Spectra of soft ring graphs

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We discuss of a ring-shaped soft quantum wire modeled by δ interaction supported by the ring of a generally nonconstant coupling strength. We derive condition which determines the discrete spectrum of such systems, and analyze the dependence of eigenvalues and eigenfunctions on the coupling and ring geometry. In particular, we illustrate that a random component in the coupling leads to a localization. The discrete spectrum is investigated also in the situation when the ring is placed into a homogeneous magnetic field or threaded by an Aharonov-Bohm flux and the system exhibits persistent currents.

1 Introduction

The aim of the present paper is to investigate spectral properties of a two-dimensional quantum particle subject to a δ interaction supported by a circular ring, plus possibly a magnetic field. Similar systems were studied for a quite a long time. One can mentioned an early paper by Kurilev [Ku] and later work [BT, BEKŠ] devoted to measure-type perturbations of the Laplacian where some general spectral results were derived[1]. Later this line of

[1]Recall also a related model of a photonic crystal studied in [?, ?]. It differs from the present by the switched roles of the coupling and spectral parameters.

study was extended to the situation when a singular zero-measure set itself supports a nontrivial dynamics [Ka, KK, Ko].

In the most part of the mentioned work the emphasis was laid on methods to treat such singular interactions. Recently the problem was put into a different perspective related to a specific physical model describing idealized “quantum wires”. These tube-like or graph-like semiconductor structures are usually modeled as strips or tubes with hard walls, or in a more simplified version by graphs supporting “one-dimensional” electrons. A weak point of such models is the assumption about the nature of the confinement. In actual quantum wires, the electrons are trapped due to interfaces between two different semiconductor materials which represents a finite potential jump. Thus electrons can be found also outside such a wire, although not “too far” since the exterior represents here a classically forbidden region.

Recent investigations revealed interesting relations between the spectrum and geometry of such a “soft” quantum wire, namely the existence of curvature-induced bound states [EI]. Moreover, it was demonstrated that in the strong-coupling limit the negative part of the spectrum of such Hamiltonians approaches that of an “ideal” curve-like wire with an effective potential determined by the curvature [EY1, EY2]. On the other hand, apart of a few very simple examples we are lacking solvable models of such systems.

In the present paper we analyze a simple model with a δ interaction supported by a circle of radius R and a generally non-constant coupling strength. In particular, we are interested in the discrete spectrum of such ring systems. First we illustrate that it is rather the geometry of the interaction curve than its topology which determines the spectral properties. To this aim we discuss in Section 2 the case of the “full” and “broken” ring, the latter being obtained by putting the coupling constant zero at its segment; we will show that the spectrum depends substantially on the ability of the particle to tunnel though the “gap”. Another natural question concerns spectral properties in the situation when the coupling strength is randomly varying. It is customary to investigate it in an infinite-length setting as a passage from absolutely continuous to dense pure point spectrum [**?**]. At the same time the localization effect can be seen in systems with perturbations of a finite length too if we observe the characteristic size of the wavefunction; we will illustrate this fact in the framework of the present model.

Another situation we are going to discuss concerns the case when the particle is exposed to a magnetic field perpendicular to the ring plane. In Section 3 we discuss two models, one with a homogeneous background field and the other with a pointlike Aharonov-Bohm flux piercing the centre of the ring. For an ideal wire these situations are, of course, equivalent because the only quantity which counts is the flux through the loop. This is no longer true in the soft case when the particle wave packet extends outside the ring.

An important question concerns the existence of current-carrying states. Recall that persistent currents in rings threaded by a magnetic flux are one of the characteristic features of mesoscopic systems – see, e.g., [CGR, CWB] and numerous other theoretical and experimental papers where they were discussed. If an electron is strictly confined to a loop Γ the effect is contained in the dependence of the corresponding eigenvalues E_j on the flux ϕ threading the loop (measured in the units of flux quanta, $2\pi\hbar c|e|^{-1}$); the persistent current I_j in the j–th state is defined as the multiple $-c\partial E_j/\partial\phi$ of the eigenvalue derivative w.r.t. the flux. In particular, if the particle motion on such a loop is free, we have

$$E_j(\phi) = \frac{\hbar^2}{2m^*}\left(\frac{2\pi}{L}\right)^2 (j+\phi)^2, \tag{1.1}$$

where L is the loop circumference, so the currents depend linearly on the applied field. For a soft quantum wire in the form of a closed curve with an attractive coupling which is constant around the loop and strong enough the existence of persistent currents was demonstrated in [?]. An illustration of this effect can be seen in a related model [?] in which a curve supported δ-coupling is replaced by an array of two-dimensional point interaction. In analogy with this paper we also expect that a randomization of the coupling will destroy a coherent transport around the ring, however, since this problem is numerically very demanding we postpone it to a subsequent paper.

2 Formulation of the problem

Let $\Gamma := \{x\,:\, |x| = R\}$ with an $R>0$ be a circle in $\mathbb{R}^2$ and $\alpha:\,\Gamma\to\mathbb{R}$ a piecewise continuous function. We are going to discuss the Hamiltonian in $L^2(\mathbb{R}^2)$ which can be formally written as

$$H_{\text{form}} = (-i\nabla - A(r))^2 - \alpha(\varphi)\delta(r-R)\,, \tag{2.1}$$

where r,φ are the polar coordinates of a point x and A is a vector potential determining the magnetic field. We suppose that the latter is rotationally symmetric. For convenience we associate positive values of the coupling strength α with the attractive interaction. As usual we also simplify the treatment by using rationalized units, $\hbar = 2m^* = e = c = 1$.

The operator (2.1) can be defined properly, and quite generally, by means of the corresponding quadratic form. As long as the function α is sufficiently regular, however, one use alternatively boundary conditions [BEKŠ, Rem. 4.1], i.e. to consider the operator $H_{\alpha,A}$ acting as

$$(H_{\alpha,A}\psi)(x) = \left((-i\nabla - A)^2\psi\right)(x)\,, \quad x \in \mathbb{R}^2 \setminus \Gamma\,, \tag{2.2}$$

for any ψ of the domain consisting of functions which belong to $W_{2,2}(\mathbb{R}^2 \setminus \Gamma)$, are continuous at Γ with the normal derivatives having a jump there,

$$\left.\frac{\partial\psi}{\partial r}(x)\right|_{r=R+} - \left.\frac{\partial\psi}{\partial r}(x)\right|_{r=R-} = -\alpha(\varphi)\psi(x)\,, \quad x = (r,\varphi) \in \Gamma\,. \tag{2.3}$$

It is straightforward to check that $H_{\alpha,A}$ is e.s.a., so the Hamiltonian of our system corresponding to the formal expression (2.1) can be identified with its closure.

2.1 A warm-up: constant δ interaction on a ring

Let us consider first the simplest case when the magnetic field is absent, the coupling strength is constant along the ring and the interaction is attractive, $\alpha(\varphi) = \alpha > 0$. We look for the discrete spectrum which is in view of [BEKŠ, Thm. 4.2] located at the negative halfline. Since the system is rotationally symmetric we can perform the partial-wave decomposition and adopt the following Ansatz for the eigenfunctions

$$\psi_m(r,\varphi) = \rho_m(r)\,\mathrm{e}^{im\varphi}\,, \quad m \in \mathbb{Z}\,.$$

Outside the ring the solution coincides with that of the free Schrödinger equation, including the behaviour at the origin and at infinity, i.e.

$$\rho_m(r) = \begin{cases} c_1 I_m(\kappa r) & \dots \quad r \le R \\ c_2 K_m(\kappa r) & \dots \quad r \ge R \end{cases}$$

corresponding to the energy $-\kappa^2$. Using the continuity of the function ρ_m at $r = R$ together with the matching condition (2.3) and the Wronskian relation $W(K_m(z), I_m(z)) = z^{-1}$ we get the spectral condition

$$(I_m K_m)(\kappa R) = \frac{1}{\alpha R}\,. \tag{2.4}$$

To find its solutions, recall some properties of the modified Bessel functions [AS, Chap. 9]; since $I_{-m}(z) = I_m(z)$ and $K_{-m}(z) = K_m(z)$, we may consider $m \ge 0$ only. The l.h.s. of (2.4) is decreasing in $(0, \infty)$ and has the following asymptotics,

$$(I_m K_m)(z) = \begin{cases} -\ln\frac{z}{2} - \gamma + o(1) & \ldots \quad m = 0 \\ \frac{1}{2m} + \mathcal{O}(z) & \ldots \quad m \ne 0 \end{cases}$$

as $z \to 0$, where $\gamma = 0.577\ldots$ is the Euler number, and

$$(I_m K_m)(z) = \frac{1}{2z}\left[1 - \frac{2\left(m^2 - \frac{1}{4}\right)}{(2z)^2} + \mathcal{O}(z^{-3})\right]$$

as $z \to \infty$. Consequently, there is at most one solution at each partial wave which we have anticipated labeling the solution by the orbital index only. The ground state corresponding to $m = 0$ exists for any $\alpha > 0$, while other partial waves exhibit a bound state provided

$$\alpha R > 2|m|\,.$$

In the weak-coupling case, $\alpha R \ll 1$, the binding energy is exponentially small,

$$E_0 \approx -\frac{4}{R^2}\,\mathrm{e}^{-2/\alpha R},$$

as expected for a two-dimensional Schrödinger operator[2], while for $\alpha R \gg 1$ the above asymptotics gives

$$E_m = -\frac{\alpha^2}{4} + \frac{m^2 - \frac{1}{4}}{R^2} + \mathcal{O}(\alpha^{-2}R^{-4})\,.$$

This can be regarded as a particular case of the theorem proven in [EY1], just the error estimate is better than in the general case. The discrete spectrum for a fixed coupling strength as a function of R is plotted in Fig. 1.

One can continue the analysis of the rotationally symmetric δ-interaction on the ring by discussing the scattering, approximations by means of scaled potentials, combinations with other interactions, etc., in analogy in the three-dimensional case considered in [?]. We will not do that, however, because we are interested mainly in the situation without the rotational symmetry.

[2]Taking the limit $R \to 0$ with a properly scaled $\alpha(R)$ we can get a two-dimensional point interaction. If α depends on the angle, such a limit can be done using the resolvent formula of [BEKŠ] in analogy with the tree-dimensional case worked out recently in [?].

2.2 General formalism

Next we want to look what happens if the rotational symmetry is broken due to non-constancy of the function $\alpha(\cdot)$. In this case the point interaction couples different partial waves and we have to look for the solution in the form of a series. Since the scheme is similar for the magnetic case, or even for operators

$$H_{\alpha,A} + V(r)$$

with a rotationally invariant potential, we shall formulate it generally adopting the following Ansatz

$$\psi(r,\varphi) = \begin{cases} \sum_{m\in\mathbb{Z}} c_m f_m(r)\, \mathrm{e}^{im\varphi} & \dots \quad r \le R \\ \sum_{m\in\mathbb{Z}} d_m g_m(r)\, \mathrm{e}^{im\varphi} & \dots \quad r \ge R \end{cases} \tag{2.5}$$

where $\{c_m\}$, $\{d_m\}$ are coefficient sequences and f_m, g_m are solutions of the "free" Schrödinger equation with the energy $-\kappa^2$ inside and outside the ring, respectively, in the m-th partial wave; recall that by assumption the system without the point interaction is rotationally symmetric. We impose the standard requirement that the solution is regular at the origin and L^2 at infinity, then the functions f_m, g_m are unique up to multiplicative constants which can be absorbed into the coefficients.

The above function has naturally to belong to the domain of the Hamiltonian. Its continuity at $r = R$ together with the orthonormality of the trigonometric basis in $L^2((0,2\pi))$ implies

$$c_m = d_m\, \frac{g_m(R)}{f_m(R)}\,. \tag{2.6}$$

At the same time, the matching condition (2.3) yields

$$\sum_m d_m g_m'(R)\, \mathrm{e}^{im\varphi} - \sum_m c_m f_m'(R)\, \mathrm{e}^{im\varphi} = -\alpha(\varphi) \sum_m d_m g_m(R)\, \mathrm{e}^{im\varphi}\,;$$

multiplying this relation by $\mathrm{e}^{-in\varphi}$ and integrating w.r.t. φ we get an expression of c_n by means of the coefficients d_m. Combing it with (2.6) and introducing

$$\alpha_{mn} := \int_0^{2\pi} \alpha(\varphi)\, \mathrm{e}^{i(m-n)\varphi}\, \mathrm{d}\varphi\,, \quad \langle\alpha\rangle := \alpha_{mm} = \int_0^{2\pi} \alpha(\varphi)\, \mathrm{d}\varphi$$

we get after a simple manipulation

$$\sum_{m\in\mathbb{Z}} \Big\{\delta_{mn}\left[\langle\alpha\rangle f_n g_n + 2\pi W(f_n,g_n)\right](R) + (1-\delta_{mn})\alpha_{mn}(g_m f_n)(R)\Big\} d_m = 0 \tag{2.7}$$

as a set of equations for the coefficients, or in other words, an operator equation in the ℓ^2 space of the coefficients. Solving it numerically, one takes a family of truncated systems; the convergence of such an approximation is checked in the same way as in [EŠTV].

2.3 A broken ring

Consider now the case when the coupling strength is a step function,

$$\alpha(\varphi) = \alpha\chi_{[\theta/2,2\pi-\theta/2]}(\varphi)$$

for some $\alpha > 0$ and $\theta \in (0, 2\pi)$. Substituting $W(I_m(\kappa R), K_m(\kappa R)) = -R^{-1}$ (recall that the derivative is taken w.r.t. r) and

$$\alpha_{mn} = \begin{cases} \alpha(2\pi-\theta) & \dots \quad m = n \\ -\frac{2\alpha}{m-n}\sin\frac{\theta(m-n)}{2} & \dots \quad m \neq n \end{cases}$$

into (2.7) we get equations for the coefficients d_m in this case.

In an ideal quantum wire described by an appropriate Schrödinger operator on Γ it is the topology rather than geometry which determines the character of the discrete spectrum. For the full ring, $\theta = 0$, the spectrum is twice degenerate with the exception of the ground state, while a broken ring spectrum looks like that of an infinite deep rectangular well. The spectrum of a soft wire will be similar if the distance between the loose ends is large so that the tunneling between them is negligible. To get an idea at what values of the parameters the regime changes, recall that the eigenfunction of the one-center point interaction of the coupling strength $-\alpha$ on line is $\mathrm{e}^{-\alpha|x|/2}$ [AGHH], and thus the characteristic "size" of the bound-state wave function measured from the ring is $2\alpha^{-1}$. Comparing this to the "gap size" we see that the switch from the full-ring-type spectrum to the infinite-well-type takes place at $\alpha \approx 2\theta R$. This is illustrated in Fig. 2 where we plot the spectrum as a function of the gap angle; it is seen that the transition takes place around the value where we expect it.

3 Magnetic rings

3.1 The homogeneous background field

Consider the magnetic field of intensity $B > 0$ perpendicular to the ring plane. In the circular gauge the corresponding vector potential is $A = \frac{1}{2}\, Br\, e_\varphi$ and the Hamiltonian without the singular interaction at the ring acts as

$$H_{0,A} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(-i\frac{\partial}{\partial \varphi} + \frac{1}{2}\, Br^2\right)^2 .$$

After a standard manipulation we find that the solutions at energy z for an interior and exterior of the ring, to be inserted into (2.5), are

$$\begin{aligned} f_m(r) &= r^{|m|}\, \mathrm{e}^{-Br^2/4}\, M\left(a_m(z), |m|+1; \frac{1}{2}\, Br^2\right) , \\ g_m(r) &= r^{|m|}\, \mathrm{e}^{-Br^2/4}\, U\left(a_m(z), |m|+1; \frac{1}{2}\, Br^2\right) , \end{aligned}$$

respectively, where

$$a_m(z) := \frac{1}{2}\left(m + |m| + 1 - \frac{z}{B}\right)$$

and M, U are the regular and singular confluent hypergeometric function, respectively. In particular, the spectrum of the free (Landau) Hamiltonian is given by the condition $a_m(z) = -n$, $n = 0, 1, 2, \dots$, i.e. consists of the Landau levels

$$z = B(m + |m| + 2n + 1)\, , \quad n = 0, 1, 2, \dots .$$

To employ (2.7) we need also the Wronskian of the above solutions which is by [AS, 13.1.22] equal to

$$W(f_m(r), g_m(r)) = -\frac{2}{r}\frac{\Gamma(|m|+1)}{\Gamma(a_m(z))}\left(\frac{B}{2}\right)^{-|m|} .$$

In particular, if the ring is rotationally symmetric, $\alpha(\varphi) = \alpha$, the condition (2.7) is reduced to

$$\frac{\left(\frac{1}{2}\, BR^2\right)^{|m|}}{2|m|!}\, \Gamma(a_m(z))\mathrm{e}^{-BR^2/2}(MU)\left(a_m(z), |m|+1; \frac{1}{2}\, BR^2\right) = \frac{1}{\alpha R} .$$

3.2 The AB flux case

Suppose now that instead of the homogeneous field considered above the ring is threaded at its centre by a magnetic flux line. The vector potential can be now chosen as $A = \frac{\Phi}{2\pi r}\, e_\varphi$, where Φ is the value of the flux. The normalized flux appearing in (1.1) equals in the present units $\phi = \Phi/2\pi$.

We assume that the flux line is not combined with a point interaction [AT, DŠ]; treating then the Schrödinger equation in a standard way [Ru] we get the interior and exterior solutions at the energy $-\kappa^2$ in the form

$$f_m(r) = I_{|m-\phi|}(\kappa r)\,, \quad g_m(r) = K_{|m-\phi|}(\kappa r)\,,$$

respectively; their Wronskian is

$$W(f_m(r), g_m(r)) = -\frac{1}{r}\,.$$

In particular, in case of a rotationally symmetric ring we get the spectral condition which is similar to (2.4), namely

$$(I_{|m-\phi|}K_{|m-\phi|})(\kappa R) = \frac{1}{\alpha R}\,. \tag{3.1}$$

Taking into account the relation

$$\lim_{z\to 0}(I_{|\nu|}K_{|\nu|})(z) = \frac{1}{2|\nu|}$$

we can find the condition

$$2|m-\phi| = \alpha R \tag{3.2}$$

that determines critical values at which the eigenvalues are absorbed in the essential spectrum.

4 Results and discussion

Let us now turn to a more detailed discussion of the results.

4.1 Non-magnetic case, constant α

We have presented already in Figs. 1,2 the spectrum for the full and broken ring, respectively. If the radius is large enough (in terms of the quantity αR) the spectrum has a distinctive one-dimensional character with the largest density at its bottom. At the same time, a gap which is large enough to prevent tunneling between the loose ends gives the spectrum the hard-wall nature. This leads us to the *conjecture* that the asymptotic formula derived in [EY1] will hold for any finite, smooth, and *non-closed* curve as well, with the comparison operator $S = -\frac{d^2}{ds^2} - \frac{1}{4}k(s)^2$, where $k(s)$ is the curvature, being now specified by *Dirichlet* boundary conditions at the curve endpoints. The one-dimensional character can be seen also from the eigenfunctions: in Fig. 3 we plot the ground state and the eighth excited state.

4.2 Non-magnetic case, localization

As we have said in the introduction one expects that irregular variations of the coupling constant may destroy the coherence of the wavefunction along the curve supporting the interaction. Since the negative spectrum of the present model is discrete in any case, one can try to see the effect through the characteristic size of the eigenfunctions. If the coupling strength α is constant over the whole ring or a substantial part of it, the latter are distributed roughly uniformly along Γ; the uniformity is ideal for the full ring and it has sinusoidal variations due to the natural quantization for excited states.

When we want to find a quantity to characterize the "size" of the eigenfunction ψ_k corresponding to the energy E_k we notice first that it can be expressed as [?]

$$\psi_k(x) = \int_\Gamma \psi(y(\varphi)) G_0(x, y(\varphi); E_k)\, \mathrm{d}\varphi\,,$$

possibly up to a normalization factor, where $y(\varphi)$ is a point of the ring and G_0 is the Green function of the two-dimensional Laplacian. Consequently, it is sufficient to characterize ψ_k by its restriction to $\psi_k(\varphi) \equiv \psi_k(y(\varphi))$ to the ring. Then we choose as the quantity of interest the second moment of the corresponding probability distribution, minimized over the choice of the reference point,

$$\langle \Delta\psi \rangle := \min_{\varphi_0} \left(\int_0^{2\pi} (\varphi - \varphi_0)^2 |\psi_k(\varphi)|^2 \mathrm{d}\varphi \right)^{1/2}.$$

Now we take a family of random step functions $\alpha(\cdot)$ and plot the above quantity for the ground-state eigenfunction in dependence on the dispersion $\Delta\alpha$; the result is plotted in Fig. 4. It is obvious that in the average a stronger coupling randomness means a "more localized" eigenfunction.

4.3 Magnetic rings

Let us pass now to the case with a non-zero magnetic field. If the latter in homogeneous, the spectrum is pure point and it accumulates at the Landau levels. In Fig. 5 we show the eigenvalues belonging to the three lowest "bands" of the full ring as functions of the magnetic field intensity. Fig. 6 shows the situation in a cut ring with B fixed and the gap angle θ changing. At the limit $\theta \to 2\pi$ the eigenvalues are, of course, absorbed in the Landau levels. What is more interesting are the avoided crossings in the higher "bands" which witness about the non-trivial character of the cyclotronic motion around such an "obstacle". Another illustration of the fact is given by the eigenfunction contour plots shown in Fig. 7.

On the other hand, in the Aharonov-Bohm case the essential spectrum covers the positive halfline, and the dependence of the negative eigenvalues on the magnetic flux shown in Fig. 8 resembles a picture corresponding to (1.1) for the ideal ring graph, that is a family of shifted parabolas. The same can be said, of course, about the lower part of the homogeneous-field spectrum in Fig. 5, but in distinction to that case the AB spectrum is periodic modulo an integer number of flux quanta. A common feature is a strong dependence of the eigenvalues on the flux, which by the formula given in the introduction means that such system exhibit non-negligible persistent currents.

Recall in this connection a close relation between such current-carrying states and the edge currents [Ha, MS]. Furthermore, to create a magnetic transport one does not need a hard wall; much more gentle perturbations like a potential "ditch", translationally invariant field modification [Iw, MP, EK] or even an array of point obstacles [EJK1] are sufficient. If a "linear" perturbation is replaced by a circular one, one can expect occurrence of localized states carrying current along the circle. Naturally, the magnetic transport may be destroyed by a disorder. The problem of stability of (a part of) absolutely continuous spectrum with respect to perturbations has been studied recently for edge currents in halfplanes and similar domains [BP, FGW, MMP], strips [EJK2] and compact domains [FM]. In a finite-length setting the localization due to disorder was shown in [**?**]; we intend to do the same in the context of the present model in a subsequent publication.

While two types of the magnetic field yield similar results, there are important differences. One of them concerns the character of the perturbation. The Aharonov-Bohm Hamiltonian is *not* an analytic perturbation of the operator with $\phi = 0$. This is obvious from the fact that the eigenvalue curves are in this case non-smooth at the integer values of the flux ϕ as one can see in Figs. 9 and 10 showing the spectrum for different α's and gap angles[3]; in the full-ring case we present a comparison with the homogeneous field having the same flux through Γ. Another difference from the homogeneous field case is that the discrete spectrum may be void for some values of the parameters which is again clear from the said figures; for the full ring the critical value is given by (**??**).

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[3]The same is true in Fig. 8 where, however, the jump in the derivative is too small to be visible.

Figure captions

Figure 1 The dependence of energy eigenvalues of a full ring on the radius R, $\alpha = 5$. Each level is labelled by m, all levels except the ground state are twice degenerated.

Figure 2 This figure shows a transition between the two regimes as the distance θ between the loose ends of the ring increases. The full-ring-type spectrum ($\theta \approx 0$) passes to the infinite-well-type around $\alpha \approx 2\theta R$. In the presented situation $R = 2$ and $\alpha = 10$ is chosen. The dotted lines show the spectrum of a Dirichlet case.

Figure 3 The contour plots of the (real) wave functions of the ground state and the $8th$ excited states of a broken ring are shown for $\alpha = 1$, $R = 10$, and $\theta = \pi/3$. The corresponding eigenenergies are $E_0 = -0.249$ and $E_8 = -0.00415$, respectively.

Figure 4 The influence of random choice of α on localization of the ground-state eigenfunction is presented. Each of the 5000 points represents a particular (random) choice of the piecewise constant function $\alpha(\varphi)$ that is formed by 10 constant pieces of the same angle length $\pi/5$ and the value of α is uniformely distributed around $\alpha_0 = 1$ with a given mean square $\Delta\alpha$. The quantity $\langle\Delta\psi\rangle$ is the (minimized) second moment of the probability distribution. The radius $R = 5$.

Figure 5 The dependence of energy eigenvalues of a full ring on intensity of homogeneous magnetic field B, $\alpha = 1$, $R = 5$. We plot only levels "setting on" the first three Landau levels.

Figure 6 The dependence of energy eigenvalues of a broken ring on the gap angle, $B = 0.2$, $\alpha = 1$, $R = 5$. Only first seven levels of each "band" are plotted. The dotted lines mark the Landau levels for the given B.

Figure 7 The contour plots of the absolute value of wave functions of broken rings, $B = 0.2$, $\alpha = 1$, $R = 5$. The upper subplot shows the ground state ($E = -0.193$) of a "almost full" ring $\theta = \pi/10$ and the lower one an excited state ($E = 0.552$) of the second "band" with $\theta = \pi$.

Figure 8 The dependence of energy eigenvalues of a full ring in the Aharonov-Bohm case on the magnetic flux ϕ, $\alpha = 1$, $R = 10$.

Figure 9 The dependence of energy eigenvalues of a full ring in the Aharonov-Bohm case on the magnetic flux ϕ, $R = 1$. The three values of $\alpha = 0.5, 1, 1.5$ are chosen so as to demonstrate both the non-smootheness of energy curves at integer values of ϕ and the possibility of a void spectrum for some intervals of ϕ.

Figure 10 The comparison of dependence of eigenenergies on the magnetic flux ϕ for a full $\theta = 0$ and broken $\theta = \pi/3$ rings in the Aharonov-Bohm case, $\alpha = 0.5$, $R = 1$. The dotted curve shows an analogical situation with the homogeneous magnetic field of the same flux.

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Figure 1

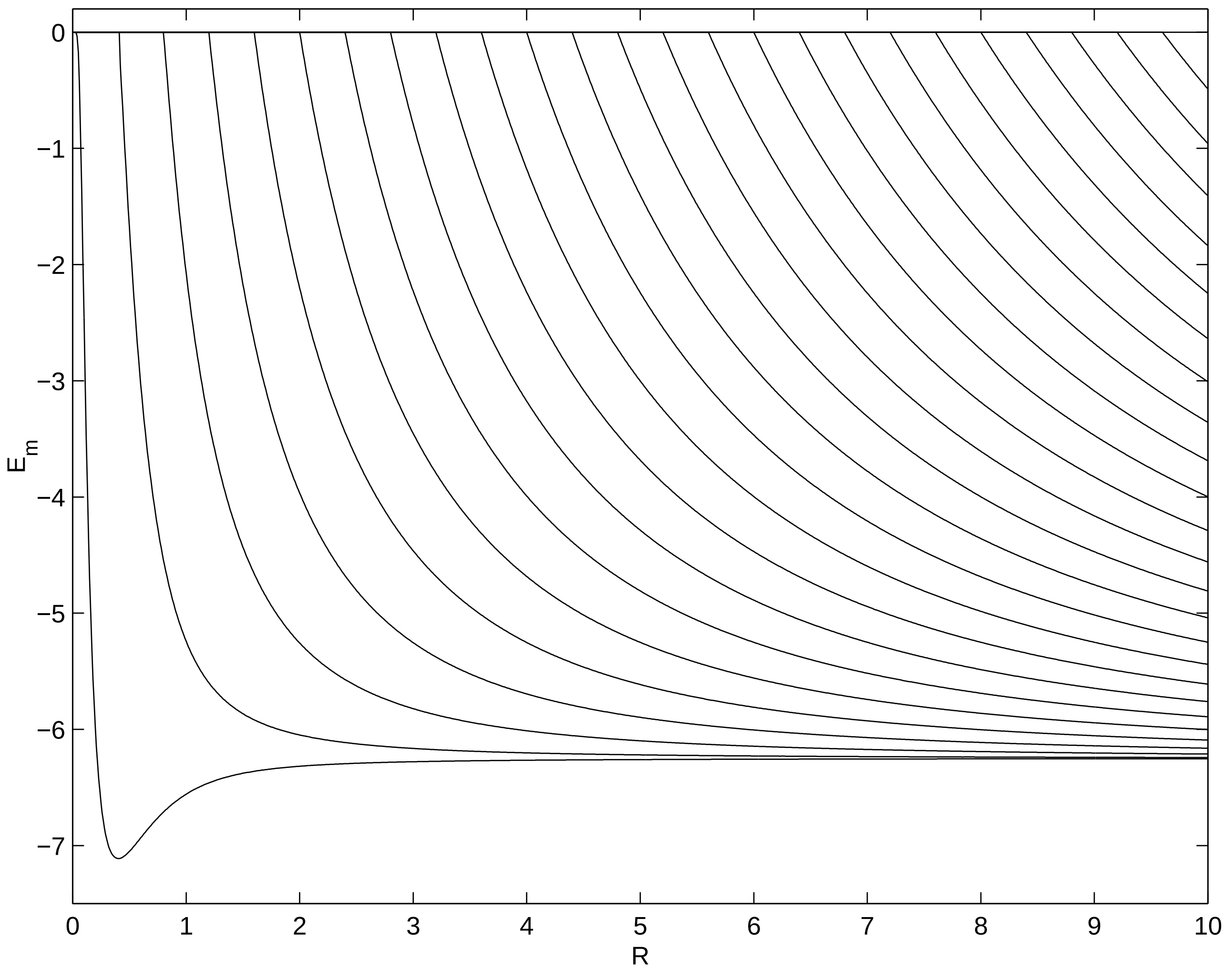
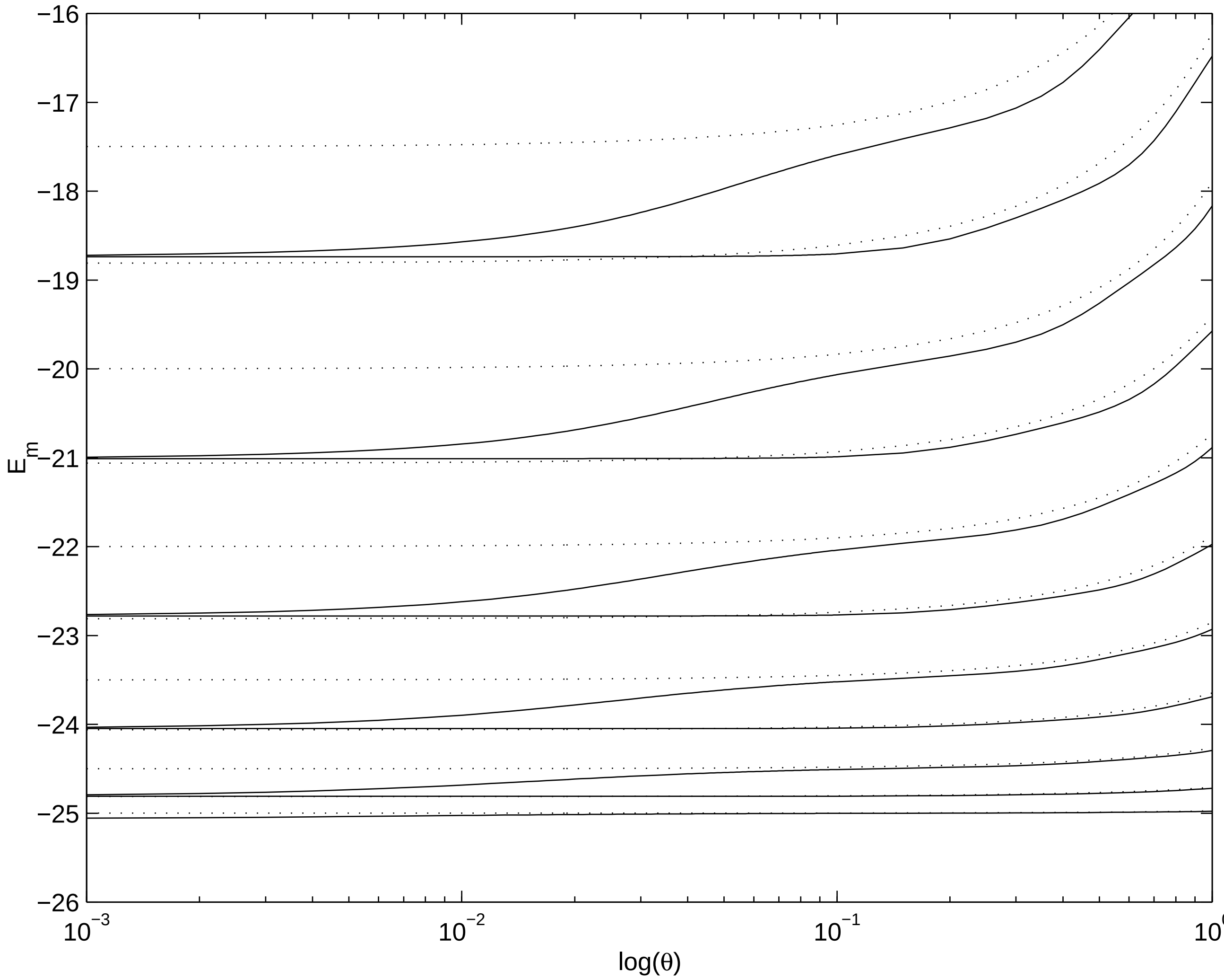


Figure 2



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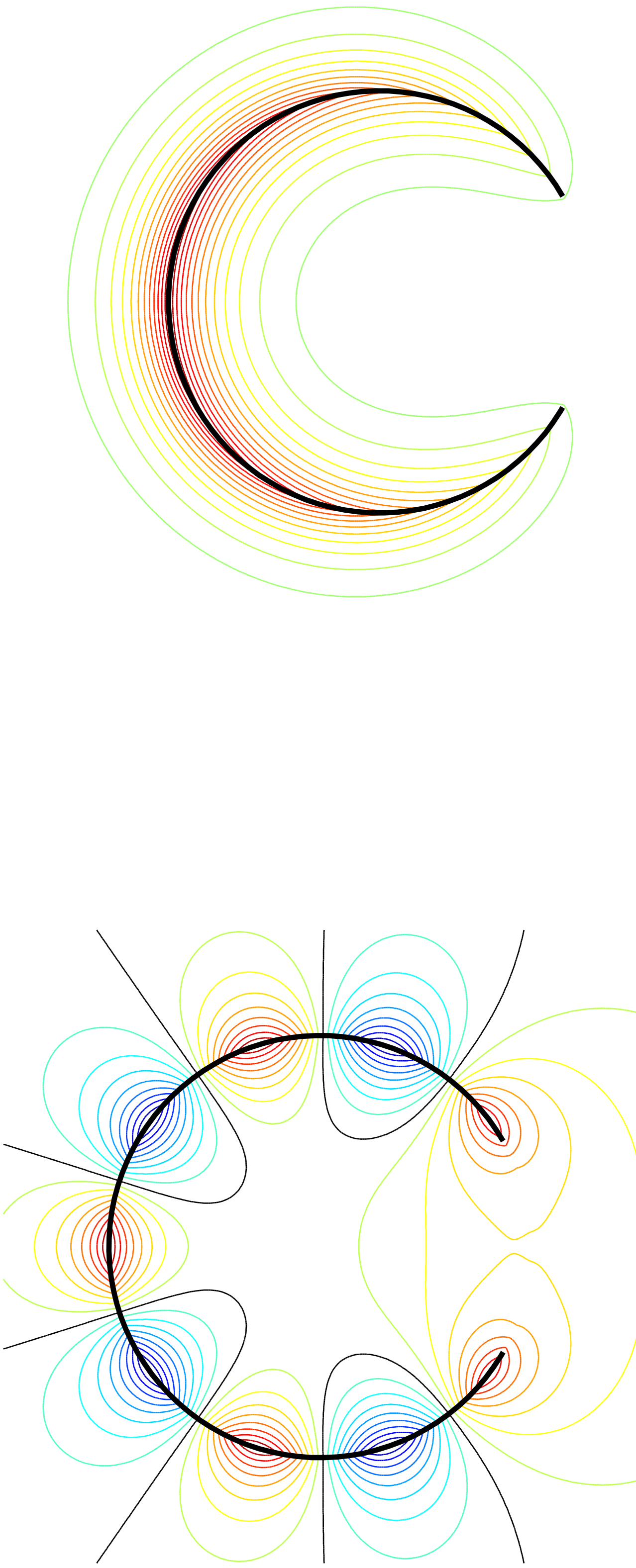

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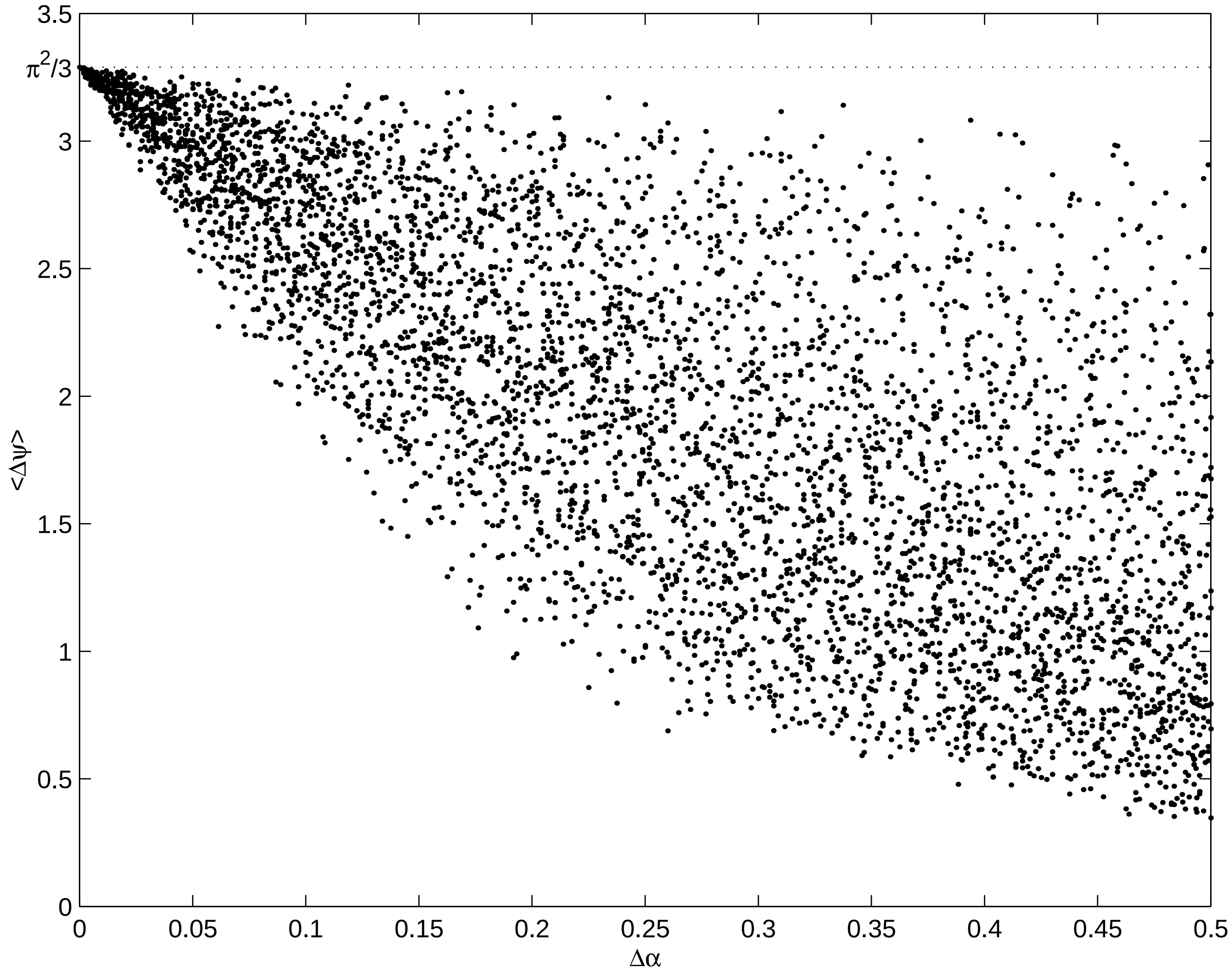

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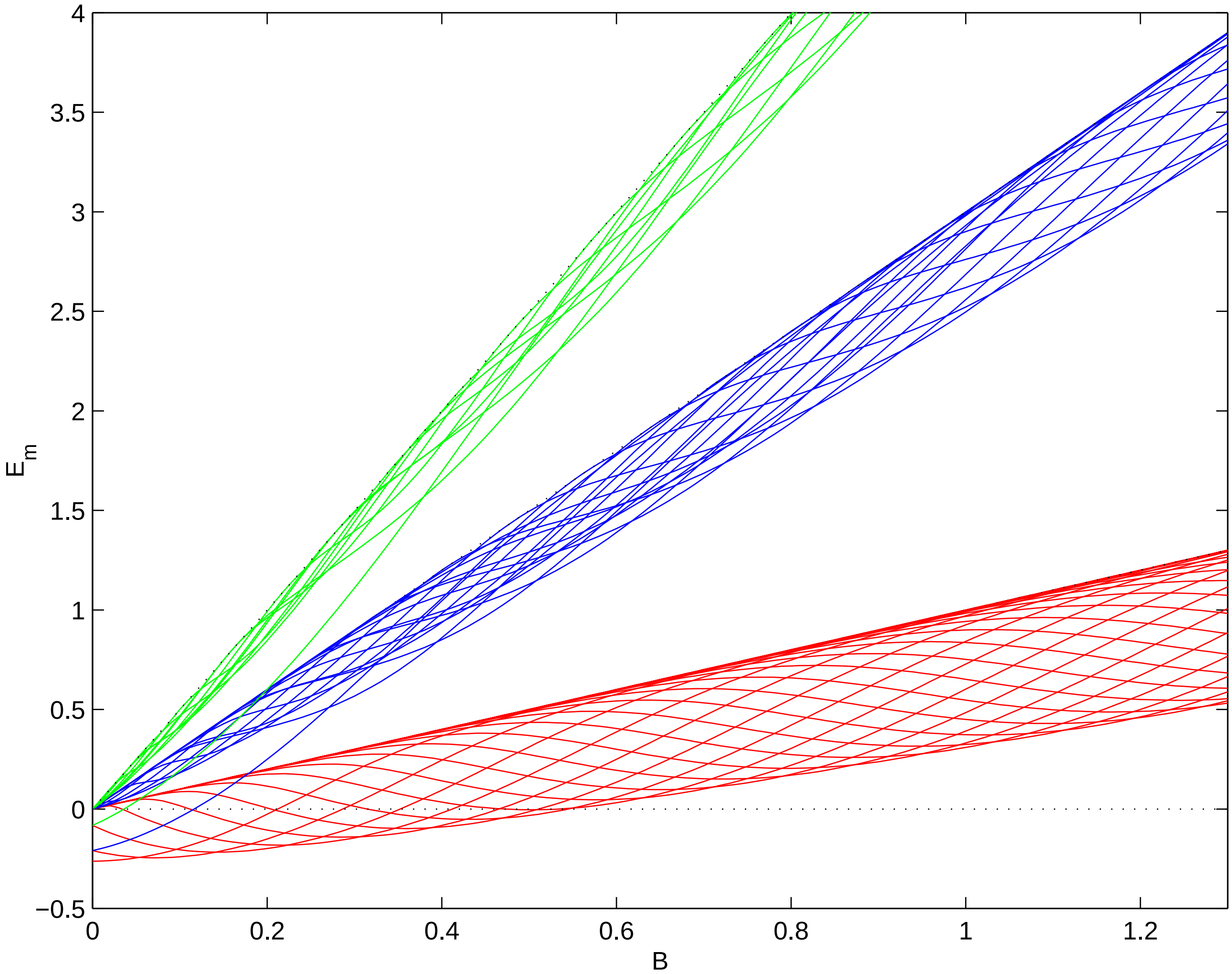

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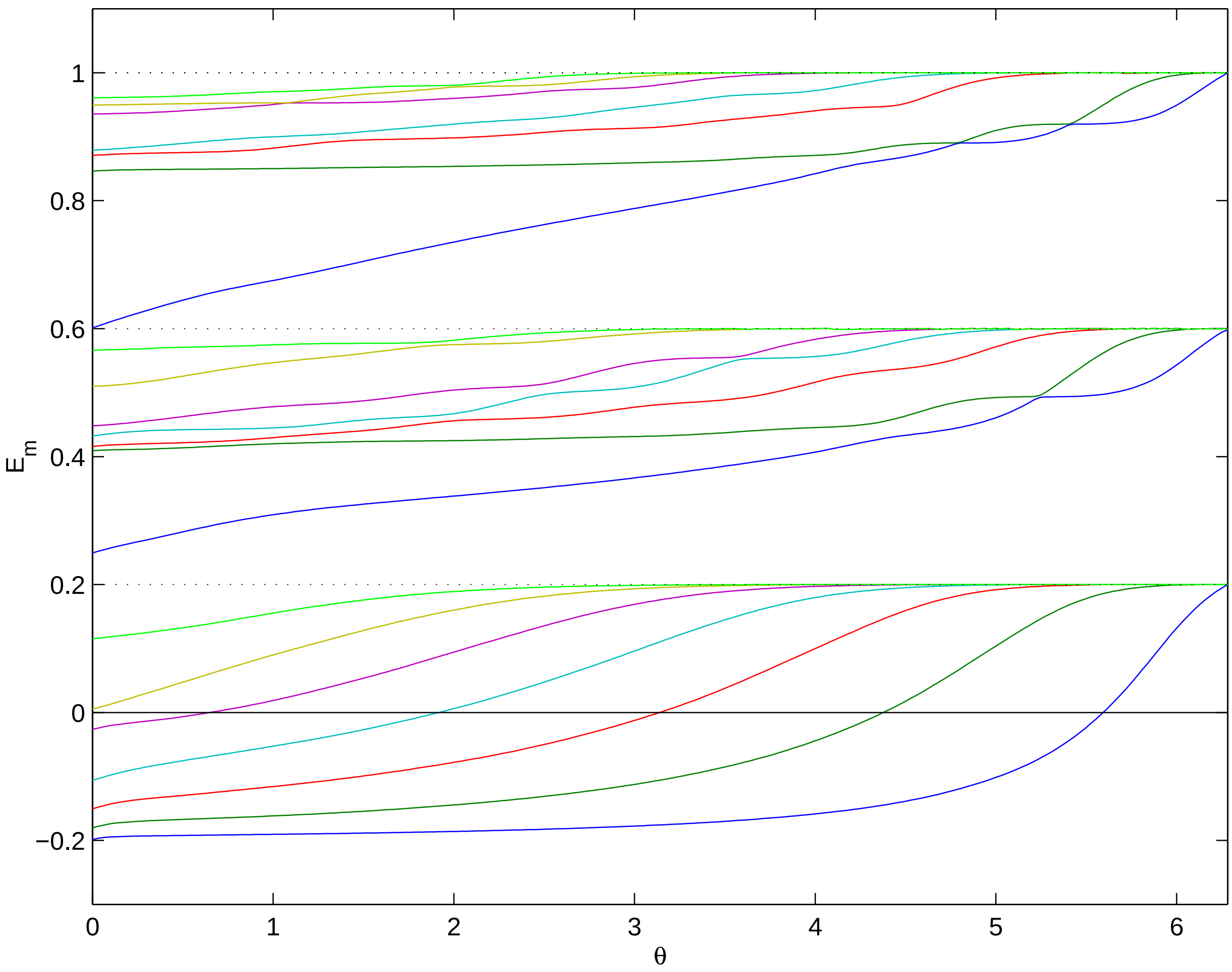

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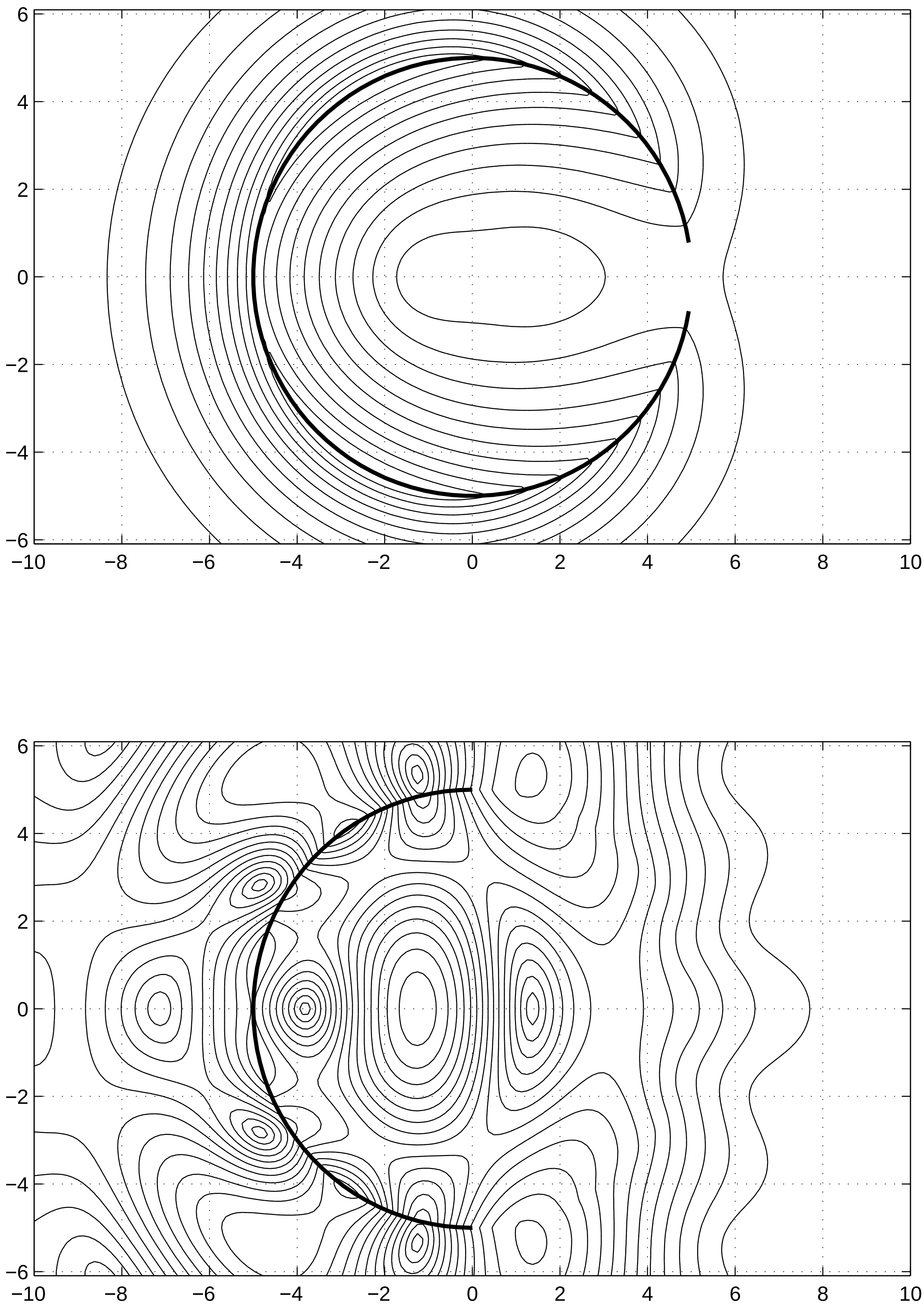

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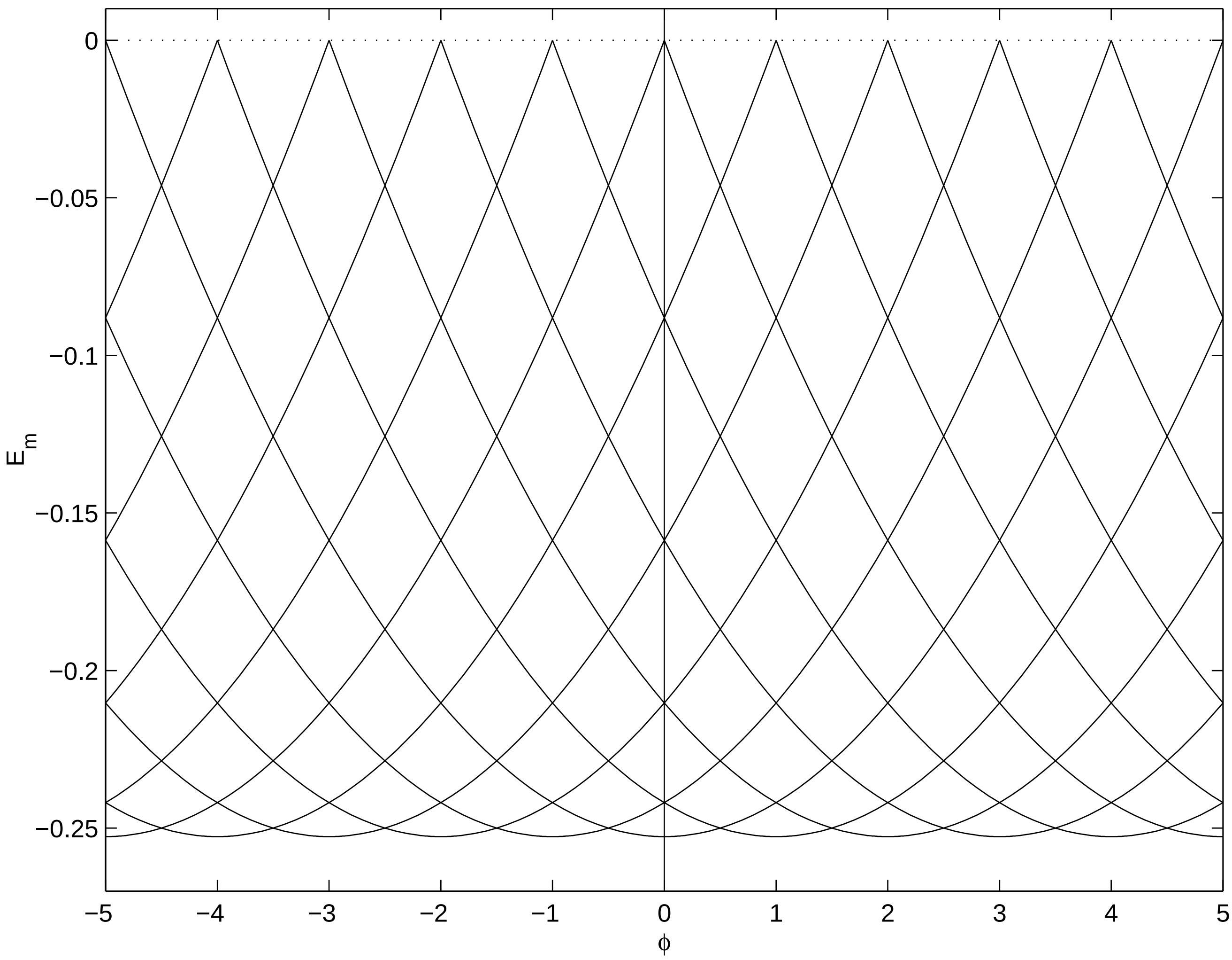

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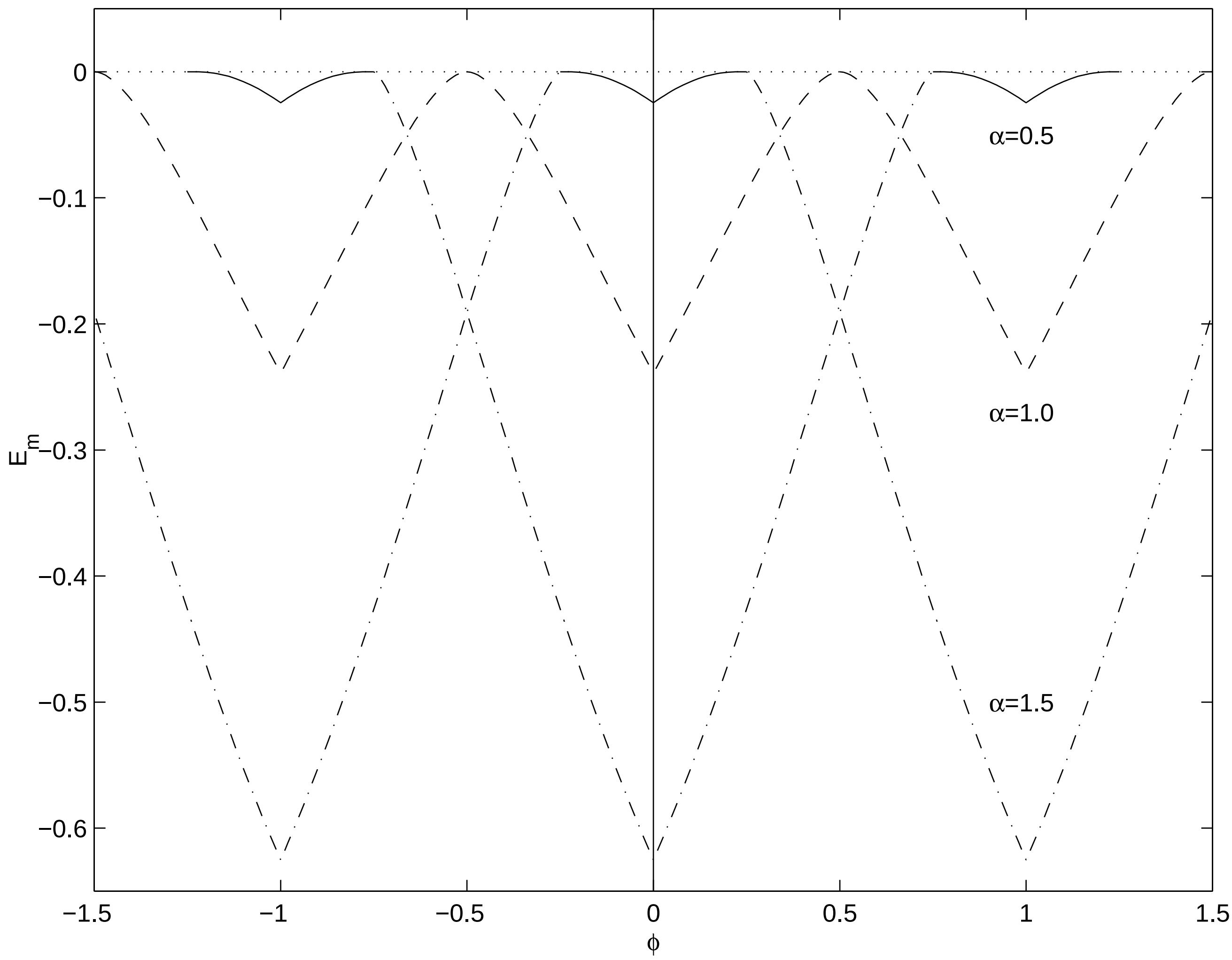
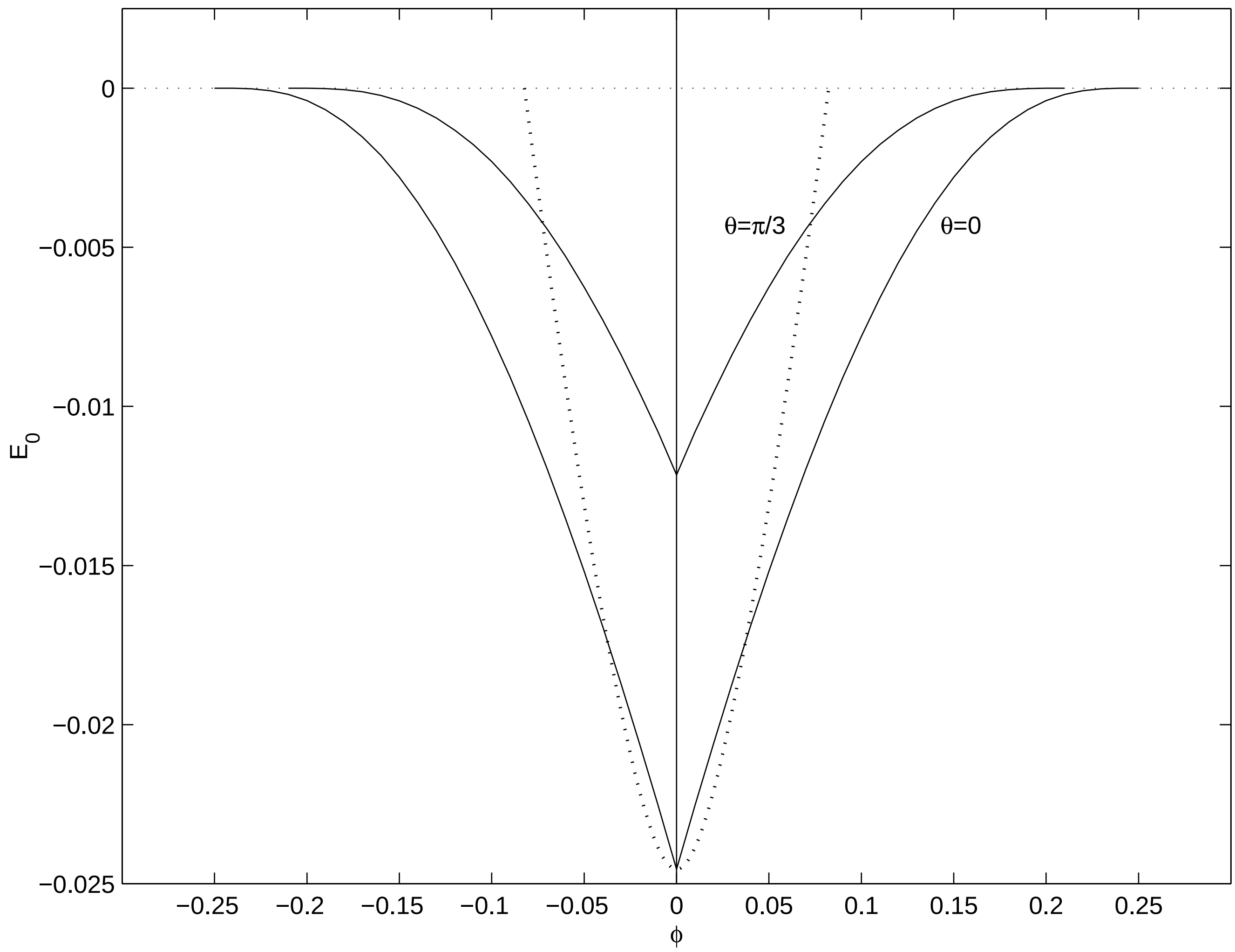


Figure 10



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0.1
0.15
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0.25
ϕ
$\theta=\pi/3$
$\theta=0$