dependence of λ on $\beta \in \mathbb{R}$, we will use the notation $\lambda(\beta)$. But otherwise, we will simply write λ. We will denote by $P, \mu_t, Z_t, \cdots$, the quantities $P^x, \mu_t^x, Z_t^x, \cdots$ with $x = 0$. Note that $(Z_t^x)_{t\geq 0}$ has the same distribution as $(Z_t)_{t\geq 0}$. For this reason, we will state and prove results on $(Z_t^x)_{t\geq 0}$ only for the case $x = 0$ with simpler notation, however without loss of generality.

1.2 Connection to the Kardar-Parisi-Zhang equation

Another strong motivation for the present model is its relation to some *stochastic partial differential equations*. To describe the connection, it is necessary to relativize the partition function, by specifying the ending point of the Brownian motion at time t. For $0 \leq s < t$, let $P_{s\to t}^{x\to y}$ be the distribution of the Brownian bridge starting at point x at time s and ending at y at time t. Define

$$Z_t^x(y) = g_t(y - x) P_{0\to t}^{x\to y}[\exp(\beta\eta(V_t))] , \tag{1.9}$$

with $g_t(x) = (2\pi t)^{-d/2} \exp\{-|x|^2/2t\}$ the Gaussian density. Then, by definition of the Brownian bridge,

$$Z_t^x = \int_{\mathbb{R}^d} Z_t^x(y) dy .$$

Similar to the Feynman-Kac formula, we will show the following stochastic heat equation (SHE) with multiplicative noise in a weak sense,

$$dZ_t^x(y) = \frac{1}{2}\Delta_y Z_t^x(y) dt + \lambda Z_{t-}^x(y)\eta(dt \times U(y)) , \quad t \geq 0, x, y \in \mathbb{R}^d , \tag{1.10}$$

where $dZ_t^x(y)$ denotes the time differential and $\Delta_y = (\frac{\partial}{\partial y^1})^2 + ... + (\frac{\partial}{\partial y^d})^2$ the Laplacian operator.

(SHE) will be properly formulated and be proved in section 10. In the literature, this equation has been extensively considered in the case of a Gaussian driving noise, instead of the Poisson process η here. Although we are able to prove (1.10) only in the weak sense, let us now pretend that (1.10) is true for all $y \in \mathbb{R}^d$. We would then see from Itô's formula that the function $h_t(y) = \ln Z_t^x(y)$ solves the Kardar-Parisi-Zhang equation (KPZ):

$$dh_t(y) = \frac{1}{2}\left(\Delta h_t(y) + |\nabla h_t(y)|^2\right) dt + \beta\, \eta(dt \times U(y)) .$$

We observe that, since h has jumps in the space variable y, the non-linearity makes the precise meaning of this equation somewhat knotty. We will not address this equation in the present paper, but we make a few comments. This equation was introduced in [15] to describe the long scale behavior of growing interfaces. More precisely, the fluctuations in the KPZ equation –driven by a δ-correlated, Gaussian noise–, are believed to be non standard, and universal, i.e., the same as in a large class of microscopic models. See [17] for a detailed review of kinetic roughening of growth models within the physics literature, in particular to Section 5 for the status of this equation. In dimension $d = 1$, Bertini and Giacomin [1] proved that the KPZ equation comes as the limit of renormalized fluctuations for two microscopic models: the weakly asymmetric exclusion process, and the related Solid-On-Solid interface model.

1.3 Other related models

The model we introduce in this paper has a number of close relatives in the literature. We now mention some of them.

• *Simple random walk model of directed polymers:* This model was originally introduced in physics literature [12] to mimic the phase boundary of Ising model subject to random impurities. Later on, the model reached the mathematics community [14, 3], where it was reformulated in terms of the d-dimensional simple random walk $(\omega_n)_{n\geq 1}$ and of i.i.d. random variables $\{\eta(n,x)\ ;\ (n,x) \in \mathbb{N} \times \mathbb{Z}^d\}$. The energy of this simple random walk model is given by

$$\beta \mathbf{H}_n^\eta(\omega) = -\beta \sum_{j=1}^{n} \eta(j, \omega_j). \tag{1.11}$$

Therefore, the Brownian motion model described by (1.7) can be thought of as a natural transposition of (1.11) into continuum setting. The model (1.11) has already been studied for more than a decade and by many authors. See for example [14, 3, 23, 22, 5, 7].

• *Gaussian random walk model of directed polymers* [21, 19]: The Hamiltonian of this model takes the same form as (1.11). However, the random walk $(\omega_n)_{n\geq 1}$ here is the summation of independent Gaussian random variables in $\mathbb{R}^d$ and the random field $\{\eta(n,x)\ ;\ (n,x) \in \mathbb{N}\times\mathbb{R}^d\}$ has certain correlation in x variables. A major technical advantage in working with the Gaussian random walk rather than the simple random walk is the applicability of a Girsanov-type path transformation, which plays a key role in analysing this model.

• *Crossing Brownian motion in a soft Poissonian potential* [25, 27, 28, 29]: The model investigated there is also described in terms of Brownian motion and of Poisson points. However, the Brownian motion there is "undirected", in other words, the d-dimensional Brownian motion travels through the Poisson points distributed in space $\mathbb{R}^d$, not in space-time as in ours.

• *First and last passage percolation* [16, 20, 18]: The first (resp. last) passage percolation can be thought of as an analogue of directed polymers at $\beta = -\infty$ (resp. $\beta = +\infty$). In fact, it is expected and even partly vindicated that the properties of the path with minimal/maximal passage time has similar feature to the typical paths under the polymer measure.

2 Main Results

2.1 The normalized partition function and its positivity in the limit

Let us begin by introducing an important martingale on $(\mathcal{M}, \mathcal{G}, Q)$ ((2.1) below). In fact, the large time behavior of this martingale somehow characterizes the phase diagram of this model and for this reason, many of results in this paper can be best understood from the viewpoint of this martingale.

For any fixed path ω, the process $\{\eta(V_t)\}_{t\geq 0}$ has independent, Poissonian increments, hence it is itself a standard Poisson process on the half-line, and $\{\exp(\beta\eta(V_t) - \lambda t)\}_{t\geq 0}$ is its exponential martingale. Therefore, the normalized partition function

$$W_t = e^{-\lambda t} Z_t, \quad t \geq 0 \tag{2.1}$$

is itself a mean-one, right-continuous and left-limited, positive martingale on $(\mathcal{M}, \mathcal{G}, Q)$, with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$ defined by (1.3). In particular, the following limit exists Q-a.s.:

$$W_\infty \stackrel{\text{def.}}{=} \lim_{t\nearrow\infty} W_t \ . \tag{2.2}$$

Since $\exp(\beta\eta(V_t)) > 0$ Q-a.s. for all $0 \leq t < \infty$ and all $\omega \in \Omega$, the event $\{W_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{t\geq 1} \sigma[\eta|_{[t,\infty)\times\mathbb{R}^d}] \ ,$$

and therefore by Kolmogorov's 0-1 law, we only have the two contrasting situations:

$$Q\{W_\infty = 0\} = 1 \ , \tag{2.3}$$

or

$$Q\{W_\infty > 0\} = 1 \ , \tag{2.4}$$

Loosely speaking, the presence of the random environment is supposed to make qualitative difference in the large time behavior the Brownian polymer in the former case (2.3) (the strong disorder phase), while it does not in the latter case (2.4) (the weak disorder phase).

The phase structure of this model is described as follows.

Theorem 2.1.1 **(a)** *For all $d \geq 1$, there is $\beta_1(d) > 0$ such that (2.3) holds for $\beta \in (\beta_1(d), \infty)$.*

(b) *For $d = 1, 2$, (2.3) holds whenever $\beta \neq 0$.*

(c) *For $d \geq 3$, there is $\beta_0(d) > 0$ such that $\lim_{d\nearrow\infty} \beta_0(d) = \infty$ and that (2.4) holds for $\beta \in (-\infty, \beta_0(d))$.*

Theorem 2.1.1 in particular shows the existence of the phase transition in $d \geq 3$, from the weak disorder phase to the strong disorder phase. In Theorem 2.2.2 below, we will capture this phase transition in terms of the quenched Lyapunov exponent. Theorem 2.1.1(a) follows from a stronger result (2.9) we present later on. The other parts of Theorem 2.1.1 are proved in section 4.

Remark 2.1.1 Theorem 2.1.1(a) and (b) for the simple random walk model can be found in [7, Theorem 2.3 (b), Proposition 2.4 (a)]. Theorem 2.1.1(c) for the simple random walk model is also known, see [3, Lemma 2] and [23, Lemma 1].

2.2 The quenched Lyapunov exponent

We now state the result on the existence of the quenched Lyapunov exponent.

Theorem 2.2.1 *Let $d \geq 1$ and $\beta \in \mathbb{R}$ be arbitrary.*

(a) *There exists $\psi(\beta) \in \mathbb{R}_+$ such that*

$$\psi(\beta) = \lim_{t \nearrow \infty} -\frac{1}{t} \ln Z_t + \lambda(\beta), \quad Q\text{-a.s. and in } L^2(Q). \tag{2.5}$$

(b) *The function $\psi(\beta) - \lambda(\beta)$ is concave on $\mathbb{R}$. Hence, the function $\beta \mapsto \psi(\beta)$ is locally Lipschitz continuous and has right and left derivatives $\psi'_+(\beta)$ and $\psi'_-(\beta)$ for all $\beta \in \mathbb{R}$ such that*

$$-\infty < \psi'_+(\beta) \leq \psi'_-(\beta) < \infty. \tag{2.6}$$

Moreover, the set of non-differentiability $\{\beta \in \mathbb{R} \; ; \; \psi'_+(\beta) < \psi'_-(\beta)\}$ is at most countable.

We call $\psi(\beta)$ the quenched Lyapunov exponent. The quantity $\psi(\beta)$ is the exponent for the decay of the martingale W_t as $t \nearrow \infty$ (cf. (2.1)). Equivalently, but on more physical ground, $\psi(\beta)$ is the difference between the annealed free energy $t^{-1} \ln Q[Z_t]$, and the quenched free energy $t^{-1} Q[\ln Z_t]$ in the thermodynamic limit $t \nearrow \infty$. It is reasonably expected, and even confirmed partly in this paper (see Theorem 2.3.1 and Theorem 2.3.2 below for example) that the positivity of ψ makes the large time behavior of the Brownian polymer dramatically different from that of the original Brownian motion, while the polymer behaves somewhat alike usual Brownian motion when $\psi = 0$.

The proof of Theorem 2.2.1 is given in section 6 and goes roughly as follows. We first show the existence of the limit

$$\psi(\beta) \stackrel{\text{def.}}{=} \lim_{t \nearrow \infty} -\frac{1}{t} Q[\ln Z_t] + \lambda(\beta) \, . \tag{2.7}$$

The existence of (2.7) is in fact a consequence of a simple super-additivity argument. We will then derive (2.5) from Theorem 2.4.1 below, which deals with the fluctuation of $\ln Z_t$.

We next study the Lyapunov exponent $\psi(\beta)$ as a function of $\beta \in \mathbb{R}$ and gather some information on the phase diagram.

Theorem 2.2.2 *Let $d \geq 1$ be arbitrary.*

(a) *The function $\psi(\beta)$ is non-decreasing in $\beta \in \mathbb{R}_+$, and is non-increasing in $\beta \in \mathbb{R}_-$. Moreover,*

$$0 \leq \psi(\beta) \leq \lambda(\beta) + \max\{0, -\beta\} \;\; \text{for all } \beta \in \mathbb{R}. \tag{2.8}$$

(b)

$$\psi(\beta) = e^\beta - \mathcal{O}(e^{\beta/2}) \quad \text{as } \beta \nearrow \infty. \tag{2.9}$$

(c) *There exist critical values $\beta_c^+ = \beta_c^+(d), \beta_c^- = \beta_c^-(d)$ with*

$$-\infty \leq \beta_c^- \leq 0 \leq \beta_c^+ < +\infty,$$

such that

$$\psi(\beta) = \psi'_\pm(\beta) = 0 \quad \text{if} \quad \beta \in [\beta_c^-, \beta_c^+] \cap \mathbb{R}, \tag{2.10}$$

$$\psi(\beta) > 0, \quad \text{if} \quad \beta \in \mathbb{R} \backslash [\beta_c^-, \beta_c^+]. \tag{2.11}$$

(d) *For $d \geq 3$, $\beta_c^-(d) = -\infty$, $\beta_c^+(d) > 0$ and $\lim_{d \nearrow \infty} \beta_c^+(d) = \infty$.*

Remark 2.2.1 • Theorem 2.2.2 shows that the sign of β makes the drastic difference in the behavior of $\psi(\beta)$; it grows exponentially fast as $\beta \nearrow \infty$, while the growth is at most linear when $\beta \searrow -\infty$. The contrast is even sharper if $d \geq 3$ where $\psi(\beta) = 0$ for all negative β.

• We see from (2.10) and (2.11) that, if $\beta_c^-(d) < \beta_c^+(d)$ (resp. $-\infty < \beta_c^-(d) < \beta_c^+(d)$), a phase transition occurs at $\beta_c^+(d)$ (resp., $\beta_c^-(d)$), in the sense that ψ is non-analytic there. Moreover, from (2.10) it will follow also that ψ is differentiable at this point with zero derivative, meaning that the phase transition there is at least of second order. Note that the phase transition does occur at $\beta_c^+(d)$ if $d \geq 3$.

The proofs of Theorem 2.2.2(a)–(c) is given in section 7. Theorem 2.2.2(d) follows from Theorem 2.1.1(c).

2.3 The replica overlap

We now characterize the critical values $\beta_c^{\pm}(d)$ in terms of replica overlaps of two independent polymers.

On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $\mu_t^{x,x} = (\mu_t^x)^{\otimes 2}(d\omega, d\widetilde{\omega})$, that we will view as the distribution of the couple $(\omega, \widetilde{\omega})$ with $\widetilde{\omega}$ an independent copy of ω with law μ_t^x. We introduce random variables I_t^x and J_t^x, $t \geq 0$, $x \in \mathbb{R}^d$, given by

$$I_t^x = \mu_t^{x,x}\left[|U(\omega_t) \cap U(\widetilde{\omega}_t)|\right], \tag{2.12}$$

$$J_t^x = \mu_t^{x,x}\left[|V_t(\omega) \cap V_t(\widetilde{\omega})|\right] = \int_0^t \mu_t^{x,x}\left[|U(\omega_s) \cap U(\widetilde{\omega}_s)|\right] ds\,. \tag{2.13}$$

(We use the same notation $|\cdot|$ for the Lebesgue measure on $\mathbb{R}^d$ and on $\mathbb{R}^{1+d}$, and also to denote the Euclidean norm in the sequel.) Then, both J_t^x and $\int_0^t I_s^x ds$ are interpreted as the expected volume of the overlap in time $[0, t]$ of tubes around two independent polymer paths in the same (fixed) environment. In particular, the fraction

$$R_t(\omega, \widetilde{\omega}) = \frac{1}{t}|V_t(\omega) \cap V_t(\widetilde{\omega})|, \tag{2.14}$$

is a natural transposition to our setting, of the so-called *replica overlap* often discussed in the context of disordered systems, e.g. mean field spin glass, and also of directed polymers on trees [9]. Its relevance to us is explained by the following results, which relates the asymptotics of the overlap to the critical values for β.

Theorem 2.3.1 *For all $\beta \in [\beta_c^-, \beta_c^+] \cap \mathbb{R}$,*

$$\lim_{t \nearrow \infty} \frac{1}{t} Q[J_t] = 0\,. \tag{2.15}$$

On the other hand, we have the equality

$$\left\{\beta \in \mathbb{R} : \lim_{t \nearrow \infty} \frac{1}{t} Q[J_t] > 0\right\} = \left\{\beta > 0\,;\ \psi'_+(\beta) > 0\right\} \cup \left\{\beta < 0\,;\ \psi'_-(\beta) < 0\right\}\,. \tag{2.16}$$

Therefore,

$$\beta_c^+ = \sup\left\{\beta' \geq 0 : \forall \beta \in [0,\beta'], \lim_{t\nearrow\infty} \frac{1}{t} Q[J_t] = 0\right\} \tag{2.17}$$

$$= \inf\left\{\beta \geq 0 : \varliminf_{t\nearrow\infty} \frac{1}{t} Q[J_t] > 0\right\} , \tag{2.18}$$

and similarly,

$$\beta_c^- = \inf\left\{\beta' \leq 0 : \forall \beta \in [\beta',0], \lim_{t\nearrow\infty} \frac{1}{t} Q[J_t] = 0\right\}$$

$$= \sup\left\{\beta \leq 0 : \varliminf_{t\nearrow\infty} \frac{1}{t} Q[J_t] > 0\right\} .$$

The proof of Theorem 2.3.1 is given in section 7.5.

Theorem 2.3.2 *Let $\beta \neq 0$. Then,*

$$\{W_\infty = 0\} = \left\{\int_0^\infty I_s ds = \infty\right\}, \quad Q\text{-a.s.} \tag{2.19}$$

Moreover, if $Q\{W_\infty = 0\} = 1$, then there exist $c_1, c_2, t_0 \in (0,\infty)$ such that

$$c_1 \int_0^t I_s ds \leq \lambda t - \ln Z_t \leq c_2 \int_0^t I_s ds \qquad \text{for } t \geq t_0, \quad Q\text{-a.s.} \tag{2.20}$$

In particular,

$$\lim_{t\nearrow\infty} \frac{1}{t} \int_0^t I_s ds = 0 \ \text{ if } \beta \in [\beta_c^-(d), \beta_c^+(d)] \cap \mathbb{R}, \tag{2.21}$$

$$\varliminf_{t\nearrow\infty} \frac{1}{t} \int_0^t I_s ds > 0 \ \text{ if } \beta \in \mathbb{R}\backslash[\beta_c^-(d), \beta_c^+(d)]. \tag{2.22}$$

The proof of Theorem 2.3.2 is given in section 8.

Remark 2.3.1 Analogous results of Theorem 2.3.2 for the simple random walk model for directed polymers can be found in [5] and in [7]; Theorem 2.3.1 has a counterpart in the case of a Gaussian environment, which can be seen from Lemma 7.1 in [5]. As in the discrete case, we can interpret the results from the present subsection, in terms of localization for the path. Indeed, we will prove in section 8, that for some constant $c_1 = c_1(d) \in (0,1)$,

$$c_1 \sup_{y\in\mathbb{R}^d} \mu_t^x \left[\omega_s \in U(y)\right]^2 \leq \mu_t^{x,x}\left[|U(\omega_s) \cap U(\widetilde{\omega}_s)|\right] \leq \sup_{y\in\mathbb{R}^d} \mu_t^x \left[\omega_s \in U(y)\right] . \tag{2.23}$$

The maximum appearing in the above bounds should be viewed as the probability of the favourite "location" for ω_s, under the polymer measure μ_t^x. Both Theorem 2.3.1 and Theorem 2.3.2 are precise statements that, the polymer localizes in the strong disorder regime in a few specific corridors of width $\mathcal{O}(1)$, but spreads out in a diffuse way in the weak disorder regime.

2.4 Fluctuation results

We now state the following estimate for the longitudinal fluctuation of the free energy.

Theorem 2.4.1 *Let $d \geq 1$ and $\beta \in \mathbb{R}$ be arbitrary.*

(a) *With $Q^{\mathcal{G}_s}$ the conditional expectation under Q given $\mathcal{G}_s$, we have*

$$\mathrm{Var}_Q(\ln Z_t) = Q \int_{[0,t]\times\mathbb{R}^d} dsdx \left[Q^{\mathcal{G}_s} \ln\left(1 + \lambda\mu_t\{U(\omega_s) \ni x\}\right)\right]^2 . \tag{2.24}$$

As a consequence, the following inequalities hold:

$$\mathrm{Var}_Q(\ln Z_t) \;\geq\; \lambda(-|\beta|)^2 Q \int_{[0,t]\times\mathbb{R}^d} dsdx \left(Q^{\mathcal{G}_s}\mu_t\{U(\omega_s) \ni x\}\right)^2, \tag{2.25}$$

$$\mathrm{Var}_Q(\ln Z_t) \;\leq\; \lambda(|\beta|)^2 Q \int_{[0,t]\times\mathbb{R}^d} dsdx \left(Q^{\mathcal{G}_s}\mu_t\{U(\omega_s) \ni x\}\right)^2 , \tag{2.26}$$

and

$$\mathrm{Var}_Q(\ln Z_t) \;\leq\; \lambda(|\beta|)^2 Q[J_t] \tag{2.27}$$

$$\;\leq\; \lambda(|\beta|)^2 t. \tag{2.28}$$

(b) *With $C = e^{|\lambda|}\lambda(|\beta|)^2$,*

$$Q\left\{|\ln Z_t - Q[\ln Z_t]| > u\right\} \leq 2\exp -\frac{1}{2}(u \wedge \frac{u^2}{Ct}) , \quad u \geq 0 . \tag{2.29}$$

The formula (2.24) is analogous to (3.2) in [20]. Here, it is obtained rather easily, thanks to the power of stochastic calculus. The formula (2.27) shows that the fluctuations of $\ln Z_t$ are small when the overlap is small (cf. (2.15)). The proof of Theorem 2.4.1 and these of following corollaries are given in section 5.

Corollary 2.4.2 *Let $d \geq 1$, $\beta \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. Then, as $t \nearrow \infty$,*

$$\ln Z_t - Q[\ln Z_t] = \mathcal{O}(t^{\frac{1+\varepsilon}{2}}), \quad Q\text{-a.s.} \tag{2.30}$$

Corollary 2.4.3 *Let $\beta \neq 0$. For $\xi > 0$ and $C > 0$, there exists $c_1 = c_1(d, C) \in (0, \infty)$ such that*

$$\varliminf_{t\nearrow\infty} t^{-(1-d\xi)}\mathrm{Var}_Q(\ln Z_t) \;\geq\; c_1 \varliminf_{t\nearrow\infty} \inf_{0\leq s\leq t} \left(Q\mu_t\{|\omega_s| \leq C + Ct^{\xi}\}\right)^2 \tag{2.31}$$

$$\;\geq\; c_1 \varliminf_{t\nearrow\infty} \left(Q\mu_t\{\sup_{0\leq s\leq t} |\omega_s| \leq C + Ct^{\xi}\}\right)^2 \tag{2.32}$$

Remark 2.4.1 Let us interpret Theorem 2.4.1 and its corollaries in terms of critical exponents. Let us write $\xi(d)$ for the "wandering exponent",i.e., the critical exponent for the transversal fluctuation of the path, and $\chi(d)$ for the the critical exponent for the longitudinal fluctuation of the free energy. Their definitions are roughly that

$$\sup_{0\leq s\leq t} |\omega_s| \approx t^{\xi(d)} \quad \text{and} \quad \mathrm{Var}_Q(\ln Z_t) \approx t^{2\chi(d)} \quad \text{as } t \nearrow \infty. \tag{2.33}$$

There are various ways to define rigorously these exponents, e.g. (0.6) and (0.10-11) in [27], (2.4) and (2.6-7-8) in [22]. Although the equivalence between these specific definitions are not always clear, the common idea behind all the definitions is described by (2.33).

We see from (2.28) that $\chi(d) \leq 1/2$, while Corollary 2.4.3 suggests that

$$\chi(d) \geq (1 - d\xi(d))/2 , \tag{2.34}$$

for non-zero β. (Note that this inequality fails to hold for $\beta = 0$.) See also Theorem 2.4.4 and Remark 2.4.4 below for further considerations on the exponents.

Remark 2.4.2 Critical exponents similar to the ones discussed above are investigated in the context of various other models and in a large number of papers. In particular, it is conjectured in physics literature that

$$\chi(d) = 2\xi(d) - 1, \quad d \geq 1, \tag{2.35}$$

$$\chi(1) = 1/3, \quad \xi(1) = 2/3 \text{ for all } \beta \neq 0. \tag{2.36}$$

See, e.g., [12],[11, (3.4),(5.11),(5.12)], [17, (5.19),(5.28)]. We now mention a few articles which we think are especially relevant to us.[4]

• M. Piza [22] studies critical exponents for the simple random walk model for directed polymers. There he proves various relations between $\chi(d)$ and $\xi(d)$ including an analogue of (2.34). He also proves that a certain curvature assumption on the free energy (page 589, "Definition" in that paper) implies bounds $\xi(d) \leq 3/4$ and $\chi(1) \geq 1/8$. It seems difficult to check this assumption in general. However, the large deviation principle (2.38)-(2.39) below with $\xi = 1$, means that the assumption is satisfied in our model: More precisely, the minimizer $\theta = 0$ of the (quadratic) rate function I in (2.38) –corresponding to the direction of the diagonal in Piza's framework–, is a "direction of curvature" in the sense of [22].

• M. Peterman [21] proves for the Gaussian random walk model of directed polymers that $\xi(1) \geq 3/5$, while O. Mejane [19] proves $\xi(d) \leq 3/4$ for all $d \geq 1$.

• M. Wüthrich studies critical exponents for crossing Brownian motion in a soft Poissonian potential [27, 28, 29]. There he obtains $\xi(d) \leq 3/4$, $\xi(1) \geq 3/5$, $\chi(1) \geq 1/8$ and various other relations between $\chi(d)$ and $\xi(d)$ including an analogue of (2.34). We stress that his model is undirected, so the dimension there corresponds to $1 + d$ in our model. Also, his techniques do not seem to be immediately transportable to our model, since they depend quite heavily on the spatial invariance under rotation, which makes very precise information on the quenched Lyapunov exponent available [25, Chapter5].

We have the following large deviation principle for the transversal fluctuation of the Brownian polymer.

Theorem 2.4.4 *Let $\chi \geq 0$ be such that*

$$\lim_{t \nearrow \infty} t^{-\chi}(\ln Z_t - Q[\ln Z_t]) = 0, \quad Q\text{-}a.s. \tag{2.37}$$

and let $\xi > (1 + \chi)/2$. Then,

[4] We warn the reader that the following quotations are quite rough, since we totally disregard the differences in the specific definitions of the exponents $\chi(d)$ and $\xi(d)$ in these articles.

(a) *The large deviation principle for* $\mu_t^x\left\{\frac{\omega_t}{t^\xi}\in\cdot\right\}$, $t\nearrow\infty$ *holds* Q*-a.s., with the rate function* $I(\theta)=|\theta|^2/2$ *and the speed* $t^{2\xi-1}$*: There exists an event* $\mathcal{M}_0^x$ *with* $Q(\mathcal{M}_0^x)=1$ *for each* $x\in\mathbb{R}^d$ *such that for any* $\eta\in\mathcal{M}_0^x$ *and for any Borel set* $B\subset\mathbb{R}^d$,

$$-\inf_{B^\circ} I - o(1) \le \frac{1}{t^{2\xi-1}}\ln\mu_t^x\left\{\frac{\omega_t}{t^\xi}\in B\right\} \le -\inf_{\overline{B}} I + o(1)\ , \quad \textit{as } t\nearrow\infty. \tag{2.38}$$

In particular, for any $\varepsilon>0$,

$$\lim_{t\nearrow\infty} -\frac{1}{t^{2\xi-1}}\ln\mu_t\left\{|\omega_t|\ge\varepsilon t^\xi\right\} = \frac{\varepsilon^2}{2}, \quad Q\textit{-a.s.} \tag{2.39}$$

(b) *For* $d\ge 1$ *and* $\beta\in\mathbb{R}$, *(2.37) holds true with any* $\chi>1/2$ *and hence (2.38) and (2.39) hold for all* $\xi>3/4$.

Remark 2.4.3 The proof of Theorem 2.4.4 given in section 9, roughly goes as follows. We first make use of Girsanov's formula to compute the cost (under the polymer measure) for the Brownian path to deviate away from the origin. Then, we use the Gärtner-Ellis-Baldi theorem [8, page 44,Theorem 2.3.6] to conclude (2.38). We also mention that our proof of Theorem 2.4.4 works for the model studied in [21, 19] (cf. Remark 2.4.2), yielding the analogues of (2.38) and (2.39).

Remark 2.4.4 In terms of critical exponents, Theorem 2.4.4 suggests that

$$\chi(d) \ \ge\ 2\xi(d)-1 \tag{2.40}$$
$$\xi(d) \ \le\ 3/4, \tag{2.41}$$

in consistency with the conjectures (2.35) and (2.36). If we combine the statements (2.34) and (2.41), we then get $\chi(1)\ge 1/8$. If we now insert this into the conjecture (2.35), we then obtain $\xi(1)\ge 9/16>1/2$, i.e., the polymer is super-diffusive in dimension $d=1$, for all non-zero β.

3 Some preliminaries from stochastic calculus

As we will see later in some proofs, it is convenient for us to assume that the Poisson random measure is canonically realized. For this reason, we henceforth specify the space $(\mathcal{M},\mathcal{G})$ as follows. We take

$$\begin{aligned}\mathcal{M} &= \left\{\eta\ ;\ \text{integer valued measures on } \mathbb{R}_+\times\mathbb{R}^d : \forall t>0, \sup_{r>0} r^{-d}\eta([0,t]\times[-r,r]^d)<\infty\right\}\ ,\\ \mathcal{G} &= \sigma[\eta(A)\ ;\ A\in\mathcal{B}(\mathbb{R}_+\times\mathbb{R}^d)\ ],\end{aligned}$$

where a generic element $\eta\in\mathcal{M}$ is regarded as the identity mapping $\mathrm{id}(\eta)=\eta$ on $\mathcal{M}$. Recall the definitions (1.2) and (1.3). For a measurable function $f(s,x,\eta)$ such that

$$\int_{[0,t]\times\mathbb{R}^d} Q[|f(s,x,\eta)|]dsdx<\infty,$$

we define the compensated integral

$$\int f(s,x,\eta)\overline{\eta}_t(dsdx) = \int f(s,x,\eta)\eta_t(dsdx) - \int_{[0,t]\times\mathbb{R}^d} f(s,x,\eta)dsdx. \tag{3.1}$$

If the integrand $(f(s,x,\eta))_{s\geq 0}$ is predictable in the sense of [13, page 61, Definition 3.3], then, (3.1) defines a martingale on $(\mathcal{M}, \mathcal{G}, Q)$ for the filtration $(\mathcal{G}_t; t \geq 0)$. If, moreover,

$$\int_{[0,t]\times\mathbb{R}^d} Q[f(s,x,\eta)^2] ds dx < \infty,$$

then, this martingale is square-integrable, and its predictable bracket

$$\langle \int f d\overline{\eta} \rangle_t = \int_{[0,t]\times\mathbb{R}^d} f^2 ds dx. \tag{3.2}$$

is such that $(\int f d\overline{\eta}_t)^2 - \langle \int f d\overline{\eta} \rangle_t$ is a martingale on $(\mathcal{M}, \mathcal{G}, Q)$. The following short hand notation will frequently be used in the sequel:

$$\zeta_t = \zeta_t(\omega, \eta) = \exp\left(\beta \eta(V_t)\right) , \tag{3.3}$$

$$\chi_{t,x} = \chi_{t,x}(\omega) = 1\{x \in U(\omega_t)\} . \tag{3.4}$$

The latter will not be confused with the exponent $\chi(d)$ from Remark 2.4.1. It is useful to note that

$$\int_{\mathbb{R}^d} \chi_{t,x} \, dx = 1 , \quad t \geq 0 . \tag{3.5}$$

With these notations, we have

Lemma 3.0.5

$$\eta(V_t) = \int \chi_{s,x} \eta_t(dsdx) \tag{3.6}$$

$$= \int \chi_{s,x} \overline{\eta}_t(dsdx) + t , \tag{3.7}$$

$$\zeta_t = 1 + \lambda \int \eta_t(dsdx) \zeta_{s-} \chi_{s,x}, \quad \textit{for all } \omega \in \Omega, \tag{3.8}$$

$$Z_t^x = 1 + \lambda \int \eta_t(dsdy) P^x[\zeta_{s-} \chi_{s,y}], \tag{3.9}$$

$$W_t = 1 + \lambda \int \overline{\eta}_t(dsdx) \, W_{s-} \, \mu_{s-}[\chi_{s,x}] . \tag{3.10}$$

Proof: The first two identities are obvious. The three last ones follow from direct applications of Itô's formula. □

4 Proof of Theorem 2.1.1 (b),(c)

4.1 Proof of part (b)

The argument we present here is based on [7, proof of Proposition 1.3 (b)]. For (2.3), it is enough to prove that some fractional moment vanishes:

$$\lim_{t \nearrow \infty} Q[W_t^\theta] = 0 \tag{4.1}$$

for some $\theta \in (0,1)$. We first state a technical lemma, which is a generalization of Gronwall's inequality.

Lemma 4.1.1 *Let $u \in C^1(\mathbb{R}_+ \to \mathbb{R})$ and $v, w \in C(\mathbb{R}_+ \to \mathbb{R})$ be such that*

$$\frac{d}{dt}u(t) \le -v(t)u(t) + w(t), \quad \textit{for all } t > 0. \tag{4.2}$$

Then, with $V(t) = \int_0^t v(s)ds$

$$u(t) \le \left(u(0) + \int_0^t w(s)e^{V(s)}ds \right) e^{-V(t)}, \quad \textit{for all } t > 0. \tag{4.3}$$

In particular, when v, w are non-negative, it holds

$$u(t) \le u(0)e^{-V(t)} + \int_0^t w(s)ds, \quad t > 0. \tag{4.4}$$

Proof: We write $\overline{u}(t)$ for the right-hand side of (4.3). Then,

$$\overline{u}(0) = u(0), \quad \frac{d}{dt}\overline{u}(t) = w(t) - v(t)\overline{u}(t), \quad \text{for all } t > 0,$$

and therefore,

$$\frac{d}{dt}\left([u(t) - \overline{u}(t)]e^{V(t)}\right) = \left[\frac{d}{dt}\left(u(t) - \overline{u}(t)\right) + v(t)\left(u(t) - \overline{u}(t)\right)\right] e^{-V(t)} \le 0 , \quad \text{for all } t > 0.$$

By integration, this implies $u(t) \le \overline{u}(t)$ for all $t > 0$. All the statements follow easily. □

We now present the following key lemma.

Lemma 4.1.2 *For $\theta \in (0, 1)$, there exists $c_0 = c_0(\theta, \beta) > 0$ such that for any $\Lambda \subset \mathbb{R}^d$,*

$$\frac{d}{dt}Q\left[W_t^\theta\right] \;\; \le \;\; -\frac{c_0}{|\Lambda|}Q\left[W_t^\theta\right] + \frac{2c_0}{|\Lambda|}P(U(\omega_t) \not\subset \Lambda)^\theta. \tag{4.5}$$

Let us postpone the proof of this lemma for a moment to complete the proof of (4.1). In what follows, c_i, $i = 1, 2$ denote universal constants.

For $d = 1$, set $\Lambda = (-t^{2/3}, t^{2/3}]$. Then,

$$P(U(\omega_t) \not\subset \Lambda) = P(|\omega_t| \ge t^{2/3} - \tfrac{1}{2}) \le c_1 \exp\left(-t^{1/3}/c_1\right),$$

and hence by (4.5), $u(t) = Q\left[W_t^\theta\right]$ satisfies

$$\frac{d}{dt}u(t) \le -\frac{c_0}{2t^{2/3}}u(t) + \exp\left(-\theta t^{1/3}/c_1\right)$$

for large t. We then have by Lemma 4.1.1 that

$$u(t) \le \exp\left(-\frac{c_0}{2}\int_{t/2}^t s^{-2/3}ds\right) + \int_{t/2}^t \exp\left(-\theta s^{1/3}/c_1\right) ds \stackrel{t \nearrow \infty}{\longrightarrow} 0,$$

which implies (4.1) for $d = 1$.

For $d = 2$, we set

$$\Lambda = \left(-\sqrt{\gamma t \ln t}, \sqrt{\gamma t \ln t}\right]^2$$

with $\gamma > 0$. We then see in a similar way as above that for large t,

$$Q\left[W_t^\theta\right] \le \exp\left(-\frac{1}{4\gamma}\int_{\ln t}^t \frac{ds}{s \ln s}\right) + \int_{\ln t}^t \exp\left(-\frac{\theta\gamma}{c_2}\ln s\right) ds,$$

which, if $\frac{\theta\gamma}{c_2} > 1$, goes to zero as $t \nearrow \infty$. This proves (4.1) for $d = 2$. □

We now turn to the proof of Lemma 4.1.2.

Lemma 4.1.3 *For $\theta \in [0,1]$ and $\Lambda \subset \mathbb{R}^d$,*

$$|\Lambda| Q\left[W_t^\theta I_t\right] \geq Q\left[W_t^\theta\right] - 2P(U(\omega_t) \not\subset \Lambda)^\theta, \tag{4.6}$$

where I_t is defined by (2.12).

Proof: Note first that

$$|\Lambda \cap U(\omega_t)| = 1 - |U(\omega_t) \backslash \Lambda| \geq 1 - 1\{U(\omega_t) \not\subset \Lambda\}.$$

We then use the Schwarz inequality and the above observation as follows

$$\begin{aligned}
|\Lambda| I_t &\geq |\Lambda| \int_\Lambda \mu_t[\chi_{t,x}]^2 dx \\
&\geq \left(\int_\Lambda \mu_t[\chi_{t,x}] dx\right)^2 \\
&= \mu_t[|\Lambda \cap U(\omega_t)|]^2 \\
&\geq (1 - \mu_t\{U(\omega_t) \not\subset \Lambda\})^2 \\
&\geq 1 - 2\mu_t\{U(\omega_t) \not\subset \Lambda\} \\
&\geq 1 - 2\mu_t\{U(\omega_t) \not\subset \Lambda\}^\theta.
\end{aligned} \tag{4.7}$$

Note also that

$$\begin{aligned}
Q\left[W_t^\theta \mu_t\{U(\omega_t) \not\subset \Lambda\}^\theta\right] &\leq Q\left[W_t \mu_t\{U(\omega_t) \not\subset \Lambda\}\right]^\theta \\
&= P\{U(\omega_t) \not\subset \Lambda\}^\theta.
\end{aligned} \tag{4.8}$$

We now use (4.7) and then (4.8) to conclude (4.6):

$$\begin{aligned}
|\Lambda| Q\left[W_t^\theta I_t\right] &\geq Q\left[W_t^\theta\right] - 2Q\left[W_t^\theta \mu_t\{U(\omega_t) \not\subset \Lambda\}^\theta\right] \\
&\geq Q\left[W_t^\theta\right] - 2P\{U(\omega_t) \not\subset \Lambda\}^\theta.
\end{aligned}$$

□

Proof of Lemma 4.1.2: From(3.9), Itô's formula and from $\int \mu_s(\chi_{s,x})dx = 1$, we have

$$\begin{aligned}
W_t^\theta &= 1 + \int W_{s-}^\theta \left([\mu_{s-}(1 + \lambda\chi_{s,x})]^\theta - 1\right) \eta_t(dsdx) - \theta\lambda \int_{(0,t]} W_s^\theta ds \\
&= 1 + \int W_{s-}^\theta \left([\mu_{s-}(1 + \lambda\chi_{s,x})]^\theta - 1\right) \overline{\eta}_t(dsdx) \\
&\quad - \int_{(0,t]\times\mathbb{R}^d} W_s^\theta f\left(\lambda\mu_s[\chi_{s,x}]\right) dsdx,
\end{aligned}$$

where we have defined a function $f : (-1, \infty) \to [0, \infty)$ by

$$f(u) = 1 + \theta u - (1 + u)^\theta.$$

Therefore,

$$Q\left[W_t^\theta\right] = 1 - \int_{(0,t]\times\mathbb{R}^d} dsdx\, Q\left[W_s^\theta f\left(\lambda\mu_s[\chi_{s,x}]\right)\right],$$

It is clear that there are constants $c_i = c_i(\theta, \beta) \in (0, \infty)$ such that

$$c_1 u^2 \le f(u) \le c_2 u^2 \quad \text{for all } u \in (-1, |\lambda|]. \tag{4.9}$$

and hence

$$\begin{aligned} \frac{d}{dt} Q\left[W_t^\theta\right] &= -\int_{\mathbb{R}^d} dx\, Q\left[W_t^\theta f\left(\lambda \mu_t[\chi_{t,x}]\right)\right] \\ &\le -c_1 \int_{\mathbb{R}^d} dx\, Q\left[W_t^\theta \mu_t[\chi_{t,x}]^2\right] \\ &= -c_1 Q\left[W_t^\theta I_t\right]. \end{aligned}$$

Now (4.5) follows from (4.6). □

4.2 Proof of part(c)

The next proposition provides a condition for the martingale $W_t = e^{-\lambda t} Z_t$ to converge in L^2, and hence for $Q\{W_\infty \neq 0\} = 1$. To state the proposition, let us introduce the Bessel function as usual,

$$J_\nu(\gamma) = (\gamma/2)^\nu \sum_{k \ge 0} \frac{(-\gamma^2/4)^k}{k!\Gamma(\nu + k + 1)}, \quad \gamma \ge 0,\ \nu > -1.$$

We write γ_d for the smallest positive zero of $J_{\frac{d-4}{2}}$. Note then that $(\gamma/2)^{-\frac{d-4}{2}} J_{\frac{d-4}{2}}(\gamma) > 0$ for $\gamma \in [0, \gamma_d)$.

Proposition 4.2.1 (a)

$$\sup_{t \ge 0} Q[W_t^2] \le P\left[\exp\left(\frac{\lambda^2}{2} \int_0^\infty \chi_{s,0} ds\right)\right]. \tag{4.10}$$

If $d \ge 3$ and

$$|\lambda| < \gamma_d / r_d, \tag{4.11}$$

where $r_d = \Gamma\left(\frac{d+2}{2}\right)^{1/d} / \sqrt{\pi}$ stands for the radius of $U(0)$, then,

$$P\left[\exp\left(\frac{\lambda^2}{2} \int_0^\infty \chi_{s,0} ds\right)\right] = \frac{1}{\Gamma\left(\frac{d-2}{2}\right) (|\lambda| r_d/2)^{-\frac{d-4}{2}} J_{\frac{d-4}{2}}(|\lambda| r_d)} < \infty. \tag{4.12}$$

In particular, $\sup_{t \ge 0} Q[W_t^2] < \infty$ if $d \ge 3$ and (4.11) holds.

(b) *$\gamma_d / r_d > 1$ for all $d \ge 3$, and hence* $\sup_{t \ge 0} Q[W_t^2] < \infty$ *if $\beta \in (-\infty, \ln(1 + \gamma_d / r_d))$. Moreover, $\lim_{d \nearrow \infty} (\gamma_d / r_d) = \infty$.*

Proof: **(a)**: Consistently with the notation $\mu_t^{x, \widetilde{x}}$ introduced in section 2.3, we let $P^{x, \widetilde{x}} = P^x \otimes P^{\widetilde{x}}$. Under this measure, ω and $\widetilde{\omega}$ are independent Brownian motions starting respectively from x and $\widetilde{x}$. As in [3, 23], we start by writing

$$W_t(\eta)^2 = e^{-2\lambda t} P^{0,0}[\zeta_t(\omega, \eta) \zeta_t(\widetilde{\omega}, \eta)],$$

so that

$$Q[W_t^2] = e^{-2\lambda t} P^{0,0} \left[Q[\zeta_t(\omega,\eta)\zeta_t(\widetilde{\omega},\eta)] \right] \tag{4.13}$$

by Fubini's theorem. For $(\omega, \widetilde{\omega}) \in \Omega^2$, we compute $e^{-2\lambda t} Q[\zeta_t(\omega,\eta)\zeta_t(\widetilde{\omega},\eta)]$, using (1.1), (1.8), and observing that $\lambda^2(\beta) = \lambda(2\beta) - 2\lambda(\beta)$,

$$\begin{aligned} e^{-2\lambda t} Q[\zeta_t(\omega,\eta)\zeta_t(\widetilde{\omega},\eta)] &= Q[\exp\left(2\beta\eta(V_t(\omega)\cap V_t(\widetilde{\omega})) + \beta\eta(V_t(\omega)\triangle V_t(\widetilde{\omega})) - 2\lambda t\right)] \\ &= \exp\left(\lambda(2\beta)|V_t(\omega)\cap V_t(\widetilde{\omega})| + \lambda|V_t(\omega)\triangle V_t(\widetilde{\omega})| - 2\lambda t\right) \\ &= \exp\left(\lambda^2 |V_t(\omega)\cap V_t(\widetilde{\omega})|\right) \\ &\nearrow \exp\left(\lambda^2 |V_\infty(\omega)\cap V_\infty(\widetilde{\omega})|\right) \end{aligned} \tag{4.14}$$

as $t \nearrow \infty$, by monotone convergence. Now,

$$\begin{aligned} |V_\infty(\omega)\cap V_\infty(\widetilde{\omega})| &= \int_0^\infty |U(\omega_t)\cap U(\widetilde{\omega}_t)| dt \\ &\le \int_0^\infty 1\{\omega_t - \widetilde{\omega}_t \in 2U(0)\} dt. \end{aligned} \tag{4.15}$$

Since $\bar{\omega} = (\omega - \widetilde{\omega})/\sqrt{2}$ is a standard Brownian motion, we get (4.10) from (4.13), (4.14), (4.15) and the Brownian scaling property.

At this point, a standard way to bound the expectation on the right-hand side of (4.10) would be via Khas'minskii's lemma (cf, Remark 4.2.1 below). Here, we take a different route to get exact formula (4.12) and thereby proceed to the part (b). Now, the exponent of the integrand on the right-hand-side of (4.10) is nothing but the occupation time for the Bessel process in the interval $[0, r_d]$. Therefore, (4.12) follows from a formula for the Laplace transform of the occupation time [4, page 376] via analytic continuation.

(b): The formula (4.12) shows that the expectation is finite if $|\lambda| < \gamma_d / r_d$. On the other hand, it is known [26, pages 486,748] that

$$\gamma_3 = \pi/2, \quad \gamma_4 = 2.404..., \quad \gamma_5 = \pi, \tag{4.16}$$

$$\gamma_d \ge \tfrac{1}{2}\sqrt{d(d-4)}, \quad d \ge 5. \tag{4.17}$$

It is then, not difficult to see from (4.16), (4.17), and the direct estimation of r_d that $1 < \gamma_d / r_d$ for all $d \ge 3$ and that $\lim_{d \nearrow \infty} (\gamma_d / r_d) = \infty$. □

Remark 4.2.1 The expectation on the right-hand-side of (4.10) can be bounded also by using Khas'minskii's lemma as follows. For $d \ge 3$, the Brownian motion is transient, and

$$c := \sup_{x \in \mathbb{R}^d} P^x \left[\int_0^\infty \chi_{s,0} ds \right] < \infty .$$

By Khas'minskii's lemma (e.g., [25, page 8,Lemma 2.1]), this implies that

$$\sup_{x \in \mathbb{R}^d} P^x \left[\exp\left(a \int_0^\infty \chi_{s,0} ds \right) \right] < (1 - ac)^{-1} , \quad \text{if } ac < 1 ,$$

from which, we see the convergence of the expectation on the right-hand-side of (4.10) when $\lambda^2/2 < c^{-1}$, i.e., when $|\beta|$ is small enough.

5 Proof of Theorem 2.4.1

5.1 Proof of part (a)

Define the functional $F : \mathcal{M} \to \mathbb{R}$, by

$$F(\eta) = \ln Z_t \ , \quad \eta \in \mathcal{M} \ ,$$

and its increment

$$\begin{aligned} D_{s,x}F(\eta) &= F(\eta + \delta_{s,x}) - F(\eta) \\ &= \ln \frac{P[e^{\beta\eta(V_t)} e^{\beta\chi_{s,x}}]}{P[e^{\beta\eta(V_t)}]} \\ &= \ln\left(1 + \lambda\mu_t[\chi_{s,x}]\right) \ . \end{aligned} \tag{5.1}$$

Let $t > 0$, that we will keep fixed all through this section. We introduce a martingale $(Y_{t,s})_{s\in[0,t]}$ by

$$Y_{t,s} = Q^{\mathcal{G}_s}[Y_t] \ , \quad \text{with } Y_t = \ln Z_t - Q[\ln Z_t] = F(\eta_t) - Q[F(\eta_t)] \ ,$$

where $Q^{\mathcal{G}_s}$ is the conditional expectation given $\mathcal{G}_s$. The random variable Y_t is the one of interest in (2.29) and (2.24), and it is natural to introduce its Doob martingale $(Y_{t,s})_{s\in[0,t]}$ for an easier approach. Similarly to the proof of Proposition 2.1 in [6] of the basic concentration property for the Sherrington-Kirkpatrick model, we will use stochastic calculus, but for jump processes instead of Brownian functionals. Observe that, by independence and homogeneity of the Poisson increments,

$$Q^{\mathcal{G}_s}[F(\eta_t)] = Q^{\mathcal{G}_s}[F(\eta_s + [\eta_t - \eta_s])] = \int_{\mathcal{M}} F(\eta_s + m)\rho_{t-s}(dm) \ ,$$

with ρ_{t-s} the law of η_{t-s}. Hence, for $0 \le s < s+h \le t$,

$$Y_{t,s+h} = \int_{\mathcal{M}} [F(\eta_{s+h} + m) - F(\eta_s + m)]\rho_{t-s-h}(dm) + \int_{\mathcal{M}} F(\eta_s + m)\rho_{t-s-h}(dm) \ ,$$

and therefore, the stochastic differential of $Y_{t,s}$ with respect to s is given by

$$dY_{t,s} = \int_{\mathbb{R}^d} \eta(dsdx) \int_{\mathcal{M}} D_{s,x}F(\eta_{s^-} + m)\rho_{t-s}(dm) - ds\frac{\partial}{\partial r}\int_{\mathcal{M}} F(\eta_s + m)\rho_r(dm)_{|r=t-s} \ .$$

But the factor of ds in the last term is equal to

$$\frac{d}{dt}Q^{\mathcal{G}_s}[F(\eta_t)] = \int_{\mathbb{R}^d} dx \int_{\mathcal{M}} D_{s,x}F(\eta_{s^-} + m)\rho_{t-s}(dm)$$

according to (3.1). Finally, the martingale $(Y_{t,s})_{s\in[0,t]}$ can be written as the compensated stochastic integral

$$Y_{t,s} = \int \overline{\eta}_s(dudx)h_t(u^-, u, x) \ , \tag{5.2}$$

where

$$h_t(s, u, x) = Q^{\mathcal{G}_s}[D_{u,x}F(\eta_t)] = Q^{\mathcal{G}_s}[\ln(1 + \lambda\mu_t[\chi_{u,x}])].$$

It follows from (3.2) and (5.2) that

$$\mathrm{Var}(\ln Z_t) = Q[Y_t^2] = Q\int_{[0,t]\times\mathbb{R}^d} dsdxh_t(s,s,x)^2\ ,$$

that is (2.24). Note that for $u \in [0,1]$ and $\beta \neq 0$,

$$\ln(1+\lambda u) = (\lambda/|\lambda|)|\ln(1+\lambda u)|\ ,\quad \lambda(-|\beta|)u \le |\ln(1+\lambda u)| \le \lambda(|\beta|)u. \tag{5.3}$$

This shows (2.25) and (2.26). Using Jensen's inequality on the right-hand-side of (2.26),

$$\begin{aligned}\mathrm{Var}(\ln Z_t) &\le \lambda(|\beta|)^2 Q\int_{[0,t]\times\mathbb{R}^d} dsdx\left(Q^{\mathcal{G}_s}\mu_t[\chi_{s,x}]\right)^2\\ &\le \lambda(|\beta|)^2 Q\int_{[0,t]\times\mathbb{R}^d} dsdx\left(\mu_t[\chi_{s,x}]\right)^2\\ &= \lambda(|\beta|)^2 Q[J_t].\end{aligned}$$

This completes the proof of (a).

5.2 Proof of part (b)

In this proof, we set $\varphi(v) = e^v - v - 1$ for the notational simplicity. For the stochastic integral (5.2), it is standard matter to see that, for $a \in \mathbb{R}$,

$$M_{t,s} = \exp\left\{aY_{t,s} - \int_{[0,s]\times\mathbb{R}^d} dudx\,\varphi\left(ah_t(u,u,x)\right)\right\},\quad 0 \le s \le t, \tag{5.4}$$

is a martingale. It follows from (5.3) and Jensen's inequality that

$$\begin{aligned}\int_{[0,t]\times\mathbb{R}^d} dudx\, h_t(u,u,x)^2 &\le \lambda(|\beta|)^2\int_{[0,t]\times\mathbb{R}^d} dudx\, Q^{\mathcal{G}_u}\left(\mu_t[\chi_{u,x}]^2\right)\\ &\le \lambda(|\beta|)^2\int_{[0,t]\times\mathbb{R}^d} dudx\, Q^{\mathcal{G}_u}\left(\mu_t[\chi_{u,x}]\right)\\ &= \lambda(|\beta|)^2 t\end{aligned}$$

Using this together with the inequality $|\varphi(v)| \le e^{|v|}v^2/2$, we obtain for $a \in [-1,1]$,

$$\left|\int_{[0,t]\times\mathbb{R}^d} dudx\,\varphi\left(ah_t(u,u,x)\right)\right| \le Ca^2t/2,$$

where $C = e^{|\lambda|}\lambda(|\beta|)^2$. By Markov inequality and the martingale property, we have for all $a \in (0,1]$ and $u > 0$,

$$Q[Y_t > u] \le \exp\left\{Ca^2t/2 - au\right\}\ ,$$

and hence,

$$Q[Y_t > u] \le \exp\left(\min_{a\in(0,1]}\left\{Ca^2t/2 - au\right\}\right) = \exp\left(-\frac{1}{2}(u \wedge \frac{u^2}{Ct})\right)\ .$$

Performing the same way with lower deviations, we obtain the desired estimate (2.29). $\Box$

5.3 Proof of Corollary 2.4.2

The asymptotic bound (2.30) along a sequence $t = 1, 2, \ldots$ can be obtained by (2.29) and the Borel-Cantelli lemma. But we see from the following lemma that we do not need to take a subsequence.

Lemma 5.3.1 *Define*

$$\delta_t(h) = \int_{(t,t+h]\times\mathbb{R}^d} \mu_{s-}[\chi_{s,x}]\eta(dsdx)$$

for a fixed $h > 0$. Then, for any $\varepsilon > 0$,

$$\delta_t(h) = \mathcal{O}(t^{\frac{1+\varepsilon}{2}}) \text{ in } Q\text{-a.s.} \tag{5.5}$$

Moreover, for $0 \le s \le h$,

$$\lambda(-|\beta|)\delta_t(h) \le \ln \frac{Z_{t+s}}{Z_t} \le \lambda(|\beta|)\delta_t(h). \tag{5.6}$$

Proof: We have $\delta_t(h) = M_{t+h} - M_t + h$ where

$$M_t = \int \eta_t(dsdx)\mu_{s-}[\chi_{s,x}] - t.$$

Then, $\langle M \rangle_t = \int_0^t I_s ds \le t$. The Q-a.s.bound in (5.5) is a consequence of the two following facts:

$$\lim_{t\nearrow\infty} M_t \text{ exists and is finite if } \langle M \rangle_\infty < \infty, \tag{5.7}$$

$$\lim_{t\nearrow\infty} M_t/\langle M \rangle_t^{\frac{1+\varepsilon}{2}} = 0 \text{ if } \langle M \rangle_\infty = \infty. \tag{5.8}$$

These facts for the discrete martingales are standard (e.g. [10, page 255, (4.9),(4.10)]). It is not difficult to adapt the proof for discrete setting to our case.

On the other hand, we have by (3.9) and Itô's formula that

$$\ln Z_t = \int \eta_t(dsdx) \ln(1 + \lambda\mu_{s-}[\chi_{s,x}]), \tag{5.9}$$

and hence that

$$\ln \frac{Z_{t+s}}{Z_t} = \int_{(t,t+s]\times\mathbb{R}^d} \eta(dsdx) \ln(1 + \lambda\mu_{s-}[\chi_{s,x}]),$$

from which (5.6) is obvious. □

5.4 Proof of Corollary 2.4.3

We set $\Lambda_t = \{x \in \mathbb{R}^d \ ; \ |x| + r \le C + Ct^\xi\}$, where r denotes the radius of $U(0)$. We then see from (2.25) and Jensen's inequality that

$$\lambda(-|\beta|)^{-2}\mathrm{Var}_Q(\ln Z_t) \ \ge \ Q\int_{[0,t]\times\mathbb{R}^d} dsdx \left(Q^{\mathcal{G}_s}\mu_t[\chi_{s,x}]\right)^2$$

$$
\begin{aligned}
&\geq \int_{[0,t]\times\mathbb{R}^d} dsdx\,(Q\mu_t[\chi_{s,x}])^2 \\
&\geq \int_0^t ds \int_{\Lambda_t} dx\,(Q\mu_t[\chi_{s,x}])^2 \\
&\geq \frac{1}{|\Lambda_t|}\int_0^t ds\,(Q\mu_t[|\Lambda_t \cap U(\omega_s)|])^2
\end{aligned}
$$

Observe at this point, that $\lambda(-|\beta|)^2 > 0$ since $\beta > 0$. Now, note that

$$
Q\mu_t[|\Lambda_t \cap U(\omega_s)|] \geq Q\mu_t\{U(\omega_s) \subset \Lambda_t\} \geq Q\mu_t\{|\omega_s| \leq C + Ct^\xi\}.
$$

Therefore, with some $c_1 = c_1(d, C) \in (0, \infty)$,

$$
\begin{aligned}
\varliminf_{t\nearrow\infty} t^{-(1-d\xi)}\mathrm{Var}_Q(\ln Z_t) &\geq c_1 \varliminf_{t\nearrow\infty} t^{-1}\int_0^t \left(Q\mu_t\{|\omega_s| \leq C + Ct^\xi\}\right)^2 ds \\
&\geq c_1 \varliminf_{t\nearrow\infty} \inf_{0\leq s\leq t} \left(Q\mu_t\{|\omega_s| \leq C + Ct^\xi\}\right)^2 .
\end{aligned}
$$

□

6 Proof of Theorem 2.2.1

6.1 Proof of part (a)

We first prove the existence of the limit (2.7). It is convenient to introduce

$$
Z^x_{(s,s+t]} = P^x\left[\exp\left(\beta\int_{(s,s+t]\times\mathbb{R}^d} \eta(dudy)\chi_{u-s,y}\right)\right], \quad t, s \geq 0. \tag{6.1}
$$

Note that $Z^x_{(s,s+t]}$ has the same law as Z_t and is independent of $\mathcal{G}_s$. We have by Markov property that

$$
Z_{s+t} = Z_s \int \mu_s\{\omega_s \in dx\} Z^x_{(s,s+t]}
$$

and then by Jensen's inequality that

$$
\ln Z_{s+t} \geq \ln Z_s + \int \mu_s\{\omega_s \in dx\} \ln Z^x_{(s,s+t]} .
$$

Taking expectation and using independence, we obtain

$$
\begin{aligned}
Q[\ln Z_{s+t}] &\geq Q[\ln Z_s] + Q\Big[\int \mu_s\{\omega_s \in dx\} Q[\ln Z^x_{(s,s+t]}|\mathcal{G}_s]\Big] \\
&= Q[\ln Z_s] + Q[\ln Z_t] ,
\end{aligned}
$$

i.e., $Q[\ln Z_t]$ is super-additive. Hence the following limit exists by the sub-additive lemma.

$$
\psi(\beta) \stackrel{\text{def.}}{=} \lim_{t\nearrow\infty} -\frac{1}{t}Q[\ln Z_t] + \lambda(\beta) = \inf_{t>0} -\frac{1}{t}Q[\ln Z_t] + \lambda(\beta).
$$

Moreover, $\psi(\beta) \geq 0$, since $Q[\ln Z_t] \leq \ln Q[Z_t] \leq \lambda t$.

We next prove (2.5). The convergence in $L^2(Q)$ follows from (2.7) and (2.28). By Lemma 5.3.1, the almost sure convergence follows from the convergence along a sequence $t = 1, 2, \ldots$. But this is a simple consequence of (2.7), (2.29) and the Borel-Cantelli lemma. □

6.2 Proof of part (b)

We now introduce short-hand notation:

$$\varphi_t(\beta) = \frac{1}{t} Q \ln Z_t \quad \text{and} \quad \psi_t(\beta) = \lambda(\beta) - \varphi_t(\beta) \; . \tag{6.2}$$

It is easy to see that $\varphi_t(\cdot)$ is convex. By (6.2), the limit $\varphi(\beta) = \lim_{t \nearrow \infty} \varphi_t(\beta)$ exists, convex in β, and $\varphi(\beta) = \lambda(\beta) - \psi(\beta)$. Thus, the function ψ inherits all the properties stated in Theorem 2.2.1 (b) from the convexity of φ. For example, we have by the convexity of φ

$$\varphi'_-(\beta) \le \varliminf_{t \nearrow \infty} \varphi'_t(\beta) \le \varlimsup_{t \nearrow \infty} \varphi'_t(\beta) \le \varphi'_+(\beta) \; ,$$

and hence

$$\psi'_+(\beta) \le \varliminf_{t \nearrow \infty} \psi'_t(\beta) \le \varlimsup_{t \nearrow \infty} \psi'_t(\beta) \le \psi'_-(\beta). \tag{6.3}$$

□

7 Proof of Theorem 2.2.2 and Theorem 2.3.1

7.1 Differentiating the averaged free energy

It is straightforward to check that, for a Poisson variable Y with parameter θ, the identity $\mathbf{E} Y f(Y) = \theta \mathbf{E} f(Y+1)$ holds for all $f : \mathbb{N} \to \mathbb{R}_+$. The following statement is the analogous property for the Poisson point process, which is useful here.

Proposition 7.1.1 *For $h : [0,t] \times \mathbb{R}^d \times \mathcal{M} \to \mathbb{R}_+$ a measurable function, we have*

$$Q\left[\int h(s,x;\eta_t)\eta_t(dsdx) \right] = \int_{[0,t]\times\mathbb{R}^d} dsdx Q\left[h(s,x;\eta_t + \delta_{s,x}) \right] \; .$$

Proof: Recall the (shifted) Palm measure $Q_{s,x}$ of the point process η_t, which can be thought of as the law of η_t "given that $\eta_t\{(s,x)\} = 1$": By definition of the Palm measure,

$$Q[\int h(s,x;\eta_t)\eta_t(dsdx)] = \int_{[0,t]\times\mathbb{R}^d} dsdx \int_{\mathcal{M}} h(s,x;\eta) Q_{s,x}(d\eta) \; .$$

By Slivnyak's theorem [24, page 50] for the Poisson point process η_t, the Palm measure $Q_{s,x}$ is the law of $\eta_t + \delta_{s,x}$, hence the right-hand-side of the above formula is equal to the right-hand-side of the desired formula. □

Recall the notation (6.2). We now use Proposition 7.1.1 to prove the following

Lemma 7.1.2 *For all $\beta \in \mathbb{R}$,*

$$t\psi'_t(\beta) = \lambda e^{\beta} \int_{[0,t]\times\mathbb{R}^d} dsdx \; Q \, \frac{[\mu_t(\chi_{s,x})]^2}{1 + \lambda \mu_t(\chi_{s,x})} \; . \tag{7.1}$$

Hence,

$$\lambda Q[J_t] \le t\psi'_t(\beta) \le e^{\beta} \lambda Q[J_t] \; . \tag{7.2}$$

Proof: We see from (6.2) that

$$\psi_t'(\beta) = \lambda'(\beta) - \varphi_t'(\beta) = e^\beta - \frac{1}{t} Q[\mu_t(\eta(V_t))]$$

By Fubini's theorem, and by Proposition 7.1.1,

$$\begin{aligned} Q[\mu_t(\eta(V_t))] &= Q \int \eta_t(dsdx)\mu_t[\chi_{s,x}] \\ &= Q \int \eta_t(dsdx) \frac{P[\chi_{s,x} e^{\beta\eta(V_t)}]}{P[e^{\beta\eta(V_t)}]} \\ &= Q \int_{[0,t]\times\mathbb{R}^d} dsdx \frac{P[\chi_{s,x} e^{\beta(\eta+\delta_{s,x})(V_t)}]}{P[e^{\beta(\eta+\delta_{s,x})(V_t)}]} \\ &= Q \int_{[0,t]\times\mathbb{R}^d} dsdx \frac{e^\beta P[\chi_{s,x} e^{\beta\eta(V_t)}]}{P[(\lambda\chi_{s,x}+1)e^{\beta\eta(V_t)}]} \\ &= e^\beta Q \int_{[0,t]\times\mathbb{R}^d} dsdx \frac{\mu_t[\chi_{s,x}]}{1+\lambda\mu_t[\chi_{s,x}]} . \end{aligned}$$

Since $t = \int_{[0,t]\times\mathbb{R}^d} dsdx\mu_t[\chi_{s,x}]$, it follows that

$$\begin{aligned} t\psi_t'(\beta) &= e^\beta Q \int_{[0,t]\times\mathbb{R}^d} dsdx \left[\mu_t[\chi_{s,x}] - \frac{\mu_t[\chi_{s,x}]}{1+\lambda\mu_t[\chi_{s,x}]} \right] \\ &= \lambda e^\beta \int_{[0,t]\times\mathbb{R}^d} dsdx \; Q \; \frac{[\mu_t(\chi_{s,x})]^2}{1+\lambda\mu_t(\chi_{s,x})} , \end{aligned}$$

i.e., (7.1). Now, since $e^{-\beta^-} \le 1 + \lambda\mu_t(\chi_{s,x}) \le e^{\beta^+}$, where $\beta^\pm = \max\{0, \pm\beta\}$, (7.1) implies (7.2) by definition of $J_t = \int_{[0,t]\times\mathbb{R}^d} dsdx[\mu_t(\chi_{s,x})]^2$. □

7.2 Proof of Theorem 2.2.2 (a)

We see from (7.2) and (6.3) that there are $c_i = c_i(\beta) \in (0,\infty)$, $i = 1, 2$ such that the following hold;

$$\text{If } \beta > 0, \text{ then} \quad c_1\psi_+'(\beta) \le \underline{\lim}_{t\nearrow\infty} \tfrac{1}{t} Q[J_t] \le \overline{\lim}_{t\nearrow\infty} \tfrac{1}{t} Q[J_t] \le c_2\psi_-'(\beta). \tag{7.3}$$

$$\text{If } \beta < 0, \text{ then} \quad -c_1\psi_-'(\beta) \le \underline{\lim}_{t\nearrow\infty} \tfrac{1}{t} Q[J_t] \le \overline{\lim}_{t\nearrow\infty} \tfrac{1}{t} Q[J_t] \le -c_2\psi_+'(\beta). \tag{7.4}$$

These imply that $0 \le \psi_-'(\beta)$ for $\beta \in \mathbb{R}_+$, and $\psi_+'(\beta) \le 0$ for $\beta \in \mathbb{R}_-$, from which the monotonicity of ψ on $\mathbb{R}_\pm$ follows.

The upper bound of $\psi(\beta)$ in (2.8) for $\beta \ge 0$ follows immediately from the definition (2.7) and that $Z_t \ge 1$. For $\beta < 0$, we see from (6.3) and (7.2) that

$$\begin{aligned} -\psi(\beta) &= \int_\beta^0 \psi_-'(\gamma)d\gamma \\ &\ge \int_\beta^0 \lambda(\gamma)d\gamma \\ &= 1 + \beta - e^\beta. \end{aligned}$$

□

7.3 Proof of Theorem 2.2.2 (b)

We first explain the strategy of the proof. With a parameter $\gamma > 0$ which we will introduce later on, we will define a set $\mathcal{M}_{t,\gamma} \subset \mathcal{M}$ such that the following hold for β large enough;

$$\lim_{t \nearrow \infty} Q[\mathcal{M}_{t,\gamma}] = 1 \tag{7.5}$$

and

$$Q[Z_t : \mathcal{M}_{t,\gamma}] \leq 2 \exp\left(C_1 t \lambda^{1/2}\right), \tag{7.6}$$

where C_1 is a constant. We can then conclude (2.9) as follows. Observe that we have

$$\begin{aligned} |\, Q\left[\ln Z_t | \mathcal{M}_{t,\gamma}\right] - Q\left[\ln Z_t\right] \,| &\leq Q\left[|\ln Z_t - Q[\ln Z_t]| \mid \mathcal{M}_{t,\gamma}\right] \\ &\leq Q\left[|\ln Z_t - Q[\ln Z_t]|\right]/Q[\mathcal{M}_{t,\gamma}] \\ &= \mathcal{O}(\sqrt{t}) \end{aligned} \tag{7.7}$$

by the concentration property (2.29) and (7.5). Therefore, by (7.5), (7.6), (7.7) and Jensen inequality,

$$\begin{aligned} \lambda(\beta) - \psi(\beta) &= \lim_{t\to\infty} \frac{1}{t} Q\left[\ln Z_t\right] \\ &= \lim_{t\to\infty} \frac{1}{t} Q\left[\ln Z_t \mid \mathcal{M}_{t,\gamma}\right] \\ &\leq \liminf_{t\to\infty} \frac{1}{t} \ln Q\left[Z_t \mid \mathcal{M}_{t,\gamma}\right] \\ &= \liminf_{t\to\infty} \frac{1}{t} \ln Q\left[Z_t \,;\, \mathcal{M}_{t,\gamma}\right] \\ &\leq C_1 \lambda^{1/2}\,. \end{aligned}$$

This, together with (2.8), shows (2.9).

Let us now turn to the construction of the set $\mathcal{M}_{t,\gamma}$ alluded to above. For $a, t > 0$ and $\{f, g\} \subset \Omega$, we denote by $\rho_t(f,g) = \sup_{s\in[0,t]} |f(s) - g(s)|$ the distance induced by the sup-norm on $C([0,t] \to \mathbb{R}^d)$ and by $\mathcal{K}_{t,a}$ the set of absolutely continuous function $f : \mathbb{R} \to \mathbb{R}^d$, such that $f(0) = 0$ and

$$\frac{1}{t}\int_0^t |\dot{f}(s)|^2 ds \leq a\,.$$

Lemma 7.3.1 *There exist a finite constant $a_0 > 0$ and a function $b(a) > 0$ defined for $a > a_0$, such that*

$$P[\rho_t(\omega, \mathcal{K}_{t,a}) \geq 1] \leq \exp -b(a)t$$

for large t, and

$$C_0 = \liminf_{a\to\infty} b(a)/a > 0\,.$$

We postpone the proof of Lemma 7.3.1, in order to finish the construction of $\mathcal{M}_{t,\gamma}$.

We first cover the set $\mathcal{K}_{t,a}$ with finite numbers of unit ρ_t-balls. The point here is that the number of the balls we need is bounded from above explicitly in terms of a and t as we explain now. By a result of Birman and Solomjak, Theorem 5.2 in [2] (taking there $p = 2, \alpha = 1, q = \infty, m = 1, \omega = 1$), for all $\varepsilon > 0$, the set $\mathcal{K}_{1,1}$ can be covered by a number smaller than $\exp\{C(d)\varepsilon^{-1}\}$ of ρ_1-balls with radius ε. Since, for $a, t > 0$, a map

$f \mapsto g$, $g(u) = (ta^{1/2})^{-1} f(ut)$ defines a bijection from $\mathcal{K}_{t,a}$ to $\mathcal{K}_{1,1}$, it follows that, we can find $f_i \in \mathcal{K}_{t,a}, 1 \le i \le i_0 \le \exp\{C(d)ta^{1/2}\}$, such that

$$\mathcal{K}_{t,a} \subset \bigcup_{i \le i_0} \Big\{ f \in \Omega : \rho_t(f, f_i) \le 1 \Big\} .$$

We next introduce a tube $V_t(f, R) \subset \mathbb{R}_+ \times \mathbb{R}^d$ ($t > 0$, $R > 0$) around the graph of a function $f : [0, t] \to \mathbb{R}^d$ by

$$V_t(f, R) = \{(s, x) \; ; \; s \in (0, t], \; |x - f(s)| \le R\}, \tag{7.8}$$

For $\gamma > 1$ to be chosen later, we consider the set $\mathcal{M}_{t,\gamma}$ of "good environments":

$$\mathcal{M}_{t,\gamma} = \bigcap_{i=1}^{i_0} \Big\{ \eta \in \mathcal{M} \; ; \; \eta(V_t(f_i, r+1)) \le \gamma \left(\tfrac{r+1}{r}\right)^d t \Big\} \; ,$$

with r the radius of $U(0)$. Since $\eta(V_t(f_i, r+1))$ has Poisson distribution with mean $\left(\frac{r+1}{r}\right)^d t$, we have, see Cramér's theorem, for all $t > 0$,

$$\begin{aligned} Q[\mathcal{M}_{t,\gamma}^c] &\le \sum_{i \le i_0} Q[\eta(V_t(f_i, r+1)) > \gamma \left(\tfrac{r+1}{r}\right)^d t] \\ &\le \exp\{-t[\left(\tfrac{r+1}{r}\right)^d \lambda^*(\gamma) - C(d) a^{1/2}]\} \; , \end{aligned} \tag{7.9}$$

where $\lambda^*(u) = \sup_{\beta \in \mathbb{R}} \{\beta u - \lambda(\beta)\} = u \ln u - u + 1, u > 0$. We will eventually make this term small by choosing our parameters. We estimate the expectation of Z_t on the set $\mathcal{M}_{t,\gamma}$ by

$$\begin{aligned} Q\Big[Z_t; \mathcal{M}_{t,\gamma}\Big] &= Q\Big[P[\zeta_t; \rho_t(\omega, \mathcal{K}_{t,a}) < 1]; \mathcal{M}_{t,\gamma}\Big] + Q\Big[P[\zeta_t; \rho_t(\omega, \mathcal{K}_{t,a}) \ge 1]; \mathcal{M}_{t,\gamma}\Big] \\ &\le Q\left[\exp\Big\{\beta \max_{i \le i_0} \eta[V_t(f_i, r+1)]\Big\}; \mathcal{M}_{t,\gamma}\right] + P\Big[Q[\zeta_t]; \rho_t(\omega, \mathcal{K}_{t,a}) \ge 1\Big] \\ &\le \exp\{\beta\gamma \left(\tfrac{r+1}{r}\right)^d t\} + \exp\{\lambda t - b(a) t\} \; , \end{aligned} \tag{7.10}$$

using Lemma 7.3.1.

Choose now $a = \lambda/(2C_0)$, with C_0 from the Lemma 7.3.1, and $\gamma = 2C(d)C_0^{-1/2}\lambda^{1/2}/\beta$. For large β, we have $a > a_0$, $\gamma > 1$ and $b(a) \ge \lambda$. Then, there exist finite, positive constants β_0, C_1, such that for $\beta > \beta_0$, we have (7.5) due to (7.9), while (7.6) can be obtained from (7.10).

□

Proof of Lemma 7.3.1: Let $\tau_0 = 0$, and $\tau_n, n \ge 1$, be the successive times when the Brownian path moves a unit distance,

$$\tau_n = \inf\{t > \tau_{n-1}; |\omega_t - \omega_{\tau_{n-1}}| = 1\} \; .$$

Then, by the strong Markov property, the sequence $\sigma_n = \tau_n - \tau_{n-1}$, $n \ge 1$ is independent identically distributed. For $\omega \in \Omega$, we let $\ell(\omega) = (\ell_t(\omega))_{t \ge 0}$ denote a "skelton" of the path ω defined by linearly interpolating $\ell_{\tau_n}(\omega) = \omega_{\tau_n}$, $n \ge 0$. By definition,

$$\rho_t(\omega, \ell(\omega)) \le 1 \; , \quad \int_0^{\tau_n} |\dot{\ell}_s(\omega)|^2 ds = \sum_{k=1}^n \sigma_k^{-1} \; . \tag{7.11}$$

With $N(t) = \inf\{n \geq 1 : \tau_n \geq t\}$ and $\alpha > 0$, it holds

$$
\begin{aligned}
\Big\{\omega\ ;\ \rho_t(\omega, \mathcal{K}_{t,a}) \geq 1\Big\} &\subset \Big\{\omega\ ;\ \ell(\omega) \notin \mathcal{K}_{t,a}\Big\} \\
&= \Big\{\omega\ ;\ \int_0^t |\dot{\ell}_s(\omega)|^2 ds > at\Big\} \\
&\subset \Big\{\omega\ ;\ N(t) > \alpha t\Big\} \bigcup \Big\{\omega\ ;\ \int_0^{\tau_{[\alpha t]+1}} |\dot{\ell}_s(\omega)|^2 ds > at\Big\} \\
&\subset \Big\{\omega\ ;\ \sum_{n \leq \alpha t} \sigma_n < t\Big\} \bigcup \Big\{\omega\ ;\ \sum_{n \leq \alpha t + 1} \sigma_n^{-1} > at\Big\}\ , \qquad (7.12)
\end{aligned}
$$

using (7.11). Clearly, $a_0 := P[\tau_1^{-1}]/P[\tau_1]$ is finite. For $a > a_0$, we will choose α such that $\alpha < a/P[\tau_1^{-1}]$ and $\alpha > 1/P[\tau_1]$. Cramér's theorem [8] holds for both sequences of variables σ_n and σ_n^{-1}, with respective rate functions I_1 and I_{-1} defined on $(0, \infty)$. Moreover,

$$
P(\tau_1 \leq u) = \exp\left(-\tfrac{1}{2}(u - o(1))\right) \qquad \text{as } u \searrow 0\ . \qquad (7.13)
$$

By (7.13), we have for $v \geq 0$,

$$
\begin{aligned}
P[\exp(v\tau_1^{-1})] &= P[\exp(v\tau_1^{-1}); \tau_1^{-1} \leq 1] + P[\exp(v\tau_1^{-1}); \tau_1^{-1} > 1] \\
&\leq \exp(v) + c_1(c_2 - v)^{-1}\ ,
\end{aligned}
$$

with constants $c_1, c_2 > 0$, which implies that, for $u \geq P[\tau_1^{-1}]$,

$$
\begin{aligned}
I_{-1}(u) &:= \sup_{v \geq 0}\{uv - \ln P[\exp(v\tau_1^{-1})]\} \\
&\geq (c_2/2)u - c_3 \qquad (7.14)
\end{aligned}
$$

by choosing $v = c_2/2$. Substituting now Cramér's bound

$$
\begin{aligned}
\frac{1}{\alpha t} \ln P\Big[\sum_{n \leq \alpha t} \sigma_n < t\Big] &\leq -I_1(\alpha^{-1}) + o(1)\ , \\
\frac{1}{\alpha t} \ln P\Big[\sum_{n \leq \alpha t + 1} \sigma_n^{-1} > at\Big] &\leq -I_{-1}(\alpha^{-1}a) + o(1)
\end{aligned}
$$

in (7.12), we get

$$
\frac{1}{t} \ln P[\rho_t(\omega, \mathcal{K}_{t,a}) \geq 1] \leq -\min\{\alpha I_1(\alpha^{-1}); c_2 a/2 - c_3\alpha\} + o(1)\ .
$$

Hence, we obtain the desired result by taking $\alpha = c_4 a$ with a small positive constant c_4. □

7.4 Proof of Theorem 2.2.2(c)

We define

$$
\beta_c^+ = \sup\{\beta \geq 0\ ;\ \psi(\beta) = 0\}, \quad \beta_c^- = \inf\{\beta \leq 0\ ;\ \psi(\beta) = 0\}
$$

Then, all the desired properties of $\beta_c^\pm$ follow from what we have already seen (ψ is monotone, continuous on $\mathbb{R}_\pm$ and diverges as $\beta \nearrow \infty$), except the differentiability at $\beta \in \{\beta_c^-, \beta_c^+\} \cap \mathbb{R}$. But this is easy to see. Suppose for example that $\beta_c^+ < \infty$. Then, $\psi'_-(\beta_c^+) \leq 0 \leq \psi'_+(\beta_c^+)$, since β_c^+ is a minimal point of ψ. This, together with (2.6) proves $\psi'_\pm(\beta_c^+) = 0$. □

7.5 Proof of Theorem 2.3.1

All the statements are consequences of (7.3), (7.4) and the following observations:

$$0 < \psi(\beta) = \int_{\beta_c^+}^{\beta} \psi'_+(\gamma)d\gamma \ \text{ for } \beta > \beta_c^+,$$
$$0 < \psi(\beta) = \int_{\beta_c^-}^{\beta} \psi'_-(\gamma)d\gamma \ \text{ for } \beta < \beta_c^-.$$

□

8 Proof of Theorem 2.3.2

To conclude (2.19) and (2.20), it is enough to show the following (8.1) and (8.2):

$$\{W_\infty = 0\} \subset \left\{ \int_0^\infty I_s ds = \infty \right\}, \quad Q\text{-a.s.} \tag{8.1}$$

There are $c_1, c_2 \in (0, \infty)$ such that

$$\left\{ \int_0^\infty I_s ds = \infty \right\} \subset \left\{ c_1 \int_0^t I_s ds \le \lambda t - \ln Z_t \le c_2 \int_0^t I_s ds \ \text{ for large enough } t \right\}, \quad Q\text{-a.s.} \tag{8.2}$$

The proof of (8.1) and (8.2) is based on Doob's decomposition with respect to the filtration $(\mathcal{G}_t, t \ge 0)$,

$$\lambda t - \ln Z_t = M_t + A_t \ .$$

In view of (5.9), the martingale part M_t and the increasing part A_t are given by

$$\begin{aligned} M_t &= -\int \overline{\eta}_t(dsdx) \ln(1 + \lambda \mu_{s-}[\chi_{s,x}]), \\ A_t &= \int_{[0,t)\times\mathbb{R}^d} \varphi(\lambda \mu_s[\chi_{s,x}]) dsdx, \end{aligned} \tag{8.3}$$

where $\varphi(u) = u - \ln(1+u) \ge 0$, $-1 < u$. It is clear that there are constants $c_i = c_i(\beta) \in (0, \infty)$ such that

$$c_1 u^2 \le \varphi(u) \le c_2 u^2, \quad \ln^2(1+u) \le c_2 u^2 \quad \text{for all } u \in [\lambda \wedge 0, \lambda \vee 0], \tag{8.4}$$

and hence

$$c_1 \int_0^t I_s ds \le A_t \ \le \ c_2 \int_0^t I_s ds, \tag{8.5}$$

$$\langle\, M \,\rangle_t \ \le \ c_2 \int_0^t I_s ds. \tag{8.6}$$

Note also that we have (5.7) and (5.8) for the martingale M_t defined by (8.3). We now conclude (8.1) from (8.5), (8.6) and (5.7) as follows (the equalities and the inclusions here being understood as Q-a.s.):

$$\begin{aligned} \left\{ \int_0^\infty I_s ds < \infty \right\} &= \{A_\infty < \infty\} \\ &= \{A_\infty < \infty, \ \langle\, M \,\rangle_\infty < \infty\} \\ &\subset \{A_\infty < \infty, \ \lim_{t \nearrow \infty} M_t \text{ exists and is finite}\} \\ &\subset \{W_\infty > 0\}. \end{aligned}$$

Finally we prove (8.2). By (8.5), it is enough to show that

$$\{A_\infty = \infty\} \subset \left\{ \lim_{t \nearrow \infty} \frac{\lambda t - \ln Z_t}{A_t} = 1 \right\}, \quad Q\text{-a.s.} \tag{8.7}$$

Thus, let us suppose that $A_\infty = \infty$. If $\langle M \rangle_\infty < \infty$, then $\lim_{t \nearrow \infty} M_t$ exists and is finite and therefore (8.7) holds. If, on the contrary, $\langle M \rangle_\infty = \infty$, then

$$\frac{\lambda t - \ln Z_t}{A_t} = \frac{M_t}{\langle M \rangle_t} \frac{\langle M \rangle_t}{A_t} + 1 \to 1 \quad Q\text{-a.s.}$$

by (8.5), (8.6) and (5.8). This completes the proof of Theorem 2.3.2.

□

We end this section with the proof of the inequalities (2.23). Since

$$\mu_t^{x,x}\left[|U(\omega_s) \cap U(\widetilde{\omega}_s)|\right] = \int_{\mathbb{R}^d} \mu_t^x \left[\omega_s \in U(z)\right]^2 dz \ ,$$

the right-hand-side inequality follows immediately from (3.5). To prove the left-hand-side, we introduce a smaller ball $\frac{1}{2}U(0) = \{\frac{1}{2}z \ ; \ z \in U(0)\}$. By the Schwarz and the triangle inequalities,

$$\begin{aligned}
\int_{\mathbb{R}^d} \mu_t^x \left[\omega_s \in U(z)\right]^2 dz &\geq \left|\tfrac{1}{2}U(0)\right|^{-1} \left(\int_{y+\frac{1}{2}U(0)} \mu_t^x \left[\omega_s \in U(z)\right] dz \right)^2 \\
&\geq 2^d \left(\int_{y+\frac{1}{2}U(0)} \mu_t^x \left[\omega_s \in y + \tfrac{1}{2}U(0)\right] dz \right)^2 \\
&= 2^{-d} \mu_t^x \left[\omega_s \in y + \tfrac{1}{2}U(0)\right]^2 \ .
\end{aligned}$$

By additivity of μ_t^x,

$$\sup_{y \in \mathbb{R}^d} \mu_t^x \left[\omega_s \in U(y)\right] \leq c_2 \sup_{y \in \mathbb{R}^d} \mu_t^x \left[\omega_s \in y + \tfrac{1}{2}U(0)\right] \ ,$$

with $c_2 = c_2(d)$ the minimal number of translates of $\frac{1}{2}U(0)$ necessary to cover $U(0)$. Combining these two inequalities, we finish the proof. □

9 Proof of Theorem 2.4.4

For $\theta \in \mathbb{R}^d$ and $\alpha \geq 1/2$, we define the transformation T_θ^α on $\mathbb{R}_+ \times \mathbb{R}^d$ by

$$T_\theta^\alpha(t, x) = (t, x + t^{2\alpha - 1}\theta) \ ,$$

and, for $t \geq 0$, the function $\Theta_t : s \mapsto (s \wedge t)\theta$. We will abuse the notation slightly and write T_θ^α also for the induced transformation $\eta \mapsto \eta \circ (T_\theta^\alpha)^{-1}$ on $\mathcal{M}$. By Girsanov's formula, the process $\overline{\omega} = \omega - \Theta_t$ is a Brownian motion under the probability $\overline{P}$, $\overline{P}(d\omega) = \exp(\theta \cdot \omega_t - t|\theta|^2/2) P(d\omega)$, and therefore

$$\begin{aligned}
P[\zeta_t(\omega, \eta) \exp(\theta \cdot \omega_t - t|\theta|^2/2)] &= \overline{P}[\exp\left(\beta\eta[V_t(\overline{\omega} + \Theta_t)]\right)] \\
&= P[\exp\left(\beta\eta[T_\theta^1(V_t)]\right)] \\
&= Z_t \circ T_{-\theta}^1 \ .
\end{aligned} \tag{9.1}$$

We see from (9.1) that

$$\ln \mu_t[\exp(\theta \cdot \omega_t)] = t|\theta|^2/2 + \ln Z_t \circ T^1_{-\theta} - \ln Z_t$$

and hence that

$$\ln \mu_t[\exp(t^{\xi-1}\theta \cdot \omega_t)] = t^{2\xi-1}|\theta|^2/2 + \ln Z_t \circ T^{\xi}_{-\theta} - \ln Z_t. \tag{9.2}$$

Observe that $T^{\xi}_{-\theta}\eta$ is a Poisson process with intensity measure $dtdx$, which implies that $(Z_t \circ T^{\xi}_{-\theta})_{t\geq 0}$ has the same distribution as $(Z_t)_{t\geq 0}$, and also $Q[\ln Z_t \circ T^{\xi}_{-\theta}] = Q[\ln Z_t]$. Combining this with (9.2) and (2.37), we get

$$\lim_{t\nearrow\infty} \frac{1}{t^{2\xi-1}} \ln \mu_t[\exp(t^{\xi-1}\theta \cdot \omega_t)] = |\theta|^2/2, \quad Q\text{-a.s.}$$

This implies (2.38) via the Gärtner-Ellis-Baldi theorem [8, page 44,Theorem 2.3.6]. □

10 The stochastic heat equation

We prove (1.10) in the following formulation.

Proposition 10.0.1 *For every compactly supported*[5] *test function* $\Psi \in C^2(\mathbb{R}^d)$*, it holds* Q*-a.s.,*

$$\begin{aligned} &\int_{\mathbb{R}^d} Z^x_t(y)\Psi(y)dy \\ &= \Psi(x) + \tfrac{1}{2}\int_0^t ds \int_{\mathbb{R}^d} Z^x_s(y)\Delta\Psi(y)dy + \lambda \int_{\mathbb{R}^d} \Psi(y)dy \int_{(0,t]} Z^x_{s-}(y)\eta(ds\times U(y)) \end{aligned} \tag{10.1}$$

for all $t \geq 0, x, y \in \mathbb{R}^d$*. In particular,* $Z^x_t(y)dy \to \delta_x$ *weakly as* $t \searrow 0$*.*

Proof: Note that $\int_{\mathbb{R}^d} Z^x_t(y)\Psi(y)dy = P^x[\zeta_t\Psi(\omega_t)]$. We obtain by first applying Itô's formula to $\zeta_t\Psi(\omega_t)$, and then by taking P^x-expectation that Q-a.s.,

$$\begin{aligned} &\int_{\mathbb{R}^d} Z^x_t(y)\Psi(y)dy \\ &= \Psi(x) + \tfrac{1}{2}\int_0^t P^x[\zeta_s\Delta\Psi(\omega_s)]ds + \lambda\int_{(0,t]\times\mathbb{R}^d} \eta(dsdz)P^x[\zeta_{s-}\chi_{s,z}\Psi(\omega_s)] \\ &= \Psi(x) + \tfrac{1}{2}\int_0^t ds \int_{\mathbb{R}^d} Z^x_s(y)\Delta\Psi(y)dy + \lambda\int \Psi(y)dy \int_{(0,t]} \eta(ds\times U(y))Z^x_{s-}(y), \end{aligned}$$

which proves (10.1). Here, we have used Fubini's theorem on the last line. But this can easily be justified. For example,

$$\begin{aligned} Q\int_{(0,t]\times\mathbb{R}^d} \eta(dsdz)P^x[\zeta_{s-}\chi_{s,z}|\Psi(\omega_s)|] &= \int_0^t Q[\zeta_{s-}]P^x[|\Psi(\omega_s)|]ds \\ &= \int_0^t e^{\lambda s}P^x[|\Psi(\omega_s)|]ds < \infty. \end{aligned}$$

□

[5]It is enough to assume that Ψ is C^2 and that it is polynomially bounded at infinity together with its first and second derivatives.

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