A General Framework for Localization of Classical Waves: II. Random Media *

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Abstract. We study localization of classical waves in random media in the general framework introduced in Part I of this work [KK]. This framework allows for two random coefficents, encompasses acoustic waves with random position dependent compressibility and mass density, elastic waves with random position dependent Lamé moduli and mass density, and electromagnetic waves with random position dependent magnetic permeability and dielectric constant, and allows for anisotropy. We show exponential localization (Anderson localization) and strong Hilbert-Schmidt dynamical localization for random perturbations of periodic media with a spectral gap.

Keywords: wave localization, random media, Anderson localization, dynamical localization, Wegner estimate

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1. Introduction

In this series of articles we provide a general framework for studying localization of acoustic waves, elastic waves, and electromagnetic waves in inhomogeneous and random media, i.e., the existence of acoustic, elastic, and electromagnetic waves such that almost all of the wave's energy remains in a fixed bounded region uniformly over time. Our general framework encompasses acoustic waves with position dependent compressibility and mass density, elastic waves with position dependent Lamé moduli and mass density, and electromagnetic waves with position dependent magnetic permeability and dielectric constant. We also allow for anisotropy.

In the first article [KK] we developed mathematical methods to study wave localization in inhomogeneous media; as an application we proved localization for local perturbations (defects) of media with a gap in the spectrum, and studied midgap eigenmodes. In this second article

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these methods are applied to prove existence of exponential localization (Anderson localization) and strong Hilbert-Schmidt dynamical localization for classical waves in random media. This phenomenum has been experimentally observed for light waves [WBLR].

Previous results on localization of classical waves in random media [FK3, FK4, FK7, CHT] considered only the case of one random coefficient. Acoustic and electromagnetic waves were treated separately. Elastic waves were not discussed.

Our results extend the work of Figotin and Klein [FK3, FK4, FK7] in several ways: 1) We study a general class of classical waves which includes acoustic, electromagnetic and elastic waves as special cases. 2) We allow for two random coefficients (e.g., electromagnetic waves in media where both the magnetic permeability and the dielectric constant are random). 3) We allow for anisotropy in our wave equations. 4) We prove strong Hilbert-Schmidt dynamical localization in random media, using the bootstrap multiscale analysis of Germinet and Klein [GK1] and the generalized eigenfunction expansion of of Klein, Koines and Seifert [KKS] for classical wave operators.

Our approach to the mathematical study of localization of classical waves is operator theoretic and reminiscent of quantum mechanics. It is based on the fact that many wave propagation phenomena in classical physics are governed by equations that can be recast in abstract Schrödinger form [Wi, SW, FK4, Kle, KKS, KK]. The corresponding self-adjoint operator, which governs the dynamics, is a first order partial differential operator, but its spectral theory may be studied through an auxiliary self-adjoint, second order partial differential operator. These second order *classical wave operators* are analogous to Schrödinger operators in quantum mechanics. The method is particularly suitable for the study of phenomena historically associated with quantum mechanical electron waves, especially Anderson localization in random media [FK3, FK4, FK7, Kle] and midgap defect eigenmodes [FK5, FK6, KK].

Physically interesting inhomogeneous and random media give rise to nonsmooth coefficients in the classical wave equations, and hence in their classical wave operators. *Thus we make no assumptions about the smoothness of the coefficients of classical wave operators.*

Classical waves do not localize in a homogeneous medium; to obtain wave localization an appropriate medium must be fabricated. We start with an underlying periodic medium (a “photonic crystal” in the case of light waves) with a spectral gap. As randomness is added to the medium, we prove that the gap in the spectrum shrinks (possibly closing), and localization occurs in the spectrum at the edges of the gap. A crucial technical result is a Wegner-type estimate for random second order classical wave operators with two random coefficients.

This paper is organized as follows: In Section 2 we review our framework for studying classical waves. In Section 3 we discuss localization of classical waves in random media. We introduce a model for random media, and consider the corresponding random classical wave operators. Exponential localization and strong Hilbert-Schmidt dynamical localization are defined. The connection between localization of a random first order classical wave operator and localization of the two associated random second order classical wave operators is described in Remarks 3.7 and 3.8. We study the effect of randomness on a spectral gap of an underlying periodic medium in Theorem 3.10. The results on localization are stated in Theorems 3.11 - 3.14. In Section 4 we show that random second order partially elliptic classical wave operators satisfy the requirements for the bootstrap multiscale analysis in Theorem 4.1; the Wegner estimate for random second order classical wave operators is given in Theorem 4.4. The results on localization are proven using the bootstrap multiscale analysis.

2. Classical wave operators

We start by reviewing the mathematical framework for classical waves introduced in the prequel [KK], to which we refer for discussion and examples.

Many classical wave equations in a linear, lossless, inhomogeneous medium can be written as first order equations of the form:

$$\begin{array}{rcl} \mathcal{K}(x)^{-1}\dfrac{\partial}{\partial t}\psi_t(x) & = & \mathbf{D}^*\phi_t(x) \\ \mathcal{R}(x)^{-1}\dfrac{\partial}{\partial t}\phi_t(x) & = & -\mathbf{D}\,\psi_t(x) \end{array}, \tag{2.1}$$

where $x \in \mathbb{R}^d$ (space), $t \in \mathbb{R}$ (time), $\psi_t(x) \in \mathbb{C}^n$ and $\phi_t(x) \in \mathbb{C}^m$ are physical quantities that describe the state of the medium at position x and time t, $\mathbf{D}$ is an $m \times n$ matrix whose entries are first order partial differential operators with constant coefficients (see Definition 2.1), $\mathbf{D}^*$ is the formal adjoint of $\mathbf{D}$, and, $\mathcal{K}(x)$ and $\mathcal{R}(x)$ are $n \times n$ and $m \times m$ positive, invertible matrices, uniformly bounded from above and away from 0, that describe the medium at position x (see Definition 2.3). In addition, $\mathbf{D}$ satisfies a partial ellipticity property (see Definition 2.2), and there may be auxiliary conditions to be satisfied by the quantities $\psi_t(x)$ and $\phi_t(x)$.

The physical quantities $\psi_t(x)$ and $\phi_t(x)$ then satisfy second order wave equations, with the same auxiliary conditions:

$$\frac{\partial^2}{\partial t^2}\psi_t(x) \;=\; -\mathcal{K}(x)\mathbf{D}^*\mathcal{R}(x)\mathbf{D}\psi_t(x) \tag{2.2}$$

$$\frac{\partial^2}{\partial t^2}\phi_t(x) \;=\; -\mathcal{R}(x)\mathbf{D}\mathcal{K}(x)\mathbf{D}^*\phi_t(x)\;. \tag{2.3}$$

Conversely, given (2.2) (or (2.3)), we may write this equation in the form (2.1) by introducing an appropriate quantity $\phi_t(x)$ (or $\psi_t(x)$), which will then satisfy equation (2.3) (or (2.2)).

The wave equation (2.1) may be rewritten in abstract Schrödinger form:

$$-i\frac{d}{dt}\Psi_t = \mathbb{W}\Psi_t\;, \tag{2.4}$$

where $\Psi_t = \begin{pmatrix}\psi_t\\ \phi_t\end{pmatrix}$ and

$$\mathbb{W} = \begin{pmatrix} 0 & -i\mathcal{K}(x)\mathbf{D}^* \\ i\mathcal{R}(x)\mathbf{D} & 0\end{pmatrix}\;. \tag{2.5}$$

The (first order) classical wave operator $\mathbb{W}$ is formally (and can be defined as) a self-adjoint operator on the Hilbert space

$$\mathcal{H} = L^2\left(\mathbb{R}^d, \mathcal{K}(x)^{-1}dx; \mathbb{C}^n\right) \oplus L^2\left(\mathbb{R}^d, \mathcal{R}(x)^{-1}dx; \mathbb{C}^m\right), \tag{2.6}$$

where, for a $k\times k$ positive invertible matrix-valued measurable function $\mathcal{S}(x)$, we set

$$L^2\left(\mathbb{R}^d, \mathcal{S}(x)^{-1}dx; \mathbb{C}^k\right) = \left\{f\colon \mathbb{R}^d \to \mathbb{C}^k;\;\; \left\langle f, \mathcal{S}(x)^{-1}f\right\rangle_{L^2\left(\mathbb{R}^d,dx;\mathbb{C}^k\right)} < \infty\right\}.$$

The auxiliary conditions to the wave equation are imposed by requiring the solutions to equation (2.4) to also satisfy

$$\Psi_t = P_{\mathbb{W}}^{\perp}\Psi_t, \tag{2.7}$$

where $P_{\mathbb{W}}^{\perp}$ denotes the orthogonal projection onto the orthogonal complement of the kernel of $\mathbb{W}$. The solutions to the equations (2.4) and (2.7) are of the form

$$\Psi_t = \mathrm{e}^{it\mathbb{W}}P_{\mathbb{W}}^{\perp}\Phi_0\;,\quad \Phi_0 \in \mathcal{H}\;. \tag{2.8}$$

The *energy density* at time t of a solution $\Psi \equiv \Psi_t(x) = (\psi_t(x), \phi_t(x))$ of the wave equation (2.1) is given by

$$\mathcal{E}_\Psi(t,x) = \frac{1}{2}\left\{\langle\psi(x)_t, \mathcal{K}(x)^{-1}\psi_t(x)\rangle_{\mathbb{C}^n} + \langle\phi_t(x), \mathcal{R}(x)^{-1}\phi_t(x)\rangle_{\mathbb{C}^m}\right\}. \tag{2.9}$$

The wave *energy*, a conserved quantity, is thus given by

$$\mathcal{E}_\Psi = \tfrac{1}{2}\|\Psi_t\|^2_{\mathcal{H}} \quad \text{for any} \quad t. \tag{2.10}$$

Note that (2.8) gives the finite energy solutions to the wave equation (2.1).

It is convenient to work on $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)$ instead of the weighted space $L^2\left(\mathbb{R}^d, \mathcal{S}(x)^{-1}dx; \mathbb{C}^k\right)$. To do so, note that the operator $V_{\mathcal{S}}$, given by multiplication by the matrix $\mathcal{S}(x)^{-1/2}$, is a unitary map from the Hilbert space $L^2\left(\mathbb{R}^d, \mathcal{S}(x)^{-1}dx; \mathbb{C}^k\right)$ to $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)$, and if we set $\widetilde{\mathbb{W}} = (V_{\mathcal{K}} \oplus V_{\mathcal{R}})\, \mathbb{W}\, (V^*_{\mathcal{K}} \oplus V^*_{\mathcal{R}})$, we have

$$\widetilde{\mathbb{W}} = \begin{pmatrix} 0 & -i\sqrt{\mathcal{K}(x)}\mathbf{D}^*\sqrt{\mathcal{R}(x)} \\ i\sqrt{\mathcal{R}(x)}\mathbf{D}\sqrt{\mathcal{K}(x)} & 0 \end{pmatrix} , \tag{2.11}$$

a formally self-adjoint operator on $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^n\right) \oplus L^2\left(\mathbb{R}^d, dx; \mathbb{C}^m\right)$.

In addition, if $S_- I \le \mathcal{S}(x) \le S_+ I$ with $0 < S_- \le S_+ < \infty$, as it will be the case in this article, it turns out that if $\widetilde{\varphi} = V_{\mathcal{S}}\varphi$, then the functions $\varphi(x)$ and $\widetilde{\varphi}(x)$ share the same decay and growth proporties (e.g., exponential or polynomial decay).

Thus it will suffice for us to work on $L^2\left(\mathbb{R}^d, dx; \mathbb{C}^k\right)$, and we will do so in the remainder of this article. We set

$$\mathcal{H}^{(k)} = L^2(\mathbb{R}^d, dx; \mathbb{C}^k) \ . \tag{2.12}$$

Given a closed densely defined operator T on a Hilbert space $\mathcal{H}$, we will denote its kernel by $\ker T$ and its range by $\operatorname{ran} T$; note $\ker T^*T = \ker T$. If T is self-adjoint, it leaves invariant the orthogonal complement of its kernel; *the restriction of T to $(\ker T)^\perp$ will be denoted by $T_\perp$*. Note that $T_\perp$ is a self-adjoint operator on the Hilbert space $(\ker T)^\perp = P_T^\perp \mathcal{H}$, where $P_T^\perp$ denotes the orthogonal projection onto $(\ker T)^\perp$.

DEFINITION 2.1. *A constant coefficient, first order, partial differential operator $\mathbf{D}$ from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$ ($CPDO^{(1)}_{n,m}$) is of the form $\mathbf{D} = D(-i\nabla)$, where, for a d-component vector k, $D(k)$ is the $m \times n$ matrix*

$$D(k) = [D(k)_{r,s}]_{\substack{r=1,\ldots,m \\ s=1,\ldots,n}} \ ; \quad D(k)_{r,s} = a_{r,s} \cdot k \ , \quad a_{r,s} \in \mathbb{C}^d \ . \tag{2.13}$$

We set

$$D_+ = \sup\{\|D(k)\|; \ \ k \in \mathbb{C}^d, \ |k| = 1\} \, , \tag{2.14}$$

so $\|D(k)\| \leq D_+|k|$ for all $k \in \mathbb{C}^d$. Note that D_+ is bounded by the norm of the matrix $[|a_{r,s}|]_{\substack{r=1,\ldots,m \\ s=1,\ldots,n}}$.

Defined on

$$\mathcal{D}(\mathbf{D}) = \{\psi \in \mathcal{H}^{(n)} : \ \mathbf{D}\psi \in \mathcal{H}^{(m)} \text{ in distributional sense}\} \ , \tag{2.15}$$

a $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ is a closed, densely defined operator, and $C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$ (the space of infinitely differentiable functions with compact support) is an operator core for $\mathbf{D}$. We will denote by $\mathbf{D}^*$ the $CPDO^{(1)}_{m,n}$ given by the formal adjoint of the matrix in (2.13).

DEFINITION 2.2. *A $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ is said to be partially elliptic if there exists a $CPDO^{(1)}_{n,q}$ $\mathbf{D}^\perp$ (for some q), satisfying the following two properties:*

$$\begin{aligned} \mathbf{D}^\perp \mathbf{D}^* \ &= \ 0 \ , & (2.16)\\ \mathbf{D}^*\mathbf{D} + (\mathbf{D}^\perp)^*\mathbf{D}^\perp \ &\geq \ \Theta \ [(-\Delta)\otimes I_n] \ , & (2.17) \end{aligned}$$

with $\Theta > 0$ being a constant. ($\Delta = \nabla\cdot\nabla$ is the Laplacian on $L^2(\mathbb{R}^d, dx)$; I_n denotes the $n\times n$ identity matrix.)

If $\mathbf{D}$ is partially elliptic, we have

$$\mathcal{H}^{(n)} = \ker \mathbf{D}^\perp \oplus \ker \mathbf{D} \ , \tag{2.18}$$

and

$$\mathbf{D}^*\mathbf{D} + (\mathbf{D}^\perp)^*\mathbf{D}^\perp = (\mathbf{D}^*\mathbf{D})_\perp \oplus ((\mathbf{D}^\perp)^*\mathbf{D}^\perp)_\perp \ . \tag{2.19}$$

Note that $\mathbf{D}$ is elliptic if and only it is partially elliptic with $\mathbf{D}^\perp = 0$. Note also that a $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ may be partially elliptic with $\mathbf{D}^*$ not being partially elliptic [KKS, Remark 1.1].

DEFINITION 2.3. *A coefficient operator $\mathcal{S}$ on $\mathcal{H}^{(n)}$ (CO_n) is a bounded, invertible operator given by multiplication by a coefficient matrix: an $n\times n$ matrix -valued measurable function $\mathcal{S}(x)$ on $\mathbb{R}^d$, satisfying*

$$S_- I_n \leq \mathcal{S}(x) \leq S_+ I_n \ , \ \ \textit{with} \ \ 0 < S_- \leq S_+ < \infty \ . \tag{2.20}$$

DEFINITION 2.4. *A multiplicative coefficient, first order, partial differential operator from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$ ($MPDO^{(1)}_{n,m}$) is of the form*

$$A = \sqrt{\mathcal{R}}\mathbf{D}\sqrt{\mathcal{K}} \ \ \textit{on} \ \ \mathcal{D}(A) = \mathcal{K}^{-\frac{1}{2}}\mathcal{D}(\mathbf{D}) \ , \tag{2.21}$$

where $\mathbf{D}$ is a $CPDO^{(1)}_{n,m}$, $\mathcal{K}$ is a CO_n , and $\mathcal{R}$ is a CO_m . (We will write $A_{\mathcal{K},\mathcal{R}}$ for A whenever it is necessary to make explicit the dependence on the on the medium, i.e., on the coefficient operators. $\mathbf{D}$ does not depend on the medium, so it will be omitted in the notation.)

An $MPDO^{(1)}_{n,m}$ A is a closed, densely defined operator with $A^* = \sqrt{\mathcal{K}}\mathbf{D}^*\sqrt{\mathcal{R}}$ an $MPDO^{(1)}_{m,n}$. Note that $\mathcal{K}^{-\frac{1}{2}}C_0^\infty(\mathbb{R}^d;\mathbb{C}^n)$ is an operator core for A. The following quantity will appear often in estimates:

$$\Xi_A \equiv D_+\sqrt{R_+K_+}\,. \tag{2.22}$$

DEFINITION 2.5. *A first order classical wave operator ($CWO^{(1)}_{n,m}$) is an operator of the form*

$$\mathbb{W}_A = \begin{bmatrix} 0 & -iA^* \\ iA & 0 \end{bmatrix} \quad on \ \ \mathcal{H}^{(n+m)} \cong \mathcal{H}^{(n)} \oplus \mathcal{H}^{(m)}\ , \tag{2.23}$$

where A is an $MPDO^{(1)}_{n,m}$. If either $\mathbf{D}$ or $\mathbf{D}^*$ is partially elliptic, $\mathbb{W}_A$ will also be called partially elliptic. If both $\mathbf{D}$ and $\mathbf{D}^*$ are partially elliptic, $\mathbb{W}_A$ will be called doubly partially elliptic.

Note that our definition of a first order classical wave operator is more restrictive than the one used in [KKS]. Our definition of partial ellipticity is also different from [KKS], where partially eliptic corresponds to our doubly partially elliptic - see [KKS, Remark 1.2].

REMARK 2.6. *The usual first order classical wave operators are doubly partially elliptic, including the operators corresponding to electromagnetic waves (Maxwell equations), acoustic waves, and elastic waves (see [KK, p. 100]). But there are examples of first order classical wave operators which are partially elliptic but not doubly partially elliptic (see [KKS, Remark 1.1]).*

The Schrödinger-like equation (2.4) for classical waves with the auxiliary condition (2.7) may be written in the form:

$$-i\frac{\partial}{\partial t}\mathbf{\Psi}_t = (\mathbb{W}_A)_\perp\mathbf{\Psi}_t\ , \quad \mathbf{\Psi}_t \in (\ker\mathbb{W}_A)^\perp = (\ker A)^\perp \oplus (\ker A^*)^\perp, \tag{2.24}$$

with $\mathbb{W}_A$ a $CWO^{(1)}_{n+m}$ as in (2.23). Its solutions are of the form

$$\Psi_t = \mathrm{e}^{it(\mathbb{W}_A)_\perp}\Psi_0\ , \quad \Psi_0 \in (\ker\mathbb{W}_A)^\perp\ , \tag{2.25}$$

which is just another way of writing (2.8).

Since

$$(\mathbb{W}_A)^2 = \begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}, \tag{2.26}$$

if $\mathbf{\Psi}_t = (\psi_t, \phi_t) \in \mathcal{H}^{(n)} \oplus \mathcal{H}^{(m)}$ is a solution of (2.24), then its components satisfy the second order wave equations (2.2) and (2.3), plus the auxiliary conditions, which may be all written in the form

$$\frac{\partial^2}{\partial t^2}\psi_t = -(A^*A)_\perp \psi_t \ , \quad \text{with} \quad \psi_t \in (\ker A)^\perp \ , \tag{2.27}$$

$$\frac{\partial^2}{\partial t^2}\phi_t = -(AA^*)_\perp \phi_t \ , \quad \text{with} \quad \phi_t \in (\ker A^*)^\perp \ . \tag{2.28}$$

The solutions to (2.27) and (2.28) may be written as

$$\psi_t = \cos\left(t(A^*A)_\perp^{\frac{1}{2}}\right)\psi_0 + \sin\left(t(A^*A)_\perp^{\frac{1}{2}}\right)\eta_0,\ \ \psi_0, \eta_0 \in (\ker A)^\perp, \tag{2.29}$$

$$\phi_t = \cos\left(t(AA^*)_\perp^{\frac{1}{2}}\right)\phi_0 + \sin\left(t(AA^*)_\perp^{\frac{1}{2}}\right)\zeta_0,\ \ \phi_0, \zeta_0 \in (\ker A^*)^\perp, \tag{2.30}$$

with a similar expression for the solutions of (2.28).

The operators $(A^*A)_\perp$ and $(AA^*)_\perp$ are unitarily equivalent (see [KK, Lemma A.1]): the operator U defined by

$$U\psi = A(A^*A)_\perp^{-\frac{1}{2}}\psi \quad \text{for} \quad \psi \in \ \operatorname{ran}(A^*A)_\perp^{\frac{1}{2}}, \tag{2.31}$$

extends to a unitary operator from $(\ker A)^\perp$ to $(\ker A^*)^\perp$, and

$$(AA^*)_\perp = U(A^*A)_\perp U^* \ . \tag{2.32}$$

In particular, $\mathbf{\Psi}_t = (\psi_t, \phi_t)$ is the solution of (2.24) given in (2.25) if and only if ψ_t and ϕ_t are the solutions (2.29) and (2.30) of (2.27) and (2.28) with $\eta_0 = U\phi_0$ and $\zeta_0 = U^*\psi_0$.

In addition, if

$$\mathbb{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} I_A & I_A \\ iU & -iU \end{bmatrix} \ , \quad \text{with } I_A \text{ the identity on } (\ker A)^\perp \ , \tag{2.33}$$

$\mathbb{U}$ is a unitary operator from $(\ker A)^\perp \oplus (\ker A)^\perp$ to $(\ker A)^\perp \oplus (\ker A^*)^\perp$, and we have the unitary equivalence:

$$\mathbb{U}^*(\mathbb{W}_A)_\perp \mathbb{U} = (A^*A)_\perp^{\frac{1}{2}} \oplus \left[-(A^*A)_\perp^{\frac{1}{2}}\right] \ . \tag{2.34}$$

Thus the operator $(A^*A)_\perp$ contains full information about the spectral theory of the operator $(\mathbb{W}_A)_\perp$. In particular

$$\sigma((\mathbb{W}_A)_\perp) = \sigma\left((A^*A)_\perp^{\frac{1}{2}}\right) \cup \left(-\sigma\left((A^*A)_\perp^{\frac{1}{2}}\right)\right), \tag{2.35}$$

and to find all eigenvalues and eigenfunctions for $(\mathbb{W}_A)_\perp$, it is necessary and sufficient to find all eigenvalues and eigefunctions for $(A^*A)_\perp$. For if $(A^*A)_\perp \psi_{\omega^2} = \omega^2 \psi_{\omega^2}$, with $\omega \neq 0$, $\psi_{\omega^2} \neq 0$, we have

$$(\mathbb{W}_A)_\perp\left(\psi_{\omega^2}, \pm\frac{i}{\omega}A\psi_{\omega^2}\right) = \pm\omega\left(\psi_{\omega^2}, \pm\frac{i}{\omega}A\psi_{\omega^2}\right). \tag{2.36}$$

Conversely, if $(\mathbb{W}_A)_\perp(\psi_{\pm\omega},\phi_{\pm\omega}) = \pm\omega(\psi_{\pm\omega},\phi_{\pm\omega})$, with $\omega \neq 0$, it follows that (see [KKS, Proposition 5.2])

$$(A^*A)_\perp\psi_{\pm\omega} = \omega^2\psi_{\pm\omega} \;\; \textit{and} \;\; \phi_{\pm\omega} = \pm\frac{i}{\omega}A\psi_{\pm\omega}\,. \tag{2.37}$$

DEFINITION 2.7. *A second order classical wave operator on $\mathcal{H}^{(n)}$ ($CWO_n^{(2)}$) is an operator $W = A^*A$, with A an $MPDO_{n,m}^{(1)}$ for some m. (We write $W_{\mathcal{K},\mathcal{R}} = A^*_{\mathcal{K},\mathcal{R}}A_{\mathcal{K},\mathcal{R}}$.) If $\mathbf{D}$ in (2.21) is partially elliptic, the $CWO_n^{(2)}$ will also be called partially elliptic.*

Note that a first order classical wave operator $\mathbb{W}_A$ is partially elliptic if and only if one of the two second order classical wave operators A^*A and AA^* is partially elliptic. It is doubly elliptic if both A^*A and AA^* are partially elliptic.

DEFINITION 2.8. *A classical wave operator (CWO) is either a $CWO_n^{(1)}$ or a $CWO_n^{(2)}$. If the operator W is a CWO, we call $W_\perp$ a proper CWO.*

REMARK 2.9. *A proper classical wave operator W has a trivial kernel by construction, so* 0 *is not an eigenvalue. But 0 is in the spectrum of $W_\perp$ [KKS, Theorem A.1], so $W_\perp$ and W have the same spectrum and essential spectrum.*

3. Wave localization in random media

The form of the wave equation (2.1) is given by a constant coefficient, first order, partial differential operator $\mathbf{D}$ from $\mathcal{H}^{(n)}$ to $\mathcal{H}^{(m)}$; the properties of the medium are encoded in the coefficients matrices $\mathcal{K}(x)$ and $\mathcal{R}(x)$. Random media is modeled by random coefficients matrices.

In this article we study random perturbations of an underlying periodic medium, i.e., a medium specified by periodic coefficients matrices (recall Definition 2.3) $\mathcal{K}_0(x)$ and $\mathcal{R}_0(x)$ with the same period q (i.e., $\mathcal{K}_0(x) = \mathcal{K}_0(x+qj)$ and $\mathcal{R}_0(x) = \mathcal{R}_0(x+qj)$ for all $j \in \mathbb{Z}^d$ - we take $q \in \mathbb{N}$ without loss of generality). We use the following model for random media:

ASSUMPTION 3.1 (Random medium). *The random medium is modeled by random matrix-valued functions $\mathcal{K}_g(x) = \mathcal{K}_{g,\omega}(x)$ and $\mathcal{R}_g(x) =$*

$\mathcal{R}_{g,\omega}(x)$ of the form

$$\mathcal{K}_{g,\omega}(x) \;=\; \gamma_{g,\omega}(x)\,\mathcal{K}_0(x),\ \text{with}\ \gamma_{g,\omega}(x) = 1 + g\sum_{i\in\mathbb{Z}^d}\omega_i u_i(x)\,, \tag{3.1}$$

$$\mathcal{R}_{g,\omega}(x) \;=\; \zeta_{g,\omega}(x)\mathcal{R}_0(x)\,,\ \text{with}\ \zeta_{g,\omega}(x) = 1 + g\sum_{i\in\mathbb{Z}^d}\omega_i v_i(x)\,, \tag{3.2}$$

where

(i) *$\mathcal{K}_o(x)$ and $\mathcal{R}_o(x)$ are $n\times n$ and $m\times m$ periodic coefficient matrices with period $q\in\mathbb{N}$.*

(ii) *$u_i(x) = u(x-i)$ and $v_i(x) = v(x-i)$ for $i\in\mathbb{Z}^d$, where u and v are real valued measurable functions on $\mathbb{R}^d$ with support in the cube centered at the origin with side $r<\infty$, with*

$$0 \le U_- \le U(x) \;=\; \sum_{i\in\mathbb{Z}^d} u_i(x) \le U_+ \tag{3.3}$$

$$0 \le V_- \le V(x) \;=\; \sum_{i\in\mathbb{Z}^d} v_i(x) \le V_+ \tag{3.4}$$

for a.e. $x\in\mathbb{R}^d$, where $U_\pm$ and $V_\pm$ are constants such that

$$0 < Z_- \le Z_+ < \infty\,, \quad \text{with}\ \ Z_\pm = \max\{U_\pm, V_\pm\}\,. \tag{3.5}$$

(iii) *$\omega = \{\omega_i; i\in\mathbb{Z}^d\}$ is a family of independent, identically distributed random variables taking values in the interval $[-1,1]$, whose common probability distribution μ has a bounded density $\rho>0$ a.e. in $[-1,1]$.*

(iv) *g, the disorder parameter, satisfies*

$$0 \le g < \frac{1}{Z_+}\,. \tag{3.6}$$

REMARK 3.2. *The use of the same random variables in (3.1) and (3.2) models the fact that the medium itself is what is random. This randomness in the medium is modeled by random coefficient matrices, which are not independent since a change in the medium leads to changes in both coefficient matrices.*

REMARK 3.3. *The results in this article are also valid for random coefficient matrices $\mathcal{K}_{g,\omega}(x)$ and $\mathcal{R}_{g,\omega}(x)$ of the form*

$$\mathcal{K}_{g,\omega}(x) = \gamma_{g,\omega}^{-1}\mathcal{K}_0(x)\,,\ \ \mathcal{R}_{g,\omega}(x) = \zeta_{g,\omega}^{-1}\mathcal{R}_0(x)\,. \tag{3.7}$$

The modifications in the proofs are obvious. This is the form used in [FK3, FK4] for acoustic and electromagnetic waves.

It follows from Assumption (3.1) that for a.e. ω the coefficient matrices $\mathcal{K}_{g,\omega}(x)$ and $\mathcal{R}_{g,\omega}(x)$ satisfy (2.20) with

$$K_{g,\omega,\pm} = K_{g,\pm} \;\equiv\; K_{0,\pm}(1 \pm gU_+)\,, \tag{3.8}$$
$$R_{g,\omega,\pm} = R_{g,\pm} \;\equiv\; R_{0,\pm}(1 \pm gV_+)\,. \tag{3.9}$$

Thus multiplication by $\mathcal{K}_{g,\omega}(x)$ and $\mathcal{R}_{g,\omega}(x)$ yield coefficient operators $\mathcal{K}_g = \mathcal{K}_{g,\omega}$ and $\mathcal{R}_g = \mathcal{R}_{g,\omega}$ as in Definition 2.3, for a.e. ω. For later use, we set

$$\Xi_g \;=\; D_+\sqrt{R_{g,+}K_{g,+}}\,, \tag{3.10}$$
$$\delta_\pm(g) \;=\; \frac{U_\pm}{1 \mp gU_+}\,, \tag{3.11}$$
$$\eta_\pm(g) \;=\; \frac{V_\pm}{1 \mp gV_+}\,. \tag{3.12}$$

The periodic operators associated with the coefficient matrices $\mathcal{K}_0(x)$ and $\mathcal{R}_0(x)$ will carry the subscript 0, i.e.,

$$A_0 = \sqrt{\mathcal{R}_0}\mathbf{D}\sqrt{\mathcal{K}_0}\,, \;\; \mathbb{W}_0 = \mathbb{W}_{A_0}\,, \;\; W_0 = A_0^* A_0\,. \tag{3.13}$$

Similarly, we write (for a.e. ω)

$$A_{g,\omega} = \sqrt{\mathcal{R}_{g,\omega}}\mathbf{D}\sqrt{\mathcal{K}_{g,\omega}}\,, \;\; \mathbb{W}_{g,\omega} = \mathbb{W}_{A_{g,\omega}}\,, \;\; W_{g,\omega} = A_{g,\omega}^* A_{g,\omega}\,. \tag{3.14}$$

We also set

$$W_{g,\omega,*} = A_{g,\omega}A_{g,\omega}^*\,, \tag{3.15}$$

and recall (2.32).

DEFINITION 3.4. *By a* random classical wave operator *we will always mean either* $\mathbb{W}_{g,\omega}$ *(first order) or* $W_{g,\omega}$ *(second order) as in (3.14), with the random coefficient matrices satisfying Assumption (3.1)*

Note that $W_{g,\omega,*}$ is also a random second order classical wave operator.

Random classical wave operators are random operators (see Appendix A; a random operator is a mapping $\omega \to H_\omega$ from a probability space to self-adjoint operators on a Hilbert space, such that the mappings $\omega \to f(H_\omega)$ are strongly measurable for all bounded measurable functions f on $\mathbb{R}$). In addition, they are $q\mathbb{Z}^d$-ergodic.

It is a consequence of ergodicity that there exist nonrandom sets $\mathbf{\Sigma}_g$ and Σ_g, such that $\sigma(\mathbb{W}_{g,\omega}) = \mathbf{\Sigma}_g$ and $\sigma(W_{g,\omega}) = \Sigma_g$ with probability one. In addition, the decompositions of $\sigma(\mathbb{W}_{g,\omega})$ and $\sigma(W_{g,\omega})$ into pure point spectrum, absolutely continuous spectrum and singular continuous spectrum are also independent of the choice of ω with probability one [KM1, CL]. (These sets are related by (2.34).)

We will use $\widehat{W}_{g,\omega}$ to denote a random classical operator of either first or second order; its almost sure spectrum will be denoted by $\widehat{\Sigma}_g$.

Random classical wave operators may exhibit the phenomenum of localization. We give two definitions: the first, spectral localization, in its stronger form, exponential localization, is sometimes called Anderson localization; the second is a stronger form of dynamical localization introduced in [GK1].

DEFINITION 3.5 (Exponential localization). *The random random classical wave operator $\widehat{W}_{g,\omega}$ exhibits* spectral localization *in an interval I if $I \cap \widehat{\Sigma}_g \neq \emptyset$ and $\widehat{W}_{g,\omega}$ has* only *pure point spectrum in $I \cap \widehat{\Sigma}_g$ with probability one. It exhibits* exponential localization *in I if it exhibits spectral localization in I and, with probability one, all the eigenfunctions corresponding to eigenvalues in I are exponentially decaying (in the sense of having exponentially decaying local L^2-norms).*

DEFINITION 3.6. *The random classical wave operator $\widehat{W}_{g,\omega}$ exhibits* strong HS-dynamical localization *in an interval I if $I \cap \widehat{\Sigma}_g \neq \emptyset$ and for any bounded region Ω and all $p \geq 0$ we have*

$$\mathbb{E}\left(\sup_{\|\!|f|\!\|\leq 1}\left\| |X|^{\frac{p}{2}} f(\widehat{W}_{g,\omega}) E_{\widehat{W}_{g,\omega}}(I)\chi_\Omega\right\|_2^2\right) < \infty\,. \tag{3.16}$$

(The supremum is taken over Borel functions f of a real variable, with $\|\!|f|\!\| = \sup_{t\in\mathbb{R}} |f(t)|$; $E_H(\ )$ denotes the spectral projection of the self-adjoint operator H; $\|B\|_2$ denotes the Hilbert-Schmidt norm of the operator B.)

REMARK 3.7. *In view of (2.32) and (2.34), spectral localization of a random second order classical wave operator $W_{g,\omega}$ in a compact interval $I \subset (0,\infty)$ is equivalent to spectral localization of $W_{g,\omega,*}$ in I, and also equivalent to spectral localization of the random first order classical wave operator $\mathbb{W}_{g,\omega}$ in one (and then both) of the compact intervals $\pm\sqrt{I}$. The same is true for exponential localization, in view of (2.36), (2.37), and the interior estimate of [KK, Lemma 3.4].*

REMARK 3.8. *In view of (2.26), strong HS-dynamical localization of* both *random second order classical wave operators $W_{g,\omega}$ and $W_{g,\omega,*}$ in a compact interval $I \subset (0,\infty)$ is equivalent to strong HS-dynamical localization of the random first order classical wave operator $\mathbb{W}_{g,\omega}$ in* both *compact intervals $\pm\sqrt{I}$.*

It follows from Remarks 3.7 and 3.8 that it suffices to prove localization for random second order classical wave operators.

To create an environment which favors localization, we follow the strategy first introduced in [FK1] and subsequently used in [FK2, FK3, FK4, KSS, CHT]: We start with an underlying periodic medium. The spectrum associated with a periodic medium has band gap structure and may have a gap in the spectrum. We assume the existence of a spectral gap for the underlying periodic medium. We randomize this periodic medium with a gap in the spectrum, prove that the gap shrinks but does not close if the disorder is not too large, and show that exponential localization and strong HS-dynamical localization occurs in a vicinity of the edges of the gap.

ASSUMPTION 3.9 (Gap in the spectrum). *There is a gap in the spectrum of the periodic second order classical wave operator W_0. More precisely, there exist $a, b \in \sigma(W_0)$, $0 < a < b$, such that*

$$(a, b) \cap \sigma(W_0) = \emptyset \,. \tag{3.17}$$

When randomness is added to the medium, the spectrum of the corresponding classical wave operator changes. The following theorem gives information on what happens to a spectral gap.

THEOREM 3.10 (Location of the spectral gap). *Let $W_{g,\omega}$ be a random second order classical wave operator satisfying Assumption 3.9. There exists g_0, with*

$$\begin{aligned} \max\left\{\tfrac{1}{U_+}, \tfrac{1}{V_+}\right\}\left(1-\left(\tfrac{a}{b}\right)^{\frac{1}{4}}\right) &\le g_0 \le \\ \min\left\{\tfrac{1}{Z_+}, \tfrac{1}{U_+}\left(\left(\tfrac{b}{a}\right)^{\frac{U_+}{4U_-}}-1\right), \tfrac{1}{V_+}\left(\left(\tfrac{b}{a}\right)^{\frac{V_+}{4V_-}}-1\right)\right\}&, \end{aligned} \tag{3.18}$$

and increasing, Lipschitz continuous real valued functions $a(g)$ and $-b(g)$ on the interval $\left[0, \frac{1}{Z_+}\right)$, with $a(0) = a$, $b(0) = b$, and $a(g) \le b(g)$, such that:

(i)

$$\Sigma_g \cap [a, b] = [a, a(g)] \cup [b(g), b] \,. \tag{3.19}$$

(ii) *If $g < g_0$ we have $a(g) < b(g)$ and $(a(g), b(g))$ is a gap in the spectrum Σ_g of the random operator $W_{g,\omega}$. Moreover, we have*

$$a(1+gU_+)^{\frac{U_-}{U_+}}(1+gV_+)^{\frac{V_-}{V_+}} \le a(g) \le \tfrac{a}{(1-gU_+)(1-gV_+)} \tag{3.20}$$

and

$$b(1-gU_+)(1-gV_+) \le b(g) \le \frac{b}{(1+gU_+)^{\frac{U_-}{U_+}}(1+gV_+)^{\frac{V_-}{V_+}}}. \tag{3.21}$$

In addition, if $0 \le g_1 < g_2 < g_0$ *we have*

$$\begin{aligned} \tfrac{1}{2}\left(\delta_-(g_2)+\eta_-(g_2)\right)\left(a(g_1)+a(g_2)\right) &\le \frac{a(g_2)-a(g_1)}{g_2-g_1} \\ &\le \tfrac{1}{2}\left(\delta_+(g_2)+\eta_+(g_2)\right)\left(a(g_1)+a(g_2)\right)\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} \tfrac{1}{2}\left(\delta_-(g_2)+\eta_-(g_2)\right)\left(b(g_1)+b(g_2)\right) &\le \frac{b(g_1)-b(g_2)}{g_2-g_1} \\ &\le \tfrac{1}{2}\left(\delta_+(g_2)+\eta_+(g_2)\right)\left(b(g_1)+b(g_2)\right)\,. \end{aligned} \tag{3.23}$$

(iii) *If* $g_0 < \frac{1}{Z_+}$ *we have* $a(g) = b(g)$ *for all* $g \in \left[g_0, \frac{1}{Z_+}\right)$, *and the random classical wave operator* $W_{g,\omega}$ *has no spectral gap inside the gap* (a,b) *of the periodic classical wave operator* W_0, *i.e., we have* $[a,b] \subset \Sigma_g$.

Theorem 3.10 is proven in Section 5.

Localization for continuous random operators is usually proved by a multiscale analysis, e.g., [HM, CH, Klo1, FK3, FK4, KSS, GD, CHT, Kle, DS, GK1, GK4, GK5]. (But note that the fractional moment method [AM, ASFH] has just been extended to the continuum [AENSS].) In this article we use the most recent and powerful version, the bootstrap multiscale analysis introduced in [GK1]. It can be applied in all cases where a multiscale analysis has been used, and it yields both exponential localization and strong HS-dynamical localization. (It gives a lot more, see [GK3].)

A random second order partially elliptic classical wave operator $W_{g,\omega}$ will be shown (Theorem 4.1) to satisfy all the requirements of the bootstrap multiscale analysis in each compact interval $I \subset (0,\infty)$. Thus, to prove exponential localization and strong HS-dynamical localization for $W_{g,\omega}$ in some interval centered at $E \in \Sigma_g \backslash \{0\}$, it suffices to verify the initial length scale estimate of the bootstrap multiscale analysis [GK1, Eq. (3.3)] at E.

We will show that if the random second order partially elliptic classical wave operator $W_{g,\omega}$ has a gap in the spectrum, the random perturbation creates localization near the edges of the gap for $g < g_0$, where g_0 is given in Theorem 3.10. To prove the initial length scale estimate for the multiscale analysis (as originally done in [FK1]), we need low probability to have spectrum near an edge of the gap in finite but large volume. This can achieved either by hypotheses on the probability distribution μ in Assumption 3.1(iii), which produce *classical tails* at the edge of the gap, or by postulating the existence of Lifshitz tails at the edges of the gap.

Lifshitz tails were originally proved for random Schrödinger operators at the bottom of the spectrum (e.g., [PF, Section 10], [CL, Section VI.2]). Holden and Martinelli [HM] used the Lifshitz tails estimate to obtain the initial length scale estimate for the Fröhlich-Spencer multiscale analysis at the bottom of the spectrum for random Schrödinger operators. The best estimates on the size of the interval of localization at the bottom of the spectrum at low disorder have been obtained from Lifshitz tails by Klopp [Klo3].

Klopp [Klo2] proved that for a random perturbation of a periodic Schrödinger operator, there are Lifshitz tails at an edge of a spectral gap if and only if the density of states of the periodic operator is nondegenerate at the same edge of the spectral gap. (This nondegeneracy has not been established for arbitrary edges of spectral gaps.) Najar [Na] extended Klopp's results to random acoustic operators with constant compressibility and smooth mass density. If Lifshitz tails are present at an edge of a spectral gap, the Holden-Martinelly argument can be used to obtain the initial length scale estimate for the bootstrap multiscale analysis. But the existence of Lifshitz tails at the edges of spectral gaps of has not been established for the random classical wave operators studied in this article.

We state our results with hypotheses on the probability distribution μ, as in [FK3, FK4]. The following two theorems achieve low probability of extremal values for the random variables in different ways. The results are formulated for the left edge of the gap, with similar results holding at the right edge. We use the notation of Theorem 3.10.

THEOREM 3.11 (Localization at the edge). *Let $W_{g,\omega}$ be a random second order partially elliptic classical wave operator satisfying Assumption 3.9. Suppose the probability distribution μ in Assumption 3.1(iii) satisfies*

$$\mu\{(1-\gamma,1]\} \leq K\gamma^{\eta} \ \text{ for all } \ 0 \leq \gamma \leq 1, \tag{3.24}$$

where $K < \infty$ and $\eta > \frac{d}{2}$. Then, for any $g < g_0$ there exists $\delta(g) > 0$, depending only on the constants $d, g, q, \mathcal{K}_{0,\pm}, \mathcal{R}_{0,\pm}, \Theta, U_{\pm}, V_{\pm}$, r, $\|\rho\|_{\infty}$, K, η, a, b, such that the random classical wave operator $W_{g,\omega}$ exhibits exponential localization and strong HS-dynamical localization in the interval $[a(g)-\delta(g), a(g)]$.

THEOREM 3.12 (Localization in a specified interval). *Let $W_{g,\omega}$ be a random second order partially elliptic classical wave operator satisfying*

Assumption 3.9. Let $g < g_0$, and fix a_1 and a_2 such that $a < a_1 < a_2 < a(g)$ and $a(g) - a_1 \leq b(g) - a(g)$. Then there exists $p_1 > 0$, depending only on the constants $d, g, q, \mathcal{K}_{0,\pm}$, $\mathcal{R}_{0,\pm}$, Θ, $U_\pm, V_\pm$, r, K, η, a, b, on the fixed a_1, a_2, and on a fixed upper bound on $\|\rho\|_\infty$, such that if

$$\mu\left(\left(\tfrac{g_1}{g}, 1\right]\right) \leq p_1\,, \tag{3.25}$$

where g_1 is defined by $a(g_1) = a_1$, the random classical wave operator $W_{g,\omega}$ exhibits exponential localization and strong HS-dynamical localization in the interval $[a_2, a(g)]$.

Theorems 3.11 and 3.12 can be extended to the situation when the gap is totally filled by the spectrum of the random classical wave operator, establishing the existence of a subinterval of the original gap where the random classical wave operator exhibits localization. Note that the extension of Theorem 3.12 says that we can arrange for localization in as much of the gap as we want.

Recall that if $g_0 < \frac{1}{Z_+}$ (see (3.18) for a necessary condition), then $(a, b) \subset \Sigma_g$ for $g \in \left[g_0, \frac{1}{Z_+}\right)$, i.e., the gaps closes.

THEOREM 3.13 (Localization at the meeting of the edges). *Let $W_{g,\omega}$ be a random second order partially elliptic classical wave operator satisfying Assumption 3.9. Suppose the probability distribution μ in Assumption 3.1(iii) satisfies*

$$\mu\{(1-\gamma, 1]\},\;\; \mu\{[-1, -1+\gamma)\} \leq K\gamma^\eta \;\; \textit{for all}\;\; 0 \leq \gamma \leq 1, \tag{3.26}$$

where $K < \infty$ and $\eta > d$. Suppose also that $g_0 < \frac{1}{Z_+}$. Then there exist $0 < \epsilon < \frac{1}{Z_+} - g_0$ and $\delta > 0$, depending only on the constants $d, g, q, \mathcal{K}_{0,\pm}, \mathcal{R}_{0,\pm}, \Theta, U_\pm, V_\pm$, r, $\|\rho\|_\infty$, K, η, a, b, such that the random classical wave operator $W_{g,\omega}$ exhibits exponential localization and strong HS-dynamical localization in the interval $[a(g_0) - \delta, a(g_0) + \delta]$ for all $g \in [g_0, g_0 + \epsilon)$.

THEOREM 3.14 (Localization in a specified interval in the closed gap). *Let $W_{g,\omega}$ be a random second order partially elliptic classical wave operator satisfying Assumption 3.9. Suppose $g_0 < \frac{1}{Z_+}$, and fix a_1, a_2, b_1 and b_2 such that $a < a_1 < a_2 < a(g_0) = b(g_0) < b_2 < b_1 < b$. For any $g \in [g_0, \frac{1}{Z_+})$ there exist $p_1, p_2 > 0$, depending only on the constants $d, g, q, \mathcal{K}_{0,\pm}$, $\mathcal{R}_{0,\pm}$, Θ, $U_\pm, V_\pm$, r, K, η, a, b, on the given a_1, a_2, b_1, b_2, and on a fixed upper bound on $\|\rho\|_\infty$, such that if*

$$\mu\left(\left(\frac{g_1}{g}, 1\right]\right) \leq p_1 \quad \textit{and} \quad \mu\left(\left[-1, -\frac{g_2}{g}\right)\right) \leq p_2\,, \tag{3.27}$$

where g_1 and g_2 are defined by $a(g_1) = a_1$ and $b(g_2) = b_1$ (notice $0 < g_1,\ g_2 < g_0 \le g$), the random classical wave operator $W_{g,\omega}$ exhibits exponential localization and strong HS-dynamical localization in the interval $[a_2, b_2]$.

Theorems 3.13 and 3.14 are proved similarly to Theorems 3.11 and 3.12, respectively, taking into account both edges of the gap.

4. The multiscale analysis and localization

The analysis requires finite volume random classical wave operators.

Throughout this paper we use two norms in $\mathbb{R}^d$ and $\mathbb{C}^d$:

$$|x| \;=\; \left(\sum_{i=1}^{d} |x_i|^2\right)^{\frac{1}{2}}, \tag{4.1}$$

$$\|x\| \;=\; \max\{|x_i|,\ i = 1, \ldots, d\}\,. \tag{4.2}$$

By $\Lambda_L(x)$ we denote the open cube in $\mathbb{R}^d$, centered at x with side $L > 0$:

$$\Lambda_L(x) = \{y \in \mathbb{R}^d;\ \ \|y - x\| < \tfrac{L}{2}\}\,, \tag{4.3}$$

by $\overline{\Lambda}_L(x)$ the closed cube, and by $\breve{\Lambda}_L(x)$ the half-open/half-closed cube, i.e.,

$$\breve{\Lambda}_L(x) = \{y \in \mathbb{R}^d;\ -\tfrac{L}{2} \le y_i - x_i < \tfrac{L}{2},\ i = 1, 2, \ldots, d\}\,. \tag{4.4}$$

We will identify a closed cube $\overline{\Lambda}_L(x)$ with a torus in the usual way, We set

$$\chi_{x,L} = \chi_{\Lambda_L(x)}\,, \tag{4.5}$$

where χ_Ω denotes the characteristic function of a set $\Omega \subset \mathbb{R}^d$.

Since we will work with an underlying periodic medium with period $q \in \mathbb{N}$, we restrict ourselves to cubes $\Lambda_L(x)$ with $x \in \mathbb{Z}^d$ and $L \in 2q\mathbb{N}$. We set

$$\mathcal{H}^{(n)}_{x,L} = \mathcal{H}^{(n)}_{\Lambda_L(x)} = L^2(\overline{\Lambda}_L(x), dx; \mathbb{C}^n)\,. \tag{4.6}$$

A $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ defines a closed densely defined operator $\mathbf{D}_{x,L}$ from $\mathcal{H}^{(n)}_{x,L}$ to $\mathcal{H}^{(m)}_{x,L}$ with periodic boundary condition; an operator core is given by $C^\infty_{\mathrm{per}}(\overline{\Lambda}_L(x), \mathbb{C}^n)$, the infinitely differentiable, periodic $\mathbb{C}^n$-valued functions on $\overline{\Lambda}_L(x)$.

If the $CPDO^{(1)}_{n,m}$ $\mathbf{D}$ is partially elliptic, then the restriction $\mathbf{D}_{x,L}$ is also partially elliptic, in the sense that equations (2.16) and (2.17) hold for $\mathbf{D}_{x,L}$, $(\mathbf{D}^{\perp})_{x,L}$, and $\Delta_{x,L}$. ($\Delta_{x,L}$ is the Laplacian on $L^2(\overline{\Lambda}_L(x), dx)$ with periodic boundary condition.) This can be easily seen by using the Fourier transform; here the use of periodic boundary condition plays a crucial role. We also have (2.18) and (2.19) with $\mathcal{H}^{(n)}_{x,L}$.

We fix a random second order classical wave operator $W_{g,\omega}$ as in (3.14). Given $\omega \in \mathbb{R}^{\mathbb{Z}^d}$, we define $\omega_{x,L} = \omega_{\Lambda_L(x)} \in \mathbb{R}^{\mathbb{Z}^d}$ by

$$\begin{aligned} \omega_{x,L,i} &= \omega_i \ \text{ for each } \ i \in \breve{\Lambda}_L(x) \cap \mathbb{Z}^d \,, \\ \omega_{x,L,i} &= \omega_{x,L,i+Lj} \ \text{ for all } \ i,j \in \mathbb{Z}^d \,. \end{aligned} \tag{4.7}$$

We set

$$A_{g,\omega,x,L} = A_{g,\omega,\Lambda_L(x)} = \sqrt{\mathcal{R}_{g,\omega_{x,L}}}\mathbf{D}_{x,L}\sqrt{\mathcal{K}_{g,\omega_{x,L}}} \tag{4.8}$$

on $\mathcal{D}(A_{g,\omega,x,L}) = \mathcal{K}^{-\frac{1}{2}}_{g,\omega_{x,L}}\mathcal{D}(\mathbf{D}_{x,L})$, a closed, densely defined operator on $\mathcal{H}^{(n)}_{x,L}$. The finite volume random classical wave operator $W_{g,\omega,x,L}$ on $\mathcal{H}^{(n)}_{x,L}$ is now defined by

$$W_{g,\omega,x,L} = W_{g,\omega,\Lambda_L(x)} = A^*_{g,\omega,x,L}A_{g,\omega,x,L} \,. \tag{4.9}$$

($W_{g,\omega,x,L}$ is a "periodic restriction" of $W_{g,\omega}$ to $\Lambda_L(x)$ with periodic boundary condition.) We have the equivalent of (2.31), (2.32), etc. We write

$$R_{g,\omega,x,L}(z) = (W_{g,\omega,x,L} - z)^{-1} \tag{4.10}$$

for the finite volume resolvent.

The multiscale analysis works with the decay of the finite volume resolvent from the center of a cube to its boundary, or more precisely, to its boundary belt. We set

$$\tilde{q} = \min\{q' \in q\mathbb{N};\ q' \geq \tfrac{q+r}{2}\} \,, \tag{4.11}$$

where r is given in Assumption (3.1)(ii). Given a cube $\Lambda_L(x)$, we set

$$\Upsilon_L(x) = \left\{ y \in \mathbb{Z}^d;\ \|y-x\| = \frac{L}{2} - \tilde{q} \right\} , \tag{4.12}$$

and define its (boundary) belt by

$$\tilde{\Upsilon}_L(x) = \bigcup_{y \in \Upsilon_L(x)} \overline{\Lambda}_q(y) \ ; \tag{4.13}$$

it has the characteristic function

$$\Gamma_{x,L} = \chi_{\tilde{\Upsilon}_L(x)} = \sum_{y \in \Upsilon_L(x)} \chi_{y,q} \ \ a.e. \tag{4.14}$$

Note that

$$|\Upsilon_L(x)| \leq d \left(\frac{L}{q}\right)^{d-1} . \tag{4.15}$$

The following theorem shows that random second order partially elliptic classical wave operators satisfy the requirements for the bootstrap multiscale analysis of [GK1]. *Note that we use the finite volume operators defined in (4.9), and the boundary belt defined in (4.13), i.e., with* $\Gamma_{x,L}$ *as in (4.14).*

THEOREM 4.1. *A random second order partially elliptic classical wave operator is a* $q\mathbb{Z}^d$*-ergodic random operator satisfying the requirements for the bootstrap multiscale analysis in any compact interval* $I_0 \subset (0,\infty)$*, i.e., it satisfies Assumptions SLI (Simon-Lieb inequality), EDI (eigenfunction decay inequality), IAD (independence at a distance), NE (number of eigenvalues), SGEE (strong generalized eigenfunction expansion), and W (Wegner's estimate) of [GK1] in* I_0*. The constants* γ_{I_0} *in Assumption SLI and* $\tilde{\gamma}_{I_0}$ *in Assumption EDI are given by* $\gamma_{I_0} = \tilde{\gamma}_{I_0} = \sup_{E\in I_0} \gamma_E$*, with*

$$\gamma_E = \tfrac{6\sqrt{d}}{q}\Xi_g \left(2E + \tfrac{100d}{q^2}\Xi_g^2\right)^{\frac{1}{2}} , \tag{4.16}$$

where Ξ_g *is given in (3.10). In addition, it satisfies the kernel polynomial decay estimate of [GK2, Theorem 2] with* $\Gamma = \frac{3\sqrt{7}}{32\Xi_g}$ *for a.e.* ω *(note that* $\Theta_2 = 0$*).*

REMARK 4.2. *Partial ellipticity is required for Assumption SGEE. Assumptions NE and W require that either* $W_{g,\omega}$ *or* $W_{g,\omega,*}$ *is partially elliptic. The other assumptions and the kernel polynomial decay estimate do not require partial ellipticity.*

REMARK 4.3. *It follows from Theorem 4.1 that the results of [GK3] on the Anderson metal-insulator transport transition apply to random second order partially elliptic classical wave operators.*

We have already proven most of Theorem 4.1. Taking into account (3.8) and (3.9), Assumptions SLI, EDI, and NE follow from Lemmas 3.8, 3.9, and 3.3 in [KK], respectively. (We used slightly different finite volume operators in [KK], where we used a boundary belt with $\tilde{q} = q$. But the proofs of Lemmas 3.8 and 3.9 in [KK] still apply with the definitions used in this article due to our choice of $\tilde{q}$ in (4.11).) IAD is true by Assumption (3.1) and the definition of finite volume operators in (4.9); note that $\varrho = 0$. Assumption SGEE is proven in [KKS], in the stronger form of the trace estimate given in [GK1, Eq. (2.36)]. The kernel polynomial estimate is just a special case of [GK2, Theorem 2].

Assumption W follows from the following theorem. The constants r, $Z_\pm$, $K_{0,-}$, $R_{0,-}$, and the probability density ρ are as in Assumption (3.1); Θ is the constant in (2.17).

THEOREM 4.4 (Wegner estimate). *Let $W_{g,\omega}$ be a random second order classical wave operator, with either $W_{g,\omega}$ or $W_{g,\omega,*}$ partially elliptic. Then for all $E>0$, cubes $\Lambda=\Lambda_L(x)$ with $x\in\mathbb{Z}^d$ and $L\in 2q\mathbb{N}$, and $0\le\eta\le E$, we have*

$$\mathbb{P}\left\{\operatorname{dist}\left(\sigma(W_{g,\omega,x,L}),E\right)\le\eta\right\}\le Q_g\|\rho\|_\infty E^{\frac{d}{2}-1}\eta L^{2d}\,, \tag{4.17}$$

where

$$Q_g=\frac{nC_d(2+r)^d}{gZ_-(1-gZ_+)^{d+1}}\left(K_{0,-}R_{0,-}\Theta\right)^{-\frac{d}{2}}\,, \tag{4.18}$$

wkth C_d a constant depending only on the dimension d.

Theorem 4.4 is proven in Section 6.

In view of Theorems 4.1 and 4.4, to prove exponential localization and strong HS-dynamical localization for $W_{g,\omega}$ in some interval centered at $E\in\Sigma_g\backslash\{0\}$, it suffices to verify the initial length scale estimate for the bootstrap multiscale analysis [GK1, Eq. (3.3)] at E. To state this estimate, we need a definition, which we state in the context of this article.

DEFINITION 4.5. *Let $W_{g,\omega}$ be a random second order classical wave operator. Given $\theta>0$, $E>0$, $x\in\mathbb{Z}^d$, and $L\in 6q\mathbb{N}$, we say that the cube $\Lambda_L(x)$ is (θ,E)-suitable for $W_{g,\omega}$ if $E\notin\sigma(W_{g,\omega,x,L})$ and*

$$\|\Gamma_{x,L}R_{g,\omega,x,L}(E)\chi_{x,L/3}\|_{x,L}\le\frac{1}{L^\theta}\,. \tag{4.19}$$

The following theorem summarizes the results of [GK1] that will be used to prove Theorems 3.11-3.14.

THEOREM 4.6. *Let $W_{g,\omega}$ be a random second order partially elliptic classical wave operator. Let $E_0\in\Sigma_g\backslash\{0\}$. Given $\theta>2d$, there exists a finite scale $\overline{\mathcal{L}}=\overline{\mathcal{L}}(d,q,\mathcal{K}_{0,\pm},\mathcal{R}_{0,\pm},\Theta,U_\pm,V_\pm,r,\|\rho\|_\infty,g,E_0,\theta)$, bounded for E_0 in compact subintervals of $(0,\infty)$, such that, if we can verify at some finite scale $\mathcal{L}\ge\overline{\mathcal{L}}$ that*

$$\mathbb{P}\{\Lambda_{\mathcal{L}}(0)\ \textit{ is } (\theta,E_0)\textit{-suitable for } W_{g,\omega}\}>1-\frac{1}{841^d}\,, \tag{4.20}$$

then there exists $\delta_0 = \delta_0(d, q, \mathcal{K}_{0,\pm}, \mathcal{R}_{0,\pm}, \Theta, U_\pm, V_\pm,\, r, \|\rho\|_\infty, g, E_0, \theta, \mathcal{L}) > 0$*, such that the random classical wave operator* $W_{g,\omega}$ *exhibits exponential localization and strong HS-dynamical localization in the interval* $[E_0 - \delta_0, E_0 + \delta_0]$*. In addition, we have the conclusions of [GK1, Theorems 3.4, 3.8 and 3.10, Corollaries 3.10 and 3.12].*

Proof of Theorem 3.11. In view of Theorem 4.6 it suffices to prove that for all $g < g_0$ and $\theta > 2d$ we have

$$\lim_{L\to\infty} \mathbb{P}\{\Lambda_L(0) \text{ is } (\theta, a(g))\text{-suitable for } W_{g,\omega}\} = 1\,. \tag{4.21}$$

We fix $g < g_0$ and $\theta > 2d$. Given $L \in 6q\mathbb{Z}^d$ we use Theorem 3.10 and, for large L, define $g_L \in (0, g)$ by

$$a(g_L) = a(g) - \left(\kappa \frac{\log L}{L}\right)^2, \tag{4.22}$$

where $\kappa > 0$ will be specified later. We define the event

$$\mathcal{E}_L = \left\{\omega \in \mathbb{R}^{\mathbb{Z}^d};\ \ \omega_i \le \frac{g_L}{g} \text{ for all } i \in \breve{\Lambda}_L(0) \cap \mathbb{Z}^d\right\}\,. \tag{4.23}$$

It follows from Theorem 3.10, Lemma 5.1 and [KK, Theorem 4.3] that

$$(a(g_L), b(g)) \subset \mathbb{R}\backslash\sigma(W_{g,\omega,0,L}) \text{ for all } \omega \in \mathcal{E}_L\,. \tag{4.24}$$

Hence it follows from [KK, Theorem 3.6] that for large L we have

$$\begin{aligned} \|\Gamma_{0,L} R_{g,\omega,0,L}(a(g))\chi_{0,L/3}\|_{0,L} \ &\le\ C_1 L^{2d-1} \left(\frac{\kappa \log L}{L}\right)^{-2} \mathrm{e}^{-C_2(\frac{\kappa \log L}{L})L} \\ &=\ \frac{C_1 L^{2d+1}}{(\kappa \log L)^2 L^{\kappa C_2}} \end{aligned} \tag{4.25}$$

for all $\omega \in \mathcal{E}_L$, where C_1 and C_2 are finite, strictly positive constants depending only on $d, q, \mathcal{K}_{0,\pm}, \mathcal{R}_{0,\pm}, U_\pm, V_\pm, g, a, b$. It follows that

$$\mathcal{E}_L \subset \{\omega \in \mathbb{R}^{\mathbb{Z}^d};\ \ \Lambda_L(0) \text{ is } (\theta, a(g))\text{-suitable for } W_{g,\omega}\} \tag{4.26}$$

for all L sufficiently large if $\kappa > \frac{\theta+2d+1}{C_2}$.

Fixing κ, denoting by $\widetilde{\mathcal{E}_L}$ the complementary event to $\mathcal{E}_L$, and using (3.24), (3.22), and (4.22), we have that

$$\mathbb{P}\left(\widetilde{\mathcal{E}_L}\right) \ \le\ L^d \mu\left(\left(\tfrac{g_L}{g}, 1\right]\right) \le K L^d \left(\tfrac{g-g_L}{g}\right)^\eta \tag{4.27}$$

$$\le\ K L^d \left(\frac{a(g) - a(g_L)}{ag\,(\delta_-(g) + \eta_-(g))}\right)^\eta \tag{4.28}$$

$$\le\ K L^d \left(\frac{\left(\kappa \frac{\log L}{L}\right)^2}{ag\,(\delta_-(g) + \eta_-(g))}\right)^\eta \to 0 \tag{4.29}$$

as $L \to \infty$ since $\eta > \frac{d}{2}$. $\square$

Proof of Theorem 3.12. Let $g < g_0$, fix $a < a_1 < a_2 < a(g)$, with $a(g) - a_1 \leq b(g) - a(g)$, and define $g_1 \in (0, g)$ by $a(g_1) = a_1$ using Theorem 3.10.

We use the notation of the the previous proof with $a_L = a_1$ and $g_L = g_1$ for all L. As before, it follows from [KK, Theorem 3.6] that for sufficiently large L we have

$$\|\Gamma_{0,L} R_{g,\omega,0,L}(E) \chi_{0,L/3}\|_{0,L} \leq C_1 L^{2d-1} (a_2 - a_1)^{-1} \mathrm{e}^{-C_2 \sqrt{a_2 - a_1}\, L}$$

for all $E \in [a_2, a(g)]$ and $\omega \in \mathcal{E}_L$, where C_1 and C_2 are the same constants as in (4.25). Thus, given $\theta > 0$, we have

$$\mathcal{E}_L \subset \bigcap_{E \in [a_2, a(g)]} \{\omega \in \mathbb{R}^{\mathbb{Z}^d};\ \Lambda_L(0) \text{ is } (\theta, E)\text{-suitable for } W_{g,\omega}\} \quad (4.30)$$

for sufficiently large L. We also have, using (3.25), that

$$\mathbb{P}\left(\widetilde{\mathcal{E}_L}\right) \leq L^d \mu\left(\left(\tfrac{g_1}{g}, 1\right]\right) \leq L^d p_1. \quad (4.31)$$

We now fix $\theta > 2d$, and pick $L_0 \in 6q\mathbb{Z}^d$, suficiently large so (4.30) holds for this θ and $L_0 \geq \overline{\mathcal{L}}$, where $\overline{\mathcal{L}}$ is given in Theorem 4.6, and take

$$p_1 < \frac{1}{841^d L_0^d}\,. \quad (4.32)$$

We then have (4.20) with $\mathcal{L} = L_0$ for all $E \in [a_2, a(g)]$, so Theorem 3.12 follows from Theorem 4.6. $\square$

Proof of Theorem 3.13. It proceeds in the same way as the proof of Theorem 3.11, but taking into account both edges of the gap. Since we will use [KK, Theorem 3.6] for an energy in the middle of a gap, we will need $\eta > d$ instead of $\eta > \frac{d}{2}$ as in Theorem 3.11. We verify (4.20) instead of (4.21).

We fix $g \in [g_0, \frac{1}{Z_+})$ and $\theta > 2d$. Recall $a(g) = a(g_0) = b(g_0) = b(g)$. Given $L \in 6q\mathbb{Z}^d$ we use Theorem 3.10 and, for large L, define $g_L^{\pm} \in (0, g_0)$ by

$$a(g_L^-) \;=\; a(g_0) - \kappa \frac{\log L}{L}\,, \quad (4.33)$$

$$b(g_L^+) \;=\; a(g_0) + \kappa \frac{\log L}{L}\,, \quad (4.34)$$

where $\kappa > 0$ will be specified later. We define the event

$$\mathcal{F}_L = \left\{\omega \in \mathbb{R}^{\mathbb{Z}^d};\ \ -\frac{g_L^+}{g} \leq \omega_i \leq \frac{g_L^-}{g} \text{ for all } i \in \breve{\Lambda}_L(0) \cap \mathbb{Z}^d \right\}. \quad (4.35)$$

It follows from Theorem 3.10, Lemma 5.1 and [KK, Theorem 4.3] that

$$(a(g_L^-), b(g_L^+)) \subset \mathbb{R}\backslash\sigma(W_{g,\omega,0,L}) \text{ for all } \omega \in \mathcal{F}_L\,. \tag{4.36}$$

Hence it follows from [KK, Theorem 3.6] that for large L we have

$$\begin{aligned}\|\Gamma_{0,L}R_{g,\omega,0,L}(a(g_0))\chi_{0,L/3}\|_{0,L} \;&\leq\; C_1' L^{2d-1}\left(\frac{\kappa\log L}{L}\right)^{-1}\mathrm{e}^{-C_2'\left(\frac{\kappa\log L}{L}\right)L} \\ &=\; \frac{C_1' L^{2d}}{\kappa(\log L)L^{\kappa C_2'}}\end{aligned} \tag{4.37}$$

for all $\omega \in \mathcal{F}_L$, where C_1' and C_2' are finite, strictly positive constants depending only on $d, q, \mathcal{K}_{0,\pm}, \mathcal{R}_{0,\pm}, U_\pm, V_\pm, g, a, b$. It follows that

$$\mathcal{F}_L \subset \{\omega \in \mathbb{R}^{\mathbb{Z}^d};\;\; \Lambda_L(0) \text{ is } (\theta, a(g_0))\text{-suitable for } W_{g,\omega}\} \tag{4.38}$$

for all L sufficiently large if $\kappa > \frac{\theta+2d}{C_2'}$.

Fixing κ, denoting by $\widetilde{\mathcal{F}_L}$ the complementary event to $\mathcal{F}_L$, and using (3.26), (3.22), (3.23), (4.33), and (4.34), we have that

$$\mathbb{P}\left(\widetilde{\mathcal{F}_L}\right) \leq L^d\left\{\mu\left(\left(\tfrac{g_L^-}{g}, 1\right]\right) + \mu\left(\left[-1, -\tfrac{g_L^+}{g}\right)\right)\right\} \tag{4.39}$$

$$\leq KL^d\left\{\left(\tfrac{g-g_L^-}{g}\right)^\eta + \left(\tfrac{g-g_L^+}{g}\right)^\eta\right\} \tag{4.40}$$

$$\begin{aligned}\leq \frac{KL^d}{g_0^\eta}\Bigg\{&\left(g - g_0 + \frac{a(g_0) - a(g_L^-)}{a\left(\delta_-(g_0) + \eta_-(g_0)\right)}\right)^\eta \\ &+ \left(g - g_0 + \frac{b(g_L^+) - a(g_0)}{a\left(\delta_-(g_0) + \eta_-(g_0)\right)}\right)^\eta\Bigg\}\end{aligned} \tag{4.41}$$

$$\leq \frac{2KL^d}{g_0^\eta}\left(g - g_0 + \frac{\kappa\frac{\log L}{L}}{a\left(\delta_-(g_0) + \eta_-(g_0)\right)}\right)^\eta. \tag{4.42}$$

We now fix $\theta > 2d$ and $\kappa > \frac{\theta+2d}{C_2'}$, and pick $L_0 \in 6q\mathbb{Z}^d$, suficiently large so (4.38) holds for this θ and $L_0 \geq \overline{\mathcal{L}}$, where $\overline{\mathcal{L}}$ is given in Theorem 4.6, and

$$\frac{2KL_0^d}{g_0^\eta}\left(\frac{2\kappa\frac{\log L_0}{L_0}}{a\left(\delta_-(g_0) + \eta_-(g_0)\right)}\right)^\eta < \frac{1}{841^d}\,, \tag{4.43}$$

what can be done since $\eta > d$. If we now set

$$\varepsilon = \min\left\{\frac{\kappa\frac{\log L_0}{L_0}}{a\left(\delta_-(g_0) + \eta_-(g_0)\right)}, \frac{1}{Z_+} - g_0\right\}\,, \tag{4.44}$$

we have (4.20) with $\mathcal{L} = L_0$ and $E = a(g_0)$ for all $g \in [g_o, g_0 + \varepsilon)$, so Theorem 3.13 follows from Theorem 4.6. □

Proof of Theorem 3.14. The proof is similar to the proof of Theorem 3.12, but taking into account both edges of the gap, as in the proof of Theorem 3.13. □

5. The location of the spectral gap

In this section we prove Theorem 3.10. We proceed as in [FK3, Theorem 3], but we must take into consideration two random coefficients. To do so, we make use of the unitary equivalence between the operators $(W_{g,\omega,\Lambda})_\perp$ and $(W_{g,\omega,*,\Lambda})_\perp$, and use [KK, Theorem 4.3].

We start by approximating the spectrum of the random operator by spectra of periodic operators. If $k, n \in \mathbb{N}$, we say that $k \preceq n$ if $n \in k\mathbb{N}$ and that $k \prec n$ if $k \preceq n$ and $k \neq n$.

Let us fix g as in (3.6) and set

$$\mathcal{T}_g = \{\tau = \{\tau_i, i \in \mathbb{Z}^d\};\ -g \leq \tau_i \leq g\} = [-g, g]^{\mathbb{Z}^d}, \tag{5.1}$$

$$\mathcal{T}_g^{(n)} = \{\tau \in \mathcal{T};\ \tau_{i+nj} = \tau_i \text{ for all } i, j \in \mathbb{Z}^d\},\ \ n \in \mathbb{N}, \tag{5.2}$$

and

$$\mathcal{T}_g^{(\infty)} = \bigcup_{n \succeq q} \mathcal{T}_g^{(n)}. \tag{5.3}$$

For $\tau \in \mathcal{T}_g$ we let

$$\mathcal{K}_\tau(x) \ = \ \gamma_\tau(x)\,\mathcal{K}_0(x), \text{ with } \gamma_\tau(x) = 1 + \sum_{i \in \mathbb{Z}^d} \tau_i u_i(x)\,, \tag{5.4}$$

$$\mathcal{R}_\tau(x) \ = \ \zeta_\tau(x)\mathcal{R}_0(x)\,, \text{ with } \zeta_\tau(x) = 1 + \sum_{i \in \mathbb{Z}^d} \tau_i v_i(x)\,, \tag{5.5}$$

$$A_\tau = \sqrt{\mathcal{R}_\tau}\mathbf{D}\sqrt{\mathcal{K}_\tau}\,,\ \ W_\tau = A_\tau^* A_\tau\,. \tag{5.6}$$

The following lemma shows that the (nonrandom) spectrum of the random classical wave operator $W_{g,\omega}$ is determined by the spectra of the periodic classical wave operators W_τ, $\tau \in \mathcal{T}_g^{(\infty)}$. The analogous result for random Schrödinger operators was proven in [KM2, Theorem 4]. It was exended to certain random classical wave operators in in [FK3, Lemma 19] and [FK4, Lemma 27].

LEMMA 5.1. *Let $W_{g,\omega}$ be a random second order classical wave operator. Its spectrum Σ_g is given by*

$$\Sigma_g = \overline{\bigcup_{\tau \in \mathcal{T}_g^{(\infty)}} \sigma\left(W_\tau\right)}\,. \tag{5.7}$$

Proof. Let Σ_g' denote the right hand side of (5.7). We start by showing that

$$\sigma(W_\tau) \subset \Sigma_g' \;\text{ for all }\; \tau \in \mathcal{T}_g\,, \tag{5.8}$$

which implies that

$$\Sigma_g \subset \Sigma_g'\,. \tag{5.9}$$

Let $\{\ell_n;\ n = 0,1,2,\ldots\}$ be a sequence in $2\mathbb{N}$ such that $\ell_0 = 2q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0,1,2,\ldots$. Given $\tau \in \mathcal{T}_g$, we specify $\tau_n \in \mathcal{T}_g^{(\ell_n)}$ by requiring $(\tau_n)_i = \tau_i$ for $i \in [-\frac{\ell_n}{2}, \frac{\ell_n}{2})^d \cap \mathbb{Z}^d$. We set $R_n = (W_{\tau_n}+1)^{-1}$, $R = (W_\tau + 1)^{-1}$. We will show that $R_n \to R$ strongly, which implies (5.8), as in [FK3, Lemma 45].

To do so, note that

$$\begin{aligned} R_n \;&=\; \left(\sqrt{\mathcal{K}_{\tau_n}}\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D}\sqrt{\mathcal{K}_{\tau_n}} + 1\right)^{-1} \\ &=\; \mathcal{K}_{\tau_n}^{-1/2}\left(\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D} + \mathcal{K}_{\tau_n}^{-1}\right)^{-1}\mathcal{K}_{\tau_n}^{-1/2}\,, \end{aligned} \tag{5.10}$$

and similarly for R. Note that we have uniform (in n) bounds on the operator norms of $\mathcal{K}_{\tau_n}^{-1}$, $\mathcal{R}_{\tau_n}$, and $\left(\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D} + \mathcal{K}_{\tau_n}^{-1}\right)^{-1}$. In addition, it is easy to see that $\mathcal{K}_{\tau_n}^{-1/2} \to \mathcal{K}_\tau^{-1/2}$, $\mathcal{K}_{\tau_n}^{-1} \to \mathcal{K}_\tau^{-1}$, and $\mathcal{R}_{\tau_n} \to \mathcal{R}_\tau$, the convergence being in the strong operator topology. Thus it suffices to show that $\left(\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D} + \mathcal{K}_{\tau_n}^{-1}\right)^{-1} \to \left(\mathbf{D}^*\mathcal{R}_\tau\mathbf{D} + \mathcal{K}_\tau^{-1}\right)^{-1}$ strongly. But this follows from the preceding remarks, the relation

$$\begin{aligned} &\left(\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D} + \mathcal{K}_{\tau_n}^{-1}\right)^{-1} - \left(\mathbf{D}^*\mathcal{R}_\tau\mathbf{D} + \mathcal{K}_\tau^{-1}\right)^{-1} = \\ &\left(\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D} + \mathcal{K}_{\tau_n}^{-1}\right)^{-1}\left(\mathbf{D}^*(\mathcal{R} - \mathcal{R}_{\tau_n})\mathbf{D} + (\mathcal{K} - \mathcal{K}_{\tau_n}^{-1})\right)\left(\mathbf{D}^*\mathcal{R}_\tau\mathbf{D} + \mathcal{K}_\tau^{-1}\right)^{-1}, \end{aligned} \tag{5.11}$$

and the fact that the operators $\mathbf{D}\left(\mathbf{D}^*\mathcal{R}_{\tau_n}\mathbf{D} + \mathcal{K}_{\tau_n}^{-1}\right)^{-1}$ are bounded with norms uniformly bounded in n, and hence also their adjoints.

To prove the opposite inclusion to (5.9), we introduce the countable sets

$$\mathcal{T}_{g,\mathbb{Q}}^{(N)} = \mathcal{T}_g^{(N)} \cap \mathbb{Q}^{\mathbb{Z}^d}, \quad N = 0,1,2,\ldots,\infty\,. \tag{5.12}$$

Since any $\tau \in \mathcal{T}_g^{(\infty)}$ can be approximated uniformly by a sequence $\tau_n \in \mathcal{T}_{g,\mathbb{Q}}^{(\infty)}$, the previous argument shows that

$$\sigma(W_\tau) \subset \overline{\bigcup_{\tau \in \mathcal{T}_{g,\mathbb{Q}}^{(\infty)}}}\sigma\left(W_\tau\right) \;\text{ for all }\; \tau \in \mathcal{T}_g^{(\infty)}\,, \tag{5.13}$$

which implies that

$$\Sigma_g' = \overline{\bigcup_{\tau \in \mathcal{T}_{g,\mathbb{Q}}^{(\infty)}} \sigma\left(W_\tau\right)}\,. \tag{5.14}$$

Thus (5.7) follows if we prove that

$$\sigma(W_\tau) \subset \Sigma_g \ \text{ for all } \ \tau \in \mathcal{T}_{g,\mathbb{Q}}^{(\infty)}\,. \tag{5.15}$$

Note that a.e. $\omega \in \Omega \equiv [-1,1]^{\mathbb{Z}^d}$. Let $\{\ell_n \in \mathbb{N};\ n = 0,1,2,\ldots\}$ be such that $\ell_0 = 2q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0,1,2,\ldots$. For each n,, $q' \succeq q$, and $\tau \in \mathcal{T}_{g,\mathbb{Q}}^{(q')}$ we consider the event

$$\Omega_{n,q',\tau} = \Big\{\omega \in \Omega;\ \text{there is } x_\omega = x_{n,q',\tau,\omega} \in q'\mathbb{Z}^d \text{ such that } \max_{i \in \left(x_\omega + [-\frac{\ell_n}{2},\frac{\ell_n}{2})^d\right)\cap\mathbb{Z}^d} |g\omega_i - \tau_i| \le \frac{1}{\ell_n^{d+1}}\Big\}\,; \tag{5.16}$$

notice $\mathbb{P}(\Omega_{n,q',\tau}) = 1$. We take the countable intersection

$$\widehat{\Omega} = \bigcap_{n=0}^{\infty} \bigcap_{q' \succeq q} \bigcap_{\tau \in \mathcal{T}_{g,\mathbb{Q}}^{(q')}} \Omega_{n,q',\tau}\,, \tag{5.17}$$

so we have $\mathbb{P}(\widehat{\Omega}) = 1$. We will show that

$$\sigma(W_\tau) \subset \sigma(W_{g,\omega}) \ \text{ for all } \ \tau \in \mathcal{T}_{g,\mathbb{Q}}^{(\infty)} \text{ and } \omega \in \widehat{\Omega}\,, \tag{5.18}$$

so (5.15) follows.

So let $\tau \in \mathcal{T}_{g,\mathbb{Q}}^{(\infty)}$, say $\tau \in \mathcal{T}_{g,\mathbb{Q}}^{(q')}$ for some $q' \succeq q$. Let $\omega \in \widehat{\Omega}$, $n \in \mathbb{N}$, and let $x_\omega = x_{n,q',\tau,\omega}$ be as in (5.16). We set $\omega^{(n)} = \{\omega_i^{(n)} = \omega_{i-x_\omega}\,;\ i \in \mathbb{Z}^d\}$, and notice that $\sigma(W_{g,\omega^{(n)}}) = \sigma(W_{g,\omega})$. We have the following inequalitie for the matrices norms:

$$\left\|\Big(\mathcal{R}_{g,\omega^{(n)}}\left(x\right) - \mathcal{R}_\tau\left(x\right)\Big)\chi_{0,\ell_n - r}\left(x\right)\right\| \le \frac{R_{0,+}V_+}{\ell_n}\chi_{0,\ell_n - r}\left(x\right), \tag{5.19}$$

$$\left\|\Big(\mathcal{K}_{g,\omega^{(n)}}\left(x\right)^{-1} - \mathcal{K}_\tau\left(x\right)^{-1}\Big)\chi_{0,\ell_n - r}\left(x\right)\right\| \le \frac{U_+}{\ell_n\, K_{0,-}(1-gU_+)^2}\,\chi_{0,\ell_n - r}\left(x\right)\,, \tag{5.20}$$

$$\left\|\Big(\mathcal{K}_{g,\omega^{(n)}}(x)^{-\frac{1}{2}} - \mathcal{K}_\tau\left(x\right)^{-\frac{1}{2}}\Big)\chi_{0,\ell_n - r}\left(x\right)\right\| \le \frac{K_{0,+}^{\frac{1}{2}}U_+(1+gU_+)}{\ell_n\, K_{0,-}(1-gU_+)^{\frac{5}{2}}}\,\chi_{0,\ell_n - r}\left(x\right)\,. \tag{5.21}$$

Using these inequalities we can proceed as before to show that

$$\lim_{n\to\infty}\left(W_{g,\omega(n)}+I\right)^{-1}=(W_\tau+I)^{-1} \tag{5.22}$$

in the strong operator topology, an hence that

$$\sigma(W(\tau))\subset\overline{\bigcup_{n=0}^{\infty}\sigma(W_{g,\omega^{(n)}})}=\sigma(W_{g,\omega})\,. \tag{5.23}$$

□

Given real numbers k,h, with $|k|,|h|<\frac{1}{Z_+}$, we set

$$\begin{aligned}\mathcal{K}_k(x) &= \mathcal{K}_0(x)\,(1+kU(x))\ \text{ and } \mathcal{R}_h(x)=\mathcal{R}_0(x)\,(1+hV(x))\ , \\ A(k,h) &= \sqrt{\mathcal{R}_h}\mathbf{D}\sqrt{\mathcal{K}_k}\,, \\ W(k,h) &= A(k,h)^*A(k,h)\,,\ \ W_*(k,h)=A(k,h)A(k,h)^*.\end{aligned} \tag{5.24}$$

LEMMA 5.2. *Let $W(k,h)$ be as in (5.24), and let $\Lambda=\Lambda_\ell(x_0)$ for some $x_0\in\mathbb{R}^d$ and $\ell\succeq q$. The positive self-adjoint operator $(W(k,h)_\Lambda)_\perp$ has compact resolvent, so let $\mu_1(k,h)\le\mu_2(k,h)\le\ldots$ be its eigenvalues, repeated according to their (finite) multiplicity. Then each $\mu_j(h)\equiv\mu_j(h,h)$, $j=1,2,\ldots$, is a Lipschitz continuous, strictly increasing function of h, with*

$$\begin{aligned}\tfrac{1}{2}\left(\delta_-(g)+\eta_-(g)\right)\left(\mu_j(h_1)+\mu_j(h_2)\right) &\le \frac{\mu_j(h_2)-\mu_j(h_1)}{h_2-h_1} \\ &\le \tfrac{1}{2}\left(\delta_+(g)+\eta_+(g)\right)\left(\mu_j(h_1)+\mu_j(h_2)\right)\end{aligned} \tag{5.25}$$

for any $h_1,h_2\in(-g,g)$, $0<g<\frac{1}{Z_+}$, where $\delta_\pm(g)$ and $\eta_\pm(g)$ are given in (3.11) and (3.12).

Proof. Let $0<g<\frac{1}{Z_+}$, $-g\le h_1<h_2\le g$. We have

$$\mathcal{R}_{h_2}(x)-\mathcal{R}_{h_1}(x)=(h_2-h_1)U(x)\mathcal{R}_0(x)\ge 0\,, \tag{5.26}$$

so $W(k,h_2)_\Lambda\ge W(k,h_1)_\lambda$ and hence each $\mu_j(k,h)$ is an increasing function of h for fixed k. It also follows from (5.26) that

$$\mathcal{R}_{h_2}(x)=\mathcal{R}_{h_1}(x)\left(1+\frac{(h_2-h_1)V(x)}{1+h_2V(x)}\right) \tag{5.27}$$

and

$$\mathcal{R}_{h_1}(x)=\mathcal{R}_{h_2}(x)\left(1-\frac{(h_2-h_1)V(x)}{1+h_1V(x)}\right)\,, \tag{5.28}$$

which gives us

$$\mathcal{R}_{h_1}(x)\,(1+\eta_-(g)(h_2-h_1)) \le \mathcal{R}_{h_2}(x) \le \mathcal{R}_{h_1}(x)\,(1+\eta_+(g)(h_2-h_1)) \tag{5.29}$$

and

$$\mathcal{R}_{h_2}(x)\,(1-\eta_+(g)(h_2-h_1)) \le \mathcal{R}_{h_1}(x) \le \mathcal{R}_{h_2}(x)\,(1-\eta_-(g)(h_2-h_1)) \tag{5.30}$$

with $\eta_\pm(g)$ as in (3.12).

From (5.29) we get that for each k we have

$$\begin{aligned}(1+\eta_-(g)(h_2-h_1))W(k,h_1)_\Lambda &\le W(k,h_2)_\Lambda \\ &\le (1+\eta_+(g)(h_2-h_1))W(k,h_1)_\Lambda\,,\end{aligned} \tag{5.31}$$

so it follows from the min-max principle that for all $j=1,2,\ldots$,

$$\begin{aligned}(1+\eta_-(g)(h_2-h_1))\,\mu_j(k,h_1) &\le \mu_j(k,h_2) \\ &\le (1+\eta_+(g)(h_2-h_1))\,\mu_j(k,h_1)\,,\end{aligned} \tag{5.32}$$

i.e.,

$$\eta_-(g)\mu_j(k,h_1) \le \frac{\mu_j(k,h_2)-\mu_j(k,h_1)}{h_2-h_1} \le \eta_+(g)\mu_j(k,h_1) \tag{5.33}$$

Similarly, using (5.30) we get

$$\eta_-(g)\mu_j(k,h_2) \le \frac{\mu_j(h_2,h_2)-\mu_j(h_2,h_1)}{h_2-h_1} \le \eta_+(g)\mu_j(k,h_2)\,. \tag{5.34}$$

Thus

$$\eta_-(g)\mu_j(k,h_2) \le \frac{\mu_j(k,h_2)-\mu_j(k,h_1)}{h_2-h_1} \le \eta_+(g)\mu_j(k,h_1)\,. \tag{5.35}$$

Since the operators $(W(k,h)_\Lambda)_\perp$ and$(W_*(k,h)_\Lambda)_\perp$ are unitarily equivalent, the $\mu_j(k,h)$ are also the egenvalues of $(W_*(k,h)_\Lambda)_\perp$, so the above argument gives

$$\delta_-(g)\mu_j(k_2,h) \le \frac{\mu_j(k_2,h)-\mu_j(k_1,h)}{k_2-k_1} \le \delta_+(g)\mu_j(k_1,h)\,, \tag{5.36}$$

where $-g \le k_1 < k_2 \le g$.

Since

$$\begin{aligned}&\mu_j(h_2,h_2)-\mu_j(h_1,h_1) \\ &\quad = (\mu_j(h_2,h_2)-\mu_j(h_2,h_1)) + (\mu_j(h_2,h_1)-\mu_j(h_1,h_1)) \qquad (5.37)\\ &\quad = (\mu_j(h_2,h_2)-\mu_j(h_1,h_2)) + (\mu_j(h_1,h_2)-\mu_j(h_1,h_1))\;, \qquad (5.38)\end{aligned}$$

we may use (5.35) and (5.36) with (5.37), repeat the procedure with (5.38) instead of (5.37), and take the average of the bounds to obtain (5.25). The properties of the functions $\mu_j(h)$ follow. □

The following lemma follows immediately from [KK, Theorem 4.3], Lemmas 5.1 and 5.2, and the min-max principle. We write $W(h)$ for $W(h,h)$ as in (5.24).

LEMMA 5.3. *Let $W_{g,\omega}$ be a random second order classical wave operator. For all sequences $\{\ell_n \in \mathbb{N};\ n = 0,1,2,\ldots\}$, with $\ell_0 = 2q$ and $\ell_n \prec \ell_{n+1}$ for each $n = 0,1,2,\ldots$, we have*

$$\Sigma_g = \overline{\bigcup_{h\in[-g,g]} \sigma\left(W(h)\right)} = \overline{\bigcup_{h\in[-g,g]} \bigcup_{n=0}^{\infty} \sigma\left(W(h)_{\Lambda_{\ell_n}(0)}\right)}. \tag{5.39}$$

In particular, Σ_g is increasing in g.

We are now ready to prove Theorem 3.10. As Σ_g is increasing in g, we expect the gap to shrink as we increase g until it either disappears at some g_0, or it remains open for all allowed g. Thus we define

$$g_0 = \sup\left\{g \in \left[0, \tfrac{1}{Z_+}\right);\ \ \Sigma_g \cap (a,b) \neq (a,b)\right\}. \tag{5.40}$$

Let $\{\ell_n;\ n = 0,1,2,\ldots\}$ be as in Lemma 5.3, $h \in [-g,g]$, and let $\mu_1^{(n)}(h) \le \mu_2^{(n)}(h) \le \ldots$ be the nonzero eigenvalues of $W(h)_{\Lambda_n}$, where $\Lambda_n = \Lambda_{\ell_n}(0)$, repeated according to their (finite) multiplicity; notice $\lim_{j\to\infty} \mu_j^{(n)}(h) = \infty$. By Lemma 5.2 each $\mu_j^{(n)}(h)$ is a strictly increasing continuous function of h, hence it follows from Lemma 5.3 that

$$\Sigma_g = \overline{\bigcup_{n=0}^{\infty} \bigcup_{h\in[-g,g]} \sigma\left(W(h)_{\Lambda_n}\right)} = \overline{\bigcup_{n=0}^{\infty} \bigcup_{j=1}^{\infty} [\mu_j^{(n)}(-g), \mu_j^{(n)}(g)]}. \tag{5.41}$$

In particular, Σ_g is a countable union of disjoint closed intervals, so for $g < g_0$ we can define $a(g)$ and $b(g)$ by (3.19). Since Σ_g is increasing in $g \in [0, \frac{1}{Z_+})$ by Lemma 5.3, it follows that $a(g)$ and $-b(g)$ are increasing functions in $[0, g_0)$.

For each n let

$$j_n = \max\{j;\ \mu_j^{(n)}(0) \le a\}, \tag{5.42}$$

so using Assumption 3.9 and [KK, Eq. (4.1) in Theorem 4.3], we have

$$j_n + 1 = \min\{j;\ \mu_j^{(n)}(0) \ge b\}. \tag{5.43}$$

If $g < g_0$, it follows from the definition of j_n, Assumption 3.9 and [KK, Theorem 4.3], that $\mu_{j_n}(-g)$ and $-\mu_{j_n+1}(g)$ are both increasing in n, and

$$\begin{aligned} a(g) &= \lim_{n\to\infty} \mu_{j_n}(g), &(5.44)\\ b(g) &= \lim_{n\to\infty} \mu_{j_n+1}(-g). &(5.45)\end{aligned}$$

Thus, given $0 \le g_1 < g_2 < g_0$, we can conclude from (5.25) that

$$\begin{aligned} \tfrac{1}{2}\left(\delta_-(g_2)+\eta_-(g_2)\right)\left(a(g_1)+a(g_2)\right) &\le \frac{a(g_2)-a(g_1)}{g_2-g_1} \qquad (5.46)\\ &\le \tfrac{1}{2}\left(\delta_+(g_2)+\eta_+(g_2)\right)\left(a(g_1)+a(g_2)\right)\,, \end{aligned}$$

$$\begin{aligned} \tfrac{1}{2}\left(\delta_-(g_2)+\eta_-(g_2)\right)\left(b(g_1)+b(g_2)\right) &\le \frac{b(g_1)-b(g_2)}{g_2-g_1} \qquad (5.47)\\ &\le \tfrac{1}{2}\left(\delta_+(g_2)+\eta_+(g_2)\right)\left(b(g_1)+b(g_2)\right)\,, \end{aligned}$$

which are exactly (3.22) and (3.23).

The Lipschitz continuity of $a(g)$ and $b(g)$ follows, and hence they are absolutely continuous functions. Their a.e. derivatives can be estimated from (5.46) and (5.47):

$$\begin{aligned} \delta_-(h)+\eta_-(h) &\le \frac{a'(h)}{a(h)} \le \delta_+(h)+\ \eta_+(h)\,, &(5.48)\\ \delta_-(h)+\eta_-(h) &\le -\frac{b'(h)}{b(h)} \le \delta_+(h)+\ \eta_+(h)\,. &(5.49)\end{aligned}$$

Using the abolute continuity, we may integrate over h obtaining

$$\int_{g_1}^{g_2}\left(\delta_-(h)+\eta_-(h)\right)\mathrm{d}h \le \log\left(\frac{a(g_2)}{a(g_1)}\right) \le \int_{g_1}^{g_2}\left(\delta_+(h)+\eta_+(h)\right)\mathrm{d}h \qquad (5.50)$$

and

$$\int_{g_1}^{g_2}\left(\delta_-(h)+\eta_-(h)\right)\mathrm{d}h \le \log\left(\frac{b(g_1)}{b(g_2)}\right) \le \int_{g_1}^{g_2}\left(\delta_+(h)+\eta_+(h)\right)\mathrm{d}h\,. \qquad (5.51)$$

Performing the integrations, we obtain (3.20) and (3.21), from which (3.18) follows.

If $g_0 < \frac{1}{Z_+}$, we must have $\lim_{g\uparrow g_0} a(g) = \lim_{g\uparrow g_0} b(g)$. This follows from (5.41), (5.44) and (5.45), since by (5.25) each $\mu_j^{(n)}(h)$ is a locally Lipschitz continuous functions of $h \in (-\frac{1}{Z_+}, \frac{1}{z_+})$, uniformly in n.

Thus, if $g \in [g_0, \frac{1}{Z_+})$ it follows that $[a,b] \subset \Sigma_g$; we set $a(g) = b(g) = \lim_{g \uparrow g_0} a(g)$.

Theorem 3.10 is proven.

6. The Wegner estimate

In this section we prove Theorem 4.4. We proceed as in [FK3, Theorem 23], but we must take into consideration two random coefficients. To do so, we make use of the unitary equivalence between the operators $(W_{g,\omega,\Lambda})_\perp$ and $(W_{g,\omega,*,\Lambda})_\perp$. We assume that $W_{g,\omega}$ is the partially elliptic operator without loss of generality.

We start by picking $\kappa \in \left(1, \frac{1}{g}\frac{1}{Z_+}\right)$, say

$$\kappa = \frac{1+gZ_+}{2gZ_+}\,. \tag{6.1}$$

We rewrite $\gamma_{g,\omega}$ and $\zeta_{g,\omega}$ in the form

$$\gamma_{g,\omega} \;=\; \hat{\gamma} + g\sum_{i\in\mathbb{Z}^d}\hat{\omega}_i u_i\,, \tag{6.2}$$

$$\zeta_{g,\omega} \;=\; \hat{\zeta} + g\sum_{i\in\mathbb{Z}^d}\hat{\omega}_i v_i\,, \tag{6.3}$$

where

$$\hat{\gamma} = 1-\kappa g\sum_{i\in\mathbb{Z}^d} u_i \;\geq\; \frac{1-gU_+}{2} > 0\,, \tag{6.4}$$

$$\hat{\zeta} = 1-\kappa g\sum_{i\in\mathbb{Z}^d} v_i \;\geq\; \frac{1-gV_+}{2} > 0\,, \tag{6.5}$$

and $\hat{\omega}_i = \omega_i + \kappa \in [\kappa-1, \kappa+1]$ for each $i \in \mathbb{Z}^d$.

We fix $\Lambda = \Lambda_L(x)$ with $x \in \mathbb{Z}^d$ and $L \in 2q\mathbb{N}$. The finite volume operators operators $(W_{g,\omega,\Lambda})_\perp$ and $(W_{g,\omega,*,\Lambda})_\perp$ are unitarily equivalent by [KK, Lemma A.1]. Since $(W_{g,\omega,\Lambda})_\perp$ has compact resolvent by [KK, Proposition 3.2], so does $(W_{g,\omega,*,\Lambda})_\perp$, and they have the same eigenvalues, say $\{\lambda_{g,\omega,n}\}_{n\in\mathbb{N}}$. We will denote by $\{\psi_{g,\omega,n}\}_{n\in\mathbb{N}}$ and $\{\varphi_{g,\omega,n}\}_{n\in\mathbb{N}}$ the corresponding orthonormal eigenfunctions for $(W_{g,\omega,\Lambda})_\perp$ and $(W_{g,\omega,*,\Lambda})_\perp$, respectively. Note that they can may chosen so they are measurable functions of ω.

Given $j \in \mathbb{Z}^d$, we set $\varepsilon^{(j)} \in \mathbb{R}^{\mathbb{Z}^d}$ by $\varepsilon^{(j)}_i = \delta_{j,i}$. Note that $\mathcal{K}_{g,\omega+s\varepsilon^{(j)}}(x)$ and $\mathcal{R}_{g,\omega+t\varepsilon^{(j)}}(x)$ are coefficient matrices for $|s|, |t|$ sufficiently small; the corresponding classical wave operators will be denoted by $W_{g,\omega}(s,t;j)$, etc. The remarks of the previous paragraph still apply. Note that we can choose each $\lambda_{g,\omega,n}(s,t;j)$ jointly analytic in s and t. If $j \in \breve{\Lambda} \cap \mathbb{Z}^d$, we have

$$\begin{aligned}
\frac{\partial}{\partial \hat{\omega}_j}\lambda_{g,\omega,n} &= \frac{\partial}{\partial s}\lambda_{g,\omega,n}(s,t;j)|_{(s,t)=(0,0)} + \frac{\partial}{\partial t}\lambda_{g,\omega,n}(s,t;j)|_{(s,t)=(0,0)} \\
&= \left\langle \varphi_{g,\omega,n}, \left(\frac{\partial}{\partial s}W_{g,\omega,*,\Lambda}(s,t;j)|_{(s,t)=(0,0)}\right)\varphi_{g,\omega,n}\right\rangle + \\
&\qquad \left\langle \psi_{g,\omega,n}, \left(\frac{\partial}{\partial t}W_{g,\omega,\Lambda}(s,t;j)|_{(s,t)=(0,0)}\right)\psi_{g,\omega,n}\right\rangle \\
&= \left\langle \varphi_{g,\omega,n}, W_{g,\omega,*,\Lambda}(gu_j^{\Lambda}\mathcal{K}_0)\,\varphi_{g,\omega,n}\right\rangle + \qquad (6.6) \\
&\qquad \left\langle \psi_{g,\omega,n}, W_{g,\omega,\Lambda}(gv_j^{\Lambda}\mathcal{R}_0)\,\psi_{g,\omega,n}\right\rangle ,
\end{aligned}$$

where we used (4.7), with

$$u_j^{\Lambda} = \sum_{i\in\mathbb{Z}^d} u_{j+Li}\,, \quad v_j^{\Lambda} = \sum_{i\in\mathbb{Z}^d} v_{j+Li}\,, \tag{6.7}$$

and $W_{g,\omega,*,\Lambda}(\mathcal{K}_1)$ and $W_{g,\omega,\Lambda}(\mathcal{R}_1)$ the finite volume operators defined by

$$W_{g,\omega,*,\Lambda}(\mathcal{K}_1) \;=\; \sqrt{\mathcal{R}_{g,\omega_\Lambda}}\mathbf{D}_\Lambda\mathcal{K}_1\mathbf{D}_\Lambda^*\sqrt{\mathcal{R}_{g,\omega_\Lambda}}\,, \tag{6.8}$$
$$W_{g,\omega,\Lambda}(\mathcal{R}_1) \;=\; \sqrt{\mathcal{K}_{g,\omega_\Lambda}}\mathbf{D}_\Lambda^*\mathcal{R}_1\mathbf{D}_\Lambda\sqrt{\mathcal{K}_{g,\omega_\Lambda}}\,. \tag{6.9}$$

Since $(W_{g,\omega,\Lambda})_\perp \geq 0$ has compact resolvent, we may define

$$N_{g,\omega,\Lambda}(E) = \operatorname{tr}\chi_{(-\infty,E]}((W_{g,\omega,\Lambda})_\perp)\,, \tag{6.10}$$

the number of eigenvalues of $(W_\Lambda)_\perp$ that are less than or equal to E. If $E \leq 0$, we have $N_{W_\Lambda}(E) = 0$, and if $E > 0$, $N_{W_\Lambda}(E)$ is the number of eigenvalues of $W_{g,\omega,\Lambda}$ (or $(W_{g,\omega,\Lambda})_\perp$) in the interval $(0,E]$. Notice that $N_{g,\omega,\Lambda}(E)$ is the distribution function of the measure $n_{g,\omega,\Lambda}(dE)$ given by

$$\int h(E)n_{g,\omega,\Lambda}(dE) = \operatorname{tr}\left(h((W_{g,\omega,\Lambda})_\perp)\right), \tag{6.11}$$

for positive continuous functions h of a real variable. Note also that

$$N_{g,\omega,\Lambda}(E) = \sum_{n\in\mathbb{N}} Y(E - \lambda_{g,\omega,n})\,, \tag{6.12}$$

where $Y(x)$ is the Heaviside function.

Let f be a positive, continuous function on the real line with $f(0) = 0$, and $j \in \breve{\Lambda} \cap \mathbb{Z}^d$. We have, using (6.6), that

$$\begin{aligned}
-\frac{\partial}{\partial \hat{\omega}_j} \int N_{g,\omega,\Lambda}(E) f(E) dE &= -\frac{\partial}{\partial \hat{\omega}_j} \sum_{n=1}^{\infty} \int Y(E - \lambda_{g,\omega,n}) f(E) dE \\
&= \sum_{n=1}^{\infty} \int \left(\frac{\partial}{\partial \hat{\omega}_j} \lambda_{g,\omega,n} \right) \delta(E - \lambda_{g,\omega,n}) f(E) dE \qquad (6.13) \\
&= \sum_{n=1}^{\infty} f(\lambda_{g,\omega,n}) \frac{\partial}{\partial \hat{\omega}_j} \lambda_{g,\omega,n} \\
&= \sum_{n=1}^{\infty} \Big\{ \Big\langle f(W_{g,\omega,*,\Lambda}) \varphi_{g,\omega,n}, W_{g,\omega,*,\Lambda}(g u_j^\Lambda \mathcal{K}_0)\, \varphi_{g,\omega,n} \Big\rangle + \\
&\qquad \Big\langle f(W_{g,\omega,\Lambda}) \psi_{g,\omega,n}, W_{g,\omega,\Lambda}(g v_j^\Lambda \mathcal{R}_0)\, \psi_{g,\omega,n} \Big\rangle \Big\} \\
&= \operatorname{tr} \Big\{ W_{g,\omega,*,\Lambda}(g u_j^\Lambda \mathcal{K}_0) f(W_{g,\omega,*,\Lambda}) \Big\} + \operatorname{tr} \Big\{ W_{g,\omega,\Lambda}(g v_j^\Lambda \mathcal{R}_0) f(W_{g,\omega,\Lambda}) \Big\} .
\end{aligned}$$

The last step used the fact that $f(0) = 0$. Thus

$$\begin{aligned}
&- \sum_{i \in \breve{\Lambda} \cap \mathbb{Z}^d} \hat{\omega}_i \frac{\partial}{\partial \hat{\omega}_i} \int N_{g,\omega,\Lambda}(E) f(E) dE = \qquad (6.14) \\
&\quad \operatorname{tr} \left\{ W_{g,\omega,*,\Lambda}((\gamma_{g,\omega_\Lambda} - \hat{\gamma}) \mathcal{K}_0) f(W_{g,\omega,*,\Lambda}) \right\} \\
&\qquad + \operatorname{tr} \Big\{ W_{g,\omega,\Lambda}((\zeta_{g,\omega_\Lambda} - \hat{\zeta}) \mathcal{R}_0) f(W_{g,\omega,\Lambda}) \Big\} \\
&= \operatorname{tr} \left\{ W_{g,\omega,*,\Lambda} f(W_{g,\omega,*,\Lambda}) \right\} - \operatorname{tr} \left\{ W_{g,\omega,*,\Lambda}(\hat{\gamma} \mathcal{K}_0) f(W_{g,\omega,*,\Lambda}) \right\} \\
&\quad + \operatorname{tr} \left\{ W_{g,\omega,\Lambda} f(W_{g,\omega,\Lambda}) \right\} - \operatorname{tr} \Big\{ W_{g,\omega,\Lambda}(\hat{\zeta} \mathcal{R}_0) f(W_{g,\omega,\Lambda}) \Big\} .
\end{aligned}$$

We have, for any $\omega \in [-1.1]^{\mathbb{Z}^d}$ (and hence also for ω_Λ), that

$$\begin{aligned}
\hat{\gamma}^{-1} \gamma_{g,\omega} &= \hat{\gamma}^{-1} \left(\hat{\gamma} + g \sum_{i \in \mathbb{Z}^d} \hat{\omega}_i u_i \right) \qquad (6.15) \\
&\geq 1 + \frac{(\kappa - 1) g U_-}{1 - \kappa g U_-} \geq 1 + \frac{(1 - g Z_+) U_-}{2 Z_+} ,
\end{aligned}$$

and similarly,

$$\hat{\zeta}^{-1} \zeta_{g,\omega} \geq 1 + \frac{(1 - g Z_+) V_-}{2 Z_+} . \qquad (6.16)$$

Since $f \geq 0$, we obtain

$$\begin{aligned}
&\operatorname{tr} \left\{ W_{g,\omega,*,\Lambda}(\hat{\gamma} \mathcal{K}_0) f(W_{g,\omega,*,\Lambda}) \right\} \leq \qquad (6.17) \\
&\qquad \left(1 + \frac{(1 - g Z_+) U_-}{2 Z_+} \right)^{-1} \operatorname{tr} \left\{ W_{g,\omega,*,\Lambda} f(W_{g,\omega,*,\Lambda}) \right\} ,
\end{aligned}$$

$$\operatorname{tr}\left\{W_{g,\omega,\Lambda}(\hat{\zeta}\mathcal{R}_0)f(W_{g,\omega,\Lambda})\right\} \leq \left(1+\frac{(1-gZ_+)V_-}{2Z_+}\right)^{-1} \operatorname{tr}\left\{W_{g,\omega,\Lambda}f(W_{g,\omega,\Lambda})\right\} . \tag{6.18}$$

In addition, using the unitary equivalence between $(W_{g,\omega,\Lambda})_\perp$ and $(W_{g,\omega,*,\Lambda})_\perp$, we get

$$\operatorname{tr}\left\{W_{g,\omega,*,\Lambda}f(W_{g,\omega,*,\Lambda})\right\} = \operatorname{tr}\left\{W_{g,\omega,\Lambda}f(W_{g,\omega,\Lambda})\right\} . \tag{6.19}$$

It follows from (6.14)-(6.19) that

$$\operatorname{tr}\left\{W_{g,\omega,\Lambda}f(W_{g,\omega,\Lambda})\right\} \leq \frac{(2Z_+ + (1-gZ_+)Z_-)^2}{2(1-gZ_+)Z_+Z_-}\left\{-\sum_{i\in\breve{\Lambda}\cap\mathbb{Z}^d}\hat{\omega}_i\frac{\partial}{\partial\hat{\omega}_i}\int N_{g,\omega,\Lambda}(E)f(E)dE\right\} , \tag{6.20}$$

where we used

$$\begin{aligned}
&\left(1-\left(1+\tfrac{(1-gZ_+)U_-}{2Z_+}\right)^{-1}\right)+\left(1-\left(1+\tfrac{(1-gZ_+)V_-}{2Z_+}\right)^{-1}\right)\\
&\quad = \frac{\frac{1-gZ_+}{2Z_+}U_-}{1+\frac{1-gZ_+}{2Z_+}U_-}+\frac{\frac{1-gZ_+}{2Z_+}V_-}{1+\frac{1-gZ_+}{2Z_+}V_-} \qquad (6.21)\\
&\quad \geq \frac{\frac{1-gZ_+}{2Z_+}(U_-+V_-)}{\left(1+\frac{1-gZ_+}{2Z_+}U_-\right)\left(1+\frac{1-gZ_+}{2Z_+}V_-\right)}\\
&\quad \geq \frac{\frac{1-gZ_+}{2Z_+}Z_-}{\left(1+\frac{1-gZ_+}{2Z_+}Z_-\right)^2} = \frac{2(1-gZ_+)Z_+Z_-}{(2Z_+ + (1-gZ_+)Z_-)^2} .
\end{aligned}$$

For given $j\in\mathbb{Z}^d$ we set $\omega^{(j)}=\{\omega_i;\, i\in\mathbb{Z}^d\backslash\{j\}\}$, and denote the corresponding expectation by $\mathbb{E}^{(j)}$. We have, for $j\in\breve{\Lambda}\cap\mathbb{Z}^d$,

$$\begin{aligned}
&\mathbb{E}\left(-\frac{\partial}{\partial\hat{\omega}_j}\int N_{g,\omega,\Lambda}(E)f(E)dE\right) \qquad (6.22)\\
&= \mathbb{E}^{(j)}\left(\int_{\kappa-1}^{\kappa+1}\left[-\frac{\partial}{\partial\hat{\omega}_j}\int N_{g,\omega,\Lambda}(E)f(E)dE\right]\rho(\hat{\omega}_j-\kappa)d\hat{\omega}_j\right)\\
&\leq \|\rho\|_\infty\mathbb{E}^{(j)}\left(\int\left|N_{g,\{\omega^{(j)},\omega_j=-1\},\Lambda}(E)-N_{g,\{\omega^{(j)},\omega_j=1\},\Lambda}(E)\right|f(E)dE\right)\\
&\leq 2nC_d'\left(K_{g,-}R_{g,-}\Theta\right)^{-\frac{d}{2}}\|\rho\|_\infty L^d\int E^{\frac{d}{2}}f(E)dE,
\end{aligned}$$

where we used [KK, Lemma 3.3] in the last step. C_d' is a constant depending only on d, and Θ is the constant in (2.17).

Now let $\bar{n}_{g,\Lambda}(dE) = \mathbb{E}(n_{g,\omega,\Lambda}(dE))$. For functions f as above, it now follows from (6.11), (6.20), and (6.22), that

$$\begin{aligned} \int E f(E) \bar{n}_{g,\Lambda}(dE) &= \mathbb{E}\left\{\operatorname{tr}\left\{W_{g,\omega,\Lambda} f(W_{g,\omega,\Lambda})\right\}\right\} \\ &\leq C \|\rho\|_\infty L^{2d} \int E^{\frac{d}{2}} f(E) dE, \end{aligned} \tag{6.23}$$

where

$$\begin{aligned} C &= 2nC'_d(2+r)^d(\kappa+1)^{\frac{(2Z_+ + (1-gZ_+)Z_-)^2}{2(1-gZ_+)Z_+Z_-}} \left(K_{g,-}R_{g,-}\Theta\right)^{-\frac{d}{2}} \\ &\leq \frac{36nC'_d(2+r)^d}{gZ_-(1-gZ_+)^{d+1}} \left(K_{0,-}R_{0,-}\Theta\right)^{-\frac{d}{2}} . \end{aligned} \tag{6.24}$$

We can now conclude that $\bar{n}_{g,\Lambda}(dE)$ is absolutely continuous with

$$\frac{\bar{n}_{g,\Lambda}(dE)}{dE} \leq C\|\rho\|_\infty E^{\frac{d}{2}-1} L^{2d} \quad \text{for} \quad E \geq 0\,. \tag{6.25}$$

The estimate (4.17) now follows by a standard argument:

$$\begin{aligned} &\mathbb{P}\{\operatorname{dist}(\sigma(W_{g,\omega,\Lambda}), E) < \eta\} \leq \mathbb{P}\left\{\int_{(E-\eta,E+\eta)} n_{g,\omega,\Lambda}(dE) \geq 1\right\} \\ &\leq \int_{(E-\eta,E+\eta)} \bar{n}_{g,\Lambda}(dE) \leq 2^{\frac{d}{2}} C\|\rho\|_\infty E^{\frac{d}{2}-1} \eta L^{2d}, \end{aligned} \tag{6.26}$$

for all $E > 0$ and $0 \leq \eta \leq E$.

Theorem 4.4 is proven.

Appendix

A. Measurability of random classical wave operators

In this appendix we prove measurability for the random classical wave operators $\mathbb{W}_{g,\omega}$ and $W_{g,\omega}$. We also prove measurability for $W_{g,\omega,*}$.

We recall that a random operator is a mapping $\omega \to H_\omega$ from a probability space to self-adjoint operators on a Hilbert space, such that the mappings $\omega \to f(H_\omega)$ are strongly measurable for all bounded measurable functions f on $\mathbb{R}$. It suffices to require weak measurability. (See [KM1, CL].)

PROPOSITION A.1. *If the random medium satisfies Assumption 3.1, then* $\mathbb{W}_{g,\omega}$, $W_{g,\omega}$, *and* $W_{g,\omega,*}$ *are random operators.*

Proof. We start by showing that $\mathbb{W}_{g,\omega}$ is a random operator. To do so, we prove that $(\mathbb{W}_{g,\omega} \mp i)^{-1}$ is strongly measurable. It then follows from the resolvent identity, continuity of the resolvent, and a connectedness argument that $(\mathbb{W}_{g,\omega} - z)^{-1}$ is strongly measurable for all nonreal z, and hence $\mathbb{W}_{g,\omega}$ is a random operator by [KM1, Theorem 3].

Note that we may write

$$\mathbb{W}_{g,\omega} = \sqrt{\mathcal{S}_{g,\omega}}\, \mathbb{W}_{\mathbf{D}} \sqrt{S_{g,\omega}}\,, \tag{A.1}$$

where $\mathcal{S}_{g,\omega} = \mathcal{K}_{g,\omega} \oplus \mathcal{R}_{g,\omega}$ and $\mathbb{W}_{\mathbf{D}}$ is given by (2.23) with $A = \mathbf{D}$. Thus

$$(\mathbb{W}_{g,\omega} \mp i)^{-1} = \mathcal{S}_{g,\omega}^{-\frac{1}{2}} \left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1} \mathcal{S}_{g,\omega}^{-\frac{1}{2}}\,, \tag{A.2}$$

so it suffices to show that $\left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1}$ is strongly measurable.

Let $\lambda > 0$; using the resolvent identity we get

$$\begin{aligned}\left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1} &= (\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1} \mp \\ &\quad i\left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1}\left(\lambda - \mathcal{S}_{g,\omega}^{-1}\right)(\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1}\,,\end{aligned} \tag{A.3}$$

hence

$$\begin{aligned}\left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1}\left(1 \pm i\left(\lambda - \mathcal{S}_{g,\omega}^{-1}\right)(\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1}\right)& \\ = (\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1}\,.&\end{aligned} \tag{A.4}$$

If $\lambda > (\min\{K_-, R_-\})^{-1}$, we have

$$\begin{aligned}\left\|\left(\lambda - \mathcal{S}_{g,\omega}^{-1}\right)(\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1}\right\| &\le \left\|1 - \lambda^{-1}\mathcal{S}_{g,\omega}^{-1}\right\| \\ &\le 1 - \lambda^{-1}(\min\{K_-, R_-\})^{-1} < 1\,,\end{aligned} \tag{A.5}$$

and hence

$$\begin{aligned}&\left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1} = \\ &\quad (\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1}\left(1 \pm i\left(\lambda - \mathcal{S}_{g,\omega}^{-1}\right)(\mathbb{W}_{\mathbf{D}} \mp i\lambda)^{-1}\right)^{-1}.\end{aligned} \tag{A.6}$$

The strong measurability of $\left(\mathbb{W}_{\mathbf{D}} \mp i\mathcal{S}_{g,\omega}^{-1}\right)^{-1}$ follows.

We have proved that $\mathbb{W}_{g,\omega}$ is a random operator. It follows that $\mathbb{W}_{g,\omega}^2$ is also a random operator since $\left(\mathbb{W}_{g,\omega}^2 - z\right)^{-1}$ is strongly measurable if $z \notin [0, \infty)$. Thus $W_{g,\omega}$ and $W_{g,\omega,*}$ are random operators in view of (2.26). □

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