BORELSPACES

S. K. Berberian

The University of Texas
at Austin

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NOTATIONS

$\mathbb{N}$	the set $\{0, 1, 2, 3, \ldots\}$ of natural numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{M}$	the product of $\aleph_0$ copies of $\{0, 1\}$ (A.4.1)
Γ	the Cantor set (Appendix, §A.4)
$\mathbb{N}^{\mathbb{N}}$	the product of $\aleph_0$ copies of $\mathbb{N}$ (§A.5)
$\mathcal{P}(\mathrm{E})$	the power set of E (1.1.3)
$\mathcal{T}(\mathcal{E})$	the tribe generated by $\mathcal{E}$ (1.1.3)
$\mathcal{B}(\mathrm{E})$	the Borel sets of the topological space E (1.2.2)
$\mathcal{B}/\mathrm{R}$	quotient Borel structure (1.4.4)
φ_{B}	the characteristic function of a subset B
$\complement\mathrm{B}$	the complement of a subset B

INTRODUCTION

Borel spaces were introduced by G.W. Mackey as a tool in the study of representations of algebraic structures as sets of linear mappings on vector spaces. {For example, representations of locally compact groups as unitary operators on a Hilbert space; or representations of C*-algebras as operators on a Hilbert space.} It is natural in representation theory to 'identify' representations that are in some sense 'equivalent'; whence a heightened interest in *quotient spaces* modulo equivalence relations.

A goal of representation theory is to associate with an algebraic object A a 'dual object' A^*, defined in terms of representations that are 'primitive' in an appropriate sense; the hope is that A^* is amenable to analysis and that properties of A and A^* are reflected in each other.

In the best of all worlds (locally compact abelian groups) the dual object is an algebraic object of the same sort, and one recovers the original object by taking the dual of its dual (in the case of locally compact abelian groups, this is the Pontrjagin duality theorem). More commonly, A^* is a set without algebraic structure; it may, grudgingly, have one or more topologies, usually with a meager supply of open sets (rarely enough for the Hausdorff separation property).

The key to Mackey's remedy for the shortage of open sets is the concept of 'Borel set'. In the classical case of the topological space $\mathbb{R}$ of real numbers, the Borel sets are the smallest class of subsets of $\mathbb{R}$ that (1) contains the open sets, (2) is closed under complementation, and (3) is closed under countable unions (such a class is called a 'tribe', or 'σ-algebra'). The definition makes sense for any topological space E, and one writes $\mathcal{B}(E)$ for the class of Borel sets of E. A topological space with few open sets may still have an adequate supply of Borel sets.

Returning to the situation of an algebraic object A, with dual object A^* suitably topologized, one observes that various natural constructions lead to subsets of A^* that are in general neither open nor closed but are Borel (for instance, the set of points of A^* corresponding to representations of A on a vector space of a particular dimension). Mackey's first key idea: why not do business with the more abundant Borel sets instead of only the open sets? (The thrust of Mackey's idea is deeper than that: namely, the thesis that Borel structure is in fact more *appropriate* to certain problems than topological structure.)

As remarked earlier, quotient spaces figure prominently in representation theory. Let E be a topological space, R an equivalence relation in E, E/R the quotient set (the set of equivalence classes), and $\pi : \mathrm{E} \to \mathrm{E/R}$ the quotient mapping. The topology of E leads to a class $\mathcal{B}(\mathrm{E})$ of Borel sets in E. The topology of E also leads to a topology on E/R (the 'quotient topology', or 'final topology for π', consisting of all subsets T of E/R whose inverse image $\pi^{-1}(\mathrm{T})$ is an open set in E), hence to a class $\mathcal{B}(\mathrm{E/R})$ of Borel sets in E/R. The catch is that the topology of E/R may be too coarse to be of any use: even in fairly structured situations, it can happen that the only open sets in E/R are the empty set and E/R, in which case these are also the only Borel sets. Mackey's second key idea: dispense with the topology of E/R and appropriate the set of all subsets T of E/R for which $\pi^{-1}(\mathrm{T})$ is a Borel set in E.

We have now arrived at the concept of Borel space: it is a pair $(\mathrm{E}, \mathcal{B})$, where E is a set and $\mathcal{B}$ is a nonempty set of subsets of E closed under complementation and countable unions; one speaks of the category of Borel spaces, whose objects are the Borel spaces $(\mathrm{E}, \mathcal{B})$, a morphism between a pair $(\mathrm{E}, \mathcal{B})$, $(\mathrm{F}, \mathcal{C})$ of Borel spaces being a mapping $f : \mathrm{E} \to \mathrm{F}$ such that the inverse image of every Borel set in F is a Borel set in E.

Classical countability restrictions (such as 'separability' of the object and of the Hilbert spaces on which it is represented) assure, for important classes of objects A, the existence of a 'separating sequence' of Borel sets in the dual object A^*. Mackey's third key idea: certain countability conditions on a Borel space $(\mathrm{E}, \mathcal{B})$ allow one to topologize E in such a way that $\mathcal{B}$ becomes the class $\mathcal{B}(\mathrm{E})$ of Borel sets associated with the topology, and E is (or is a subspace of) a topological space with the 'right kind' of properties.

The topological spaces that arise in this way are the complete, separable metric spaces, and their close cousins. Why these are the 'right kind' of space becomes apparent from experience with their 'categorical' properties. For the present, it suffices to report that such spaces have a long history and a rich theory; what is needed (and more) is exposed in Kuratowski's *Topologie*, the work on which Mackey based his exposition of Borel spaces. The main objective of these notes is to give an exposition based on Chapter IX, §6 of Bourbaki's *Topologie générale* (more accessible to the contemporary reader). Some needed complementary material on general topology, and some instructive examples, are worked out in the Appendices; granted Chapter IX, §6, the exposition is self-contained.

I am indebted to Professor J. Dixmier for his incisive comments on an earlier informal version of these notes; and to the International Center for Pure and Applied Mathematics (C.I.M.P.A.), whose invitation to participate in its International Summer School at Nice prompted me to review the subject and to rewrite the notes.

S.K.B.

Austin, Texas
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Added May, 1997: These notes, as distributed at the 1986 C.I.M.P.A. Summer School in Nice, were prepared using Microsoft Word and printed on a 9-pin dot-matrix printer, with hundreds of handwritten mathematical characters (mostly script capitals). They were technologically primitive by today's standards, but they were neatly prepared and intensively proofread. The notes were subsequently published in a C.I.M.P.A. Proceedings volume [*Functional analysis and its applications* (Proc. International Autumn School, Nice, 25 August–20 September, 1986), pp. 134–197, World Scientific Publ. Co., Singapore, 1988; MR **90a**:54101]. Regrettably, the notes had been rekeyboarded without my knowledge and submitted to the publisher without my (or anyone else, as far as I can tell) having the opportunity to proofread them. I first learned that the notes had been rekeyboarded when the publisher sent me a complimentary copy of the finished book. Although the published version looks more professional (no handwritten symbols) it is riddled with misprints (hundreds? thousands? after a page or two, it was too painful to go on reading), utterly unfit to be put before a reader's eyes. The published version of the notes is their coffin, the corpse in no condition to be viewed. I have grieved for the notes ever since. Eventually I graduated to TEX and have always kept in the back of my mind that, when I had the time, it would be nice to spruce up these old notes for the eyes of future archaeologists. The present version is the result; it is my closure, my farewell to a manuscript over which I labored so affectionately. R.I.P.

S.K.B.

1. BOREL SPACES

§1.1. Tribes

1.1.1. Definition. Let E be a set, $\mathcal{B}$ a nonempty set of subsets of E. One calls $\mathcal{B}$ a *tribe* (or 'σ-algebra') of subsets of E if it is closed under the operations of complementation and denumerable unions:

1° $\mathrm{A} \in \mathcal{B} \Rightarrow \mathrm{A}' \in \mathcal{B}$ (where $\mathrm{A}' = \complement \mathrm{A} = \mathrm{E} - \mathrm{A}$).

2° $\mathrm{A}_n \in \mathcal{B}$ $(n = 1, 2, 3, \ldots) \Rightarrow \bigcup_{n=1}^{\infty} \mathrm{A}_n \in \mathcal{B}$.

It follows that $\mathcal{B}$ is closed under finite unions and denumerable intersections, hence also finite intersections; if $\mathrm{A} \in \mathcal{B}$ then $\mathrm{E} = \mathrm{A} \cup \mathrm{A}' \in \mathcal{B}$ and $\varnothing = \mathrm{E}' \in \mathcal{B}$.

{Convention: "denumerable" means bijective with $\mathbb{N}$; "countable" means "finite or denumerable".}

1.1.2. Let E and F be sets, $f : \mathrm{E} \to \mathrm{F}$ any function. If $\mathcal{B}$ is a tribe of subsets of E, then the set

$$\{\mathrm{T} \subset \mathrm{F} : \ f^{-1}(\mathrm{T}) \in \mathcal{B}\}$$

is a tribe of subsets of F. If $\mathcal{C}$ is a tribe of subsets of F, then the set

$$\{f^{-1}(\mathrm{C}) : \ \mathrm{C} \in \mathcal{C}\}$$

is a tribe of subsets of E, usually denoted $f^{-1}(\mathcal{C})$.

1.1.3. The 'power set' $\mathcal{P}(\mathrm{E})$ of all subsets of E is a tribe, and the intersection of any family of tribes is a tribe. It follows that for any set $\mathcal{E}$ of subsets of E, there is a smallest tribe containing $\mathcal{E}$ (namely, the intersection of the family of all tribes that contain $\mathcal{E}$); it is called the tribe *generated* by $\mathcal{E}$, and will be denoted $\mathcal{T}(\mathcal{E})$.

1.1.4. A set $\mathcal{M} \subset \mathcal{P}(\mathrm{E})$ is said to be *monotone* (or to be a 'monotone class') if it is closed under denumerable monotone unions and intersections; that is, if $\mathrm{A}_n \in \mathcal{M}$ $(n = 1, 2, 3, \ldots)$ and if $\mathrm{A}_n \uparrow \mathrm{A}$ or $\mathrm{A}_n \downarrow \mathrm{A}$, then $\mathrm{A} \in \mathcal{M}$.

The power set $\mathcal{P}(\mathrm{E})$ is monotone, and the intersection of any family of monotone classes is monotone; it follows that for every set $\mathcal{E}$ of subsets of E, there exists a smallest monotone class containing $\mathcal{E}$; it is called the monotone class *generated* by $\mathcal{E}$, denoted $\mathcal{M}(\mathcal{E})$.

1.1.5. A nonempty set $\mathcal{A} \subset \mathcal{P}(\mathrm{E})$ is called a (Boolean) *algebra* of subsets of E if it is closed under complementation and finite unions (hence under finite intersections). For $\mathcal{E} \subset \mathcal{P}(\mathrm{E})$, the algebra *generated* by $\mathcal{E}$ is defined in the obvious way, and is denoted $\mathcal{A}(\mathcal{E})$. It is elementary that an algebra $\mathcal{A}$ is a tribe if and only if it is a monotone class.

1.1.6. Example. If $\mathcal{E}$ is the set of all intervals of $\mathbb{R}$, then $\mathcal{A}(\mathcal{E})$ is the set of all finite disjoint unions of intervals of $\mathbb{R}$.

1.1.7. Proposition. (Lemma on Monotone Classes) *If a monotone class contains an algebra* $\mathcal{A}$, *then it contains the tribe generated by* $\mathcal{A}$. [H1, p. 27, Th. B of §6]

It follows that if $\mathcal{A}$ is an algebra of sets, then $\mathcal{T}(\mathcal{A}) = \mathcal{M}(\mathcal{A})$.

1.1.8. If $\mathcal{A}$ is the algebra described in 1.1.6, the sets in $\mathcal{T}(\mathcal{A})$ are called the *Borel sets* of $\mathbb{R}$. Since every open set in $\mathbb{R}$ is the union of a sequence of intervals, it is clear that $\mathcal{T}(\mathcal{A})$ is also the tribe generated by the set of all open sets of $\mathbb{R}$. In this form, the concept of 'Borel set' extends to general topological spaces.

1.1.9. Exercise. Find a surjective mapping $f : \mathrm{E} \to \mathrm{F}$ and a tribe $\mathcal{B}$ of subsets of E such that $f(\mathcal{B}) = \{f(\mathrm{B}) : \mathrm{B} \in \mathcal{B}\}$ is not a tribe of subsets of F.

1.1.10. Exercise. Find a class $\mathcal{A}$ of subsets of $\mathbb{R}$ such that $\varnothing, \mathbb{R} \in \mathcal{A}$, $\mathcal{A}$ is closed under countable unions and intersections, but $\mathcal{A}$ is not a tribe.

§1.2. Borel spaces

1.2.1. Definition. A *Borel space* is a pair $(\mathrm{E}, \mathcal{B})$, where E is a set and $\mathcal{B}$ is a tribe of subsets of E. The tribe $\mathcal{B}$ is said to define a *Borel structure* on the set E, and the sets in $\mathcal{B}$ are called the *Borel sets* of E for this structure. One also refers to $\mathcal{B}$ itself as the 'Borel structure' on E.

1.2.2. Definition. When E is a topological space, there is a distinguished Borel structure derived from the topology: if $\mathcal{O}$ is the set of all open sets in E, the tribe $\mathcal{T}(\mathcal{O})$ generated by $\mathcal{O}$ is denoted $\mathcal{B}(\mathrm{E})$, and the sets in $\mathcal{B}(\mathrm{E})$ are called the *Borel sets* of (the topological space) E.

1.2.3. Definition. If $\mathcal{B}_1, \mathcal{B}_2$ are tribes of subsets of a set E such that $\mathcal{B}_1 \subset \mathcal{B}_2$, one says that the Borel structure defined by $\mathcal{B}_2$ is *finer* than that defined by $\mathcal{B}_1$ (or that $\mathcal{B}_1$ defines a *coarser* Borel structure than does $\mathcal{B}_2$). Thus $\mathcal{P}(\mathrm{E})$ defines the finest Borel structure on E, and $\{\varnothing, \mathrm{E}\}$ defines the coarsest.

1.2.4. Definition. Let $(\mathrm{E}, \mathcal{B})$ and $(\mathrm{F}, \mathcal{C})$ be Borel spaces. A mapping $f : \mathrm{E} \to \mathrm{F}$ is called a *Borel mapping* if $f^{-1}(\mathcal{C}) \subset \mathcal{B}$, that is, if $\mathrm{C} \in \mathcal{C} \Rightarrow f^{-1}(\mathrm{C}) \in \mathcal{B}$. Such a mapping is also called a *morphism* for the Borel structures. If f is bijective and if both f and f^{-1} are Borel mappings, then f is called a *Borel isomorphism* and the Borel spaces are said to be *isomorphic* (via f).

1.2.5. Proposition. *If* $(\mathrm{E}, \mathcal{B})$, $(\mathrm{F}, \mathcal{C})$, $(\mathrm{G}, \mathcal{D})$ *are Borel spaces, and if* $f : \mathrm{E} \to \mathrm{F}$, $g : \mathrm{F} \to \mathrm{G}$ *are Borel mappings, then* $g \circ f$ *is Borel.*

It is enough to test the condition of 1.2.4 on a system of generators:

1.2.6. Proposition. *Let* $(\mathrm{E}, \mathcal{B})$ *and* $(\mathrm{F}, \mathcal{C})$ *be Borel spaces, and suppose that* $\mathcal{C} = \mathcal{T}(\mathcal{E})$ *is the tribe generated by a set* $\mathcal{E}$ *of subsets of* F. *In order that a mapping* $f : \mathrm{E} \to \mathrm{F}$ *be Borel, it is necessary and sufficient that* $f^{-1}(\mathrm{C}) \in \mathcal{B}$ *for all* $\mathrm{C} \in \mathcal{E}$.

Proof. If the inverse image of every set in $\mathcal{E}$ is Borel, then the set

$$\{\mathrm{C} \subset \mathrm{F} : \ f^{-1}(\mathrm{C}) \in \mathcal{B}\}$$

is a tribe (1.1.2) containing $\mathcal{E}$, therefore containing $\mathcal{T}(\mathcal{E}) = \mathcal{C}$; thus $f^{-1}(\mathcal{C}) \subset \mathcal{B}$, in other words f is Borel. ♢

1.2.7. Corollary. *If* E *and* F *are topological spaces and* $f : \mathrm{E} \to \mathrm{F}$ *is continuous, then* f *is Borel* (*for the Borel structures derived from the topologies*).

Proof. Writing $\mathcal{O}_{\mathrm{E}}$ and $\mathcal{O}_{\mathrm{F}}$ for the topologies of E and F, we have

$$f^{-1}(\mathcal{O}_{\mathrm{F}}) \subset \mathcal{O}_{\mathrm{E}} \subset \mathcal{B}(\mathrm{E}),$$

therefore $f^{-1}\big(\mathcal{B}(\mathrm{F})\big) \subset \mathcal{B}(\mathrm{E})$ by 1.2.6. ♢

1.2.8. Exercise. The Borel sets of $\mathbb{R}$ are the tribe generated by the (countable) set of all intervals (a, b) with rational endpoints. {Similarly for intervals $[a, b)$, etc.}

§1.3. Initial Borel structures

1.3.1. Definition. Let E be a set, $(\mathrm{F},\mathcal{C})$ a Borel space, $f:\mathrm{E}\to\mathrm{F}$ a mapping. Then $f^{-1}(\mathcal{C})$ is a tribe (1.1.2), defining the coarsest Borel structure on E for which f is a Borel mapping; it is called the *initial Borel structure* for the mapping $f:\mathrm{E}\to(\mathrm{F},\mathcal{C})$ (or, briefly, for the mapping f, it being understood that F bears the given Borel structure).

1.3.2. Proposition. *With notations as in* 1.3.1, *let* $(\mathrm{G},\mathcal{D})$ *be a Borel space and* $g:\mathrm{G}\to\mathrm{E}$ *a mapping,*

$$\begin{array}{ccccc} \mathrm{G} & \xrightarrow{\quad g \quad} & \mathrm{E} & \xrightarrow{\quad f \quad} & \mathrm{F} \\ \mathcal{D} & & f^{-1}(\mathcal{C}) & & \mathcal{C} \end{array}$$

Then: g *Borel* $\Leftrightarrow$ $f\circ g$ *Borel.*

Proof. $\Rightarrow$: Immediate from 1.2.5.

$\Leftarrow$: By assumption, $(f\circ g)^{-1}(\mathcal{C})\subset\mathcal{D}$, thus $g^{-1}\big(f^{-1}(\mathcal{C})\big)\subset\mathcal{D}$, whence g is Borel. ♢

1.3.3. Exercise. The initial Borel structure for f is characterized by the property of 1.3.2; that is, if $\mathcal{B}$ is a Borel structure on E such that, for any Borel space $(\mathrm{G},\mathcal{D})$, a mapping $g:(\mathrm{G},\mathcal{D})\to(\mathrm{E},\mathcal{B})$ is Borel if and only if $f\circ g$ is Borel, then $\mathcal{B}=f^{-1}(\mathcal{C})$. {Hint: Try $\mathrm{G}=\mathrm{E}$, $\mathcal{D}=\mathcal{B}$, $g=\mathrm{id}$; then try $\mathrm{G}=\mathrm{E}$, $\mathcal{D}=f^{-1}(\mathcal{C})$, $g=\mathrm{id}$.}

1.3.4. Definition. Let $(\mathrm{E},\mathcal{B})$ be a Borel space, S a subset of E, $i:\mathrm{S}\to\mathrm{E}$ the insertion mapping. The initial Borel structure for i is defined by the tribe

$$i^{-1}(\mathcal{B})=\{\mathrm{B}\cap\mathrm{S}:\ \mathrm{B}\in\mathcal{B}\}=\mathcal{B}\cap\mathrm{S}$$

(the 'trace' of $\mathcal{B}$ on S); $\mathcal{B}\cap\mathrm{S}$ is called the *relative Borel structure on* S *induced by that of* E, and $(\mathrm{S},\mathcal{B}\cap\mathrm{S})$ is said to be a *sub-Borel space* of $(\mathrm{E},\mathcal{B})$. {CAUTION: S need not be a Borel set in E, so we avoid the equivocal term 'Borel subspace'.}

1.3.5. Proposition. *Let* E *be a topological space,* $\mathrm{S}\subset\mathrm{E}$ *a topological subspace of* E. *Then*

$$\mathcal{B}(\mathrm{S})=\mathcal{B}(\mathrm{E})\cap\mathrm{S};$$

that is, the Borel structure derived from the relative topology on S *coincides with the sub-Borel structure induced by the Borel structure* $\mathcal{B}(\mathrm{E})$.

Proof. Write $\mathcal{O}$ for the topology of E, $\mathcal{O}_{\mathcal{S}}$ for the topology of S, $i:\mathrm{S}\to\mathrm{E}$ for the insertion mapping; thus $\mathcal{O}_{\mathcal{S}}=\mathcal{O}\cap\mathrm{S}$ is the initial topology for i [B, p. I.17]. Since i is continuous it is Borel (1.2.7), thus

$$\mathcal{B}(\mathrm{S})\supset i^{-1}\big(\mathcal{B}(\mathrm{E})\big)=\mathcal{B}(\mathrm{E})\cap\mathrm{S};$$

on the other hand,

$$\mathcal{O}_{\mathrm{S}} = i^{-1}(\mathcal{O}) \subset i^{-1}\big(\mathcal{B}(\mathrm{E})\big)\,,$$

therefore $\mathcal{B}(\mathrm{S}) = \mathcal{T}(\mathcal{O}_{\mathrm{S}}) \subset i^{-1}\big(\mathcal{B}(\mathrm{E})\big)$. ♢

1.3.6. Corollary. *Let* E *and* F *be topological spaces,* $f : \mathrm{E} \to \mathrm{F}$ *a mapping. View* $f(\mathrm{E})$ *as a topological subspace of* F, *and let* $f_0 : \mathrm{E} \to f(\mathrm{E})$ *have the graph of* f. *Then*:

$$f \textit{ Borel} \;\Leftrightarrow\; f_0 \textit{ Borel.}$$

Proof. By 1.3.5, $f(\mathrm{E})$ bears the initial Borel structure for the insertion mapping $i : f(\mathrm{E}) \to \big(\mathrm{F}, \mathcal{B}(\mathrm{F})\big)$, thus the corollary is a special case of 1.3.2 (note that $f = i \circ f_0$). ♢

It is useful to extend Definition 1.3.1 to families of mappings:

1.3.7. Definition. Let E be a set, $(\mathrm{F}, \mathcal{C}_i)_{i\in\mathrm{I}}$ a family of Borel spaces, $f_i : \mathrm{E} \to \mathrm{F}_i$ $(i \in \mathrm{I})$ a family of mappings. Let

$$\mathcal{B}_i = f_i^{-1}(\mathcal{C}_i)$$

and let

$$\mathcal{B} = \mathcal{T}\big(\textstyle\bigcup_{i\in\mathrm{I}} \mathcal{B}_i\big)$$

be the tribe generated by the union of the $\mathcal{B}_i$. Then $\mathcal{B}$ defines the coarsest Borel structure on E for which every f_i is Borel; it is called the *initial Borel structure* for the *family* of mappings $(f_i)_{i\in\mathrm{I}}$.

1.3.8. Proposition. *With notations as in* 1.3.7, *let* $(\mathrm{G}, \mathcal{D})$ *be a Borel space and* $g : \mathrm{G} \to \mathrm{E}$ *a mapping*:

$$\begin{array}{ccccc} \mathrm{G} & \xrightarrow{\quad g \quad} & \mathrm{E} & \xrightarrow{\quad f_i \quad} & \mathrm{F}_i \\ \mathcal{D} & & \mathcal{B} & & \mathcal{C}_i \end{array}$$

Then: g *Borel* $\Leftrightarrow f_i \circ g$ *Borel for all* $i \in \mathrm{I}$.

Proof. Straightforward. ♢

1.3.9. Exercise. The property of 1.3.8 characterizes the initial Borel structure for the family $(f_i)_{i\in\mathrm{I}}$ (cf. 1.3.3).

1.3.10. Definition. Let $(\mathrm{E}_i, \mathcal{B}_i)_{i\in\mathrm{I}}$ be a family of Borel spaces, $\mathrm{E} = \prod_{i\in\mathrm{I}} \mathrm{E}_i$ the product set, $\mathrm{pr}_i : \mathrm{E} \to \mathrm{E}_i$ $(i \in \mathrm{I})$ the family of projection mappings. The initial Borel structure $\mathcal{B}$ for the family of mappings $(\mathrm{pr}_i)_{i\in\mathrm{I}}$ is called the *product Borel structure on* E, and the Borel space $(\mathrm{E}, \mathcal{B})$ is called the *product* of the Borel spaces $(\mathrm{E}_i, \mathcal{B}_i)$. If $j \in \mathrm{I}$ and $\mathrm{B}_j \in \mathcal{B}_j$, then

$$\mathrm{pr}_j^{-1}(\mathrm{B}_j) = \mathrm{B}_j \times \prod_{i\neq j} \mathrm{E}_i\,;$$

$\mathcal{B}$ is the tribe generated by all such sets (as j varies over I, and B_j varies over $\mathcal{B}_j$).

One writes
$$\mathcal{B} = \prod_{i \in \mathrm{I}} \mathcal{B}_i$$
and $\mathcal{B} = \mathcal{B}_1 \times \ldots \times \mathcal{B}_n$ when $\mathrm{I} = \{1, \ldots n\}$; this is an abuse of notation (even if $\mathrm{I} = \{1, 2\}$), for $\mathcal{B}$ is not a product set.

1.3.11. With notations as in 1.3.10, let $(\mathrm{G}, \mathcal{D})$ be another Borel space and $g : \mathrm{G} \to \mathrm{E}$ a mapping:

$$\begin{array}{ccccc} \mathrm{G} & \xrightarrow{\quad g \quad} & \prod_{i \in \mathrm{I}} \mathrm{E}_i & \xrightarrow{\quad \mathrm{pr}_i \quad} & \mathrm{E}_i \\ \mathcal{D} & & \mathcal{B} & & \mathcal{B}_i \end{array}$$

Setting $g_i = \mathrm{pr}_i \circ g$, one writes $g = (g_i)_{i \in \mathrm{I}}$. Thus (1.3.8), g is Borel if and only if every g_i is Borel.

1.3.12. CAUTION: Let E be a topological space, $\mathrm{E} \times \mathrm{E}$ the product topological space. It can happen that $\mathcal{B}(\mathrm{E} \times \mathrm{E}) \neq \mathcal{B}(\mathrm{E}) \times \mathcal{B}(\mathrm{E})$; an example is given in the Appendix (§A.6).

1.3.13. Lemma. $\mathcal{B}(\mathbb{R} \times \mathbb{R}) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$.

Proof. The projection mappings $\mathrm{pr}_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$ are continuous, therefore Borel (1.2.7), consequently
$$\mathcal{B}(\mathbb{R} \times \mathbb{R}) \supset \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$$
by the definition of product Borel structure (1.3.10). On the other hand, if W is any open set in $\mathbb{R} \times \mathbb{R}$, then there exist sequences $(\mathrm{U}_n), (\mathrm{V}_n)$ of open sets in $\mathbb{R}$ (open intervals with rational endpoints, if we like) such that W is the union of the $\mathrm{U}_n \times \mathrm{V}_n$, therefore W belongs to $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$; it follows that $\mathcal{B}(\mathbb{R} \times \mathbb{R}) \subset \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$. ♢

1.3.14. Proposition. *If* $(\mathrm{E}, \mathcal{B})$ *is any Borel space, then the set of all Borel functions* $f : \mathrm{E} \to \mathbb{R}$ *is an algebra for the pointwise operations.*

Proof. (It is understood that $\mathbb{R}$ has the Borel structure derived from its topology.) Write $\mathcal{A}$ for the set of all such functions f, and $\alpha : \mathbb{R}^2 \to \mathbb{R}$, $\mu : \mathbb{R}^2 \to \mathbb{R}$ for the continuous mappings
$$\alpha(s, t) = s + t\,, \quad \mu(s, t) = st\,.$$
If $f, g \in \mathcal{A}$ then, by 1.3.11, the mapping
$$h = (f, g) : \mathrm{E} \to \mathbb{R} \times \mathbb{R}$$
is Borel for the structures $\mathcal{B}$ and $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, in other words (1.3.13) for $\mathcal{B}$ and $\mathcal{B}(\mathbb{R}^2)$. On the other hand, the mappings α, μ are Borel for $\mathcal{B}(\mathbb{R}^2)$ and $\mathcal{B}(\mathbb{R})$

(1.2.7), therefore $f + g = \alpha \circ h$, $fg = \mu \circ h$ are Borel for $\mathcal{B}$ and $\mathcal{B}(\mathbb{R})$ (1.2.5). {Note: The real field $\mathbb{R}$ can obviously be replaced by the complex field $\mathbb{C}$ (or by any separable normed algebra).} $\diamondsuit$

1.3.15. Exercise. Let $(\mathrm{E}, \mathcal{B})$ be any Borel space, $(\mathrm{B}_i)_{i \in \mathrm{I}}$ any indexing of all the sets of $\mathcal{B}$, $f_i : \mathrm{E} \to \{0,1\}$ the characteristic function of B_i $(i \in \mathrm{I})$. Equip $\{0,1\}$ with the 'discrete' Borel structure (defined by its power set). Then $\mathcal{B}$ is the initial Borel structure for the family $(f_i)_{i \in \mathrm{I}}$. In particular, if $(\mathrm{G}, \mathcal{D})$ is a Borel space and $g : \mathrm{G} \to \mathrm{E}$ is a mapping such that $f \circ g$ is Borel for every Borel function $f : \mathrm{E} \to \mathbb{R}$, then g is Borel; it follows (1.3.9) that $\mathcal{B}$ is also the initial Borel structure for the family of all Borel functions $f : \mathrm{E} \to \mathbb{R}$.

1.3.16. Exercise. Let E be a set, F a topological space with a countable base for open sets, $f : \mathrm{E} \to \mathrm{F}$ a mapping. Equip E with the initial topology for f. Then E has a countable base for open sets, and $\mathcal{B}(\mathrm{E})$ is the initial Borel structure for the mapping $f : \mathrm{E} \to \big(\mathrm{F}, \mathcal{B}(\mathrm{F})\big)$.

§1.4. Final Borel structures

1.4.1. Definition. Let $(\mathrm{E}, \mathcal{B})$ be a Borel space, F a set, $f : \mathrm{E} \to \mathrm{F}$ a mapping. The set

$$\mathcal{C} = \{\mathrm{C} \subset \mathrm{F} : \ f^{-1}(\mathrm{C}) \in \mathcal{B}\}$$

is a tribe (1.1.2); $\mathcal{C}$ defines the finest Borel structure on F for which f is Borel, called the *final Borel structure* for the mapping $f : (\mathrm{E}, \mathcal{B}) \to \mathrm{F}$.

1.4.2. Proposition. *With notations as in* 1.4.1, *let* $(\mathrm{G}, \mathcal{D})$ *be a Borel space and* $g : \mathrm{F} \to \mathrm{G}$ *a mapping*:

$$\begin{array}{ccccc} \mathrm{E} & \xrightarrow{\ f\ } & \mathrm{F} & \xrightarrow{\ g\ } & \mathrm{G} \\ \mathcal{B} & & \mathcal{C} & & \mathcal{D} \end{array}$$

Then: g *Borel* $\Leftrightarrow g \circ f$ *Borel.*

Proof. $\Rightarrow$: Immediate from 1.2.5.

$\Leftarrow$: By assumption, $(g \circ f)^{-1}(\mathcal{D}) \subset \mathcal{B}$, thus $f^{-1}\big(g^{-1}(\mathcal{D})\big) \subset \mathcal{B}$, therefore $g^{-1}(\mathcal{D}) \subset \mathcal{C}$ by the definition of $\mathcal{C}$; thus g is Borel. ♢

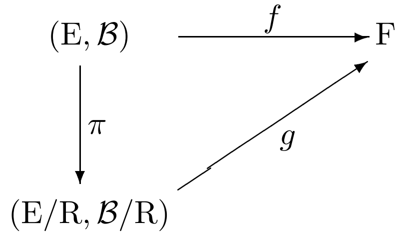
1.4.3. Exercise. The final Borel structure for f is characterized by the property of 1.4.2 (cf. 1.3.3).

1.4.4. Definition. Let $(\mathrm{E}, \mathcal{B})$ be a Borel space, R an equivalence relation in E, $\pi : \mathrm{E} \to \mathrm{E}/\mathrm{R}$ the quotient mapping. The final Borel structure for π will be denoted $\mathcal{B}/\mathrm{R}$; one calls $(\mathrm{E}/\mathrm{R}, \mathcal{B}/\mathrm{R})$ the *quotient Borel space* determined by R . For $\mathrm{T} \subset \mathrm{E}/\mathrm{R}$,

$$\begin{aligned} \mathrm{T} \in \mathcal{B}/\mathrm{R} \ &\Leftrightarrow \ \pi^{-1}(\mathrm{T}) \in \mathcal{B} \\ &\Leftrightarrow \ \pi^{-1}(\mathrm{T}) \ \text{ is a Borel set in E that is} \\ &\qquad\qquad \text{saturated for R}. \end{aligned}$$

{A set $\mathrm{S} \subset \mathrm{E}$ is said to be *saturated* for R if $x \in \mathrm{S}$ and $x'\mathrm{R}x$ imply $x' \in \mathrm{S}$; equivalently, $\mathrm{S} = \pi^{-1}\big(\pi(\mathrm{S})\big)$.} Thus $\mathcal{B}/\mathrm{R}$ is the set of all sets $\pi(\mathrm{B})$, where B is a Borel set in E that is saturated for R.

1.4.5. Exercise. Let $(\mathrm{E}, \mathcal{B})$ be a Borel space, F a set, $f : \mathrm{E} \to \mathrm{F}$ a mapping, R the equivalence relation in E defined by f ($x\mathrm{R}x'$ means $f(x) = f(x')$). Let $\pi : \mathrm{E} \to \mathrm{E}/\mathrm{R}$ be the quotient mapping and let $g : \mathrm{E}/\mathrm{R} \to \mathrm{F}$ be the mapping such that $f = g \circ \pi$:



Then f and g define the same final Borel structure on F (a special case of 'transitivity of final structures' [cf. B, p. I.14]). If $g_0 : \mathrm{E}/\mathrm{R} \to f(\mathrm{E})$ has the graph of g, then g_0 is an isomorphism of $(\mathrm{E}/\mathrm{R}, \mathcal{B}/\mathrm{R})$ onto the sub-Borel space $\big(f(\mathrm{E}), \mathcal{C} \cap f(\mathrm{E})\big)$, where $\mathcal{C}$ is the final Borel structure for f. {Hint: Look at the diagram

$$\begin{array}{ccc}
(\mathrm{E}, \mathcal{B}) & \xrightarrow{\quad f \quad} & (\mathrm{F}, \mathcal{C}) \\
\Big\downarrow \pi & \nearrow g & \Big\uparrow i \\
(\mathrm{E}/\mathrm{R}, \mathcal{B}/\mathrm{R}) & \xrightarrow[\quad g_0 \quad]{} & \big(f(\mathrm{E}), \mathcal{C} \cap f(\mathrm{E})\big)
\end{array}$$

where i is the insertion mapping and $f = i \circ g_0 \circ \pi$ is the canonical factorization of f.}

1.4.6. Exercise. Let E be a topological space, R an equivalence relation in E, $\pi : \mathrm{E} \to \mathrm{E}/\mathrm{R}$ the quotient mapping; equip E/R with the quotient topology (i.e., the final topology for π). Then $\mathcal{B}(\mathrm{E}/\mathrm{R}) \subset \mathcal{B}(\mathrm{E})/\mathrm{R}$; the inclusion may be proper (1.4.7).

1.4.7. Exercise. Let $\mathbb{Q}$ be the rational subgroup of $\mathbb{R}$, $\mathbb{R}/\mathbb{Q}$ the quotient group, $f : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ the canonical homomorphism. Equip $\mathbb{R}/\mathbb{Q}$ with the quotient topology; its only open sets are $\varnothing$ and $\mathbb{R}/\mathbb{Q}$ (the 'coarse topology'), therefore $\mathcal{B}(\mathbb{R}/\mathbb{Q}) = \{\varnothing, \mathbb{R}/\mathbb{Q}\}$.

One can also regard $\mathbb{R}/\mathbb{Q}$ as the quotient set for the equivalence relation «$f(x) = f(x')$», in other words «$x - x' \in \mathbb{Q}$»; call this equivalence relation S, thus $\mathbb{R}/\mathrm{S} = \mathbb{R}/\mathbb{Q}$. The saturation of every Borel set in $\mathbb{R}$ is a Borel set.

Every countable subset of $\mathbb{R}/\mathrm{S}$ belongs to $\mathcal{B}(\mathbb{R})/\mathrm{S}$, whereas $\mathcal{B}(\mathbb{R}/\mathrm{S}) = \{\varnothing, \mathbb{R}/\mathrm{S}\}$.

1.4.8. Definition. Let $(\mathrm{E}_i, \mathcal{B}_i)_{i \in \mathrm{I}}$ be a family of Borel spaces, F a set, and $f_i : \mathrm{E}_i \to \mathrm{F}$ $(i \in \mathrm{I})$ a family of mappings. Define

$$\begin{aligned}
\mathcal{C} &= \{\mathrm{C} \subset \mathrm{F} : \ f_i^{-1}(\mathrm{C}) \in \mathcal{B}_i \ \text{ for all } i \in \mathrm{I}\} \\
&= \bigcap_{i \in \mathrm{I}} \{\mathrm{C} \subset \mathrm{F} : \ f_i^{-1}(\mathrm{C}) \in \mathcal{B}_i\};
\end{aligned}$$

then $\mathcal{C}$ is a tribe, and it defines the finest Borel structure on F that makes every f_i Borel; it is called the *final Borel structure* on F for the family $(f_i)_{i \in \mathrm{I}}$.

1.4.9. Proposition. *With notations as in* 1.4.8, *let* $(\mathrm{G}, \mathcal{D})$ *be a Borel space and* $g : \mathrm{F} \to \mathrm{G}$ *a mapping*:

$$\begin{array}{ccccc}
\mathrm{E}_i & \xrightarrow{\quad f_i \quad} & \mathrm{F} & \xrightarrow{\quad g \quad} & \mathrm{G} \\
\mathcal{B}_i & & \mathcal{C} & & \mathcal{D}
\end{array}$$

Then: g *Borel* $\Leftrightarrow$ $g \circ f_i$ *Borel for all* $i \in \mathrm{I}$.

Proof. $\Leftarrow$: By hypothesis,

$$f_i^{-1}\big(g^{-1}(\mathcal{D})\big) = (g \circ f_i)^{-1}(\mathcal{D}) \subset \mathcal{B}_i$$

for all $i \in \mathrm{I}$, therefore $g^{-1}(\mathcal{D}) \subset \mathcal{C}$ by the definition of $\mathcal{C}$. ♢

1.4.10. Example. Let $(\mathrm{E}_i, \mathcal{B}_i)$ be a family of Borel spaces, E the direct union (or 'sum-set') of the sets E_i. Regard $(\mathrm{E}_i)_{i\in\mathrm{I}}$ as a partition of E. {CAUTION: It is conceivable that the original sets E_i are all equal; but after identifying E_i with a subset of E, the E_i are pairwise disjoint.}

Write $f_i : \mathrm{E}_i \to \mathrm{E}$ for the insertion mappings. If $\mathcal{B}$ is the final Borel structure for the family $(f_i)_{i\in\mathrm{I}}$, one calls $(\mathrm{E}, \mathcal{B})$ the *direct union* (or 'sum') of the Borel spaces $(\mathrm{E}_i, \mathcal{B}_i)$. For a set $\mathrm{S} \subset \mathrm{E}$, $f_i^{-1}(\mathrm{S}) = \mathrm{S} \cap \mathrm{E}_i$; thus

$$\mathcal{B} = \{\mathrm{S} \subset \mathrm{E} : \ \mathrm{S} \cap \mathrm{E}_i \in \mathcal{B}_i \ \text{ for all } i \in \mathrm{I}\}.$$

Evidently $\mathcal{B} \cap \mathrm{E}_i = \mathcal{B}_i$, thus $(\mathrm{E}_i, \mathcal{B}_i)$ is a sub-Borel space of $(\mathrm{E}, \mathcal{B})$ (1.3.4), and $\mathcal{B}_i = \{\mathrm{B} \in \mathcal{B} : \ \mathrm{B} \subset \mathrm{E}_i\}$.

1.4.11. Exercise. With notations as in 1.4.10, let R be the equivalence relation in E for which the E_i are the equivalence classes. Then the quotient Borel structure $\mathcal{B}/\mathrm{R}$ is the set of *all* subsets of E/R. (In a sense, $\mathrm{E}/\mathrm{R} = \mathrm{I}$.)

2. BOREL STRUCTURE IN TOPOLOGICAL SPACES

§2.1. Borel sets in topological spaces

In a topological space E, the term 'Borel' refers to the Borel structure $\mathcal{B}(\mathrm{E})$ derived from the topology (1.2.2), unless expressly mentioned to the contrary. The Borel structure is far from determining the topology:

2.1.1. Example. If E is a countable topological space in which every singleton $\{x\}$ is a closed set, then $\mathcal{B}(\mathrm{E}) = \mathcal{P}(\mathrm{E})$. For example, let

$$\mathrm{E} = \{0, 1, 1/2, 1/3, \ldots\}, \quad \mathrm{F} = \mathbb{N} = \{0, 1, 2, \ldots\},$$

both viewed as topological subspaces of $\mathbb{R}$; then E is compact, F is discrete, and every bijection $f : \mathrm{E} \to \mathrm{F}$ is a Borel isomorphism.

The extent to which abstract Borel spaces and mappings may be derived from topological situations is a recurring theme in what follows.

If E is a topological space and S is a topological subspace of E, then (1.3.5)

$$\mathcal{B}(\mathrm{S}) = \mathcal{B}(\mathrm{E}) \cap \mathrm{S};$$

in other words, if $i : \mathrm{S} \to \mathrm{E}$ is the insertion mapping of the *set* S into the topological space E, then the Borel structure derived from the initial topology for i coincides with the initial Borel structure for i derived from the Borel structure of E. Another way of putting it: the Borel structure of a subspace is a sub-Borel structure of the space.

For products, even finite products, the situation is more complicated: the Borel structure of a product topological space may differ from the product Borel structure of the factors (1.3.12). The next series of results explores this theme.

2.1.2. Proposition. *If* $(\mathrm{E}_i)_{i\in\mathrm{I}}$ *is a family of topological spaces and* $\mathrm{E} = \prod \mathrm{E}_i$ *is the product topological space, then*

$$\mathcal{B}(\mathrm{E}) \supset \prod_{i\in\mathrm{I}} \mathcal{B}(\mathrm{E}_i).$$

Proof. The meaning of the expression on the right side is explained in 1.3.10. Let $\mathrm{pr}_i : \mathrm{E} \to \mathrm{E}_i$ $(i \in \mathrm{I})$ be the family of projection mappings. For every i, pr_i is continuous, hence Borel (1.2.7), so the asserted inclusion follows from the definition of product Borel structure (1.3.10). ♢

The restrictive hypotheses in the following proposition stem from the fact that, while the class of open sets is closed under arbitrary unions, only countable unions are permitted in the class of Borel sets:

2.1.3. Proposition. *Let* (E_n) *be a countable family of topological spaces each of which has a countable base for open sets, and let* $\mathrm{E} = \prod \mathrm{E}_n$ *be the product topological space. Then*

$$\mathcal{B}(\mathrm{E}) = \prod \mathcal{B}(\mathrm{E}_n)\,.$$

Proof. For each n, choose a countable base for the open sets of E_n; the sets of the form

$$\mathrm{U} = \prod_{i=1}^{m} \mathrm{U}_i \times \prod_{i>m} \mathrm{E}_i\,,$$

where m is any index of the family and U_i belongs to the chosen base for E_i $(1 \le i \le m)$, form a countable base for the open sets of E; thus every open set in E is the union of a sequence of sets in $\prod \mathcal{B}(\mathrm{E}_n)$, consequently $\mathcal{B}(\mathrm{E}) \subset \prod \mathcal{B}(\mathrm{E}_n)$ by the definition of Borel set (1.2.2). ♢

2.1.4. Corollary. *With assumptions as in the Proposition, let* G *be a topological space,* $g : \mathrm{G} \to \mathrm{E}$ *a mapping, and write* $g = (g_n)$, *where* $g_n = \mathrm{pr}_n \circ g$:

$$\mathrm{G} \xrightarrow{\ g\ } \mathrm{E} \xrightarrow{\ \mathrm{pr}_n\ } \mathrm{E}_n\,.$$

Then: g *Borel* ⇔ g_n *Borel for all* n.

Proof. By 2.1.3, $\mathcal{B}(\mathrm{E})$ is the initial Borel structure for the family (pr_n). ♢

PROBLEM (easy? hard?): Do there exist topological spaces $\mathrm{E}_1, \mathrm{E}_2, \mathrm{G}$ and a mapping $g : \mathrm{G} \to \mathrm{E}_1 \times \mathrm{E}_2$ such that $\mathrm{pr}_i \circ g$ is Borel for $i = 1, 2$ but g is not Borel? {My guess: yes.}

2.1.5. Proposition. *Let* (E_n) *be a countable family of topological spaces,* E *the topological direct union (or 'topological sum') of the* E_n. *Then* $\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)$ *is the direct union of the Borel spaces* $\big(\mathrm{E}_n, \mathcal{B}(\mathrm{E}_n)\big)$.

Proof. Write $(\mathrm{E}, \mathcal{B})$ for the direct union of the Borel spaces $\big(\mathrm{E}_n, \mathcal{B}(\mathrm{E}_n)\big)$, thus (1.4.10)

$$\mathcal{B} = \{\mathrm{S} \subset \mathrm{E} :\ \mathrm{S} \cap \mathrm{E}_n \in \mathcal{B}(\mathrm{E}_n) \text{ for all } n\}\,.$$

Since E_n is a topological subspace of E, $\mathcal{B}(\mathrm{E}) \cap \mathrm{E}_n = \mathcal{B}(\mathrm{E}_n)$ by 1.3.5, consequently $\mathcal{B}(\mathrm{E}) \subset \mathcal{B}$.

On the other hand, E_n is an open set in E, thus $\mathrm{E}_n \in \mathcal{B}(\mathrm{E})$ and

$$\mathcal{B}(\mathrm{E}_n) = \mathcal{B}(\mathrm{E}) \cap \mathrm{E}_n = \{\mathrm{B} \in \mathcal{B}(\mathrm{E}) :\ \mathrm{B} \subset \mathrm{E}_n\}\,;$$

thus if $\mathrm{S} \in \mathcal{B}$ then $\mathrm{S} = \bigcup \mathrm{S} \cap \mathrm{E}_n$ is the union of a countable family of sets in $\mathcal{B}(\mathrm{E})$, therefore $\mathrm{S} \in \mathcal{B}(\mathrm{E})$; thus $\mathcal{B} \subset \mathcal{B}(\mathrm{E})$. $\diamondsuit$

2.1.6. Exercise. Let (F_n) be a countable family of topological spaces, each with a countable base for open sets; let E be a set and, for each n, let $f_n : \mathrm{E} \to \mathrm{F}_n$ be a mapping. Equip E with the initial topology for the family (f_n). Then $\mathcal{B}(\mathrm{E})$ is the initial Borel structure for the family of mappings $f_n : \mathrm{E} \to \big(\mathrm{F}_n, \mathcal{B}(\mathrm{F}_n)\big)$.

§2.2. Matters Polish, Lusin and Souslin

A topological space P is said to be *Polish* if it has a countable base for open sets and is homeomorphic to a complete metric space. A separated topological space E is said to be *Lusin* if there exist a Polish space P and a continuous bijection $P \to E$, and *Souslin* if there exist a Polish space P and a continuous surjection $P \to E$. Obviously, Polish ⇒ Lusin ⇒ Souslin. The basic properties of such spaces and their interrelations are summarized in the Appendix (§A.1).

Another way of putting the matter is as follows. Only separated spaces are eligible to be Polish, Lusin or Souslin. Start with the class of complete metric spaces having a countable dense subset, the so-called 'separable complete metric spaces'. The homeomorphic images are the Polish spaces; the separated, continuous bijective images are the Lusin spaces; and the separated, continuous images are the Souslin spaces.

Here are some examples of the power of such hypotheses in questions of Borel sets:

Theorem A. Two Polish spaces are Borel-isomorphic if and only if they have the same cardinality (3.1.2). If B is an uncountable Borel set in a Polish space P, then the sub-Borel space $\big(B, \mathcal{B}(P) \cap B\big)$ is isomorphic to $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ (3.1.1).

Theorem B. Every bijective Borel mapping between Souslin spaces with a countable base for open sets is a Borel isomorphism (2.2.11), and the countability hypothesis can be dropped for regular Lusin spaces (2.2.13).

The core results on such spaces and their Borel sets are summarized in the Appendix (§A.2). The present section is devoted to complements to this circle of ideas.

2.2.1. Proposition. *Every Lusin space is Borel-isomorphic to a Polish space* (*dispersed*, *if we like*).

Proof. If E is a Lusin space, there exist a dispersed Polish space P and a continuous bijection $f : P \to E$ (A.1.5); by A.2.7, f is a Borel isomorphism. ♢

2.2.2. Corollary. *If* E *is a Lusin space and* B *is a Borel set in* E, *then* B *is Borel-isomorphic to a dispersed Polish space.*

Proof. By A.2.8, B is a Lusin subspace of E, therefore $(B, \mathcal{B}(E) \cap B) = \big(B, \mathcal{B}(B)\big)$ is isomorphic to $\big(P, \mathcal{B}(P)\big)$ for some dispersed Polish space P by 2.2.1. ♢

2.2.3. Proposition. [cf. B, p. IX.68] *If* (E_n) *is a countable family of Lusin spaces and* $E = \prod E_n$ *is the product topological space*, *then* $\mathcal{B}(E) = \prod \mathcal{B}(E_n)$.

Proof. For each n let P_n be a Polish space and $f_n : P_n \to E_n$ a continuous bijection, and let $P = \prod P_n$ be the product topological space. Since each f_n is a Borel isomorphism (A.2.7), the bijection $f : P \to E$ defined by $f\big((x_n)\big) = \big(f_n(x_n)\big)$ transforms the tribe $\prod \mathcal{B}(P_n)$ into the tribe $\prod \mathcal{B}(E_n)$. Also, the bijection f is continuous and P is Polish, therefore E is Lusin and f transforms $\mathcal{B}(\mathrm{P})$ into $\mathcal{B}(\mathrm{E})$ by A.2.7. Finally, $\mathcal{B}(\mathrm{P}) = \prod \mathcal{B}(\mathrm{P}_n)$ by 2.1.3, therefore

$$\mathcal{B}(\mathrm{E}) = f\big(\mathcal{B}(\mathrm{P})\big) = f\big(\prod \mathcal{B}(\mathrm{P}_n)\big) = \prod \mathcal{B}(\mathrm{E}_n)\,. \ \diamondsuit$$

2.2.4. Proposition. *Let* E *and* F *be Souslin spaces*, $f : \mathrm{E} \to \mathrm{F}$ *a continuous surjection*, *and* R *the equivalence relation* « $f(x) = f(x')$ » *in* E. *The natural bijection* $\mathrm{E}/\mathrm{R} \to \mathrm{F}$ *is an isomorphism for the Borel structures* $\mathcal{B}(\mathrm{E})/\mathrm{R}$ *and* $\mathcal{B}(\mathrm{F})$.

Proof. By A.2.5, $\mathcal{B}(\mathrm{F})$ is the final Borel structure for $f : \big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big) \to \mathrm{F}$, so the proposition follows from 1.4.5. $\diamondsuit$

2.2.5. Corollary. *If* F *is any Souslin space*, *then* $\big(\mathrm{F}, \mathcal{B}(\mathrm{F})\big)$ *is isomorphic to a quotient Borel space of* $\big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big)$ *for some Polish space* P.

Proof. There exist a Polish space P and a continuous surjection $\mathrm{P} \to \mathrm{F}$. $\diamondsuit$

{CAUTION: A quotient Borel space of $\big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big)$ is not to be confused with the Borel space associated with a quotient topological space of P (cf. 1.4.6).}

2.2.6. Definition. [M, p. 137] A Borel space $(\mathrm{E}, \mathcal{B})$ is said to be *countably separated* if there exists a sequence (B_n) of Borel sets with the following property: for each pair of distinct points x, y of E, there is an index n such that either $x \in \mathrm{B}_n$ and $y \in \complement \mathrm{B}_n$, or $x \in \complement \mathrm{B}_n$ and $y \in \mathrm{B}_n$. Such a sequence (B_n) is said to be *separating.*

If E is a separated topological space with a countable base for open sets (for example, a Polish space), it is clear that the Borel space $\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)$ is countably separated.

2.2.7. Proposition. [A, p. 69] *If* $(\mathrm{E}, \mathcal{B})$ *is a Borel space*, $(\mathrm{F}, \mathcal{C})$ *is a countably separated Borel space*, *and* $f : \mathrm{E} \to \mathrm{F}$ *is a Borel mapping*, *then the graph* $\mathrm{G} = \{\big(x, f(x)\big) : x \in \mathrm{E}\}$ *of* f *is a Borel set in* $\mathrm{E} \times \mathrm{F}$, *that is*, $\mathrm{G} \in \mathcal{B} \times \mathcal{C}$.

Proof. Define $g : \mathrm{E} \times \mathrm{F} \to \mathrm{F} \times \mathrm{F}$ by $g(x, y) = \big(f(x), y)\big)$. Since $\mathrm{pr}_1 \circ g = f \circ \mathrm{pr}_1$ and $\mathrm{pr}_2 \circ g = \mathrm{pr}_2$ are Borel for the structures $\mathcal{B} \times \mathcal{C}$, $\mathcal{C}$ it follows from 1.3.11 that g is Borel for the structures $\mathcal{B} \times \mathcal{C}$, $\mathcal{C} \times \mathcal{C}$. If $\Delta = \{(y, y) : y \in \mathrm{F}\}$ (the 'diagonal' of $\mathrm{F} \times \mathrm{F}$), then $g^{-1}(\Delta) = \mathrm{G}$, thus it will suffice to show that Δ is a Borel set in $\mathrm{F} \times \mathrm{F}$. Let (C_n) be a separating sequence in $\mathcal{C}$. If y, z are distinct points of F, there is an index n such that $(y, z) \in \mathrm{C}_n \times \complement \mathrm{C}_n$ or $(y, z) \in \complement \mathrm{C}_n \times \mathrm{C}_n$; it follows that

$$\complement \Delta = \bigcup [(\mathrm{C}_n \times \complement \mathrm{C}_n) \cup (\complement \mathrm{C}_n \times \mathrm{C}_n)]\,,$$

thus $\complement \Delta$ belongs to $\mathcal{C} \times \mathcal{C}$, therefore so does Δ. $\diamondsuit$

A result in the reverse direction:

2.2.8. Proposition. *If* E *and* F *are topological spaces*, *with* E *separated*, *and if* $f : \mathrm{E} \to \mathrm{F}$ *is a mapping whose graph* G *is a Souslin subspace of* $\mathrm{E} \times \mathrm{F}$, *then* f *is a Borel mapping.*

Proof. Let C be a closed set in F; it will suffice to show that $f^{-1}(\mathrm{C})$ is a Borel set in E (1.2.6). For this, it is enough to show that $f^{-1}(\mathrm{C})$ and its complement are both Souslin subspaces of E (A.2.4). Note that if $\mathrm{T} \subset \mathrm{F}$ then

$$\begin{aligned}\mathrm{pr}_1[\mathrm{G} \cap (\mathrm{E} \times \mathrm{T})] &= \{\mathrm{pr}_1(x, y) : \quad y = f(x) \text{ and } y \in \mathrm{T}\} \\ &= \{x \in \mathrm{E} : \ f(x) \in \mathrm{T}\} \\ &= f^{-1}(\mathrm{T}) .\end{aligned}$$

Since $\mathrm{G} \cap (\mathrm{E} \times \mathrm{C})$ is a closed subspace of the Souslin space G, it is also a Souslin space (A.1.7), hence its image in the separated space E, under the continuous mapping pr_1, is a Souslin subspace of E (A.1.3); that is, $f^{-1}(\mathrm{C})$ is a Souslin subspace of E. The foregoing argument is applicable with C replaced by $\complement\mathrm{C}$, and "closed" by "open"; thus $\complement f^{-1}(\mathrm{C}) = f^{-1}(\complement\mathrm{C})$ is also a Souslin subspace of E. ◊

2.2.9. Proposition. *Let* E *and* F *be topological spaces,* $f : \mathrm{E} \to \mathrm{F}$ *a Borel mapping. Assume that* F *is Souslin and has a countable base for open sets. If* A *is any Souslin subspace of* E, *then* $f(\mathrm{A})$ *is a Souslin subspace of* F.

Proof. Consider

$$\mathrm{A} \xrightarrow{\ i\ } \mathrm{E} \xrightarrow{\ f\ } \mathrm{F},$$

where i is the insertion mapping. Since both i and f are Borel, so is $f|\mathrm{A} = f \circ i$; dropping down to A and changing notation, we can suppose that E is a Souslin space, and the problem is to show that $f(\mathrm{E})$ is a Souslin subspace of F.

Let G be the graph of f. By 2.2.7, G is a Borel set in $\mathrm{E} \times \mathrm{F}$. Since $\mathrm{E} \times \mathrm{F}$ is a Souslin space (A.1.7), it follows that G is Souslin (A.2.1) hence so is its image under the continuous mapping pr_2; this image is $f(\mathrm{E})$. ◊

2.2.10. Corollary. *Let* E *and* F *be topological spaces,* $f : \mathrm{E} \to \mathrm{F}$ *a Borel mapping. Assume that* F *is metrizable and has a countable base for open sets. If* A *is a Souslin subspace of* E, *then* $f(\mathrm{A})$ *is a Souslin subspace of* F.

Proof. We can suppose that F is a metric space, with metric d, having a countable dense subset [B, p. IX.18, Prop. 12]. If (F', d') is the completion of (F, d) then f, regarded as a mapping $\mathrm{E} \to \mathrm{F}'$, is also Borel (1.3.6); changing notations, we can suppose that F is Polish, therefore Souslin, and the corollary then follows from 2.2.9. ◊

2.2.11. Proposition. *Let* E *be a Souslin space,* F *a Souslin space with a countable base for open sets. If* $f : \mathrm{E} \to \mathrm{F}$ *is a bijective Borel mapping, then* f *is a Borel isomorphism.*

Proof. If A is any closed set in E, it will suffice to show that $f(\mathrm{A})$ is a Borel set in F (1.2.6). Since both A and $\complement\mathrm{A}$ are Souslin (A.1.7), so are $f(\mathrm{A})$ and $f(\complement\mathrm{A}) = \complement f(\mathrm{A})$ (2.2.9), therefore $f(\mathrm{A})$ is a Borel set in F (A.2.4). ◊

2.2.12. Corollary. *Let* $f : \mathrm{E} \to \mathrm{F}$ *be an injective Borel mapping, where* E *is a Souslin space and* F *is a Souslin space* (resp. *metrizable space*) *with a* *countable base for open sets. Then* $f(\mathrm{E})$ *is a Souslin subspace of* F *and the mapping* $f_0 : \mathrm{E} \to f(\mathrm{E})$ *having the graph of* f *is a Borel isomorphism.*

Proof. $f(\mathrm{E})$ is Souslin by 2.2.9 (resp. 2.2.10), and the bijective mapping f_0 is Borel (1.3.6), therefore f_0 is a Borel isomorphism by 2.2.11. ♢

2.2.13. Proposition. *Let* $f : \mathrm{E} \to \mathrm{F}$ *be an injective Borel mapping, where* E *is a Lusin space and* F *is a regular Lusin space. Then, for every Borel set* A *in* E, $f(\mathrm{A})$ *is a Borel set in* F. *In particular,* $f(\mathrm{E})$ *is a Borel set in* F; $f(\mathrm{E})$ *is a Lusin subspace of* F, *and the mapping* $f_0 : \mathrm{E} \to f(\mathrm{E})$ *having the graph of* f *is a Borel isomorphism.*

Proof. The first assertion is A.2.10; in particular, $f(\mathrm{E})$ is a Borel set in F, hence is a Lusin subspace of F (A.2.8). Since the bijection f_0 is Borel (1.3.6) and since $f(\mathrm{E})$ is regular [B, p. I.56], we can suppose that f is bijective; then f is a Borel isomorphism by the first assertion. ♢

3. STANDARD BOREL SPACES

§3.1. The Borel structure of Polish spaces

If $(\mathrm{E}, \mathcal{B})$ is a Borel space such that E is countable and $\mathcal{B}$ contains every singleton $\{x\}$, then $\mathcal{B} = \mathcal{P}(\mathrm{E})$; any bijection between such spaces is trivially a Borel isomorphism. In particular, any two countable Polish spaces with the same cardinality are Borel-isomorphic. It is a remarkable fact (to be proved shortly) that every uncountable Polish space is Borel-isomorphic to $\mathbb{R}$ (in particular, cardinalities between $\aleph_0$ and $c = 2^{\aleph_0}$ are excluded without having to resort to the Continuum Hypothesis). Thus, *the Borel structure of a Polish space is completely determined by its cardinality* (and only two infinite cardinalities occur); consequently, if a Borel space can be shown to be derived from a Polish topological space, then it inherits structures (for instance measures) from familiar spaces. The proof to be given here exploits an exercise in Bourbaki [B, p. IX.120, Exer. 9c].

3.1.1. Theorem [K, p. 358]. *Suppose* B_1, B_2 *are uncountable Borel sets in the Polish spaces* P_1, P_2. *Equip* B_i $(i = 1, 2)$ *with the relative topology. Then the Borel spaces* $\big(\mathrm{B}_1, \mathcal{B}(\mathrm{B}_1)\big)$ *and* $\big(\mathrm{B}_2, \mathcal{B}(\mathrm{B}_2)\big)$ *are isomorphic.*

In particular, if B *is an uncountable Borel set in a Polish space* P, *then* $\big(\mathrm{B}, \mathcal{B}(\mathrm{P}) \cap \mathrm{B}\big)$ *is Borel-isomorphic to* $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, *thus* B *has cardinality* $c = 2^{\aleph_0}$.

Proof. Since B_i is a Lusin subspace of P_i (A.2.8), it suffices to show that any two uncountable Lusin spaces are Borel-isomorphic.

Every Lusin space is Borel-isomorphic to a dispersed Polish space (2.2.1), thus the theorem is reduced to proving that *any two uncountable, dispersed Polish spaces are Borel-isomorphic.*

Let P_1 and P_2 be two such spaces. By Theorem 2 of §A.5, there exists a topological subspace A_i of P_i such that A_i is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and $\complement \mathrm{A}_i$ is countable. Then A_1 and A_2 are homeomorphic, say via $f : \mathrm{A}_1 \to \mathrm{A}_2$. Let $\mathrm{D}_i = \mathrm{P}_i - \mathrm{A}_i$ $(i = 1, 2)$. Conceivably D_1 and D_2 have different cardinalities. To overcome this, let B_1 be a denumerably infinite subset of A_1 and let $\mathrm{B}_2 = f(\mathrm{B}_1)$ be its homeomorphic image in A_2; we then have partitions

$$\mathrm{P}_i = (\mathrm{A}_i - \mathrm{B}_i) \cup (\mathrm{B}_i \cup \mathrm{D}_i) \quad (i = 1, 2),$$

where $\mathrm{A}_1 - \mathrm{B}_1$ and $\mathrm{A}_2 - \mathrm{B}_2$ are homeomorphic and their complements $\mathrm{B}_1 \cup \mathrm{D}_1$, $\mathrm{B}_2 \cup \mathrm{D}_2$ are both denumerably infinite. Any bijection of P_1 onto P_2 that maps $\mathrm{A}_1 - \mathrm{B}_1$ homeomorphically onto $\mathrm{A}_2 - \mathrm{B}_2$ is easily seen to be a Borel isomorphism. $\diamondsuit$

3.1.2. Corollary. *Two Polish spaces are Borel-isomorphic if and only if they have the same cardinality.*

We thus have a complete list of models for Borel spaces $\big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big)$, P Polish:

$$\begin{aligned}&\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)\,, \quad \big(\mathbb{N}, \mathcal{P}(\mathbb{N})\big) \quad \text{and}\\ &\Big(\{1, \ldots, n\}, \mathcal{P}\big(\{1, \ldots, n\}\big)\Big) \;\; (n = 1, 2, 3, \ldots)\,.\end{aligned}$$

A Borel space that is isomorphic to one of these models is called a 'standard Borel space'. The concept is limited by the meager list of models; the inventory is greatly enlarged by considering the class of sub-Borel spaces of standard Borel spaces (too large!) and then dropping down to those that are Borel images of Polish spaces (just right). Such spaces are explored in the sections that follow.

§3.2. Standard and substandard Borel spaces

3.2.1. Definition. A Borel space is said to be *standard* if it is isomorphic to $\big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big)$ for some Polish space P.

3.2.2. Proposition. (i) *If* $(\mathrm{E}, \mathcal{B})$ *is a standard Borel space and* $\mathrm{B} \in \mathcal{B}$, *then* $\big(\mathrm{B}, \mathcal{B} \cap \mathrm{B}\big)$ *is standard.*

(ii) *The product and direct union of a countable family of standard Borel spaces are standard.*

Proof. (i) Immediate from 2.2.2.

(ii) Immediate from 2.1.3 and 2.1.5. ♢

3.2.3. Proposition. *For a Borel space* $(\mathrm{E}, \mathcal{B})$, *the following conditions are equivalent*:

(a) $(\mathrm{E}, \mathcal{B})$ *is standard*;

(b) $(\mathrm{E}, \mathcal{B})$ *is isomorphic to* $\big(\mathrm{Q}, \mathcal{B}(\mathrm{Q})\big)$ *for some dispersed Polish space* Q;

(c) $(\mathrm{E}, \mathcal{B})$ *is isomorphic to* $\big(\mathrm{L}, \mathcal{B}(\mathrm{L})\big)$ *for some Lusin space* L.

Proof. Immediate from 2.2.1. ♢

3.2.4. Proposition. *Two standard Borel spaces are isomorphic if and only if they have the same cardinality.*

Proof. Immediate from 3.1.2. ♢

A full list of models for standard Borel spaces is given following 3.1.2.

A sharper result is the following:

3.2.5. Proposition. *If* $(\mathrm{E}, \mathcal{B})$ *and* $(\mathrm{F}, \mathcal{C})$ *are standard Borel spaces, then every bijective Borel mapping* $f : \mathrm{E} \to \mathrm{F}$ *is a Borel isomorphism.*

Proof. Immediate from A.2.10. ♢

3.2.6. Definition. A Borel space $(\mathrm{F}, \mathcal{C})$ will be called *substandard* if it is isomorphic to a sub-Borel space of a standard Borel space, that is, if there exist a standard Borel space $(\mathrm{E}, \mathcal{B})$ and a subset T of E such that $(\mathrm{F}, \mathcal{C})$ is isomorphic to $(\mathrm{T}, \mathcal{B} \cap \mathrm{T})$. {Note that T is not required to belong to $\mathcal{B}$.}

Thus, $(\mathrm{F}, \mathcal{C})$ is substandard if and only if there exist a Polish space P (dispersed, if we like) and a subset T of P such that $(\mathrm{F}, \mathcal{C})$ is isomorphic to $\big(\mathrm{T}, \mathcal{B}(\mathrm{P}) \cap \mathrm{T}\big)$, in other words (1.3.5) F is Borel-isomorphic to a topological subspace of a Polish space.

3.2.7. Theorem. [M, p. 139, Th. 3.2] *If* $(\mathrm{E}, \mathcal{B})$ *is a standard Borel space,* $(\mathrm{F}, \mathcal{C})$ *is a substandard Borel space, and* $f : \mathrm{E} \to \mathrm{F}$ *is an injective Borel mapping, then* $f(\mathrm{E}) \in \mathcal{C}$ *and the mapping* $f_0 : \mathrm{E} \to f(\mathrm{E})$ *having the graph of* f *is an isomorphism of* $(\mathrm{E}, \mathcal{B})$ *onto* $\big(f(\mathrm{E}), \mathcal{C} \cap f(\mathrm{E})\big)$.

Proof. We can suppose that E is a Polish space and $\mathcal{B} = \mathcal{B}(\mathrm{E})$; we can also regard F as a subset of a Polish space Q such that $\mathcal{C} = \mathcal{B}(\mathrm{Q}) \cap \mathrm{F}$. Consider

$$\mathrm{E} \xrightarrow{\quad f \quad} \mathrm{F} \xrightarrow{\quad i \quad} \mathrm{Q},$$

where i is the insertion mapping, and define $g = i \circ f$. Since F has the initial Borel structure for i, g is the composite of injective Borel mappings, hence is an injective Borel mapping.

If $\mathrm{A} \in \mathcal{B} = \mathcal{B}(\mathrm{E})$ then $g(\mathrm{A}) \in \mathcal{B}(\mathrm{Q})$ by A.2.10, that is, $f(\mathrm{A}) \in \mathcal{B}(\mathrm{Q})$; also $f(\mathrm{A}) \subset \mathrm{F}$, so

$$f(\mathrm{A}) = f(\mathrm{A}) \cap \mathrm{F} \in \mathcal{B}(\mathrm{Q}) \cap \mathrm{F} = \mathcal{C}\,.$$

In particular $f(\mathrm{E}) \in \mathcal{C}$, and for all $\mathrm{A} \in \mathcal{B}$ we have

$$f(\mathrm{A}) \in \mathcal{C} \cap f(\mathrm{E})\,;$$

it follows that f_0 is an isomorphism of $(\mathrm{E}, \mathcal{B})$ onto $\big(f(\mathrm{E}), \mathcal{C} \cap f(\mathrm{E})\big)$. {Incidentally, since $f(\mathrm{E}) \in \mathcal{C}$, $\mathcal{C} \cap f(\mathrm{E})$ is the set of all $\mathrm{C} \in \mathcal{C}$ such that $\mathrm{C} \subset f(\mathrm{E})$.} ♢

In particular, a bijective Borel mapping of a standard Borel space onto a substandard Borel space is a Borel isomorphism. Another way of expressing this is that a standard Borel structure is 'minimal-substandard', that is, if $(\mathrm{E}, \mathcal{B})$ is a standard Borel space and $\mathcal{C}$ is a tribe on E such that $\mathcal{C} \subset \mathcal{B}$ and $(\mathrm{E}, \mathcal{C})$ is substandard, then necessarily $\mathcal{C} = \mathcal{B}$. {More generally, see 4.6 and 4.11.}

3.2.8. Corollary. [M, p. 139, Cor. 1] *Let* $(\mathrm{E}, \mathcal{B})$ *be a standard Borel space, and let* $\mathrm{S} \subset \mathrm{E}$. *In order that the (substandard) Borel space* $(\mathrm{S}, \mathcal{B} \cap \mathrm{S})$ *be standard, it is necessary and sufficient that* $\mathrm{S} \in \mathcal{B}$.

Proof. Necessity: The insertion mapping

$$i : \big(\mathrm{S}, \mathcal{B} \cap \mathrm{S}\big) \to (\mathrm{E}, \mathcal{B})$$

is Borel, so $\mathrm{S} = i(\mathrm{S}) \in \mathcal{B}$ by 3.2.7.

Sufficiency: Immediate from 3.2.2, (i). ♢

3.2.9. Corollary. [M, p. 139, Cor. 3] *Let* $(\mathrm{E}, \mathcal{B})$ *be a substandard Borel space,* S *a subset of* E. *If the Borel space* $(\mathrm{S}, \mathcal{B} \cap \mathrm{S})$ *is standard, then* $\mathrm{S} \in \mathcal{B}$.

Proof. Let $(\mathrm{F}, \mathcal{C})$ be a standard Borel space such that $\mathrm{E} \subset \mathrm{F}$ and $\mathcal{B} = \mathcal{C} \cap \mathrm{E}$ (3.2.6). Then

$$\mathcal{B} \cap \mathrm{S} = \mathcal{C} \cap \mathrm{E} \cap \mathrm{S} = \mathcal{C} \cap \mathrm{S}\,,$$

thus $(\mathrm{S}, \mathcal{B} \cap \mathrm{S})$ is a standard sub-Borel space of the standard space $(\mathrm{F}, \mathcal{C})$, therefore $\mathrm{S} \in \mathcal{C}$ by 3.2.8; thus $\mathrm{S} \in \mathcal{C} \cap \mathrm{E} = \mathcal{B}$. ♢

3.2.10. [M, p. 139, Cor. 2] If $(\mathrm{E}, \mathcal{B})$ is standard and S is a subset of E such that $\mathrm{S} \notin \mathcal{B}$, then $(\mathrm{S}, \mathcal{B} \cap \mathrm{S})$ is substandard but not standard (3.2.8).

§3.3. Mackey's countability conditions

In Mackey's fundamental paper [M], in which the systematic study of Borel spaces was initiated (and largely concluded), the concepts of standard space (Mackey's terminology) and substandard space (Mackey is not to blame) are characterized in remarkably simple set-theoretic terms; these characterizations are the object of the present section.

3.3.1. Definition. Let E be a set, $\mathcal{E}$ a set of subsets of E. One says that $\mathcal{E}$ is *separating* (or 'separates E') if the set of characteristic functions $\{\varphi_{\mathrm{A}} : \mathrm{A} \in \mathcal{E}\}$ separates the points of E; that is, if $x, y \in \mathrm{E}$ and $x \neq y$, then there exists $\mathrm{A} \in \mathcal{E}$ such that either $x \in \mathrm{A}$ and $y \in \complement\mathrm{A}$, or $y \in \mathrm{A}$ and $x \in \complement\mathrm{A}$.

If $\mathcal{E}^* = \{\mathrm{A} \subset \mathrm{E} : \mathrm{A} \in \mathcal{E}$ or $\complement\mathrm{A} \in \mathcal{E}\}$, then $\mathcal{E}$ is separating if and only if $\mathcal{E}^*$ is separating, if and only if: for $x, y \in \mathrm{E}$ with $x \neq y$, there exists $\mathrm{A} \in \mathcal{E}^*$ such that $x \in \mathrm{A}$ and $y \in \complement\mathrm{A}$. Note also that $\mathcal{E}$ is countable if and only if $\mathcal{E}^*$ is countable.

3.3.2. Definition. A Borel space $(\mathrm{E}, \mathcal{B})$ is said to be *separated* if $\mathcal{B}$ is a separating set, and *countably separated* if $\mathcal{B}$ has a countable subset that is separating (2.2.6).

3.3.3. Definition. A Borel space $(\mathrm{E}, \mathcal{B})$ is said to be *countably generated* if (i) it is separated, and (ii) it has a countable system of generators, that is, there exists a countable set $\mathcal{D} \subset \mathcal{B}$ such that $\mathcal{B} = \mathcal{T}(\mathcal{D})$.

3.3.4. Proposition. *Let* E *be a set,* $\mathcal{E}$ *a set of subsets of* E. *In order that* $\mathcal{E}$ *be separating, it is necessary and sufficient that* $\mathcal{T}(\mathcal{E})$ *be separating.*

Proof. Assuming $\mathcal{E}$ is not separating, let us show that the tribe $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ is not separating. By assumption, there exists a pair of distinct points x, y in E such that, for every $\mathrm{A} \in \mathcal{E}$,

$$(*) \qquad \{x, y\} \subset \mathrm{A} \quad \text{or} \quad \{x, y\} \subset \complement\mathrm{A}\,.$$

Let $\mathcal{A}$ be the set of all $\mathrm{A} \subset \mathrm{E}$ satisfying $(*)$. Obviously $\mathrm{E} \in \mathcal{A}$ and $\mathcal{A}$ is closed under complementation. If (A_i) is a family of sets in $\mathcal{A}$ and $\mathrm{A} = \bigcup \mathrm{A}_i$, then $\mathrm{A} \in \mathcal{A}$; for if $\{x, y\} \subset \mathrm{A}_j$ for some j then $\{x, y\} \subset \mathrm{A}$, whereas if $\{x, y\} \subset \complement\mathrm{A}_i$ for all i, then $\{x, y\} \subset \bigcap \complement\mathrm{A}_i = \complement\mathrm{A}$. In particular, $\mathcal{A}$ is a tribe. By hypothesis, $\mathcal{E} \subset \mathcal{A}$, therefore $\mathcal{T}(\mathcal{E}) \subset \mathcal{A}$, thus $\mathcal{T}(\mathcal{E})$ is not separating. ◇

3.3.5. Corollary. *If* $\mathcal{E}$ *is a countable separating set of subsets of* E, *then the Borel space* $\big(\mathrm{E}, \mathcal{T}(\mathcal{E})\big)$ *is countably generated.*

Proof. Immediate from the definitions and the trivial half of 3.3.4. ◇

More interesting is the following:

3.3.6. Corollary. *If* $(\mathrm{E}, \mathcal{B})$ *is a countably generated Borel space, and if* $\mathcal{B} = \mathcal{T}(\mathcal{E})$, *then* $\mathcal{E}$ *is separating; in particular,* $\mathcal{B}$ *admits a set of generators that is both countable and separating.*

The above considerations make no mention of Polish (or even topological) spaces, so the following observation of Mackey is striking [cf. M, p. 137, Th. 2.1]:

3.3.7. Theorem. *A Borel space is substandard if and only if it is countably generated.*

Proof. If T is a separated topological space with a countable base for open sets, it is obvious that the Borel space $\big(\mathrm{T}, \mathcal{B}(\mathrm{T})\big)$ is countably generated, therefore so are all of its sub-Borel spaces (cf. the proof of 1.3.5). Applying this remark with T Polish, we see that substandard Borel spaces are countably generated.

Conversely, assuming $(\mathrm{E}, \mathcal{B})$ is countably generated, it will be shown to be a sub-Borel space of $\big(\mathbb{M}, \mathcal{B}(\mathbb{M})\big)$, where $\mathbb{M} = 2^{\aleph_0}$ (cf. §A.4 of the Appendix). Let (B_n) be a sequence of sets that generates the tribe $\mathcal{B}$. Define $f : \mathrm{E} \to \mathbb{M}$ by the formula

$$f(x) = \big(\varphi_{\mathrm{B}_n}(x)\big)\,,$$

where φ_{B_n} is the characteristic function of B_n. The B_n separate E by 3.3.6, therefore f is injective. For $j = 1, 2, 3, \ldots$ let

$$\begin{aligned}\mathrm{F}_j &= \{(z_n) \in \mathbb{M} :\ z_j = 1\,\} = \mathrm{pr}_j^{-1}(\{1\}) \\ &= \{1\}_j \times \prod_{n \neq j} \{0,1\}_n\end{aligned}$$

(abuse of notation—the factors on the right should be permuted); then

$$\complement \mathrm{F}_j = \{0\}_j \times \prod_{n \neq j} \{0,1\}_n = \mathrm{pr}_j^{-1}(\{0\})\,.$$

Evidently $\mathrm{F}_j \in \mathcal{B}(\mathbb{M})$ (indeed, F_j is clopen). The sequence (F_j) generates the tribe $\mathcal{B}(\mathbb{M})$; for, if $\mathcal{E}$ is the set of all F_j and their complements, then the finite intersections of sets in $\mathcal{E}$ form a countable base for the open sets of $\mathbb{M}$, therefore $\mathcal{T}(\mathcal{E}) = \mathcal{B}(\mathbb{M})$.

Since the F_j generate $\mathcal{B}(\mathbb{M})$, the sets $\mathrm{F}_j \cap f(\mathrm{E})$ generate $\mathcal{B}(\mathbb{M}) \cap f(\mathrm{E})$ (cf. the proof of 1.3.5). Note that

$$f(\mathrm{B}_j) = \mathrm{F}_j \cap f(\mathrm{E})\,;$$

for, if $z = (z_n) \in \mathbb{M}$ then

$$\begin{aligned}z \in \mathrm{F}_j \cap f(\mathrm{E}) \quad &\Leftrightarrow \quad z_j = 1 \text{ and } z = f(x) = \big(\varphi_{\mathrm{B}_n}(x)\big) \text{ for some } x \in \mathrm{E} \\ &\Leftrightarrow \quad z = f(x) \quad \text{for some } x \in \mathrm{B}_j\,.\end{aligned}$$

If $f_0 = \mathrm{E} \to f(\mathrm{E})$ is the bijection having the graph of f, we thus have

$$\begin{aligned}\mathcal{B}\big(f(\mathrm{E})\big) &= \mathcal{T}\big(\{\mathrm{F}_j \cap f(\mathrm{E}) :\ j = 1,2,3,\ldots\}\big) \\ &= \mathcal{T}\big(\{f_0(\mathrm{B}_j) :\ j = 1,2,3,\ldots\}\big) \\ &= f_0\big(\mathcal{T}(\{\mathrm{B}_j :\ j = 1,2,3,\ldots\})\big) \\ &= f_0(\mathcal{B})\,;\end{aligned}$$

thus f_0 is an isomorphism of $(\mathrm{E}, \mathcal{B})$ onto $\Big(f(\mathrm{E}), \mathcal{B}\big(f(\mathrm{E})\big)\Big) = \big(f(\mathrm{E}), \mathcal{B}(\mathbb{M}) \cap f(\mathrm{E})\big)$ (cf. 1.3.5). ♢

3.3.8. Corollary. *A Borel space is substandard if and only if it is isomorphic to a sub-Borel space of* $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$.

Proof. A substandard Borel space is isomorphic to a sub-Borel space of $\mathbb{M}$ (proof of 3.3.7), hence of the Cantor set (§A.4), hence of $\mathbb{R}$. {Alternatively, note that $\mathbb{M}$ is Borel-isomorphic to $\mathbb{R}$ (3.1.1).} ♢

3.3.9. Corollary. *If* $(\mathrm{E}, \mathcal{B})$ *is a standard Borel space and* (B_n) *is any separating sequence of Borel sets, then the* B_n *generate* $\mathcal{B}$.

Proof. Let $\mathcal{C}$ be the tribe generated by the B_n; the Borel space $(\mathrm{E}, \mathcal{C})$ is countably generated (3.3.5), therefore substandard (3.3.7). Since $\mathcal{C} \subset \mathcal{B}$, the identity mapping $(\mathrm{E}, \mathcal{B}) \to (\mathrm{E}, \mathcal{C})$ is Borel, hence a Borel isomorphism (3.2.7), therefore $\mathcal{C} = \mathcal{B}$; thus the B_n generate $\mathcal{B}$. ♢

3.3.10. Corollary. *Let* E *be a set,* (B_n) *a separating sequence of subsets of* E. *There exists at most one standard Borel structure on* E *for which the* B_n *are Borel sets.*

Proof. If there exists a tribe $\mathcal{B}$ of subsets of E such that $(\mathrm{E}, \mathcal{B})$ is standard and $\mathrm{B}_n \in \mathcal{B}$ for all n, then the B_n generate $\mathcal{B}$ by 3.3.9. Thus the only candidate is the tribe generated by the B_n. {It may be disqualified (3.3.11).} ♢

3.3.11. Example. In 3.3.10, such a structure may not exist. For example, let (B_n) be any separating sequence of Borel sets in $\mathbb{R}$ and let S be a subset of $\mathbb{R}$ that is not a Borel set. If the separating sequence $\mathrm{S}, \mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3, \ldots$ belonged to a tribe $\mathcal{B}$ for which $(\mathbb{R}, \mathcal{B})$ is standard, one would have $\mathcal{B} = \mathcal{B}(\mathbb{R})$ by 3.3.10, contrary to the choice of S.

The method of proof of 3.3.7 yields a topological model for Borel mappings between substandard spaces:

3.3.12. Theorem. *Let* $(\mathrm{E}_1, \mathcal{B}_1)$ *and* $(\mathrm{E}_2, \mathcal{B}_2)$ *be substandard Borel spaces,* $f : \mathrm{E}_1 \to \mathrm{E}_2$ *a Borel mapping.*

There exist Borel isomorphisms f_1, f_2 *of* $(\mathrm{E}_1, \mathcal{B}_1)$, $(\mathrm{E}_2, \mathcal{B}_2)$ *onto sub-Borel spaces of* $\big(\mathbb{M}, \mathcal{B}(\mathbb{M})\big)$ *such that the mapping* $g_0 = f_2 \circ f \circ f_1^{-1} : f_1(\mathrm{E}_1) \to f_2(\mathrm{E}_2)$ *is the restriction to* $f_1(\mathrm{E}_1)$ *of a continuous mapping* $g : \mathbb{M} \to \mathbb{M}$.

$$
\begin{array}{ccc}
\mathbb{M} & \xrightarrow{\;g\;} & \mathbb{M} \\
\cup & & \cup \\
f_1(\mathrm{E}_1) & \xrightarrow{\;g_0\;} & f_2(\mathrm{E}_2) \\
{\scriptstyle f_1}\big\uparrow & & \big\uparrow{\scriptstyle f_2} \\
\mathrm{E}_1 & \xrightarrow[\;f\;]{} & \mathrm{E}_2
\end{array}
$$

Proof. Let (B_n) be a generating sequence for $\mathcal{B}_2$. Let (A_n) be a generating sequence for $\mathcal{B}_1$ such that $f^{-1}(\mathrm{B}_n) = \mathrm{A}_{2n}$ for all n. {For example, choose a generating sequence $\mathrm{A}_1, \mathrm{A}_3, \mathrm{A}_5, \dots$ for $\mathcal{B}_1$ and define $\mathrm{A}_{2n} = f^{-1}(\mathrm{B}_n)$ for all n.} As in the proof of 3.3.7, define mappings

$$f_1 : \mathrm{E}_1 \to \mathbb{M}\,, \quad f_2 : \mathrm{E}_2 \to \mathbb{M}$$

by the formulas

$$f_1(x) = \big(\varphi_{\mathrm{A}_n}(x)\big)\,, \quad f_2(y) = \big(\varphi_{\mathrm{B}_n}(y)\big)$$

for $x \in \mathrm{E}_1$, $y \in \mathrm{E}_2$; as shown there, f_i is a Borel isomorphism of $(\mathrm{E}_i, \mathcal{B}_i)$ onto

$$\big(f_i(\mathrm{E}_i), \mathcal{B}(\mathbb{M}) \cap f_i(\mathrm{E}_i)\big) = \Big(f_i(\mathrm{E}_i), \mathcal{B}\big(f_i(\mathrm{E}_i)\big)\Big)\,,$$

where $f_i(\mathrm{E}_i)$ bears the relative topology (1.3.5). Define $g : \mathbb{M} \to \mathbb{M}$ as follows: if $z = (z_n) \in \mathbb{M}$, define

$$g(z) = (z_2, z_4, z_6, \dots) = (z_{2n})\,.$$

Since $\mathrm{pr}_n \circ g = \mathrm{pr}_{2n}$ is continuous for all n, g is continuous. It remains to show that $g \circ f_1 = f_2 \circ f$. If $x \in \mathrm{E}_1$, then for any n one has

$$\begin{aligned}
\mathrm{pr}_n\Big(g\big(f_1(x)\big)\Big) = 1 \quad &\Leftrightarrow \quad \mathrm{pr}_{2n}\big(f_1(x)\big) = 1 \\
&\Leftrightarrow \quad \varphi_{\mathrm{A}_{2n}}(x) = 1 \\
&\Leftrightarrow \quad x \in \mathrm{A}_{2n} = f^{-1}(\mathrm{B}_n) \\
&\Leftrightarrow \quad f(x) \in \mathrm{B}_n \\
&\Leftrightarrow \quad \mathrm{pr}_n\Big(f_2\big(f(x)\big)\Big) = 1\,;
\end{aligned}$$

since 1 and 0 are the only coordinate values in $\mathbb{M}$, this shows that $g\big(f_1(x)\big) = f_2\big(f(x)\big)$. ◇

There is a striking corollary:

3.3.13. Corollary. *If* $(\mathrm{E}_1, \mathcal{B}_1)$ *and* $(\mathrm{E}_2, \mathcal{B}_2)$ *are substandard Borel spaces and* $f : \mathrm{E}_1 \to \mathrm{E}_2$ *is a Borel mapping, then* E_i $(i = 1, 2)$ *can be topologized in such a way that* (1) *it is a dispersed, metrizable space with a countable base for open sets,* (2) $\mathcal{B}_i$ *is the set of all Borel sets for this topology, and* (3) f *is a continuous mapping.*

A corollary of the corollary: If $\mathrm{E}_1, \mathrm{E}_2$ are separated topological spaces having a countable base for open sets, and if $f : \mathrm{E}_1 \to \mathrm{E}_2$ is a Borel mapping, then E_1 and E_2 can be retopologized with topologies of the sort described in 3.3.13, so that the Borel sets of E_i are unchanged and f becomes continuous. {Apply 3.3.13 with $\mathcal{B}_i = \mathcal{B}(\mathrm{E}_i)$, after taking into account 3.3.7.}

We close the section with a set-theoretic characterization of the tribe of Borel sets of a Polish space [cf. M, p. 138]:

3.3.14. Theorem. *The following conditions on a Borel space* $(\mathrm{E},\mathcal{B})$ *are equivalent*:

(a) $(\mathrm{E},\mathcal{B})$ *is standard*;

(b) $(\mathrm{E},\mathcal{B})$ *is separated, and either* (i) E *is countable* (*in which case* $\mathcal{B}$ *is the set of all subsets of* E), *or* (ii) E *is uncountable and* $\mathcal{B}$ *contains a generating sequence* (B_n) *such that for each set* J *of positive integers, the condition*

$$\text{«}\, x \in \mathrm{B}_j \;\Leftrightarrow\; j \in \mathrm{J} \,\text{»}$$

is satisfied by exactly one point x *of* E .

Proof. (a) ⇒ (b): Assuming $(\mathrm{E},\mathcal{B})$ is standard and E is uncountable, let us construct a sequence of the desired sort. By 3.1.1 we can suppose that $\mathrm{E} = \mathbb{M}$ and $\mathcal{B} = \mathcal{B}(\mathbb{M})$, $\mathbb{M} = 2^{\aleph_0}$ being the space employed in the proof of 3.3.7. Let F_n be the set of points of $\mathbb{M}$ whose n'th coordinate is 1 ; as noted in the proof of 3.3.7, $\mathcal{B}$ is the tribe generated by the F_n .

Let J be a set of positive integers and let $x = (x_1, x_2, x_3, \ldots)$ be the element of $\mathbb{M}$ such that x_j is 1 for all $j \in \mathrm{J}$, and 0 otherwise. Then $j \in \mathrm{J} \Leftrightarrow x \in \mathrm{F}_j$. Since an element of $\mathbb{M}$ is determined by its coordinates with entry 1 , an x with this property is unique.

(b) ⇒ (a): Let $(\mathrm{E},\mathcal{B})$ be a separated Borel space.

Suppose first that $(\mathrm{E},\mathcal{B})$ satisfies condition (ii) of (b). Define $f : \mathrm{E} \to \mathbb{M}$ by $f(x) = \big(\varphi_{\mathrm{B}_n}(x)\big)$. Since (B_n) is separating (3.3.6), f is injective. Also, f is surjective: if $y = (y_n) \in \mathbb{M}$ and $\mathrm{J} = \{j : y_j = 1\}$, then the element $x \in \mathrm{E}$ determined by J according to the condition (ii) satisfies $f(x) = y$.

Let (F_n) be the sequence of subsets of $\mathbb{M}$ described in the first part of the proof. For $x \in \mathrm{E}$ and n a positive integer, we have

$$\begin{aligned} f(x) \in \mathrm{F}_n \quad &\Leftrightarrow \quad \mathrm{pr}_n\big(f(x)\big) = 1 \\ &\Leftrightarrow \quad \varphi_{\mathrm{B}_n}(x) = 1 \quad \Leftrightarrow \quad x \in \mathrm{B}_n \,, \end{aligned}$$

thus $f(\mathrm{B}_n) = \mathrm{F}_n$; since the B_n generate $\mathcal{B}$ and the F_n generate $\mathcal{B}(\mathbb{M})$, it follows that the bijection f is a Borel isomorphism, consequently $(\mathrm{E},\mathcal{B})$ is standard.

Finally, suppose E is countable. Since $(\mathrm{E},\mathcal{B})$ is separated and the set of ordered pairs of elements of E is countable, it is clear that $(\mathrm{E},\mathcal{B})$ is countably separated. Let (C_n) be a separating sequence in $\mathcal{B}$. Since $\big(\mathrm{E},\mathcal{P}(\mathrm{E})\big)$ is standard (3.2.1), it follows from 3.3.9 that (C_n) is generating for $\mathcal{P}(\mathrm{E})$, therefore $\mathcal{B} = \mathcal{P}(\mathrm{E})$. ♢

4. ANALYTIC BOREL SPACES

An ‘analytic’ Borel space is at the crossroads of two standard spaces:

4.1. Definition. [M, p. 140] A Borel space $(\mathrm{E},\mathcal{B})$ is said to be *analytic* if (i) it is substandard (3.2.6), and (ii) there exist a standard Borel space $(\mathrm{F},\mathcal{C})$ and a surjective Borel mapping $\mathrm{F}\to\mathrm{E}$. (Cf. Appendix, A.3.3.)

So to speak, a Borel space $(\mathrm{E},\mathcal{B})$ is analytic if and only if it is a sub-Borel space of one standard space and a ‘Borel image’ of another. Equivalently, there exist Polish spaces P and Q such that $(\mathrm{E},\mathcal{B})$ is a sub-Borel space of $\big(\mathrm{P},\mathcal{B}(\mathrm{P})\big)$ and a Borel image of $\big(\mathrm{Q},\mathcal{B}(\mathrm{Q})\big)$.

4.2. Definition. Let $(\mathrm{E},\mathcal{B})$ be a standard Borel space, A a subset of E. One says that A is an *analytic subset* of E if $\big(\mathrm{A},\mathcal{B}\cap\mathrm{A}\big)$ is an analytic Borel space.

4.3. Proposition. *Let* $(\mathrm{E},\mathcal{B})$ *be a standard Borel space,* A *a subset of* E. *In order that* A *be an analytic subset of* E, *it is necessary and sufficient that there exist a standard Borel space* $(\mathrm{F},\mathcal{C})$ *and a Borel mapping* $f:\mathrm{F}\to\mathrm{E}$ *such that* $f(\mathrm{F})=\mathrm{A}$.

Proof. Since $\big(\mathrm{A},\mathcal{B}\cap\mathrm{A}\big)$ is the initial Borel structure for the insertion mapping $\mathrm{A}\to\mathrm{E}$ (1.3.4), the existence of a Borel surjection $(\mathrm{F},\mathcal{C})\to\big(\mathrm{A},\mathcal{B}\cap\mathrm{A}\big)$ is equivalent to the existence of a Borel mapping $f:\mathrm{F}\to\mathrm{E}$ with $f(\mathrm{F})=\mathrm{A}$ (1.3.2). ◇

4.4. Corollary. *Let* P *be a Polish space,* $\mathrm{A}\subset\mathrm{P}$. *In order that* A *be an analytic subset of* P, *it is necessary and sufficient that* A *be a Souslin subspace of* P.

Proof. It is understood that P has the Borel structure derived from its topology. Suppose A is an analytic subset of P. By 4.3, there exist a Polish space Q and a Borel mapping $f:\mathrm{Q}\to\mathrm{P}$ with $f(\mathrm{Q})=\mathrm{A}$. Then A is a Souslin subspace of P by 2.2.10.

Conversely, if A is a Souslin subspace of P, there exist a Polish space Q and a continuous surjection $g:\mathrm{Q}\to\mathrm{A}$. Composing g with the insertion mapping $\mathrm{A}\to\mathrm{P}$ yields a continuous (hence Borel) mapping $f:\mathrm{Q}\to\mathrm{P}$ with $f(\mathrm{Q})=\mathrm{A}$, thus A is an analytic subset of P (4.3). ◇

4.5. Corollary. *A Borel space* $(\mathrm{E},\mathcal{B})$ *is analytic if and only if it is Borel-isomorphic to a Souslin subspace of* $\mathbb{R}$.

Proof. "If": Suppose $(\mathrm{E},\mathcal{B})$ is isomorphic to $\big(\mathrm{A},\mathcal{B}(\mathrm{A})\big) = \big(\mathrm{A},\mathcal{B}(\mathbb{R})\cap \mathrm{A}\big)$ for some Souslin subspace A of $\mathbb{R}$. By 4.4, A is an analytic subset of $\mathbb{R}$, therefore $\big(\mathrm{A},\mathcal{B}(\mathbb{R})\cap \mathrm{A}\big)$ is an analytic Borel space (4.2), therefore so is the isomorphic space $(\mathrm{E},\mathcal{B})$.

"Only if": Suppose $(\mathrm{E},\mathcal{B})$ is analytic. In particular, $(\mathrm{E},\mathcal{B})$ is substandard, hence isomorphic to $\big(\mathrm{A},\mathcal{B}(\mathbb{R})\cap \mathrm{A}\big)$ for some subset A of $\mathbb{R}$ (3.3.8). Then $\big(\mathrm{A},\mathcal{B}(\mathbb{R})\cap \mathrm{A}\big)$ is also analytic, in other words A is an analytic subset of $\mathbb{R}$ (4.2), so A is a Souslin subspace of $\mathbb{R}$ by 4.4. ♢

4.6. Corollary. *If* $(\mathrm{E},\mathcal{B})$ *is an analytic Borel space,* $(\mathrm{F},\mathcal{C})$ *is substandard and* $f:\mathrm{E}\to\mathrm{F}$ *is a bijective Borel mapping, then* f *is a Borel isomorphism.*

Proof. Since $(\mathrm{E},\mathcal{B})$ is the Borel image of a standard space (4.1), so is $(\mathrm{F},\mathcal{C})$; therefore $(\mathrm{F},\mathcal{C})$ is analytic. By 4.5, we can suppose that E and F are Souslin spaces with a countable base for open sets; then f is a Borel isomorphism by 2.2.11. ♢

A way of looking at 4.6: an analytic Borel space $(\mathrm{E},\mathcal{B})$ is 'minimal substandard' in the sense that no strictly coarser Borel structure $\mathcal{C}$ on E can be substandard. A more general result is given in 4.11 below.

4.7. Corollary. *Let* $(\mathrm{E},\mathcal{B})$ *be an analytic Borel space,* $(\mathrm{F},\mathcal{C})$ *a substandard Borel space, and* $f:\mathrm{E}\to\mathrm{F}$ *an injective Borel mapping. The mapping* $f_0:\mathrm{E}\to f(\mathrm{E})$ *having the graph of* f *is an isomorphism of* $(\mathrm{E},\mathcal{B})$ *onto* $\big(f(\mathrm{E}),\mathcal{C}\cap f(\mathrm{E})\big)$.

Proof. Since $\mathcal{C}\cap f(\mathrm{E})$ is the initial Borel structure for the insertion mapping $i:f(\mathrm{E})\to\mathrm{F}$ and since $i\circ f_0 = f$ is Borel, f_0 is Borel by 1.3.2. Also $\big(f(\mathrm{E}),\mathcal{C}\cap f(\mathrm{E})\big)$ is substandard (cf. the proof of 3.2.9), therefore f_0 is a Borel isomorphism by 4.6. ♢

The message of 4.7: $\mathcal{B}$ is the initial Borel structure for the mapping $f:\mathrm{E}\to(\mathrm{F},\mathcal{C})$.

4.8. In 4.7 one can't conclude in general that $f(\mathrm{E})\in\mathcal{C}$ (cf. 3.2.7). For example, let A be a Souslin subspace of $\mathbb{R}$ that is not a Borel set in $\mathbb{R}$ (see 4.19 below) and let $f:\mathrm{A}\to\mathbb{R}$ be the insertion mapping; then f is Borel but $f(\mathrm{A})=\mathrm{A}\notin\mathcal{B}(\mathbb{R})$. The same example shows that the 'analytic' analogue of 3.2.9 fails: $\big(\mathrm{A},\mathcal{B}(\mathrm{A})\big) = \big(\mathrm{A},\mathcal{B}(\mathbb{R})\cap \mathrm{A}\big)$ is a sub-Borel space of $\big(\mathbb{R},\mathcal{B}(\mathbb{R})\big)$, both are analytic (4.5), but $\mathrm{A}\notin\mathcal{B}(\mathbb{R})$.

The following corollaries generalize 3.3.9 and 3.3.10:

4.9. Corollary. *In an analytic Borel space, any separating sequence of Borel sets is generating.*

Proof. Let $(\mathrm{E},\mathcal{B})$ be an analytic Borel space, (B_n) a separating sequence of Borel sets (3.3.1). If $\mathcal{C}$ is the tribe generated by the B_n, then $(\mathrm{E},\mathcal{C})$ is countably generated (3.3.5), therefore substandard (3.3.7); the identity mapping $(\mathrm{E},\mathcal{B})\to(\mathrm{E},\mathcal{C})$ is Borel, hence a Borel isomorphism (4.7), therefore $\mathcal{C}=\mathcal{B}$. ♢

4.10. Corollary. *Let* E *be a set,* (B_n) *a separating sequence of subsets of* E. *There exists at most one analytic Borel structure on* E *for which the* B_n *are Borel sets.*

Proof. This follows from 4.9 in the same way that 3.3.10 follows from 3.3.9: the only candidate is the tribe generated by the B_n. {Such a structure need not exist: review Example 3.3.11 with "standard" replaced by "analytic".} ♢

The next result is a kind of 'open mapping theorem' for Borel spaces; it clearly includes 4.6:

4.11. Theorem. *If* $(\mathrm{E},\mathcal{B})$ *is an analytic Borel space,* $(\mathrm{F},\mathcal{C})$ *is countably separated, and* $f:\mathrm{E}\to\mathrm{F}$ *is a surjective Borel mapping, then* $(\mathrm{F},\mathcal{C})$ *is also analytic; moreover,* $\mathcal{C}$ *coincides with the final Borel structure for the mapping* $f:(\mathrm{E},\mathcal{B})\to\mathrm{F}$, *thus* $(\mathrm{F},\mathcal{C})$ *is isomorphic to the quotient Borel space* $\big(\mathrm{E}/\mathrm{R},\mathcal{B}(\mathrm{E})/\mathrm{R}\big)$, *where* $x\mathrm{R}x'$ *is the equivalence relation in* E *defined by* « $f(x)=f(x')$ » (1.4.5).

Proof. Let (C_n) be a separating sequence in $\mathcal{C}$ and let $\mathcal{C}'$ be the tribe generated by the C_n; thus $(\mathrm{F},\mathcal{C}')$ is countably generated (3.3.5), hence substandard (3.3.7). Since

$$f^{-1}(\mathcal{C}')\subset f^{-1}(\mathcal{C})\subset\mathcal{B},$$

f is Borel for the structures $\mathcal{B}$ and $\mathcal{C}'$; thus $(\mathrm{F},\mathcal{C}')$ is substandard and is the Borel image of an analytic space (hence of a standard space), therefore it is analytic (4.1).

We assert that $\mathcal{C}'=\mathcal{C}$. Let $\mathrm{C}\in\mathcal{C}$ and let $\mathcal{C}''$ be the tribe generated by $\mathcal{C}'$ and C. Since $\mathcal{C}''\subset\mathcal{C}$ and $\mathcal{C}''$ is countably generated, the above argument shows that $(\mathrm{F},\mathcal{C}'')$ is analytic. But then $\mathcal{C}'=\mathcal{C}''$ by 4.10 applied to the sequence (C_n), whence $\mathrm{C}\in\mathcal{C}''=\mathcal{C}'$.

We have thus shown that $(\mathrm{F},\mathcal{C})$ is analytic. Let $\mathcal{D}$ be the final Borel structure for the mapping $f:(\mathrm{E},\mathcal{B})\to\mathrm{F}$, that is (1.4.1),

$$\mathcal{D}=\{\mathrm{T}\subset\mathrm{F}:\ f^{-1}(\mathrm{T})\in\mathcal{B}\};$$

since $f^{-1}(\mathcal{C})\subset\mathcal{B}$ we have $\mathcal{C}\subset\mathcal{D}$. Since $(\mathrm{F},\mathcal{C})$ is countably separated, so is $(\mathrm{F},\mathcal{D})$. Thus $(\mathrm{F},\mathcal{D})$ is countably separated and is the Borel image of an analytic Borel space; therefore $(\mathrm{F},\mathcal{D})$ is analytic (by the same argument as used for $\mathcal{C}$). But $(\mathrm{F},\mathcal{C})$ is also analytic; applying 4.10 to the sequence (C_n), we conclude that $\mathcal{C}=\mathcal{D}$, that is, $\mathcal{C}$ is the final Borel structure for f. It is then immediate from 1.4.5 that $(\mathrm{F},\mathcal{C})$ is isomorphic to the quotient Borel space of $(\mathrm{E},\mathcal{B})$ by the relation « $f(x)=f(x')$ ». ♢

4.12. Corollary. *If a quotient space of an analytic Borel space is countably separated, then it is analytic.*

Proof. Immediate from the first statement of 4.11. ♢

4.13. What does it mean for a quotient Borel space to be countably separated? Let $(\mathrm{E},\mathcal{B})$ be a Borel space, R an equivalence relation in E, $\pi:\mathrm{E}\to\mathrm{E}/\mathrm{R}$ the quotient mapping, and $\mathcal{B}/\mathrm{R}$ the final Borel structure for π (1.4.4). To say that

$(\mathrm{E/R}, \mathcal{B}/\mathrm{R})$ is countably separated means that there exists a sequence (S_n) in the tribe $\mathcal{B}/\mathrm{R}$ for which the sequence of characteristic functions (φ_{S_n}) separates the points of $\mathrm{E/R}$. The $\mathrm{B}_n = \pi^{-1}(\mathrm{S}_n)$ are Borel sets in E saturated for R, and $\varphi_{\mathrm{B}_n} = \varphi_{\mathrm{S}_n} \circ \pi$, thus the φ_{B_n} separate inequivalent points of E. The complement of a saturated Borel set is a saturated Borel set. Thus, E/R is countably separated if and only if there exists a sequence (B_n) of saturated Borel sets in E such that, whenever x and y are inequivalent points of E, there exists an n such that $x \in \mathrm{B}_n$ and $y \in \complement\mathrm{B}_n$; in other words, given any two distinct equivalence classes $[x]$ and $[y]$, there exists an n such that $[x] \subset \mathrm{B}_n$ and $[y] \subset \complement\mathrm{B}_n$.

The following corollary covers an item in [D1, Appendix, B.22]:

4.14. Corollary. *Let* $(\mathrm{E}, \mathcal{B})$ *and* $(\mathrm{F}, \mathcal{C})$ *be Borel spaces,* $f : \mathrm{E} \to \mathrm{F}$ *a bijective Borel mapping. Suppose that* (i) $(\mathrm{E}, \mathcal{B})$ *is a quotient space of an analytic Borel space, and* (ii) $(\mathrm{F}, \mathcal{C})$ *is substandard. Then* $(\mathrm{F}, \mathcal{C})$ *is analytic and* f *is a Borel isomorphism.*

Proof. Since $(\mathrm{F}, \mathcal{C})$ is countably generated (3.3.7), there exists a sequence (C_n) in $\mathcal{C}$ that is both generating and separating (3.3.6). Since f is bijective and Borel, $f^{-1}(\mathrm{C}_n)$ is a separating sequence in $\mathcal{B}$, therefore $(\mathrm{E}, \mathcal{B})$ is analytic by 4.12; it then follows from 4.6 that f is a Borel isomorphism, so $(\mathrm{F}, \mathcal{C})$ is also analytic. ◇

4.15. Corollary. *Let* E *be a Souslin space,* $(\mathrm{F}, \mathcal{C})$ *a countably separated Borel space, and* $f : \mathrm{E} \to \mathrm{F}$ *a surjective Borel mapping. Then* $(\mathrm{F}, \mathcal{C})$ *is analytic and* $\mathcal{C}$ *is the final Borel structure for* $f : \big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big) \to \mathrm{F}$.

Proof. Let $g : \mathrm{P} \to \mathrm{E}$ be a continuous surjection with P Polish. Then $f \circ g : \mathrm{P} \to \mathrm{F}$ is Borel, so by 4.11, $(\mathrm{F}, \mathcal{C})$ is analytic and $\mathcal{C}$ is the final Borel structure for $f \circ g : \big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big) \to \mathrm{F}$; also $\mathcal{B}(\mathrm{E})$ is the final Borel structure for $g : \big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big) \to \mathrm{E}$ by A.2.5, therefore $\mathcal{C}$ is the final Borel structure for f ('transitivity of final structures'). ◇

It follows from 1.4.5 that if R is the equivalence relation « $f(x) = f(x')$ » on E, then $\big(\mathrm{E/R}, \mathcal{B}(\mathrm{E})/\mathrm{R}\big)$ is isomorphic to $(\mathrm{F}, \mathcal{C})$.

4.16. If E and F are Souslin spaces and $f : \mathrm{E} \to \mathrm{F}$ is a continuous surjection, then $\mathcal{B}(\mathrm{F})$ is the final Borel structure for $f : \big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big) \to \mathrm{F}$ (A.2.5). PROBLEM: Can "continuous" be replaced by "Borel"? By 4.15, the answer is yes when $\big(\mathrm{F}, \mathcal{B}(\mathrm{F})\big)$ is countably separated—as is the case when F is metrizable (hence has a countable base for open sets).

For the record (but are they all useful?), here are some variations on the theme of analyticity:

4.17. Theorem. *The following conditions on a Borel space* $(\mathrm{E}, \mathcal{B})$ *are equivalent*:

(a) $(\mathrm{E}, \mathcal{B})$ *is analytic*;
(b) E *is Borel-isomorphic to a metrizable Souslin space*;
(c) $(\mathrm{E}, \mathcal{B})$ *is substandard and* E *is Borel-isomorphic to a Souslin space*;

(d) $(\mathrm{E}, \mathcal{B})$ *is substandard and* E *is the Borel image of a Souslin space*;

(e) $(\mathrm{E}, \mathcal{B})$ *is substandard and* E *is the Borel image of a Polish space*;

(f) $(\mathrm{E}, \mathcal{B})$ *is substandard and* E *is the Borel image of a Lusin space.*

Proof. (a) $\Leftrightarrow$ (e): Noted following 4.1.

(e) $\Rightarrow$ (f) $\Rightarrow$ (d) because Polish $\Rightarrow$ Lusin $\Rightarrow$ Souslin. {Incidentally, (e) $\Leftrightarrow$ (f) by 2.2.1.}

(d) $\Rightarrow$ (a): Let $f : \mathrm{S} \to \mathrm{E}$ be a Borel surjection with S Souslin. There exist a Polish space P and a continuous (hence Borel) surjection $g : \mathrm{P} \to \mathrm{S}$. Then $f \circ g : \mathrm{P} \to \mathrm{E}$ is a Borel surjection; since $(\mathrm{E}, \mathcal{B})$ is by assumption substandard, and $\big(\mathrm{P}, \mathcal{B}(\mathrm{P})\big)$ is standard, it follows that $(\mathrm{E}, \mathcal{B})$ is analytic (4.1).

Thus (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f).

(a) $\Rightarrow$ (b): Immediate from 4.5.

(b) $\Rightarrow$ (c): Suppose E is Borel-isomorphic to the metrizable Souslin space S. Since S has a countable dense subset (it is the continuous image of a Polish space) and is metrizable, it has a countable base for open sets; thus $\big(\mathrm{S}, \mathcal{B}(\mathrm{S})\big)$ is countably generated (3.3.3), hence substandard (3.3.7), therefore the isomorphic space $(\mathrm{E}, \mathcal{B})$ is also substandard.

(c) $\Rightarrow$ (d): Trivial. ♢

4.18. An analytic Borel space need not be standard. For example, let $\mathrm{Q} = \mathbb{N}^{\mathbb{N}}$ be the Polish space considered in §A.5 of the Appendix. There exists a subset T of Q such that T is a Souslin subspace of Q but $\complement\mathrm{T}$ is not [B, p. IX.120, Exer. 8]. It follows that $\complement\mathrm{T}$ is not a Borel set in Q (A.2.1), therefore T is not a Borel set. Thus $\big(\mathrm{T}, \mathcal{B}(\mathrm{T})\big) = \big(\mathrm{T}, \mathcal{B}(\mathrm{Q}) \cap \mathrm{T}\big)$ is analytic (4.17) but it is not standard since $\mathrm{T} \notin \mathcal{B}(\mathrm{Q})$ (3.2.8). Incidentally, T also provides an example of a Souslin space that is not Lusin (A.2.8 or 2.2.1).

4.19. In $\mathbb{R}$ every Borel set is a Souslin subspace (A.2.1), but the converse is false. For example, with notations as in 4.18, by 4.5 there exists a Souslin subspace S of $\mathbb{R}$ such that $\big(\mathrm{T}, \mathcal{B}(\mathrm{T})\big)$ is isomorphic to $\big(\mathrm{S}, \mathcal{B}(\mathrm{S})\big) = \big(\mathrm{S}, \mathcal{B}(\mathbb{R}) \cap \mathrm{S}\big)$. If S were Borel then $\big(\mathrm{S}, \mathcal{B}(\mathrm{S})\big)$ would be standard (3.2.2), therefore $\big(\mathrm{T}, \mathcal{B}(\mathrm{T})\big) = \big(\mathrm{T}, \mathcal{B}(\mathrm{Q}) \cap \mathrm{T}\big)$ would be standard, which would imply that $\mathrm{T} \in \mathcal{B}(\mathrm{Q})$ by 3.2.8, a contradiction.

Theorem 4.17 characterizes analyticity in the context of 'countably generated' (that is, substandard) Borel spaces; here are two characterizations involving the weaker condition 'countably separated' (cf. A.3.3 of the Appendix):

4.20. Theorem. *The following conditions on a Borel space* $(\mathrm{E}, \mathcal{B})$ *are equivalent*:

(a) $(\mathrm{E}, \mathcal{B})$ *is analytic*;

(c′) $(\mathrm{E}, \mathcal{B})$ *is countably separated and* E *is Borel-isomorphic to a Souslin space*;

(g) $(\mathrm{E}, \mathcal{B})$ *is countably separated and is isomorphic to a quotient space of a standard Borel space.*

Proof. With (c) as in the statement of Theorem 4.17, we have (a) ⇒ (c) ⇒ (c′); (c′) ⇒ (g) by 2.2.5; and (g) ⇒ (a) by 4.12. ♢

4.21. Corollary. *For a Souslin space* E, *the following conditions are equivalent*:

(1) $\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)$ *is analytic*;
(2) $\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)$ *is substandard* (*that is*, ‘*countably generated*’);
(3) $\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)$ *is countably separated.*

Proof. (1) ⇒ (2) ⇒ (3): Immediate from the definitions.
(3) ⇒ (1) by 4.20. ♢

APPENDIX

The first two sections of this Appendix are a résumé of results from general topology (flavored Polish) pertinent to Borel spaces. The classical references are the books of Kuratowski, Sierpinski and Hausdorff. Modern expositions may be found in the books of Bourbaki [B], Arveson [A] and Takesaki [T]; it is to these that we shall refer the reader. The results are arranged for convenient reference; it is not proposed that they be *proved* in this order (they are not, in the cited references). Probably the most efficient preparation for the study of Borel spaces is to read Chapter IX, §6 of [B] 'from cover to cover'; the exposition in [A] takes some shortcuts but is rich in applications.

§A.1. Polish spaces, Lusin spaces, Souslin spaces

A.1.1. Definition. A topological space P is said to be *Polish* if it has a countable base for open sets and is metrizable by a complete metric. (In other words, P is homeomorphic to a separable complete metric space.) [B, p. IX.57], [A, p. 61], [T, p. 375]

A.1.2. Definition. A separated topological space E is said to be *Lusin* if there exist a Polish space P and a continuous bijection $\mathrm{P} \to \mathrm{E}$. [B, p. IX.62]

A.1.3. Definition. A separated topological space E is said to be *Souslin* if there exist a Polish space P and a continuous surjection $\mathrm{P} \to \mathrm{E}$. [B, p. IX.59], cf. [A, p. 64]

Obviously, Polish $\Rightarrow$ Lusin $\Rightarrow$ Souslin. Lusin spaces appear to play mainly an auxiliary role in the Borel theory of Polish and Souslin spaces. The key concept is that of Souslin space.

A.1.4. Definition. A topological space is said to be *dispersed* if it is separated and if every point has a fundamental system of 'clopen' (simultaneously closed and open) neighborhoods.

A.1.5. Proposition. *A separated topological space* E *is Lusin if and only if there exist a dispersed Polish space* P *and a continuous bijection* $\mathrm{P} \to \mathrm{E}$. [B, p. IX.62]

A.1.6. Proposition. (i) *The product of a countable family of Polish spaces is Polish.*

(ii) *The direct union* (*or* '*topological sum*') *of a countable family of Polish spaces is Polish.*

(iii) *Every closed subspace and every open subspace of a Polish space is Polish.* [B, p. IX.57], [A, p. 61]

A.1.7. Corollary. *Proposition* A.1.6 *remains true with 'Polish' replaced by 'Lusin' or by 'Souslin'.* [B, pp. IX.59, IX.62]

A.1.8. Corollary. *The projective limit of a projective sequence of Polish spaces is Polish* (*and the same is true with 'Polish' replaced by 'Lusin' or by 'Souslin'*). [B, p. IX.63]

A.1.9. Proposition. *If* E *is a separated space and* (A_n) *is a sequence of Polish subspaces of* E, *then* $\bigcap_{n=1}^{\infty} \mathrm{A}_n$ *is also a Polish subspace of* E. *The same is true with 'Polish' replaced by 'Lusin' or by 'Souslin'.* [B, pp. IX.58, IX.62, IX.60]

A.1.10. Proposition. *If* E *is a separated space and* (A_n) *is a sequence of Souslin subspaces of* E, *then* $\bigcup_{n=1}^{\infty} \mathrm{A}_n$ *is also a Souslin subspace of* E. *The same is true with 'Souslin' replaced by 'Lusin'.* [B, pp. IX.58, IX.68]

A.1.11. However, the union of a sequence of Polish subspaces need not be Polish: for example, $\mathbb{Q}$ is not a Polish subspace of $\mathbb{R}$. {For, as a topological subspace of $\mathbb{R}$, $\mathbb{Q}$ is the union of a sequence of closed sets without interior point (its singletons), therefore $\mathbb{Q}$ is not a Baire space, hence, by the Baire category theorem [B, p. IX.55] it cannot be a Polish space.}

A.1.12. Theorem. *Let* E *be a Polish space,* $\mathrm{A} \subset \mathrm{E}$. *In order that* A *be a Polish subspace of* E, *it is necessary and sufficient that* A *be a* G_δ *in* E (*that is, the intersection of a sequence of open sets in* E). [B, p. IX.58], [A, p. 62]

A.1.13. Proposition. (i) *If* E *and* F *are Polish spaces,* $f : \mathrm{E} \to \mathrm{F}$ *is a continuous mapping, and* T *is a Polish subspace of* F, *then* $f^{-1}(\mathrm{T})$ *is a Polish subspace of* E.

(ii) *If* E *is a Souslin space,* F *is a separated space,* $f : \mathrm{E} \to \mathrm{F}$ *is a continuous mapping, and* T *is a Souslin subspace of* F, *then* $f^{-1}(\mathrm{T})$ *is a Souslin subspace of* E.

(iii) *If* E *is a Lusin space,* F *is a regular space,* $f : \mathrm{E} \to \mathrm{F}$ *is a continuous mapping, and* T *is a Lusin subspace of* F, *then* $f^{-1}(\mathrm{T})$ *is a Lusin subspace of* E. [B, pp. IX.59, IX.66–67]

In (iii), f need only be a 'Borel mapping' in the sense of §1.2 (see A.2.9 below).

Some topological odds and ends.

(1) Every Souslin space has a countable dense subset (it is the continuous image of a separable metric space).

(2) Every Souslin space is a Lindelöf space (that is, every open covering has a countable subcovering). [B, p. IX.76]

(3) Every regular Souslin space is paracompact (hence normal, hence uniformizable). [B, p. IX.76]

(4) Every compact Souslin space is metrizable. [B, p. IX.77]

(5) A topological space is Polish if and only if it is homeomorphic to a G_δ subspace of the cube $[0,1]^{\mathbb{N}}$. [B, p. IX. 58]

(6) For every Souslin space S, there exists a continuous surjection $\mathbb{N}^{\mathbb{N}} \to S$. [B, p. IX.120, Exer. 8]

(7) For every Polish space P, there exists an *open* continuous surjection $\mathbb{N}^{\mathbb{N}} \to P$. [A, p. 63]

§A.2. Borel sets in Polish, Lusin and Souslin spaces

A.2.1. Proposition. *In a Souslin space, every Borel set is a Souslin subspace.* [B, p. IX.61], [A, p. 65]

A.2.2. Corollary. *If* E *is a Souslin space,* F *is a separated space and* $f : \mathrm{E} \to \mathrm{F}$ *is a continuous mapping, then for every Borel set* B *of* E, $f(\mathrm{B})$ *is a Souslin subspace of* F. [B, p. IX.61]

A.2.3. Theorem. *If* E *is a separated space and* (A_n) *is a countable family of pairwise disjoint Souslin subspaces of* E, *then there exist pairwise disjoint Borel sets* B_n *in* E *such that* $\mathrm{A}_n \subset \mathrm{B}_n$ *for all* n. [B, p. IX.65], [A, p. 66]

A.2.4. Corollary. *If a separated space* E *is the union of a countable family* (A_n) *of pairwise disjoint Souslin subspaces, then the* A_n *are Borel sets in* E. *In particular, in a separated space* E, *if* A *is a subset of* E *such that both* A *and* $\complement\mathrm{A}$ *are Souslin subspaces of* E, *then* A *is a Borel set.* [B, p. IX.66], [S, p. 230]

The second assertion of A.2.4 is attributed by Sierpinski to Souslin (for the case that E is a Polish space).

A.2.5. Corollary. *Let* E *and* F *be Souslin spaces,* $f : \mathrm{E} \to \mathrm{F}$ *continuous and surjective. For a subset* B *of* F *to be a Borel set, it is necessary and sufficient that* $f^{-1}(\mathrm{B})$ *be a Borel set in* E. [B, p. IX.66]

Thus, in A.2.5, regarding f as a mapping of the Borel space $\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)$ into the set F, the final Borel structure for f (1.4.1) coincides with the Borel structure $\mathcal{B}(\mathrm{F})$ derived from the topology of F (1.2.2). This yields a sort of 'first isomorphism theorem':

A.2.6. Corollary. *Let* E *be a Souslin space,* F *a separated space,* $f : \mathrm{E} \to \mathrm{F}$ *a continuous mapping. Let* $x\mathrm{R}x'$ *be the equivalence relation in* E *defined by* $f(x) = f(x')$, *and let* $f = i \circ g \circ \pi$ *be the canonical factorization of* f,

$$\begin{array}{ccc} \mathrm{E} & \xrightarrow{\;f\;} & \mathrm{F} \\ {\scriptstyle\pi}\big\downarrow & & \big\uparrow{\scriptstyle i} \\ \mathrm{E}/\mathrm{R} & \xrightarrow[\;g\;]{} & f(\mathrm{E}) \end{array}$$

where π *is the quotient mapping,* i *is the insertion mapping, and* g *is bijective. Then* g *is a Borel isomorphism* (1.2.4)

$$\big(\mathrm{E}/\mathrm{R}, \mathcal{B}(\mathrm{E})/\mathrm{R}\big) \to \big(f(\mathrm{E}), \mathcal{B}(\mathrm{F}) \cap f(\mathrm{E})\big)\,.$$

Sketch of proof: Replacing F by $f(\mathrm{E})$, one can suppose that F is Souslin and f is surjective. Since E/R bears the final Borel structure for π (1.4.4) and F bears the final Borel structure for $f = g \circ \pi$ (A.2.5), it follows that F bears the final Borel structure for g ('transitivity of final structures'; cf. [B, p. I.14]).

A.2.7. Corollary. *If* E *and* F *are Souslin spaces and* $f : \mathrm{E} \to \mathrm{F}$ *is a continuous bijection, then* f *is a Borel isomorphism.*

A.2.8. Theorem. (i) *In a Lusin space, every Borel set is a Lusin subspace.*
(ii) *In a regular space, every Lusin subspace is a Borel set.* [B, pp. IX.66–67]

A.2.9. Corollary. *If* E *is a Lusin space,* F *is a regular space,* $f : \mathrm{E} \to \mathrm{F}$ *is a Borel mapping, and* T *is a Lusin subspace of* F, *then* $f^{-1}(\mathrm{T})$ *is a Lusin subspace of* E.

A.2.10. Proposition. *Let* E *be a Lusin space,* F *a regular Lusin space,* $f : \mathrm{E} \to \mathrm{F}$ *an injective Borel mapping. Then for every Borel set* A *in* E, $f(\mathrm{A})$ *is a Borel set in* F. [B, p. IX.68]

If, moreover, f is bijective, then f is a Borel isomorphism.

A.2.11. Theorem. *Let* P *be a Polish space,* R *an equivalence relation in* P *for which*
(i) *the equivalence classes are closed sets, and*
(ii) *the saturation of every open set is a Borel set.*
Then there exists a Borel set S *in* P *such that* S *contains exactly one element of each equivalence class.* [B, p. IX. 70]

A.2.12. Corollary. *Let* P *be a Polish space,* R *an equivalence relation in* P *for which the saturation of every closed set is a Borel set. Same conclusion as for* A.2.11. [B, p. IX.71]

A.2.13. Exercise. The following conditions on a Polish space P are equivalent: (a) Every Souslin subspace of P is a Borel set; (b) P is countable; (c) every subset of P is a Borel set; (d) for every Borel set B in the product topological space $\mathrm{P} \times \mathrm{P}$, the projection $\mathrm{pr}_1 \mathrm{B}$ is a Borel set in P; (e) for every Borel set B in $\mathrm{P} \times \mathrm{P}$, the 'vertical saturation' $\mathrm{pr}_1{}^{-1}(\mathrm{pr}_1 \mathrm{B})$ is also a Borel set in $\mathrm{P} \times \mathrm{P}$.

{A convenient schema for the proof: (b) ⇒ (c) ⇒ (a) ⇒ (e) ⇒ (d) ⇒ (b). For (b) ⇒ (c), see 3.3.14. For (a) ⇒ (e) note, via A.2.1, that $\mathrm{pr}_1 \mathrm{B}$ is Souslin. For (e) ⇒ (d), use A.2.5. For (d) ⇒ (b), see 3.1.1 and 4.19.}

§A.3. Concordance with other expositions

A.3.1. In [A, p. 64], a subset A of a Polish space P is said to be an *analytic subset* of P if there exist a Polish space Q and a continuous mapping $f : \mathrm{Q} \to \mathrm{P}$ with $f(\mathrm{Q}) = \mathrm{A}$.

It is the same to say that there exist a Polish space Q and a continuous surjection $\mathrm{Q} \to \mathrm{A}$ (because A has the initial topology for the insertion mapping $\mathrm{A} \to \mathrm{P}$), in other words, that A is a Souslin subspace of P. In view of 4.4, A is analytic in the sense of [A] if and only if it is analytic in the sense of 4.2.

The general topological results in [A] pertaining to analytic sets can now be related easily to the corresponding results in [B] as sketched in §A.2.

The following elementary proposition gives some insight into why conditions like (b) keep popping up in hypotheses (cf. 2.2.9, 4.17):

A.3.2. Proposition. *The following conditions on a topological space* E *are equivalent:*

(a) E *is homeomorphic to an analytic subset of a Polish space;*

(b) E *is a metrizable Souslin space.*

Proof. (a) ⇒ (b): Obvious.

(b) ⇒ (a): We can suppose that E is a metric space. Let P be the metric space completion of E and regard E as a subspace of P. Since E has a countable dense subset, P is a Polish space; thus E is a Souslin subspace of a Polish space. ◊

A.3.3. In [T, p. 381], a Borel space $(\mathrm{F}, \mathcal{C})$ is called a *Borel-Souslin space* if (1) it is countably separated, and (2) there exist a standard Borel space $(\mathrm{E}, \mathcal{B})$ and a Borel surjection $f : \mathrm{E} \to \mathrm{F}$. Such a Borel space is analytic (4.11) and, conversely, an analytic Borel space has the properties (1) and (2) (cf. 4.1, 3.3.7, 3.3.3). Thus the terms 'Borel-Souslin' and 'analytic' mean the same thing.

§A.4. A topological representation of the Cantor set

A.4.1. Definition. We denote by $\mathbb{M}$ the product topological space $\prod_{n=1}^{\infty} \mathrm{E}_n$, where $\mathrm{E}_n = \{0,1\}$ is the 2-point discrete space, for all n. (The notation $\mathbb{M} = 2^{\aleph_0}$ is also used.)

Theorem. *The Cantor set* Γ *is homeomorphic to* $\mathbb{M}$.

Before undertaking the proof, let us review the construction of Cantor's 'middle third' set Γ. If $\mathrm{A} = [a,b]$ is a nondegenerate closed interval, let us write $r(\mathrm{A})$ ('the rest of A') for what is left of A after deleting the 'open middle third':

$$r(\mathrm{A}) = [a, a + (b-a)/3] \cup [b - (b-a)/3, b] .$$

More generally, if A is a finite disjoint union of nondegenerate closed intervals $\mathrm{A}_1, \dots, \mathrm{A}_n$, we define

$$r(\mathrm{A}) = r(\mathrm{A}_1) \cup \dots \cup r(\mathrm{A}_n) .$$

The operation r can then be iterated: we define $r^n(\mathrm{A})$ $(n = 1,2,3,\dots)$ recursively by the formulas

$$r^1(\mathrm{A}) = r(\mathrm{A}) , \quad r^{n+1}(\mathrm{A}) = r\big(r^n(\mathrm{A})\big) .$$

The following properties are easily verified:

(1) $\mathrm{A} \supset r(\mathrm{A}) \supset r^2(\mathrm{A}) \supset \dots$.

(2) $\lambda\big(r^n(\mathrm{A})\big) = (2/3)^n \cdot \lambda(\mathrm{A})$, where λ is Lebesgue measure.

(3) $r^n(\mathrm{A})$ is a closed set (the union of finitely many closed intervals).

(4) The set $r^\infty(\mathrm{A}) = \bigcap_{n=1}^{\infty} r^n(\mathrm{A})$ is closed and negligible: $\lambda\big(r^\infty(\mathrm{A})\big) = 0$.

A.4.2. Definition. $\Gamma = r^\infty([0,1])$, regarded as a compact subspace of $\mathbb{R}$.

Write $\mathrm{I} = [0,1]$. We have

$$r(\mathrm{I}) = [0,1/3] \cup [2/3,1] = \mathrm{I}_0 \cup \mathrm{I}_1 ,$$

where I_0 is the 'left third' of I, and I_1 is the 'right third' of I. In turn,

$$r^2(\mathrm{I}) = \mathrm{I}_{00} \cup \mathrm{I}_{01} \cup \mathrm{I}_{10} \cup \mathrm{I}_{11} ,$$

where, for example, I_{10} is the left third of I_1. For every n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_1, \dots, \alpha_n \in \{0,1\}$, we recursively define

$$\mathrm{I}_\alpha = \mathrm{I}_{\alpha_1 \alpha_2 \dots \alpha_n}$$

to be the left third of $\mathrm{I}_{\alpha_1 \alpha_2 \dots \alpha_{n-1}}$ if $\alpha_n = 0$, and the right third if $\alpha_n = 1$.

Let us call $\alpha = (\alpha_1, \dots, \alpha_n)$ an *index of rank* n and write $|\alpha| = n$. For indices α, β, let us write $\alpha \le \beta$ in case $|\alpha| \le |\beta|$ and $\alpha_i = \beta_i$ for $i = 1,2,\dots,|\alpha|$. The following properties are easily verified:

(5) If $|\alpha| = n$ then I_α is one of the 2^n intervals that make up $r^n(\mathrm{I})$, and $\lambda(\mathrm{I}_\alpha) = (1/3)^n$.

(6) If $\alpha \le \beta$ then $\mathrm{I}_\alpha \supset \mathrm{I}_\beta$.

(7) If $\alpha^1, \alpha^2, \alpha^3, \ldots$ is a sequence of indices with $\alpha^1 \le \alpha^2 \le \alpha^3 \le \ldots$ and if $|\alpha^n| = n$ for all n, then

$$\bigcap_{n=1}^{\infty} \mathrm{I}_{\alpha^n} = \{y\}$$

for some $y \in \Gamma$. {Sketch of proof: The intersection is a singleton $\{y\}$ by the 'theorem on nested intervals', and $\mathrm{I}_{\alpha^n} \subset r^n(\mathrm{I})$ for all n, therefore $y \in r^\infty(\mathrm{I}) = \Gamma$.}

(8) If $|\alpha| = |\beta|$ and $\alpha \ne \beta$ then $\mathrm{I}_\alpha \cap \mathrm{I}_\beta = \varnothing$. {Consider the first coordinate in which α and β differ; clearly I_α and I_β are contained in disjoint 'thirds' of some interval.}

We are now ready to define the promised homeomorphism

$$f : \mathbb{M} \to \Gamma .$$

Let $x \in \mathbb{M}$, say $x = (x_1, x_2, x_3, \ldots)$. Write

$$\alpha^n(x) = (x_1, \ldots, x_n)$$

for every positive integer n. Then

$$\alpha^1(x) \le \alpha^2(x) \le \alpha^3(x) \le \ldots$$

and $|\alpha^n(x)| = n$ for all n. Let y be the unique point of Γ such that

$$\bigcap_{n=1}^{\infty} \mathrm{I}_{\alpha^n(x)} = \{y\}$$

(see (7) above) and define $f(x) = y$. The rest of the proof continues the above series of remarks.

(9) f is injective. {If $x \ne x'$ then $\alpha^n(x) \ne \alpha^n(x')$ for some n, and $f(x) \ne f(x')$ follows at once from (8).}

(10) f is surjective. {Let $y \in \Gamma$. Construct $x = (x_1, x_2, x_3, \ldots) \in \mathbb{M}$ as follows: $y \in r(\mathrm{I}) = \mathrm{I}_0 \cup \mathrm{I}_1$, say $y \in \mathrm{I}_{x_1}$; since $y \in r^2(\mathrm{I})$, we have $y \in \mathrm{I}_{x_1 0}$ or $y \in \mathrm{I}_{x_1 1}$, say $y \in \mathrm{I}_{x_1 x_2}$; etc. Briefly, one pursues y through the sequence of (left or right) 'thirds' to which it belongs. Clearly $f(x) = y$.}

(11) f is a homeomorphism. {Since $\mathbb{M}$ is compact (Tychonoff's theorem) and Γ is separated, it remains only to show that f is continuous. If $x = (x_1, x_2, x_3, \ldots)$ is a point of $\mathbb{M}$, then a fundamental sequence of neighborhoods of x is given by

$$\mathrm{V}_n = \{x_1\} \times \{x_2\} \times \ldots \times \{x_n\} \times \{0,1\} \times \{0,1\} \times \ldots .$$

If $y \in \mathrm{V}_n$, say

$$y = (x_1, \ldots, x_n, y_{n+1}, y_{n+2}, y_{n+3}, \ldots),$$

then $f(x)$ and $f(y)$ both belong to $\mathrm{I}_{\alpha^n(x)}$, therefore

$$|f(x) - f(y)| \le \operatorname{diam} \mathrm{I}_{\alpha^n(x)} = (1/3)^n .$$

Given $\epsilon > 0$, choose n so that $(1/3)^n < \epsilon$. Then $|f(x) - f(y)| < \epsilon$ for all $y \in \mathrm{V}_n$.}

§A.5. Complements on dispersed Polish spaces

Recall that a topological space is said to be *dispersed* if it is separated and if every point has a neighborhood base consisting of clopen sets (A.1.4). Examples are the Cantor set $\Gamma = 2^{\aleph_0}$ (§A.4) and the product $\mathbb{N}^{\mathbb{N}}$ of denumerably many copies of the discrete space $\mathbb{N} = \{0, 1, 2, \ldots\}$. Our aim is to prove the following theorem:

Theorem 1. *If* P *is a nonempty dispersed Polish space whose only compact open subset is* $\varnothing$, *then* P *is homeomorphic to* $\mathbb{N}^{\mathbb{N}}$.

Such spaces P are (is!) at the opposite pole from dispersed compact metric spaces (such as the Cantor set Γ), in which every point has a fundamental sequence of compact open neighborhoods.

We prepare the way with two lemmas.

Lemma 1. *If* (E, d) *is a dispersed separable metric space, then for every* $\epsilon > 0$ *there exists a sequence (finite or infinite) of pairwise disjoint, nonempty clopen sets* P_n *such that* $\mathrm{E} = \bigcup \mathrm{P}_n$ *and* $\operatorname{diam} \mathrm{P}_n < \epsilon$ *for all* n.

Proof. For every $x \in \mathrm{E}$ the open ball with center x and radius $\epsilon/2$ contains a clopen neighborhood P_x of x. The family (P_x) is an open covering of E, and E has a countable base for open sets, so by Lindelöf's theorem [B, p. IX.75] there is a countable subcovering; 'disjointify', and discard any empty terms. ♢

Lemma 2. *Let* P *be a noncompact, dispersed Polish space and let* d *be a compatible complete metric on* E. *Given any* $\eta > 0$, *there exists an infinite sequence* (P_n) *of pairwise disjoint, nonempty clopen sets such that* $\mathrm{P} = \bigcup \mathrm{P}_n$ *and* $\operatorname{diam} \mathrm{P}_n < \eta$ *for all* n.

Proof. Since (P, d) is complete and noncompact, it cannot be totally bounded (i.e., 'precompact'). This means [B, p. II.29] that for some $\epsilon > 0$, P is *not* the union of finitely many sets of diameter $< \epsilon$. We can suppose that $\epsilon < \eta$. For such an ϵ, let (P_n) be as given by Lemma 1; since $\mathrm{P} = \bigcup \mathrm{P}_n$ and $\operatorname{diam} \mathrm{P}_n < \epsilon$, the number of terms of the union must be infinite. ♢

Proof of Theorem 1. Let $\mathrm{P} \neq \varnothing$ be a dispersed Polish space whose only compact open subset is $\varnothing$. Note that every nonempty clopen subset of P, equipped with the relative topology, also satisfies the hypotheses on P.

Let d be a compatible complete metric on P. By Lemma 2, P is the union of an infinite sequence of pairwise disjoint, nonempty clopen sets,

$$\mathrm{P} = \bigcup_{n=0}^{\infty} \mathrm{P}_n \,,$$

such that $\operatorname{diam} \mathrm{P}_n < 1$ for all n. For each n, apply Lemma 2 to P_n: there is a denumerable partition of P_n into nonempty clopen (in P_n or in P—it comes to the same thing) sets,

$$\mathrm{P}_n = \bigcup_{m=0}^{\infty} \mathrm{P}_{nm} \,,$$

such that $\operatorname{diam} \mathrm{P}_{nm} < 1/2$ for all m. Continuing recursively one obtains, for each finite sequence $(n_0, n_1, \ldots, n_r)$ of nonnegative integers, a nonempty clopen set $\mathrm{P}_{n_0 n_1 \ldots n_r}$, of diameter $< 1/(r+1)$, in such a way that

$$\mathrm{P}_{n_0 n_1 \ldots n_{r-1}} = \bigcup_{n_r=0}^{\infty} \mathrm{P}_{n_0 n_1 \ldots n_r},$$

the terms of the union on the right side being pairwise disjoint.

We define a mapping $f : \mathrm{P} \to \mathbb{N}^{\mathbb{N}}$ as follows. Let $x \in \mathrm{P}$. There exists a unique integer $n_0(x)$ such that $x \in \mathrm{P}_{n_0(x)}$, then a unique integer $n_1(x)$ such that $x \in \mathrm{P}_{n_0(x) n_1(x)}$, and so on. In this way one constructs a well-defined sequence of integers $n_0(x), n_1(x), n_2(x), \ldots$ such that

$$x \in \mathrm{P}_{n_0(x) n_1(x) \ldots n_r(x)} \quad \text{for all } r \in \mathbb{N}.$$

Define $f(x) = \bigl(n_0(x), n_1(x), n_2(x), \ldots\bigr)$. We now prove that f is a homeomorphism of P onto $\mathbb{N}^{\mathbb{N}}$. For brevity, let us write $\mathrm{Q} = \mathbb{N}^{\mathbb{N}}$.

f is injective: Suppose $f(x) = f(y)$. Write $n_k = n_k(x) = n_k(y)$ for $k \in \mathbb{N}$. Then

$$x, y \in \mathrm{P}_{n_0 n_1 \ldots n_r}$$

for all r; since $\operatorname{diam} \mathrm{P}_{n_0 n_1 \ldots n_r} \to 0$ as $r \to \infty$, we conclude that $x = y$.

f is surjective: Let $(n_0, n_1, n_2, \ldots) \in \mathrm{Q}$. Then

$$\mathrm{P}_{n_0} \supset \mathrm{P}_{n_0 n_1} \supset \mathrm{P}_{n_0 n_1 n_2} \supset \ldots,$$

a decreasing sequence of nonempty closed sets in the complete metric space P, with diameters tending to 0; if x is the unique point of intersection, clearly $f(x) = (n_0, n_1, n_2, \ldots)$.

From the definition of f, it is clear that

$$(*) \qquad f\bigl(\mathrm{P}_{n_0 n_1 \ldots n_r}\bigr) = \{n_0\} \times \{n_1\} \times \ldots \{n_r\} \times \prod_{i>r} \mathbb{N}$$

for every finite sequence $(n_0, n_1, \ldots, n_r)$ of nonnegative integers. The sets on the right side of $(*)$ evidently form a base for the topology of Q. Thus, to show that f is a homeomorphism, it will suffice to show that the sets $\mathrm{P}_{n_0 n_1 \ldots n_r}$ form a base for the topology of P.

Suppose U is an open set in P, and $x \in \mathrm{U}$. Then

$$\bigcap_{r=0}^{\infty} \mathrm{P}_{n_0(x) n_1(x) \ldots n_r(x)} = \{x\} \subset \mathrm{U}$$

thus

$$\bigcap_{r=0}^{\infty} (\complement \mathrm{U}) \cap \mathrm{P}_{n_0(x) n_1(x) \ldots n_r(x)} = \varnothing.$$

The factors of the latter intersection form a decreasing sequence of closed sets with diameter tending to 0; since P is complete, some factor must be empty, say

$$\complement \mathrm{U} \cap \mathrm{P}_{n_0(x)n_1(x)\ldots n_r(x)} = \varnothing ,$$

whence $x \in \mathrm{P}_{n_0(x)n_1(x)\ldots n_r(x)} \subset \mathrm{U}$. ◊

The following theorem is crucial for the Borel–classification of Polish spaces (§3.1):

Theorem 2. *If* P *is an uncountable dispersed Polish space*, *then* $\mathbb{N}^{\mathbb{N}}$ *is homeomorphic to a subspace of* P *whose complement is countable.* [K, p. 350], [B, p. IX.120, Exer. 9c]

The proof employs the concept of 'condensation point': if E is a topological space and A is a subset of E, a point $x \in \mathrm{E}$ is said to be a *condensation point of* A if every neighborhood of x has uncountable intersection with A. If E is a Lindelöf space and A is an uncountable subset of E, then E contains at least one condensation point of A. {The alternative is that every $x \in \mathrm{E}$ has an open neighborhood U_x whose intersection with A is countable; but then the existence of a countable subcovering of the open covering $(\mathrm{U}_x)_{x\in\mathrm{E}}$ would entail the countability of A, a contradiction.}

A subset A of a topological space E is said to be *dense-in-itself* if, for every $x \in \mathrm{A}$, x is adherent to $\mathrm{A} - \{x\}$; a subset of E is said to be *perfect* if it is closed and dense-in-itself.

We begin with a lemma [B, p. I.108, Exer. 17] whose classical prototype is known as the Cantor–Bendixson theorem:

Lemma. *Let* E *be a topological space in which* (i) *every singleton* $\{x\}$ *is a closed set*, *and* (ii) *every open set is a Lindelöf subspace. If* A *is any subset of* E *and* C *is the set of all condensation points of* A, *then* C *is perfect and* $\mathrm{A} \cap \complement \mathrm{C}$ *is countable.*

Proof. If A is countable then $\mathrm{C} = \varnothing$ and there is nothing to prove.

Assuming A uncountable, C is nonempty by the above remarks.

C is closed: If x is adherent to C and U is an open neighborhood of x, then $\mathrm{U} \cap \mathrm{C} \neq \varnothing$; if $y \in \mathrm{U} \cap \mathrm{C}$ then U is a neighborhood of the condensation point y of A, therefore $\mathrm{U} \cap \mathrm{A}$ is uncountable. Thus $x \in \mathrm{C}$.

C is dense-in-itself: Let $x \in \mathrm{C}$ and let U be an open neighborhood of x. Then $\mathrm{U} - \{x\} = \mathrm{U} \cap \complement\{x\}$ is open by (i), and a Lindelöf subspace of E by (ii), and $\mathrm{A} \cap (\mathrm{U} - \{x\})$ is uncountable (because $x \in \mathrm{C}$), so by the earlier remarks there exists $y \in \mathrm{U} - \{x\}$ such that y is a condensation point of $\mathrm{A} \cap (\mathrm{U} - \{x\})$ in the space $\mathrm{U} - \{x\}$. If V is any neighborhood of y in E, then $\mathrm{V} \cap (\mathrm{U} - \{x\})$ is a neighborhood of y in $\mathrm{U} - \{x\}$, therefore the set

$$[\mathrm{V} \cap (\mathrm{U} - \{x\})] \cap [\mathrm{A} \cap (\mathrm{U} - \{x\})]$$

is uncountable, whence $\mathrm{V} \cap \mathrm{A}$ is uncountable; thus $y \in \mathrm{C}$. Also $y \in \mathrm{U} - \{x\}$, so $y \in \mathrm{U} \cap (\mathrm{C} - \{x\})$, and we have shown that every open neighborhood of x intersects $\mathrm{C} - \{x\}$, that is, C is dense-in-itself.

$A \cap \complement C$ is countable: Let $D = A \cap \complement C$ and assume to the contrary. Then D is an uncountable subset of the Lindelöf subspace $\complement C$ of E (assumption (ii)), so by the earlier remarks D has a condensation point z in the space $\complement C$. If V is a neighborhood of z in E, then $V \cap \complement C$ is a neighborhood of z in $\complement C$, therefore $V \cap \complement C \cap D$ is uncountable, and

$$V \cap A \supset V \cap D \supset V \cap \complement C \cap D$$

shows that $V \cap A$ is uncountable; thus z is a condensation point of A in E, that is, $z \in C$, a contradiction. ◇

Proof of Theorem 2. Write $Q = \mathbb{N}^{\mathbb{N}}$ and let P be an uncountable dispersed Polish space; we seek a subset A of P such that $\complement A$ is countable and A, equipped with the relative topology, is homeomorphic to Q. Let C be the set of all condensation points of P, that is, the set of all $x \in P$ such that every neighborhood of x is uncountable. By the Lemma, C is closed and its complement is countable. Equipped with the relative topology, C is Polish (A.1.6), dispersed and uncountable; discarding its complement and dropping down to C, *we can suppose that every point of* P *is a condensation point of* P.

Let D be a countable dense subset of P and let A be its complement. Since D is the union of a sequence of (one-point) closed sets, A is the intersection of a sequence of open sets, therefore A is a Polish subspace of P (A.1.6 and A.1.9), also dispersed and uncountable. The proof will be completed by showing that A is homeomorphic to Q; by Theorem 1, it will suffice to show that $\varnothing$ is the only compact open subset of the topological space A.

Since $\complement A = D$ is dense, A has empty interior in P. Note that A is also dense in P (since all points of P are condensation points, every nonempty open set in P is uncountable hence must intersect A).

Suppose $K \subset A$ is compact and open for the relative topology on A. Then K is a closed set in P, and $K = U \cap A$ for some open set U in P, whence

$$A \supset K = \overline{K} = \overline{U \cap A} \supset U \cap \overline{A} = U \cap P = U$$

(the bar means closure in P); since A has empty interior in P, we conclude that $U = \varnothing$, therefore $K = \varnothing$. ◇

Corollary. *An uncountable Polish space has the cardinality of the continuum.*

What makes the corollary interesting is that the continuum hypothesis is not invoked.

§A.6. Borel sets need not multiply

We sketch here an example given by R.A. Johnson [*Amer. Math. Monthly* **77** (1970), pp. 172-176].

Proposition. *Let* E *be a nonmetrizable compact space in which every closed set is a* G_δ . *Then* $\mathcal{B}(E \times E) \neq \mathcal{B}(E) \times \mathcal{B}(E)$.

Proof. Arguing contrapositively, suppose E is a compact space in which every closed set is a G_δ and such that $\mathcal{B}(E \times E) = \mathcal{B}(E) \times \mathcal{B}(E)$; let us show that E is metrizable.

Since E is separated, the diagonal Δ is a closed subset of $E \times E$. By [H1, p. 222, Th. 51.E], $\mathcal{B}(E) \times \mathcal{B}(E)$ is the tribe generated by the compact G_δ 's in $E \times E$, and since $\Delta \in \mathcal{B}(E \times E) = \mathcal{B}(E) \times \mathcal{B}(E)$ it follows that Δ is a G_δ [H1, p. 221, Th. 51D]. Say $\Delta = \bigcap U_n$, where (U_n) is a sequence of open sets in $E \times E$. Since $\Delta \subset U_n$ and Δ is compact, there exists a closed neighborhood C_n of Δ with $C_n \subset U_n$; replacing C_n by $C_1 \cap \ldots \cap C_n$, we can suppose that $C_n \supset C_{n+1}$ for all n . It follows that (C_n) is a base for the neighborhoods of Δ ; for, if V is an open neighborhood of Δ , then $\bigcap C_n = \Delta \subset V$ shows that $E \times E$ is covered by V and the $\complement C_n$, and a finite subcovering yields $E \times E = V \cup \complement C_n$ for some n , whence $C_n \subset V$.

We thus have a countable base for the neighborhoods of Δ , in other words (compactness) for the filter of entourages for the uniformity of E [B, p. II.27, Th. 1], and this implies that E is metrizable [B, p. IX.15, Th. 1]. $\diamondsuit$

Johnson's example of a compact space E satisfying the conditions of the theorem is the following: E is the interval $[-1, 1)$ of $\mathbb{R}$, equipped with the topology for which the sets

$$[-b, -a) \cup [a, b)$$

$(0 \leq a < b \leq 1)$ form a base for the open sets.

§A.7. Borel structure and measures

A.7.1. The following terminology is taken from the book of Halmos [H1]. If $(\mathrm{E}, \mathcal{B})$ is a Borel space, a *measure* μ on $\mathcal{B}$ is a function $\mu : \mathcal{B} \to [0, +\infty]$ such that (i) $\mu(\varnothing) = 0$, and (ii) for every sequence (B_n) of pairwise disjoint sets in $\mathcal{B}$, $\mu(\bigcup \mathrm{B}_n) = \sum \mu(\mathrm{B}_n)$. The *outer measure* μ^* derived from μ is the function $\mu^* : \mathcal{P}(\mathrm{E}) \to [0, +\infty]$ defined by

$$\mu^*(\mathrm{S}) = \inf\{\mu(\mathrm{B}) :\ \mathrm{S} \subset \mathrm{B} \in \mathcal{B}\}\,.$$

A subset A of E is said to be *μ^*-measurable* if it splits every subset S of E additively, in the sense that

$$\mu^*(\mathrm{S}) = \mu^*(\mathrm{S} \cap \mathrm{A}) + \mu^*(\mathrm{S} \cap \complement\mathrm{A})\,.$$

The set of all μ^*-measurable subsets of E is a tribe $\mathcal{M}$ containing $\mathcal{B}$, and the restriction of μ^* to $\mathcal{M}$ is a measure that extends μ. One calls the measure μ *finite* if $\mu(\mathrm{E}) < +\infty$, and *$\sigma$-finite* if E is the union of a sequence (B_n) of sets in $\mathcal{B}$ such that $\mu(\mathrm{B}_n) < +\infty$ for all n. When μ is σ-finite, $\mathcal{M}$ may be described as the class of all sets $\mathrm{B} \cup \mathrm{S}$, where $\mathrm{B} \in \mathcal{B}$ and $\mathrm{S} \subset \mathrm{N} \in \mathcal{B}$ with $\mu(\mathrm{N}) = 0$ [H1, p. 56, Th. 13.C]; in this case the restriction of μ^* to $\mathcal{M}$ is called the *completion* of μ.

A.7.2. Theorem. [cf. A, p. 67] *If* E *is a metrizable topological space and* μ *is a σ-finite measure on the tribe* $\mathcal{B}(\mathrm{E})$ *of Borel sets of* E, *then for every Souslin subspace* A *of* E, *there exist Borel sets* B *and* N *in* E *such that*

$$\mathrm{A} \subset \mathrm{B}\,, \quad \mathrm{B} - \mathrm{A} \subset \mathrm{N} \quad \textit{and} \quad \mu(\mathrm{N}) = 0\,.$$

In particular, A *is μ^*-measurable.*

Proof. Let (B_n) be a sequence of Borel sets such that $\mathrm{E} = \bigcup \mathrm{B}_n$ and $\mu(\mathrm{B}_n) < +\infty$ for all n. We can suppose that the B_n are pairwise disjoint. Let $\alpha_n = \mu(\mathrm{B}_n)$. Define

$$\beta_n = \begin{cases} 0 & \text{if } \alpha_n = 0 \\ 2^{-n}/\alpha_n & \text{if } \alpha_n > 0 \end{cases}$$

and let μ' be the function on $\mathcal{B}(\mathrm{E})$ defined by

$$\mu'(\mathrm{B}) = \sum_{n=1}^{\infty} \beta_n\, \mu(\mathrm{B}_n \cap \mathrm{B})\,.$$

Since $0 \le \beta_n \cdot \mu(\mathrm{B}_n \cap \mathrm{B}) \le \beta_n \cdot \alpha_n \le 2^{-n}$ for all n, it is clear that μ' is a finite measure and that, for $\mathrm{C} \in \mathcal{B}$, $\mu'(\mathrm{C}) = 0 \Leftrightarrow \mu(\mathrm{C}) = 0$. Changing notation, we can suppose that μ is a finite measure.

Let $f = \mu^*$ be the outer measure derived from μ. The following properties hold:

(1) $\mathrm{S} \subset \mathrm{T} \subset \mathrm{E} \;\Rightarrow\; f(\mathrm{S}) \le f(\mathrm{T})$.

(2) $\mathrm{S}_n \uparrow \mathrm{S} \;\Rightarrow\; f(\mathrm{S}_n) \uparrow f(\mathrm{S})$.

(3) If S is a closed subset of E then

$$f(\mathrm{S}) = \inf\{f(\mathrm{U}) : \ \mathrm{S} \subset \mathrm{U}, \ \ \mathrm{U} \text{ open in } \mathrm{E}\}.$$

Property (1) is obvious, and (2) is not difficult [H1, p. 53, Exer. 5]. To prove (3), note that a closed set S in a metrizable space is a G_δ, thus there exists a sequence (U_n) of open sets such that $\mathrm{U}_n \downarrow \mathrm{S}$, and $\mu(\mathrm{U}_n) \downarrow \mu(\mathrm{S})$ results from the finiteness of μ.

It follows from (1)–(3) that f is a 'continuous capacity' on E in the sense of [B, p. IX.72, Def. 9] (the cited definition requires only that (3) hold for compact subsets S). Thus, if A is any Souslin subspace of E, then A is 'capacitable' [B, p. IX.73, Th. 6]:

$$f(\mathrm{A}) = \sup\{f(\mathrm{K}) : \ \mathrm{K} \subset \mathrm{A}, \ \mathrm{K} \text{ compact}\},$$

that is,

$$\mu^*(\mathrm{A}) = \sup\{\mu(\mathrm{K}) : \ \mathrm{K} \subset \mathrm{A}, \ \mathrm{K} \text{ compact}\}.$$

Let $\alpha = f(\mathrm{A})$ and let (K_n) be a sequence of compact subsets of A such that $\alpha = \sup \mu(\mathrm{K}_n)$. Then $\mathrm{C} = \bigcup \mathrm{K}_n$ is a Borel set such that $\mathrm{K}_n \subset \mathrm{C} \subset \mathrm{A}$ for all n, therefore $f(\mathrm{C}) = \alpha$. On the other hand, by the definition of f, there exists a sequence (B_n) of Borel sets with $\mathrm{B}_n \supset \mathrm{A}$ such that $f(\mathrm{B}_n) \downarrow f(\mathrm{A})$, thus $\mathrm{B} = \bigcap \mathrm{B}_n$ is a Borel set such that $f(\mathrm{B}) = f(\mathrm{A})$. Thus

$$\mathrm{C} \subset \mathrm{A} \subset \mathrm{B} \quad \text{and} \quad f(\mathrm{C}) = f(\mathrm{A}) = f(\mathrm{B}).$$

Then $\mathrm{N} = \mathrm{B} - \mathrm{C}$ is a Borel set, and

$$\mu(\mathrm{N}) = \mu(\mathrm{B}) - \mu(\mathrm{C}) = f(\mathrm{B}) - f(\mathrm{C}) = 0,$$

thus the pair B, N meets the requirements of the theorem. ♢

A.7.3. Definition. Let $(\mathrm{E}, \mathcal{B})$ be a Borel space. A measure μ on $\mathcal{B}$ is said to be *standard* if (i) μ is σ-finite, and (ii) there exists $\mathrm{B} \in \mathcal{B}$ such that $\mu(\mathrm{E} - \mathrm{B}) = 0$ and $(\mathrm{B}, \mathcal{B} \cap \mathrm{B})$ is standard. {So to speak, $(\mathrm{E}, \mathcal{B})$ is 'almost standard' relative to μ.}

A.7.4. Theorem. [M, p. 142, Th. 6.1] *Every σ-finite measure on an analytic Borel space is standard.*

Proof. Let $(\mathrm{E}, \mathcal{B})$ be an analytic Borel space, μ a σ-finite measure on $\mathcal{B}$. By the argument in the proof of A.7.2, we can suppose that μ is finite. We seek a set $\mathrm{B} \in \mathcal{B}$ such that $\mu(\mathrm{E} - \mathrm{B}) = 0$ and $(\mathrm{B}, \mathcal{B} \cap \mathrm{B})$ is standard.

By 4.1, we can suppose that E is a topological subspace of a Polish space P and that $\mathcal{B} = \mathcal{B}(\mathrm{E}) = \mathcal{B}(\mathrm{P}) \cap \mathrm{E}$. Since E is an analytic subset of P (4.2), it follows that E is a Souslin subspace of P (4.4).

Let ν be the finite measure on $\mathcal{B}(\mathrm{P})$ defined by the formula $\nu(\mathrm{C}) = \mu(\mathrm{C} \cap \mathrm{E})$ (permissible since $\mathcal{B}(\mathrm{P}) \cap \mathrm{E} = \mathcal{B}$). By A.7.2, there exist Borel sets C and N in P such that

$$\mathrm{E} \subset \mathrm{C}, \quad \mathrm{C} - \mathrm{E} \subset \mathrm{N} \text{ and } \nu(\mathrm{N}) = 0 .$$

Let $\mathrm{B} = \mathrm{C} - \mathrm{N} = \mathrm{C} \cap \complement \mathrm{N}$; then B is a Borel set in P and $\mathrm{B} \subset \mathrm{E}$, thus $\mathrm{B} = \mathrm{B} \cap \mathrm{E} \in \mathcal{B}(\mathrm{P}) \cap \mathrm{E} = \mathcal{B}$. Then $\mathrm{E} - \mathrm{B} \in \mathcal{B}$ and $\mathrm{E} - \mathrm{B} \subset \mathrm{N}$, therefore

$$\mu(\mathrm{E} - \mathrm{B}) = \mu\big((\mathrm{E} - \mathrm{B}) \cap \mathrm{E}\big) = \nu(\mathrm{E} - \mathrm{B}) \le \nu(\mathrm{N}) = 0 .$$

Finally, $\mathcal{B} \cap \mathrm{B} = \big(\mathcal{B}(\mathrm{P}) \cap \mathrm{E}\big) \cap \mathrm{B} = \mathcal{B}(\mathrm{P}) \cap \mathrm{B}$, and $\big(\mathrm{B}, \mathcal{B}(\mathrm{P}) \cap \mathrm{B})\big)$ is standard by 3.2.2. ♢

REFERENCES

[A] Arveson, W., *An invitation to* C*-*algebras*, Springer–Verlag, New York, 1976.

[B] Bourbaki, N., *Topologie générale*, 2 vols., Masson, Paris, 1971, 1974.

[D1] Dixmier, J., C*-*algebras*, translated by F. Jellett from the 2nd French edition, North–Holland, Amsterdam, 1977.

[D2] Dixmier, J., *Von Neumann algebras*, translated by F. Jellett from the 2nd French edition, North–Holland, Amsterdam, 1981.

[H1] Halmos, P. R., *Measure theory*, Springer–Verlag, New York, 1974.

[H2] Halmos, P. R., *Lectures on Boolean algebras*, Springer–Verlag, New York, 1974.

[H] Hausdorff, F., *Set theory*, 3rd edition, Chelsea, New York, 1957.

[K] Kuratowski, K., *Topologie.I*, Monografie Matematyczne, 2nd edition, Warsaw, 1948.

[M] Mackey, G. W., "Borel structure in groups and their duals", *Trans. Amer. Math. Soc.* **85** (1957), 134–165.

[S] Sierpinski, W., *General topology*, translated from the Polish by Cecilia Krieger, University of Toronto Press, Toronto, 1952.

[T] Takesaki, M., *Theory of operator algebras.I.*, Springer–Verlag, New York, 1979.

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STRUCTURES BORÉLIENNES

S. K. Berberian

Avant-propos: Le texte des 5 conférences faites par l'auteur à l'École d'automne 1986 du C.I.M.P.A. est reproduit ici. Les conférences ont été supplémentées par un texte intitulé *Borel spaces*; ce texte (en anglais) est cité dans les conférences comme 'la documentation'. Les titres des conférences:

1. Pourquoi les espaces boréliens?
2. Qu'est-ce qu'un espace borélien?
3. Topologie générale un peu bizarre
4. L'idée clef: espaces analytiques
5. A quoi bon les espaces boréliens?

1. Pourquoi les espaces boréliens?

L'idée d'ensemble borélien remonte au siècle dernier. Elle est vite devenue, et reste encore, un outil indispensable de la théorie de mesure.

Qu'est-ce-que c'est, un ensemble borélien? Intuitivement, ce sont les ensembles de nombres réels qu'on peut construire, à partir des ensembles ouverts, par des procédés dénombrables. Plus précisement, ce qu'on veut, c'est une classe d'ensembles, comprenant les ensembles ouverts, qui est 'fermée' pour les opérations de réunion et d'intersection des familles dénombrables; de telles classes existent—par exemple la classe de tous les ensembles de nombres réels—l'intersection de telles classes en est une, donc il y en a une la plus petite. On l'appelle la classe d'ensembles *boréliens* de $\mathbb{R}$, notée $\mathcal{B}(\mathbb{R})$. Cette classe contient toute intersection dénombrable d'ouverts

$$\bigcap_{n=1}^{\infty} \mathrm{U}_n ,$$

c'est-à-dire, tout ensemble G_δ ; chaque ensemble fermé A de $\mathbb{R}$ est un G_δ (pensez aux voisinages de A d'ordre $1/n$) donc est un ensemble borélien. Les ensembles fermées sont les compléments des ouverts; par miracle, le complément de *tout* ensemble borélien en est un aussi.

L'idee qu'on dégage de tout cela est celle de *tribu*: étant donné un ensemble E, on appelle 'tribu' sur E tout ensemble non vide $\mathcal{T}$ de parties A de E ayant les propriétés suivantes:

1° $\mathcal{T}$ est fermé pour l'opération de réunion des familles dénombrables: $A_n \in \mathcal{T} \Rightarrow \bigcup A_n \in \mathcal{T}$;

2° $\mathcal{T}$ est fermé pour l'opération de complémentation: $A \in \mathcal{T} \Rightarrow \complement A \in \mathcal{T}$.

Il en résulte que $\mathcal{T}$ est aussi fermé pour les intersections dénombrables, ainsi que les réunions et les intersections finies. {On utilise aussi le terme 'σ-algèbre' au lieu de 'tribu'.}

Par exemple, l'ensemble $\mathcal{P}(E)$ de toutes les parties de E est une tribu; l'intersection d'une famille de tribus est une tribu; donc, pour chaque classe $\mathcal{E}$ de parties de E,

$$\mathcal{E} \subset \mathcal{P}(E),$$

il existe une plus petite tribu contenant E, qu'on appelle la tribu *engendrée* par E, notée $\mathcal{T}(\mathcal{E})$.

L'exemple le plus important: soit E un espace topologique, $\mathcal{O}$ l'ensemble des ouverts de E, $\mathcal{T}(\mathcal{O})$ la tribu engendrée par $\mathcal{O}$; les éléments de cette tribu s'appellent les ensembles *boréliens* de l'espace topologique E, et on écrit

$$\mathcal{B}(E) = \mathcal{T}(\mathcal{O}).$$

Tout cela est bien classique, au point de faire partie de l'enseignment traditionnel de l'analyse réelle. S'il n'y avait que ça, nous ne serions pas ici aujourd'hui. Mais, en effet, le sujet a reçu un second souffle, il y a trente ans, aux mains de G.W. Mackey, qui l'a proposé comme outil dans l'étude des représentations des groupes topologiques et des algèbres d'opérateurs. Trente ans, ça fait longtemps aussi; on peut se demander comment cette application garde-t-elle toujours son intérêt? D'abord, parce que l'idée de Mackey est tombée sur le sol des théories en plein essor; ensuite, parce que cette nouvelle donnée dans la théorie des représentations a soulevée des problèmes très intéressants et fort difficiles.

Sans se laisser noyer dans les détails techniques, essayons de saisir l'essentiel de l'idée de Mackey.

Supposons que A est la structure algébrique dont on veut étudier les représentations, c'est-à-dire les homomorphismes de A dans des algèbres d'applications linéaires. Les cas les plus importants: A un groupe topologique localement compact; ou A une algèbre 'stellaire'—en anglais, "C*-algebra". (La définition précise ne nous occupe pas pour l'instant.) Le cas des groupes localement compacts se ramène au cas des algèbres stellaires; pour fixer le langage, supposons que A est une algèbre. Les représentations de A dont il s'agit sont les homomorphismes

$$\pi : A \to \mathcal{L}(H),$$

où H est un espace hilbertien et $\mathcal{L}(\mathrm{H})$ est l'algèbre de tous les opérateurs linéaires continus dans H. Si K est un espace hilbertien isomorphe à H, soit

$$u : \mathrm{H} \to \mathrm{K} \quad (u \text{ isomorphisme}),$$

on obtient une représentation

$$\rho : \mathrm{A} \to \mathcal{L}(\mathrm{K})$$

de A sur K, par moyen de la formule

$$\rho(x) = u\pi(x)u^{-1} \quad (x \in \mathrm{A}).$$

On écrit $\rho \cong \pi$, et on dit que la représentation ρ est *équivalente* à π (ou 'unitairement équivalente'), et, d'habitude, on identifie les représentations équivalentes. Plus précisément, étant donné un ensemble E de représentations de A, cette relation entre représentations est une véritable 'relation d'équivalence' sur E, donc amène à un ensemble quotient; d'où un intérêt accru aux *espaces quotients.*

Pour des raisons techniques, on se borne au cas où l'algèbre A et les espaces hilbertiens H vérifient des conditions de dénombrabilité (plus précisément, qu'ils sont 'séparables'); en particulier, si n est la dimension hilbertienne de H, on a

$$1 \leq n \leq \aleph_0$$

(l'espace nul est sans intérêt). Notons par H_n l'exemple canonique de dimension n; pour n fini, ce sont les espaces 'unitaires' (ou 'euclidiens complexes'); pour n égal à $\aleph_0$, il s'agit des suites (ξ_n) de nombres complexes, telles que

$$\sum |\xi_n|^2 < +\infty .$$

H_n s'appelle l'espace hilbertien 'type' de dimension n. Chaque espace H de dimension n est isomorphe à H_n, donc chaque représentation de A sur H est équivalente à une représentation de A sur H_n.

L'un des buts principaux de la théorie est d'exprimer toute représentation de A par moyen des représentations qui sont 'élémentaires' dans un sens convenable; pour fixer les idées, supposons qu'il s'agit des représentations dites 'irréductibles' (c'est-à-dire, pas la somme directe d'autres représentations). Notons

$$\mathrm{Irr}_n\,\mathrm{A}$$

l'ensemble de toutes les représentations irréductibles de A sur l'espace type H_n, et soit

$$\mathrm{Irr}\,\mathrm{A}$$

la somme des ensembles (disjoints) $\mathrm{Irr}_n\,\mathrm{A}$. Comme déjà remarqué, l'ensemble Irr A est muni d'une relation d'équivalence (l'équivalence unitaire); notons par $\hat{\mathrm{A}}$ l'ensemble quotient:

$$\hat{\mathrm{A}} = \mathrm{Irr}\,\mathrm{A}/\cong ,$$

et soit γ l'application quotient:

$$\gamma : \mathrm{Irr\,A} \to \hat{\mathrm{A}}\,.$$

D'autre part, si une représentation π de A est irréductible, il en est de même pour toute représentation équivalente à π ; il en résulte que π est équivalente à au moins un élément $\rho \in \mathrm{Irr\,A}$, et la classe de ρ ne dépend que sur π ; on écrit

$$\hat{\pi}$$

pour cette classe, de sorte que $\hat{\pi} \in \hat{\mathrm{A}}$. Donc

$$\hat{\mathrm{A}} = \{\hat{\pi} :\ \pi \text{ irréductible}\}\,.$$

{On a évité de parler de 'l'ensemble de toutes les représentations irréductibles de A '.} En particulier, l'application quotient

$$\gamma : \mathrm{Irr\,A} \to \hat{\mathrm{A}}$$

peut s'écrire $\pi \mapsto \hat{\pi}$.

C'est à ce point que nous courrons le plus grand risque d'être débordés par la multitude des idées qui convergent sur cette théorie. Disons tout court qu'il existent sur Irr A et sur $\hat{\mathrm{A}}$ des topologies naturelles, qui rendent continue l'application γ. J'aurai plus à dire à ce sujet dans la cinquième conférence. Mais pour l'instant, pour arriver plus directement à l'idée qu'on veut dégager, abandonnons les représentations et considérons une situation topologique beaucoup plus simple. Soit E un espace topologique, F un ensemble,

$$f : \mathrm{E} \to \mathrm{F}$$

une application surjective. On envisage de transporter à F la structure de E, par moyen de l'application f. En effet, on peut définir sur F une topologie, et une plus fine que toute autre, qui rende continue l'application, en déclarant 'ouvert' tout sous-ensemble B de F dont l'image réciproque $f^{-1}(\mathrm{B})$ est ouverte dans E ; cette topologie s'appelle la 'topologie finale' de l'application f, ou la 'topologie quotient' de E par l'application.

Ce procédé peut conduire à des déceptions. Par exemple, soit $\mathbb{Q}$ le sous-groupe rationnel du groupe additif de la droite réelle $\mathbb{R}$, et soit

$$f : \mathbb{R} \to \mathbb{R}/\mathbb{Q}$$

l'homomorphisme canonique de $\mathbb{R}$ sur le groupe quotient. Soit $\mathrm{T} \subset \mathbb{R}/\mathbb{Q}$ une partie du quotient. Dire que T est ouvert pour la topologie quotient signifie que l'ensemble $\mathrm{U} = f^{-1}(\mathrm{T})$ est à la fois ouvert et stable pour toute translation rationnelle. Grâce à la densité des nombres rationnels, il n'y a que deux possibilités:

U est soit vide, soit égal à $\mathbb{R}$. Donc la topologie quotient est la topologie dite grossière:

$$\mathcal{O}_{\mathbb{R}/\mathbb{Q}} = \{\varnothing, \mathbb{R}/\mathbb{Q}\}\,.$$

Cette topologie est évidemment une tribu, donc pas davantage d'ensembles boréliens:

$$\mathcal{B}(\mathbb{R}/\mathbb{Q}) = \{\varnothing, \mathbb{R}/\mathbb{Q}\}\,.$$

Pourtant, si T est une partie dénombrable du quotient,

$$\mathrm{T} = \{u_n :\ n = 1, 2, \ldots\}\,,$$

alors son image réciproque

$$f^{-1}(\mathrm{T}) = \bigcup_{n=1}^{\infty} f^{-1}(\{u_n\})$$

est la réunion d'une suite de classes suivant $\mathbb{Q}$ donc est dénombrable et par conséquent borélien. Ceci suggère considérer l'ensemble

$$\mathcal{C} = \{\mathrm{T} \subset \mathbb{R}/\mathbb{Q} :\ f^{-1}(\mathrm{T}) \in \mathcal{B}(\mathbb{R})\,\}$$

de toutes les parties du quotient dont l'image réciproque est borélienne. Or, cet ensemble est une tribu, comme on voit facilement (d'après le principe que l'opération de prendre l'image réciproque conserve toutes les opérations élémentaires sur les sous-ensembles). Nous sommes ainsi conduits à l'idée de 'structure borélienne': c'est un ensemble E dans lequel on a distingué une tribu $\mathcal{B}$ de sous-ensembles; le couple $(\mathrm{E}, \mathcal{B})$ s'appelle un *espace borelien:*

$$(\mathrm{E}, \mathcal{B})$$

$$\mathrm{E} \text{ ensemble}, \quad \mathcal{B} \subset \mathcal{P}(\mathrm{E}) \text{ une tribu.}$$

Dans l'exemple précédent, on dit que la tribu $\mathcal{C}$ définit la 'structure borélienne quotient' sur le groupe quotient. La structure topologique quotient est pauvre; la structure borélienne quotient est relativement riche; pour ainsi dire, la structure borélienne survie mieux le passage aux quotients que la structure topologique.

Cette explication simpliste ne donne justice au profondeur de l'idée de Mackey, mais on peut le résumer comme suit: d'abord, les ensembles boréliens sont plus abondants que les ensembles ouverts; ensuite, ce qui est plus important, pour certains problèmes des représentations, surtout quand il s'agit de passage au quotients, la structure borélienne est plus appropriée que la structure topologique. D'où la décision de Mackey de formaliser le concept de structure borélienne et d'élaborer systématiquement les propriétés de ces structures. A première vue, les structures boréliennes sont trop générales—de structure trop faible—pour être intéressantes;

en fait, en introduisant des hypothèses supplémentaires simples et naturelles, on arrive à des résultats surprenants, beaux, et utiles.

Dans la prochaine conférence, nous aborderons l'étude systématique de ces structures.

Pour conclure, je vous pose un tout petit problème: Trouver, une application surjective $f : \mathrm{E} \to \mathrm{F}$ et une tribu $\mathcal{B}$ sur E telle que l'ensemble $f(\mathcal{B}) = \{f(\mathrm{B}) : \mathrm{B} \in \mathcal{B}\}$ n'est pas une tribu sur F.

2. Qu'est-ce qu'un espace borélien?

Rappelons qu'un *espace borélien* est un couple $(\mathrm{E},\mathcal{B})$, où E est un ensemble et $\mathcal{B}$ est une tribu de parties de E. On dit que $\mathcal{B}$ définit sur E une *structure borélienne*, aussi que $\mathcal{B}$ est égal à cette structure. Les éléments de $\mathcal{B}$ s'appellent les *ensembles boréliens* de E. Si $\mathcal{B}_1, \mathcal{B}_2$ sont deux tribus sur un même ensemble E, telles que $\mathcal{B}_1 \subset \mathcal{B}_2$, on dit que la structure borélienne définie par $\mathcal{B}_2$ est *plus fine* que celle définie par $\mathcal{B}_1$, ou que $\mathcal{B}_1$ définit une structure *moins fine* que celle définie par $\mathcal{B}_2$.

Pour chaque ensemble $\mathcal{E}$ de parties de E, il existe une plus petite tribu contenant $\mathcal{E}$, qu'on appelle la tribu *engendrée* par $\mathcal{E}$, notée $\mathcal{T}(\mathcal{E})$.

L'exemple le plus important: E un espace topologique, $\mathcal{B}(\mathrm{E}) = \mathcal{T}(\mathcal{O}_\mathrm{E})$ la tribu engendrée par l'ensemble $\mathcal{O}_\mathrm{E}$ des ouverts de E; je propose de l'appeler un *espace borélien topologique.*

Voici deux constructions fondamentales de tribus: Soit $f : \mathrm{E} \to \mathrm{F}$ une application quelconque. Si $\mathcal{C}$ est une tribu de parties de F, l'ensemble

$$f^{-1}(\mathcal{C}) = \{f^{-1}(\mathrm{C}) :\ \mathrm{C} \in \mathcal{C}\}$$

est une tribu de parties de E; d'autre part, si $\mathcal{B}$ est une tribu de parties de E, alors l'ensemble

$$\{\mathrm{T} \subset \mathrm{F} :\ f^{-1}(\mathrm{T}) \in \mathcal{B}\}$$

est une tribu de parties de F (pas de notation proposée).

Une application entre espaces boréliens

$$f :\ (\mathrm{E},\mathcal{B}) \to (\mathrm{F},\mathcal{C})$$

s'appelle *application borélienne* (ou un *morphisme* pour les structures boréliennes) si l'image réciproque de chaque partie borélienne de F est partie borélienne de E:

$$f^{-1}(\mathcal{C}) \subset \mathcal{B},$$

c'est-à-dire que l'image réciproque de $\mathcal{C}$ est moins fine que $\mathcal{B}$. Si f est une bijection et si f et f^{-1} sont toutes les deux boréliennes, on dit que f est un *isomorphisme* de E sur F, et que les espaces boréliens E et F sont *isomorphes.*

Par exemple, soient E, F des espaces topologiques, $f : \mathrm{E} \to \mathrm{F}$ une application continue; alors f est une application borélienne pour les structures boréliennes topologiques:

$$(*) \qquad f^{-1}\big(\mathcal{B}(\mathrm{F})\big) \subset \mathcal{B}(\mathrm{E}).$$

En effet, grâce à la continuité, on a

$$f^{-1}(\mathcal{O}_{\mathrm{F}}) \subset \mathcal{O}_{\mathrm{E}} \subset \mathcal{B}(\mathrm{E}),$$

donc

$$\mathcal{O}_{\mathrm{F}} \subset \{\mathrm{T} \subset \mathrm{F} : \; f^{-1}(\mathrm{T}) \in \mathcal{B}(\mathrm{E})\};$$

vu que le second membre est une tribu, elle contient forcément la tribu engendrée par $\mathcal{O}_{\mathrm{F}}$, c'est-à-dire $\mathcal{B}(\mathrm{F})$, d'où la relation $(*)$.

Soit maintenant E un ensemble, $(\mathrm{F},\mathcal{C})$ un espace borélien, $f : \mathrm{E} \to (\mathrm{F},\mathcal{C})$ une application quelconque. La tribu $f^{-1}(\mathcal{C})$ définit sur E une structure borélienne rendant f borélienne, évidemment la structure la moins fine ayant cette propriété; elle s'appelle la structure borélienne *initiale* pour l'application f.

Ce concept se généralise immédiatement pour une famille d'applications d'un ensemble dans des espaces boréliens

$$f_i : \mathrm{E} \to (\mathrm{F}_i, \mathcal{C}_i) \qquad (i \in \mathrm{I});$$

on dit que la tribu engendrée par les images réciproques de tous les $\mathcal{C}_i$

$$\mathcal{T}\big(\textstyle\bigcup_i {f_i}^{-1}(\mathcal{C}_i)\big)$$

définit la structure borélienne *initiale* pour la famille $(f_i)_{i\in\mathrm{I}}$; c'est la structure la moins fine rendant borélienne toutes les f_i.

Il y a deux exemples principaux:

1) $(\mathrm{E},\mathcal{B})$ un espace borélien, $\mathrm{S} \subset \mathrm{E}$, $f : \mathrm{S} \to \mathrm{E}$ l'insertion de S dans E. Alors,

$$f^{-1}(\mathcal{B}) = \{\mathrm{B} \cap \mathrm{S} : \; \mathrm{B} \in \mathcal{B}\} = \mathcal{B} \cap \mathrm{S},$$

et $(\mathrm{S},\mathcal{B}\cap\mathrm{S})$ s'appelle la *sous-structure borélienne* induite par $\mathcal{B}$ sur S. (À éviter, autant que possible, le terme équivoque 'sous-espace borélien', parce que S n'est pas supposé partie borélienne de E.)

2) $(\mathrm{E}_i,\mathcal{B}_i)_{i\in\mathrm{I}}$ une famille d'espaces boréliens, $\mathrm{E} = \prod \mathrm{E}_i$ l'ensemble produit,

$$\mathrm{pr}_i : \mathrm{E} \to \mathrm{E}_i \qquad (i \in \mathrm{I})$$

la famille des projections. La structure initiale pour cette famille s'appelle la structure *produit* des $\mathcal{B}_i$, notée

$$\prod_{i\in\mathrm{I}} \mathcal{B}_i$$

(mais attention—ce n'est pas le produit des ensembles $\mathcal{B}_i$). Cette tribu est engendrée par les ensembles

$${\mathrm{pr}_j}^{-1}(\mathrm{B}_j) = \mathrm{B}_j \times \prod_{i\neq j} \mathrm{E}_i$$

$(j \in \mathrm{I}\,, \mathrm{B}_j \in \mathcal{B}_j)$, ou également par les intersections dénombrables de tels ensembles,

$$\bigcap_{n=1}^{\infty} \mathrm{pr}_{j_n}{}^{-1}(\mathrm{B}_{j_n}) = (\mathrm{B}_{j_1} \times \mathrm{B}_{j_2} \times \ldots) \times \prod_{i \notin \{j_1, j_2, \ldots\}} \mathrm{E}_i \,.$$

Dans le cas où $\mathrm{I} = \{1, 2\}$, on écrit

$$(\mathrm{E}_1 \times \mathrm{E}_2, \mathcal{B}_1 \times \mathcal{B}_2)$$

pour l'espace borélien produit.

Soient maintenant E, F deux espaces topologiques, $\mathrm{E}\times\mathrm{F}$ l'espace topologique produit, dont la topologie est engendrée par les 'rectangles ouverts'

$$\mathrm{U} \times \mathrm{V} \quad (\mathrm{U} \in \mathcal{O}_{\mathrm{E}},\ \mathrm{V} \in \mathcal{O}_{\mathrm{F}})\,.$$

Il est facile de voir que ces rectangles engendrent la tribu produit $\mathcal{B}(\mathrm{E}) \times \mathcal{B}(\mathrm{F})$, d'où la relation

$$\mathcal{B}(\mathrm{E}) \times \mathcal{B}(\mathrm{F}) \subset \mathcal{B}(\mathrm{E} \times \mathrm{F})\,.$$

Il se peut que cette inclusion soit stricte (pour un exemple, voire l'Appendice de la documentation).

Mais si les topologies de E, F possèdent des bases dénombrables (U_n), (V_n) , alors chaque ouvert W de l'espace produit est réunion

$$\mathrm{W} = \bigcup \mathrm{U}_{m_i} \times \mathrm{V}_{n_i}$$

d'une suite d'éléments de la tribu produit $\mathcal{B}(\mathrm{E}) \times \mathcal{B}(\mathrm{F})$, donc appartient à cette tribu; par conséquent,

$$\mathcal{B}(\mathrm{E} \times \mathrm{F}) \subset \mathcal{B}(\mathrm{E}) \times \mathcal{B}(\mathrm{F})\,,$$

d'où l'égalité de ces tribus.

À cause de cette circonstance, les espaces topologiques de base dénombrable jouent un rôle majeur dans la théorie borélienne.

Il y a aussi les 'structures finales'. Encore, deux exemples principaux:

1) $(\mathrm{E}, \mathcal{B})$ un espace borélien, F un ensemble, $f : (\mathrm{E}, \mathcal{B}) \to \mathrm{F}$ une application surjective (par exemple, l'application quotient déduite d'une relation d'équivalence dans E). La tribu

$$\{\mathrm{T} \subset \mathrm{F} :\ f^{-1}(\mathrm{T}) \in \mathcal{B}\}$$

est la plus fine rendant f borélienne; la structure borélienne qu'elle définit sur F s'appelle la structure borélienne *finale* pour l'application f (dans le cas des espaces quotients—qui comprend, bien entendu, le cas 'général'—on parle de la structure borélienne *quotient*).

2) Quant au second exemple, il s'agit de la *somme directe* d'une famille d'espaces boréliens; on trouvera les détails dans la documentation.

Revenons encore au cas fécond d'un espace borélien topologique:

$$\big(\mathrm{E}, \mathcal{B}(\mathrm{E})\big)\,.$$

Supposons que E est séparé et possède une base dénombrable (U_n). Pour chaque couple x, y de points distincts de E, il existe un indice n tel que $x \in \mathrm{U}_n$ et $y \notin \mathrm{U}_n$.

Plus généralement, on dit qu'un espace borélien $(\mathrm{E}, \mathcal{B})$ est *dénombrablement séparé* s'il existe une suite d'ensembles boréliens (B_n) qui est séparante dans le sens évident suggéré par l'exemple précédent. Supposons que ce soit le cas. On va voir qu'on peut introduire sur E une topologie, ayant une base dénombrable, dont la tribu borélienne est moins fine que $\mathcal{B}$ mais qui contient tous les B_n. Soit

$$\mathbb{M} = \prod_{n=1}^{\infty} \{0, 1\}$$

le produit topologique d'une famille dénombrable d'espaces discrets à deux points; cet espace est compact, métrisable, et possède une base dénombrable, donc est complet de type dénombrable pour chaque distance compatible avec la topologie. (Dans un Appendice de la documentation, il est démontré que $\mathbb{M}$ est homéomorphe à l'ensemble triadique Γ de Cantor.)

Soit $f : \mathrm{E} \to \mathbb{M}$ l'application

$$f(x) = \big(\varphi_{\mathrm{B}_n}(x)\big)\,,$$

où φ_{B_n} est la fonction caractéristique de B_n. La suite (B_n) étant séparante, l'application f est injective. Notons par $\mathrm{F}_j = \mathrm{pr}_j{}^{-1}(\{1\})$ l'ensemble des points de $\mathbb{M}$ ayant 1 pour coordonné j; grâce à la continuité des projections, F_j est partie ouverte (aussi fermée), donc borélienne, de $\mathbb{M}$. On a la relation

$$(*) \qquad f(\mathrm{B}_j) = \mathrm{F}_j \cap f(\mathrm{E})\,.$$

En effet, pour $z = (z_n) \in \mathbb{M}$, on a

$$\begin{aligned} z \in \mathrm{F}_j \cap f(\mathrm{E}) \quad &\Leftrightarrow \quad z_j = 1 \ \ \& \ \ \exists\, x \in \mathrm{E} \ \ni \ f(x) = z \\ &\Leftrightarrow \quad \exists\, x \in \mathrm{E} \ \ni \ f(x) = z \ \ \& \ \ \varphi_{\mathrm{B}_j}(x) = 1 \\ &\Leftrightarrow \quad \exists\, x \in \mathrm{E} \ \ni \ f(x) = z \ \ \& \ \ x \in \mathrm{B}_j \\ &\Leftrightarrow \quad z \in f(\mathrm{B}_j)\,. \end{aligned}$$

Comme f est injective, la relation $(*)$ s'écrit

$$\mathrm{B}_j = f^{-1}(\mathrm{F}_j)\,.$$

En particulier, $f^{-1}(\mathrm{F}_j) \in \mathcal{B}$ pour tout j; comme les F_j engendrent $\mathcal{B}(\mathbb{M})$, il en résulte que

$$f^{-1}\big(\mathcal{B}(\mathbb{M})\big) \subset \mathcal{B}\,,$$

c'est-à-dire que f est borélienne.

Supposons en plus que la suite séparante (B_n) engendre la tribu $\mathcal{B}$; alors on tire de la relation $(*)$ que

$$f(\mathcal{B}) = \mathcal{B}(\mathbb{M}) \cap f(\mathrm{E}) \,,$$

ce qui signifie que l'espace borélien $(\mathrm{E}, \mathcal{B})$ est isomorphe à la sous-structure borélienne

$$\big(f(\mathrm{E}), \mathcal{B}(\mathbb{M}) \cap f(\mathrm{E})\big)$$

de $\mathbb{M}$. Or, l'ensemble des ouverts du sous-espace topologique $f(\mathrm{E})$ de $\mathbb{M}$ est

$$\mathcal{O}_{f(\mathrm{E})} = \mathcal{O}_{\mathbb{M}} \cap f(\mathrm{E}) \,,$$

par conséquent

$$\mathcal{B}\big(f(\mathrm{E})\big) = \mathcal{B}(\mathbb{M}) \cap f(\mathrm{E}) \,.$$

Donc $(\mathrm{E}, \mathcal{B})$ est isomorphe à l'espace borélien topologique

$$\Big(f(\mathrm{E}), \mathcal{B}\big(f(\mathrm{E})\big)\Big) \,,$$

où $f(\mathrm{E})$ est un espace métrisable de type dénombrable.

De tout cela, on dégage les idées suivantes.

On considère les espaces métriques complets de type dénombrable; tout espace homéomorphe à un tel espace s'appelle un *espace polonais*.

Un espace borélien est dit *standard* s'il est isomorphe à l'espace borélien topologique déduit d'un espace polonais.

Je propose d'appeler *sous-standard* tout espace borélien isomorphe à une sous-structure borélienne d'un espace standard. Nous venons de démontrer que tout espace borélien dont la tribu des ensembles boréliens est engendrée par une suite séparante, est sous-standard. La proposition inverse est évidente.

On en voit que les espaces polonais vont jouer un rôle particulièrement important dans la théorie des espaces boréliens; dans la prochaine conférence, nous aurons plus à dire de ces espaces et de leurs proches banlieues un peu bizarres.

Terminons avec deux exemples.

1) Soit $(\mathrm{E}, \mathcal{B})$ un espace borélien ayant une suite séparante (B_n), et tel que l'ensemble E est dénombrable.

Conclusion: toute partie de E est borélienne, c'est-à-dire, $\mathcal{B} = \mathcal{P}(\mathrm{E})$.

En effet, si $x \in \mathrm{E}$, on a

$$\{x\} = \bigcap_{x \in \mathrm{B}_n} \mathrm{B}_n \in \mathcal{B} \,;$$

donc toute partie de E est réunion dénombrable de parties boréliennes, par conséquent est borélienne. En particulier, $(\mathrm{E}, \mathcal{B})$ est standard.

2) D'autre part, soit $(\mathrm{E}, \mathcal{B})$ standard et *non* dénombrable.

Théorème (pas facile!): $(\mathrm{E}, \mathcal{B})$ est isomorphe à $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$.

En particulier, les espaces boréliens $\big(\Gamma, \mathcal{B}(\Gamma)\big)$, $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, $\big(\mathbb{R}_n, \mathcal{B}(\mathbb{R}_n)\big)$, $\big([0,1], \mathcal{B}([0,1])\big)$, etc. *sont tous isomorphes*!

3. Topologie générale un peu bizarre

Nous allons faire maintenant l'inventaire des faits divers topologiques sous-jacents à notre sujet. Ils se trouvent tous dans le livre de Bourbaki, donc il nous est permis (même obligatoire) d'omettre la plupart des détails.

1. Tout d'abord, un espace topologique est dit *polonais* s'il est homéomorphe à un espace métrique complet séparable. À remarquer que chaque espace polonais est un espace de Baire, donc inexprimable comme réunion d'une suite de parties fermées d'intérieur vide.

2. Soit P un espace polonais, S un sous-espace de P, c'est-à-dire, un sous-ensemble de P muni de la topologie relative.

Question: Quand S est-il polonais?

Réponse: Précisément quand il est l'intersection d'une suite d'ouverts de P: $\mathrm{S} = \bigcap_{n=1}^{\infty} \mathrm{U}_n$.

3. En particulier, dans un espace polonais, chaque sous-espace polonais est un ensemble borélien. Mais la réciproque est inexacte: par exemple, dans la droite réelle $\mathbb{R}$, le sous-ensemble $\mathbb{Q}$ des nombres rationnels est borélien mais pas un sous-espace polonais. (Raison: Un espace topologique qui est dénombrable, séparé, et sans point isolé, ne peut pas être un espace de Baire.)

4. Soit P un espace polonais, B un sous-ensemble borélien de P. Supposons que B n'est pas un sous-espace polonais, de sorte qu'il n'existe pas un espace polonais Q et une bijection $\mathrm{Q} \to \mathrm{B}$ bicontinue. Néanmoins, on peut démontrer qu'il existe un espace polonais Q et une bijection $\mathrm{Q} \to \mathrm{B}$ *continue.*

5. Définition: Un espace topologique séparé qui est l'image continue bijective d'un espace polonais s'appelle un espace *lusinien.*

6. Soit P un espace polonais, B un ensemble borélien de P. On sait qu'il existe un espace polonais Q et une bijection continue $f : \mathrm{Q} \to \mathrm{B}$.

Théorème: f est un isomorphisme borélien.

La portée de cette théorème est que l'image de tout ensemble borélien de Q est ensemble borélien de B (donc de P).

Il y a réciproque: Si P est un espace polonais et S est un sous-espace de P pour lequel l'espace borélien topologique $\big(\mathrm{S}, \mathcal{B}(\mathrm{S})\big)$ est isomorphe à $\big(\mathrm{Q}, \mathcal{B}(\mathrm{Q})\big)$ avec Q polonais, alors S est ensemble borélien de P.

7. Rappelons qu'un espace borélien $(\mathrm{E}, \mathcal{B})$ est dit *standard* si E est borel-isomorphe à un espace polonais. Les sous-structures boréliennes $(\mathrm{S}, \mathcal{B} \cap \mathrm{S})$ sont alors appelées *sous-standards.* La signification de ce qui précède: si $(\mathrm{E}, \mathcal{B})$ est

standard, les sous-ensembles de E définissant une sous-structure standard sont précisément les ensembles boréliens de E :

$$(\mathrm{S}, \mathcal{B} \cap \mathrm{S}) \ \text{standard} \quad \Leftrightarrow \quad \mathrm{S} \in \mathcal{B}\,.$$

8. On a remarqué que si P , Q sont des espaces polonais et si $f : \mathrm{P} \to \mathrm{Q}$ est continue et bijective, alors l'image de chaque ensemble borélien de P est ensemble borélien de Q .

Pourtant, il existe des espaces polonais P , Q et une application continue $f : \mathrm{P} \to \mathrm{Q}$ telle que l'image $f(\mathrm{P})$ de P n'est pas ensemble borélien de Q . En considérant cette image comme sous-espace topologique de Q , f définit une surjection continue

$$\mathrm{P} \to f(\mathrm{P})\,.$$

Un espace topologique séparé qui est l'image continue d'un espace polonais est dit *souslinien.* Évidemment,

$$\text{polonais} \quad \Rightarrow \quad \text{lusinien} \quad \Rightarrow \quad \text{souslinien.}$$

Dans un espace topologique quelconque, on appelle *ensemble souslinien* tout sous-ensemble définissant un sous-espace topologique souslinien.

On peut démontrer que, dans un espace souslinien, tout ensemble ouvert et tout ensemble fermé est un sous-espace souslinien.

9. Pour résumer, soit P un espace polonais, S un sous-ensemble de P .

i) Si S est sous-ensemble borélien de P , alors S est sous-ensemble souslinien (même lusinien):

$$\mathrm{S}\ \text{borélien} \quad \Rightarrow \quad \mathrm{S}\ \text{souslinien}\,.$$

ii) Il se peut que S soit souslinien sans être borélien: S souslinien $\not\Rightarrow$ S borélien.

Mais il y a un beau critère dû à Souslin: S est borélien si et seulement si cet ensemble et son complément sont tous les deux sousliniens:

$$\mathrm{S}\ \text{borélien} \quad \Leftrightarrow \quad \mathrm{S}\ \text{et}\ \complement\mathrm{S}\ \text{souslinien.}$$

10. Citons un exemple de Mazurkewicz:

$$\begin{aligned}
&\mathrm{I} = [0,1] \\
&\mathcal{C} = \mathcal{C}_{\mathbb{R}}([0,1]) \\
&(\forall\ x \in \mathrm{I})\ \ \mathcal{D}_x = \{ f \in \mathcal{C} :\ f \ \text{est dérivable à}\ x\,\} \\
&\mathcal{D} = \bigcap_{x \in \mathrm{I}} \mathcal{D}_x = \{ f \in \mathcal{C} :\ f \ \text{est partout dérivable}\,\} \\
&\mathcal{C} - \mathcal{D} = \{ f \in \mathcal{C} :\ f \ \text{est quelque part non-dérivable}\,\}
\end{aligned}$$

Théorème (Mazurkewicz): $\mathcal{C} - \mathcal{D}$ est souslinien, mais $\mathcal{D}$ ne l'est pas; donc $\mathcal{C} - \mathcal{D}$ est souslinien mais pas borélien.

11. Soient E, F des espaces sousliniens, $f : \mathrm{E} \to \mathrm{F}$ continue et surjective. Soit $\mathrm{T} \subset \mathrm{F}$.

Si T est ensemble borélien, il en est de même pour son image réciproque (parce qu'une application continue est toujours borélienne).

L'une des propriétés fondamentales des espaces sousliniens est la proposition inverse: si l'image réciproque de T est borélien, alors T est forcément borélien.

En particulier: *toute application continue bijective entre espaces sousliniens est un isomorphisme borélien.*

En nous basant sur ces propriétés, nous allons démontrer quelques-unes de leurs conséquences importantes.

12. Théorème: Soit (E_n) une famille dénombrable d'espaces lusiniens, E leur produit topologique. Alors

$$\mathcal{B}(\mathrm{E}) = \prod \mathcal{B}(\mathrm{E}_n)\,.$$

Preuve. Notons d'abord que cette égalité est valable si les E_n sont polonais, grâce à l'existence des bases dénombrables pour les topologies.

En général, l'inclusion $\mathcal{B}(\mathrm{E}) \supset \prod \mathcal{B}(\mathrm{E}_n)$ est valable pour toute famille d'espaces topologiques, grâce à la continuité des projections.

D'autre part, d'après l'hypothèse, il existe pour chaque n un espace polonais P_n et une bijection continue $f_n : \mathrm{P}_n \to \mathrm{E}_n$. Soit $\mathrm{P} = \prod \mathrm{P}_n$ l'espace topologique produit, et soit $f : \mathrm{P} \to \mathrm{E}$ l'application

$$f\big((x_n)\big) = \big(f_n(x_n)\big)\,,$$

évidemment bijective et continue. Il en résulte que E est un espace lusinien et que f est un isomorphisme borélien:

$$\text{(a)} \qquad f\big(\mathcal{B}(\mathrm{P})\big) = \mathcal{B}(\mathrm{E})\,.$$

D'autre part f_n est, pour chaque n, un isomorphisme borélien:

$$f_n\big(\mathcal{B}(\mathrm{P}_n)\big) = \mathcal{B}(\mathrm{E}_n)\,,$$

donc

$$\text{(b)} \qquad f\Big(\prod \mathcal{B}(\mathrm{P}_n)\Big) = \prod f_n\big(\mathcal{B}(\mathrm{P}_n)\big) = \prod \mathcal{B}(\mathrm{E}_n)\,.$$

Enfin, comme remarqué au début de la démonstration,

$$\text{(c)} \qquad \mathcal{B}(\mathrm{P}) = \prod \mathcal{B}(\mathrm{P}_n)\,,$$

donc

$$\mathcal{B}(\mathrm{E}) \overset{\text{(a)}}{=} f\big(\mathcal{B}(\mathrm{P})\big) \overset{\text{(c)}}{=} f\Big(\prod \mathcal{B}(\mathrm{P}_n)\Big) \overset{\text{(b)}}{=} \prod \mathcal{B}(\mathrm{E}_n)\,. \ \diamondsuit$$

13. Nous allons maintenant caractériser certaines applications boréliennes par moyen de leurs graphes. Le suivant n'est pas, bien entendu, le résultat le plus général.

Théorème: Soient P, Q des espaces polonais (plus généralement, des espaces sousliniens métrisables), $f : \mathrm{P} \to \mathrm{Q}$ une application, et

$$\mathrm{G} = \{(x, y) \in \mathrm{P} \times \mathrm{Q} : \ y = f(x)\}$$

son graphe. Les conditions suivantes sont équivalentes:

(a) f est une application borélienne;

(b) G est un sous-ensemble borélien de $\mathrm{P} \times \mathrm{Q}$;

(c) G est un sous-ensemble souslinien de $\mathrm{P} \times \mathrm{Q}$.

Preuve. (a) $\Rightarrow$ (b): Soit $g : \mathrm{P} \times \mathrm{Q} \to \mathrm{Q} \times \mathrm{Q}$ l'application $g(x, y) = \big(f(x), y\big)$, c'est-à-dire, $g = (f, \mathrm{id}_{\mathrm{Q}})$. D'après l'hypothèse, f est borélienne, donc il en est de même de g. Si Δ est le diagonal de $\mathrm{Q} \times \mathrm{Q}$, alors

$$\mathrm{G} = \{(x, y) : \ \big(f(x), y\big) \in \Delta\} = g^{-1}(\Delta),$$

de sorte que, pour démontrer que G est borélien, il suffit de démontrer que Δ est borélien.

En effet, soit (C_n) une suite séparante d'ensembles boréliens de Q (par exemple, une base dénombrable pour la topologie); alors

$$\complement\Delta = \bigcup_{n=1}^{\infty} \mathrm{C}_n \times \complement\mathrm{C}_n,$$

car, pour $(y, z) \in \mathrm{Q} \times \mathrm{Q}$, dire que $y \neq z$ signifie qu'il existe un indice n tel que $y \in \mathrm{C}_n$ et $z \notin \mathrm{C}_n$. En particulier, $\complement\Delta$ est un ensemble borélien, donc il en est de même pour son complément.

(b) $\Rightarrow$ (c) d'après 9,(i).

(c) $\Rightarrow$ (a): Supposons G souslinien. Si C est sous-ensemble borélien de Q, il faut démontrer que son image réciproque $f^{-1}(\mathrm{C})$ est borélien, et pour cela, il suffit de vérifier que cet ensemble et son complément sont tous les deux sousliniens. On peut d'ailleurs supposer que C est fermé (parce que les ensembles fermés engendrent la tribu borélienne).

Pour chaque sous-ensemble T de Q, on a

$$f^{-1}(\mathrm{T}) = \mathrm{pr}_1[\mathrm{G} \cap (\mathrm{P} \times \mathrm{T})];$$

en effet,

$$\begin{aligned} x \in f^{-1}(\mathrm{T}) &\Leftrightarrow f(x) \in \mathrm{T} \\ &\Leftrightarrow \big(x, f(x)\big) \in \mathrm{G} \cap (\mathrm{P} \times \mathrm{T}) \\ &\Leftrightarrow x \in \mathrm{pr}_1[\mathrm{G} \cap (\mathrm{P} \times \mathrm{T})]. \end{aligned}$$

Dans le sous-espace souslinien G, l'ensemble $G \cap (P \times C)$ est fermé, donc souslinien, donc il en est de même de son image par l'application continue pr_1 ; c'est-à-dire, $f^{-1}(C)$ est souslinien.

De même, $f^{-1}(\complement C)$ est souslinien (remplacez 'fermé' par 'ouvert' dans le raisonnement ci-dessus); c'est-à-dire, $\complement f^{-1}(C)$ est souslinien, ce qui achève la démonstration. $\diamondsuit$

Drôle de résultat!: G est borélien si et seulement s'il est souslinien. Il y a là une intuition fondamentale: c'est que le graphe d'une application est un sous-ensemble pas comme un autre.

Pour conclure, rappelons que les espaces boréliens standards sont, à isomorphisme près, les structures boréliennes topologiques déduites des espaces polonais. Mais, du point de vue borélien, il y a peu d'exemples: seule $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$ et les structures déduites des espaces discrètes dénombrables ou finies. Pour rendre utile ces structures, il faut leur affaiblir.

D'autre part, les structures sous-standards sont trop faibles.

Le compromis juste, c'est de considérer les espaces boréliens topologiques déduits des espaces sousliniens métrisables. Ce sont les espaces boréliens dits *analytiques*, en conformité avec le terme classique pour les ensembles sousliniens; nous allons explorer leurs secrets dans la prochaine conférence.

4. L'idée clef: espaces analytiques

1. Il existe plusieurs façons équivalentes de définir les espaces boréliens dits 'analytiques'; entre la dizaine des formes données dans la documentation, choisissons celle-là qui est la plus facile à souvenir:

Définition: Un espace borélien est dit *analytique* s'il est isomorphe à l'espace borélien topologique déduit d'un espace souslinien métrisable. (Je vois dans cette forme la justification de ce que Bourbaki ne considère que les espaces métrisables dans la première édition du Ch. IX, §6 de Topologie générale.)

C'est là, la forme peut-être la plus 'forte', donc la plus difficile à vérifier.

2. La forme peut-être la plus 'faible', donc plus facile à vérifier, est la suivante: un espace borélien est analytique si et seulement s'il est l'image borélien d'un espace standard et s'il possède une suite séparante d'ensembles boréliens.

3. La caractérisation la plus concrète: un espace borélien est analytique si et seulement s'il est isomorphe à la structure borélienne topologique déduite d'un sous-espace souslinien de la droite réelle.

4. Enfin, une autre perspective intéressante: un espace borélien est analytique si et seulement s'il est l'image borélien d'un espace standard et s'il est sous-structure borélienne d'un autre espace standard—en bref, c'est un espace borélien sous-standard qui est l'image borélien d'un espace standard. Pour ainsi dire, un espace analytique est au carrefour de deux espaces standards.

5. Au lieu de nous occuper avec le jeu, fort amusant, d'établir l'équivalence des formes diverses de la définition d'espace analytique, résumons quelques-unes des propriétés qui les rendent utiles. Au fond, il y a le théorème suivant.

Théorème: Soit $(\mathrm{E}, \mathcal{B})$, $(\mathrm{F}, \mathcal{C})$ des espaces boréliens, $f : \mathrm{E} \to \mathrm{F}$ une application borélienne surjective, telle que

1° $(\mathrm{E}, \mathcal{B})$ est analytique,
2° $(\mathrm{F}, \mathcal{C})$ est dénombrablement séparé.

Alors $(\mathrm{F}, \mathcal{C})$ est analytique, et $\mathcal{C}$ est la structure borélienne finale pour f, c'est-á-dire, pour $\mathrm{T} \subset \mathrm{F}$ on a

$$\mathrm{T} \in \mathcal{C} \;\Leftrightarrow\; f^{-1}(\mathrm{T}) \in \mathcal{B}\,.$$

6. Une conséquence immédiate du théoréme: une bijection borélienne entre espaces analytiques est un isomorphisme borélien.

Il y a là une analogie avec les espaces compacts et les bijections continues; ou avec les espaces de Banach et les bijections linéaires continues. Dans ces deux situations analogues, il s'agit des espaces uniformes complets; dans le théorème ci-dessus, qui revient au cas des espaces sousliniens métrisables, ce qui nous frappe c'est que ces espaces ne sont supposés ni complets, ni même espaces de Baire.

7. Soit E un espace souslinien métrisable. On sait qu'il existe une surjection continue $\mathrm{P} \to \mathrm{E}$ avec P polonais. Comme P est séparable, il en est de même de son image continue E, donc le complété de E pour toute distance compatible avec sa topologie est polonais. En résumé: nous avons à faire avec des espaces métriques qui sont à la fois image continue d'un espace polonais et sous-espace d'un autre. Vu de cette optique, un espace analytique est 'presque standard'.

8. Une autre conséquence remarquable du théorème: Soit $(\mathrm{E},\mathcal{B})$ analytique, (B_n) une suite séparante d'ensembles boréliens; alors la tribu $\mathcal{B}$ est *engendrée* par les B_n.

En effet, soit $\mathcal{C}$ la tribu engendrée par les B_n; alors l'application identique

$$(\mathrm{E},\mathcal{B}) \to (\mathrm{E},\mathcal{C})$$

satisfait aux hypothèses du théorème, donc est un isomorphisme borélien, c'est-à-dire, $\mathcal{B}$ est égal à $\mathcal{C}$.

9. Tout espace polonais est souslinien et métrisable, donc tout espace borélien standard est analytique.

La réciproque est inexacte. Par exemple, on sait qu'il existe un sous-espace souslinien A de la droite réelle $\mathbb{R}$ qui n'est pas un ensemble borélien de $\mathbb{R}$. L'espace borélien $\big(\mathbb{R},\mathcal{B}(\mathbb{R})\big)$ est standard, et la sous-structure borélienne

$$\big(\mathrm{A},\mathcal{B}(\mathbb{R})\cap \mathrm{A}\big) = \big(\mathrm{A},\mathcal{B}(\mathrm{A})\big)$$

induite sur A est analytique; si cette dernière était standard, on aurait $\mathrm{A}\in\mathcal{B}(\mathbb{R})$ (n° 7 de la 3ème conférence), ce qui n'est pas le cas.

10. Revenons à la case départ: le rôle des ensembles boréliens dans la théorie de mesure.

Soit E un espace topologique métrisable, μ une mesure définie sur la tribu $\mathcal{B}=\mathcal{B}(\mathrm{E})$ des ensembles boréliens de E. Supposons μ finie (ce qui est le cas essentiel); donc,

$$\begin{aligned} &\mu : \mathcal{B} \to [0,+\infty)\,,\\ &\mu(\varnothing) = \varnothing\,,\\ &\mu\big(\dot{\bigcup}_{n=1}^{\infty}\mathrm{B}_n\big) = \textstyle\sum_{n=1}^{\infty}\mu(\mathrm{B}_n) \qquad (\text{les } \mathrm{B}_n \text{ disjoints}). \end{aligned}$$

Soit A un sous-espace souslinien de E. Il se peut que A ne soit pas borélien, donc pas question de parler de $\mu(\mathrm{A})$. Mais A est *presque* borélien dans le sens suivant: il existe des ensembles boréliens B, C tels que

$$\mathrm{B}\subset\mathrm{A}\subset\mathrm{C} \quad \text{et} \quad \mu(\mathrm{C}-\mathrm{B}) = 0\,.$$

La démonstration, pas difficile, est donnée dans l'Appendice de la documentation. Il s'agit d'un cas particulier du théorème de Choquet sur la 'capacitabilité' des ensembles sousliniens.

11. En conservant ces notations, supposons en plus que l'espace E est polonais, de sorte que l'espace borélien $(\mathrm{E},\mathcal{B})$ est standard. La restriction de la mesure μ à la tribu $\mathcal{B}\cap \mathrm{C}$ sert comme modèle d'une mesure sur A, ou sur $\mathcal{B}\cap \mathrm{A}$; d'ailleurs, la sous-structure borélienne $(\mathrm{C},\mathcal{B}\cap \mathrm{C})$ est elle-même standard, vu que l'ensemble borélien C est sous-espace lusinien de E. Pour ainsi dire, la structure borélienne $(\mathrm{A},\mathcal{B}\cap \mathrm{A})$ induite sur A est 'presque standard' par rapport à la mesure μ.

12. Soit maintenant $(\mathrm{E},\mathcal{B})$ un espace borélien, μ une mesure finie sur $\mathcal{B}$. On dit que la mesure μ est *standard* s'il existe un ensemble borélien $\mathrm{C}\in\mathcal{B}$ de complément négligeable par rapport à μ, tel que la sous-structure $(\mathrm{C},\mathcal{B}\cap \mathrm{C})$ induite sur C est standard. {Pour ainsi dire, $(\mathrm{E},\mathcal{B})$ est 'presque standard' par rapport à μ.}

Un théorème de Mackey: Sur un espace borélien analytique, toute mesure finie est standard.

Ce qui veut dire qu'un espace analytique est presque standard par rapport à toute mesure finie sur la tribu borélienne. {Ce résultat reste valable pour les mesures σ-finies, c'est-à-dire, pour lesquelles l'espace entier est réunion d'une suite d'ensembles boréliens de mesure finie.}

13. Ces résultats sont, pour la plupart, relativement facile, mais leur application et leur développement dans la théorie des représentations ne l'est pas du tout. Il est hors de question d'aborder ici cette étude, ce qui dépasserait d'ailleurs ma compétence. Néanmoins, j'essaierai, dans la conférence finale, d'esquisser la genèse de quelques-uns des espaces boréliens qui se présentent dans l'étude des algèbres stellaires, ainsi que le rôle de l'analycité pour ces espaces.

Comme préparation, terminons avec quelques mots sur le concept d'algèbre stellaire. D'abord, un exemple. Soit H un espace hilbertien, $\mathcal{L}(\mathrm{H})$ l'ensemble des applications linéaires continues dans H—en bref, les 'opérateurs' sur H. Avec les opérations linéaires ponctuelles et la composition comme produit, $\mathcal{L}(\mathrm{H})$ est une algèbre complexe. L'opération

$$x \mapsto x^* \quad (x^* \text{ l'opérateur 'adjoint'})$$

fait de $\mathcal{L}(\mathrm{H})$ une algèbre involutive, c'est-à-dire,

$$\begin{aligned} (x^*)^* &= x \\ (x+y)^* &= x^* + y^* \\ (\lambda x)^* &= \overline{\lambda} x^* \\ (xy)^* &= y^* x^* . \end{aligned}$$

La norme

$$\|x\| = \sup_{\|\xi\|\le 1} \|x\xi\|$$

fait de $\mathcal{L}(\mathrm{H})$ un espace de Banach, telle que

$$\|xy\| \leq \|x\| \cdot \|y\|$$

(donc $\mathcal{L}(\mathrm{H})$ est une algèbre de Banach), et

$$\|x^*\| = \|x\|$$

(on dit que $\mathcal{L}(\mathrm{H})$ est une algèbre de Banach involutive). En fait, on a l'identité

$$(*) \qquad \|x^*x\| = \|x\|^2\,;$$

ceci résulte de l'identité

$$(x^*x\xi|\xi) = \|x\xi\|^2$$

et du fait que l'opérateur x^*x est auto-adjoint.

On appelle *algèbre stellaire* toute algèbre de Banach involutive satisfaisant à l'identité $(*)$.

Après $\mathcal{L}(\mathrm{H})$, l'exemple le plus important: la sous-algèbre $\mathcal{LC}(\mathrm{H})$ des opérateurs compacts.

Plus généralement, si A est une sous-algèbre de $\mathcal{L}(\mathrm{H})$, fermée pour la topologie normique, et stable pour l'involution, alors A est une algèbre stellaire.

À isomorphisme près, il n'y a plus d'exemples: d'aprés le théorème célèbre de Gelfand et Naimark, toute algèbre stellaire A (dite 'abstraite') est isomorphe à un des exemples opératoriels (dits 'concrets'). Il s'agit de construire un espace hilbertien H et une application injective $\pi : \mathrm{A} \to \mathcal{L}(\mathrm{H})$ qui conserve les opérations algébriques, l'involution, et la norme—pour ainsi dire, une représentation *fidèle* de A sur H. Dans la théorie que nous allons esquisser, on n'exige plus la fidélité de la représentation π (c'est-à-dire, l'application π n'est plus supposée injective), mais on impose sur son image $\pi(\mathrm{A})$ des conditions restrictives: pour illustrer, que l'image ne laisse stable aucun sous-espace linéaire de H autre que 0 et H. Les représentations de ce type sont dites *irréductibles*; nous aurons plus à dire à leur égard dans la prochaine conférence.

5. À quoi bon les espaces borélien?

À quoi bon les espaces boréliens?

Pour donner un exemple de poids, il faut souffrir un peu—mais pas trop!

Il s'agit des représentations des algèbres dites 'stellaires'. {Rappelons que les algèbres stellaires sont, à isomorphisme près, les sous-algèbres des $\mathcal{L}(\mathrm{H})$, stables pour l'adjonction et fermées pour la topologie normique.} Tout ce que je vais dire—du moins la partie qui est vraie—se trouve dans le livre de Dixmier sur les algèbres stellaires.

Soit A une algèbre stellaire. Une *représentation* de A sur un espace hilbertien H est un homomorphisme $\pi : \mathrm{A} \to \mathcal{L}(\mathrm{H})$ d'algèbres involutives:

$$\begin{aligned}\pi(x+y) &= \pi(x)+\pi(y)\\ \pi(\lambda x) &= \lambda\pi(x)\\ \pi(xy) &= \pi(x)\pi(y)\\ \pi(x^*) &= \pi(x)^* .\end{aligned}$$

La dimension hilbertienne de H s'appelle la *dimension* de la représentation π. On dit que la représentation π est *irréductible* si H n'est pas nul et si les seuls sous-espaces linéaires de H stables pour tous les opérateurs $\pi(x)$ ($x \in \mathrm{A}$) sont 0 et H. Admise sous cette définition est la représentation nulle sur un espace hilbertien de dimension 1, une représentation d'ailleurs sans intéret (sauf qu'elle assure l'existence des représentations irréductibles); pour simplifier le langage par la suite, ce cas sera exclus, de sorte que irréductible' signifiera que π est non-nulle et que H n'admet que 0 et H comme sous-espaces linéaires stables.

On suppose l'algèbre A séparable (c'est-à-dire, qu'elle contient une suite partout dense). Il en résulte que ses représentations irréductibles sont de dimension soit finie, soit dénombrablement infinie:

$$1 \le \dim \mathrm{H} \le \aleph_0 .$$

On écrit H_n pour l'espace hilbertien type de dimension n, et

$$\mathrm{Irr}_n\,\mathrm{A}$$

pour l'ensemble (éventuellement vide) des représentations irréductibles de A sur H_n; l'ensemble somme de ces ensembles s'écrit

$$\mathrm{Irr}\,\mathrm{A} = \dot{\bigcup}_{n=1}^{\infty} \mathrm{Irr}_n\,\mathrm{A} .$$

Chacun des ensembles $\mathrm{Irr}_n\,\mathrm{A}$ est muni d'une topologie naturelle, où la convergence $\pi_i \to \pi$ signifie que

$$\|\pi_i(x)\xi - \pi(x)\xi\| \to 0 \quad \text{pour tout } x \in \mathrm{A}\,,\ \xi \in \mathrm{H}_n\,.$$

Pour cette topologie, $\mathrm{Irr}_n\,\mathrm{A}$ est un espace polonais; par conséquent $\mathrm{Irr}\,\mathrm{A}$, muni de la topologie somme, est aussi un espace polonais. D'où une structure borélienne standard sur $\mathrm{Irr}\,\mathrm{A}$.

Rappelons que deux représentations $\rho\,,\pi$ sont dites *équivalentes* s'il éxiste un isomorphisme u entre les espaces hilbertiens sur lesquels ils agissent, qui transforme $\pi(x)$ en $\rho(x)$ pour chaque élément x de A:

$$\rho(x) = u\pi(x)u^* \qquad (x \in \mathrm{A})\,;$$

on écrit alors $\rho \cong \pi$. L'ensemble quotient de $\mathrm{Irr}\,\mathrm{A}$ pour cette relation s'écrit $\hat{\mathrm{A}}$,

$$\hat{\mathrm{A}} = \mathrm{Irr}\,\mathrm{A}\,/\cong,$$

et on note $\hat{\pi}$ la classe de π. Notons

$$\gamma : \mathrm{Irr}\,\mathrm{A} \to \hat{\mathrm{A}}$$

l'application quotient $\pi \mapsto \hat{\pi}$, qui fait correspondre à la représentation π sa classe $\hat{\pi}$.

$\mathrm{Irr}\,\mathrm{A}$ étant un espace topologique, on pourrait définir sur $\hat{\mathrm{A}}$ la topologie finale pour γ, mais ce n'est pas la bonne. D'autre part, $\mathrm{Irr}\,\mathrm{A}$ est un espace borélien standard, donc on peut munir $\hat{\mathrm{A}}$ de la structure borélienne finale pour γ; c'est la bonne, et s'appelle la *structure borélienne de Mackey.* On va voir bientôt son importance. Notons par $\mathcal{M}(\hat{\mathrm{A}})$ le tribu pour cette structure:

$$\mathcal{M}(\hat{\mathrm{A}}) = \{\mathrm{T} \subset \hat{\mathrm{A}} :\ \gamma^{-1}(\mathrm{T}) \in \mathcal{B}(\mathrm{Irr}\,\mathrm{A})\,\}\,.$$

En bref, la structure borélienne de Mackey

$$\big(\hat{\mathrm{A}}, \mathcal{M}(\hat{\mathrm{A}})\big)$$

est le quotient de la structure standard $\big(\mathrm{Irr}\,\mathrm{A}, \mathcal{B}(\mathrm{Irr}\,\mathrm{A})\big)$ par la relation d'équivalence unitaire.

La topologie usuelle de $\hat{\mathrm{A}}$ est définie comme suit. Pour chaque représentation irréductible π, soit $\mathrm{Ker}\,\pi$ son noyau:

$$\mathrm{Ker}\,\pi = \{x \in \mathrm{A} :\ \pi(x) = 0\,\}\,;$$

les idéaux de ce type s'appellent les idéaux *primitifs* de A, et on écrit

$$\mathrm{Prim}\,\mathrm{A} = \{\mathrm{Ker}\,\pi :\ \pi \in \mathrm{Irr}\,\mathrm{A}\,\}$$

pour la totalité de ces idéaux. Des représentations équivalentes ont le même noyau, c'est-à-dire, le noyau de π ne dépend que sur sa classe; d'où une application surjective

$$\theta : \hat{\mathrm{A}} \to \mathrm{Prim}\,\mathrm{A}\,,$$

définie par $\theta(\hat{\pi}) = \mathrm{Ker}\,\pi$. Sur $\mathrm{Prim}\,\mathrm{A}$, il y a la topologie de Jacobson—dire qu'un ideal $\mathrm{I} \in \mathrm{Prim}\,\mathrm{A}$ est adhérent à une partie de $\mathrm{Prim}\,\mathrm{A}$ signifie que I contient l'intersection des idéaux appartenant à cette partie. On munit $\hat{\mathrm{A}}$ de la topologie initiale pour l'application θ , de sorte que les ouverts de $\hat{\mathrm{A}}$ sont, par définition, les images réciproques des ouverts de $\mathrm{Prim}\,\mathrm{A}$; muni de cette topologie, $\hat{\mathrm{A}}$ s'appelle le *spectre* (ou le 'dual') de l'algèbre A .

On a maintenant deux applications

$$\mathrm{Irr}\,\mathrm{A} \xrightarrow{\ \gamma\ } \hat{\mathrm{A}} \xrightarrow{\ \theta\ } \mathrm{Prim}\,\mathrm{A}\,,$$

toutes les deux surjectives. Les premier et troisième membres ont des topologies intrinsèques (donc des structures boréliennes): $\mathrm{Irr}\,\mathrm{A}$ est un espace polonais (donc un espace de Baire ayant une base dénombrable pour la topologie), tandis que $\mathrm{Prim}\,\mathrm{A}$ est un espace de Baire ayant une base dénombrable pour la topologie (mais pas toujours séparé).

Par contre, $\hat{\mathrm{A}}$ chipe sa structure borélienne de $\mathrm{Irr}\,\mathrm{A}$, et sa topologie de $\mathrm{Prim}\,\mathrm{A}$: plus précisément, la structure borélienne de Mackey

$$\mathcal{M}(\hat{\mathrm{A}})$$

est la structure borélienne finale pour l'application γ , tandis que la topologie de $\hat{\mathrm{A}}$ est la topologie initiale pour l'application θ . Par la définition de sa topologie, $\hat{\mathrm{A}}$ est un espace de Baire ayant une base dénombrable, et il va sans dire que θ est continue; il ne va pas sans dire que γ est continue, mais elle l'est quand même. La topologie de $\hat{\mathrm{A}}$ donne lieu à une structure borélienne topologique $\mathcal{B}(\hat{\mathrm{A}})$, en concurrence avec la structure de Mackey $\mathcal{M}(\hat{\mathrm{A}})$; vu que γ est continue, la structure de Mackey est la plus fine des deux:

$$\mathcal{M}(\hat{\mathrm{A}}) \supset \mathcal{B}(\hat{\mathrm{A}})\,.$$

Or, on attend à ce que quelque chose d'intéressante se produise quand ces deux structures coïncident.

Mais quoi? Pas facile à dire—il faut encore quelques préparatifs.

De toute façon, on peut reformuler le problème comme suit:

Théorème: Les conditions suivantes sur A (algèbre stellaire séparable) sont équivalentes:

(a) $\mathcal{M}(\hat{\mathrm{A}}) = \mathcal{B}(\hat{\mathrm{A}})$;
(b) $\big(\hat{\mathrm{A}}, \mathcal{M}(\hat{\mathrm{A}})\big)$ est standard;
(c) $\big(\hat{\mathrm{A}}, \mathcal{M}(\hat{\mathrm{A}})\big)$ est analytique;
(d) $\big(\hat{\mathrm{A}}, \mathcal{M}(\hat{\mathrm{A}})\big)$ est dénombrablement séparé;

et une dizaine de variations mineures.

La condition la plus faible est (d); selon la terminologie de Mackey, si une algèbre stellaire vérifie cette condition, on dit que son spectre est *lisse.* {Pourquoi 'lisse'? Je n'en sais rien.}

Or, l'équivalence de ces conditions sur le spectre est impressionante, mais ne nous dit rien de l'algèbre A elle-même. Pour cela il faut introduire d'autres concepts. Disons tout court: les algèbres satisfaisant à ces conditions sont précisément les algèbres 'postliminaires'—mais qu'est-ce-que c'est, une algèbre postliminaire?

Commençons avec les algèbres dites 'liminaires': on dit que l'algèbre stellaire A est *liminaire* si, pour chaque représentation irréductible π de A, tous les opérateurs $\pi(x)$ sont des opérateurs compacts.

Pour que A (toujours séparable) soit liminaire, il faut et il suffit que chaque point de son spectre $\hat{\mathrm{A}}$ soit ensemble fermé.

Dire que l'algèbre A est *postliminaire* signifie qu'elle possède une abondance d'images homomorphes liminaires; plus précisément, pour chaque idéal fermé $\mathrm{I} \neq \mathrm{A}$ de A, l'algèbre A/I possède un idéal fermé nonzéro J/I qui est liminaire. Pour le mettre dans une forme plus pittoresque, il y a beaucoup d'opérateurs compacts dans l'avenir de A.

On a alors le théorème suivant:

Théorème: L'algèbre stellaire séparable A est postliminaire si et seulement si son spectre est lisse.

Très bien—mais c'est un peu difficile de se rappeler qu'est-ce qu'une algèbre postliminaire et qu'est-ce qu'un spectre lisse. Il y a, heureusement, une caractérisation mémorable: A est postliminaire si et seulement s'il est de type I.

Mais qu'est-ce qu'une algèbre de type I? On a souffert assez—je ne vais pas le définir. Mais si tout cela vous pique l'intérêt, vous aurez tous les renseignements dans le beau livre de Dixmier.

En conclusion, je vous remercie pour votre audition tolérante et pour les questions intéressantes que vous m'avez posées; bonne chance, et adieu ...