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Resonance Theory for Schrödinger Operators

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Abstract: Resonances which result from perturbation of embedded eigenvalues are studied by time dependent methods. A general theory is developed allowing threshold eigenvalues and higher order Fermi Golden rule. The question of exponential decay rate of resonances is addressed; its uniqueness in the time dependent picture is shown is certain cases. The relation to the existence of meromorphic continuation of the properly weighted Green's function to time dependent resonance is further elucidated.

Key words. Resonances; Time-dependent Schrödinger equation

1. Introduction and results

Resonances may be defined in different ways, but usually refer to metastable behavior (in time) of the corresponding system. The standard physics definition would be as "bumps" in the scattering cross section, or exponentially decaying states in time, or poles of the analytically continued S matrix (when such an extension exists).

Mathematically, in the last 25 years one uses a definition close to the above, by defining λ to be a resonance (energy) if it is the pole of the meromorphic continuation of the weighted Green's function

$$\chi(H-Z)^{-1}\chi$$

with suitable weights χ (usually, in the Schrödinger Theory context, χ will be a C_0^∞ function). H is the Hamiltonian of the system. In many cases the equivalence of some of the above definitions has been shown [1–3]. However, the exponential behavior in time, and the correct estimates on the remainder are difficult to produce in general [21]. It is also not clear how to relate the time behavior to a resonance, uniquely, and what is the fundamental role of "analytic continuation"; see the review [5]. Important progress on such relations has recently been obtained; Orth [6] considered the time dependent behavior of states which can be related to resonances without the assumption of analytic continuation and established some preliminary estimates on the remainder terms. Then, Hunziker [7] was able to develop a quite general relation between resonances defined via poles of analytic continuations in the context of Balslev-Combes theory, to exponential decay in time, governed by the standard Fermi Golden rule. Here the resonances were small perturbations of embedded eigenvalues. In [1] a definition of resonance in a time dependent way is given and it is shown to agree with the one resulting from analytic continuation when it exists, in the Balslev-Combes theory. They also get exponential decay and estimates on the remainder terms.

Exact results, including the case ε large, for time dependent potentials have recently been obtained in [8]. Further notable results on the time dependent behavior of the wave equation were proved by Tang and Zworski [9]. The construction of states which resemble resonances, and thus decay approximately exponentially was accomplished e.g. in [10].

For resonance theory based on Balslev-Combes method the reader is referred to the book [2] and its comprehensive bibliography on the subject.

Then in a time-dependent approach to perturbation of embedded eigenvalues developed in [11] exponential decay and dispersive estimates on the remainder terms were proved in a general context, without the assumption of analytic continuation. The main condition required that a local decay estimate holds on the continuous spectrum near the embedded eigenvalue for the *unperturbed* Hamiltonian. The required decay is $O(t^{-2-\eta})$ and the Fermi Golden rule was required:

$$\Gamma \geq C\varepsilon^2 > 0.$$

where ε is measure of the size of the perturbation. The analysis in this work utilizes in some ways this framework, but generalizes the results considerably: the required time decay is $O(t^{-1-\eta})$ and we remove here the assumption of lower bound on Γ; it is replaced by

$$\Gamma \geq C\varepsilon^{\frac{2}{1-\eta}}$$

when $\eta < 1$, and

$$\Gamma > 0, \text{ arbitrary}$$

when $\eta > 1$. This may be important in applications where the lower bound on Γ is hard to prove [4].

Whenever a meromorphic continuation of the S-matrix or Green's function exists, the poles give an unambiguous definition of "resonance".

A time dependent approach or other definitions are less precise, not necessarily unique, as was observed in [6], but usually apply in more general situations, where analytic continuation is either hard to prove or not available.

The aim of this work is to generalize the results of [11] and to provide some clues about the definition of resonance by time dependent methods and its relation to the existence of "analytic continuation".

In particular, we will show that in general one can find the exponential decay rate up to higher order corrections depending on η and Γ.

In case it is known that analytic continuation exists, our approach provides a definition of a unique resonance corresponding to the perturbed eigenvalue. It is given by the solution of some transcendental equation in the complex plane and it also corresponds to a pole of the weighted Green's function. Finally, we analyze the connection between existence of analytic continuation and the Borel summability of the correction to the exponential behavior. Typically the remainder term looks like a stretched exponential times a Borel summable (incompletely) power series.

Our approach follows the setup of the time dependent theory of [11], combined with Laplace transform techniques. It is expected to generalize to the N-body case following [12].

We will follow, in part, the notation of [11].

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Given H_0 a self-adjoint operator on a Hilbert space $\mathcal{H}$, we assume that H_0 has a simple eigenvalue λ_0 with normalized eigenvector ψ_0:

$$H_0\psi_0 = \lambda_0\psi_0, \|\psi_0\| = 1 \tag{1}$$

Our interest is to describe the behavior of solutions of

$$i\frac{\partial\phi}{\partial t} = H\phi, \quad H \equiv H_0 + \epsilon W \tag{2}$$

with $\phi(0) = E_{\Delta}\phi_0$, where E_{Δ} is the spectral projection of H on the interval Δ and Δ is a small interval around λ_0. Furthermore, we will describe, in some cases, the analytic structure of $(H-z)^{-1}$ in a neighborhood of λ_0. W is a symmetric perturbation of H_0, such that H is self-adjoint with same domain as H_0.

1.1. Definitions. In this section we also introduce certain necessary terminology and notation. We then state the hypotheses (**H**) and (**W**) on the unperturbed Hamiltonian H_0 and on the perturbation W.

For an operator L, $\|L\|$ denotes its norm as an operator from L^2 to itself. We interpret functions of a self-adjoint operator as being defined by the spectral theorem. In the special case where the operator is H_0, we omit the argument, i.e., $g(H_0) = g$.

For an open interval Δ, we denote an appropriate smoothed characteristic function of Δ by $g_{\Delta}(\lambda)$. In particular, we shall take $g_{\Delta}(\lambda)$ to be a nonnegative C^{∞} function, which is equal to one on Δ and zero outside a neighborhood of Δ. The support of its derivative is furthermore chosen to be small compared to the size of Δ. We further require that $|g^{(n)}(\lambda)| \le c_n|\Delta|^{-n}, n \ge 1$.

P_0 denotes the projection on ψ_0, i.e., $P_0 f = (\psi_0, f)\psi_0$. P_{1b} denotes the spectral projection on $\mathcal{H}_{pp} \cap \{\psi_0\}^{\perp}$, the pure point spectral part of H_0 orthogonal to ψ_0. That is, P_{1b} projects onto the subspace of $\mathcal{H}$ spanned by all the eigenstates other than ψ_0. In our treatment, a central role is played by the subset of the spectrum of the operator H_0, $T^{\sharp}$ on which a sufficiently rapid local decay estimate holds. For a decay estimate to hold for $e^{-iH_0 t}$, one must certainly project out the bound states of H_0, but there may be other obstructions to rapid decay. In scattering theory these are called threshold energies. Examples of thresholds are: (i) points of stationary phase of a constant coefficient principal symbol for two body Hamiltonians and (ii) for N–body Hamiltonians, zero and eigenvalues of subsystems. We will not give a precise definition of thresholds. For us it is sufficient to say that away from thresholds the favorable local decay estimates for H_0 hold.

Let Δ_* be a union of intervals, disjoint from Δ, containing a neighborhood of infinity and all thresholds of H_0 except possibly those in a small neighborhood of λ_0. We then let

$$P_1 = P_{1b} + g_{\Delta_*}$$

where $g_{\Delta_*} = g_{\Delta_*}(H_0)$ is a smoothed characteristic function of the set Δ_*. We also define

$$\begin{aligned} \langle x \rangle^2 &= 1 + |x|^2 \\ \overline{Q} &= I - Q, \text{ and} \\ P_c^\sharp &= I - P_0 - P_1 \end{aligned} \tag{3}$$

Thus, $P_c^\sharp$ is a smoothed out spectral projection of the set $T^\sharp$ defined as

$$T^\sharp = \sigma(H_0) \setminus \{\text{eigenvalues, real neighborhoods of thresholds and infinity}\} \tag{4}$$

We expect $e^{-iH_0 t}$ to satisfy good local decay estimates on the range of $P_c^\sharp$; (see (**H4**) below).

Next we state our **hypotheses** on H_0.

(**H1**) H_0 is a self-adjoint operator with dense domain $\mathcal{D}$, in $L^2(\mathbb{R}^n)$.

(**H2**) λ_0 is a simple embedded eigenvalue of H_0 with (normalized) eigenfunction ψ_0.

(**H3**) There is an open interval Δ containing λ_0 and no other eigenvalue of H_0.

(**H4**) *Local decay estimate*: Let $r \geq 1 + \eta$ and $\eta > 0$. There exists $\sigma > 0$ such that if $\langle x \rangle^\sigma f \in L^2$ then

$$\| \langle x \rangle^{-\sigma} e^{-iH_0 t} P_c^\sharp f \|_2 \leq C \langle t \rangle^{-r} \| \langle x \rangle^\sigma f \|_2, \tag{5}$$

(**H5**) By appropriate choice of a real number c, the L^2 operator norm of $\langle x \rangle^\sigma (H_0 + c)^{-1} \langle x \rangle^{-\sigma}$ can be made sufficiently small.

Remarks:

(i) We have assumed that λ_0 is a simple eigenvalue to simplify the presentation. Our methods can be easily adapted to the case of multiple eigenvalues.

(ii) Note that Δ does not have to be small and that Δ_* can be chosen as necessary, depending on H_0.

(iii) In certain cases, the above local decay conditions can be proved even when λ_0 is a threshold; see [13].

(iv) Regarding the verification of the local decay hypothesis, one approach is to use techniques based on the Mourre estimate [14–16]. If Δ contains no threshold values, then quite generally, the bound (5) holds with r arbitrary and positive.

We now specify the conditions we require of the perturbation, W.
Conditions on W.
(**W1**) W is symmetric and $H = H_0 + W$ is self-adjoint on $\mathcal{D}$ and there exists $c \in \mathbb{R}$ (which can be used in (**H5**)), such that c lies in the resolvent sets of H_0 and H.
(**W2**) For the same σ as in (**H4**) and (**H5**) we have :

$$|||W||| \equiv \|\langle x\rangle^{2\sigma} W g_{\Delta}(H_0)\| + \|\langle x\rangle^{\sigma} W g_{\Delta}(H_0)\langle x\rangle^{\sigma}\| + \|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{-\sigma}\| < \infty$$

and

$$\|\langle x\rangle^{\sigma} W (H_0+c)^{-1}\langle x\rangle^{\sigma}\| < \infty \tag{6}$$

(**W3**) *Resonance condition–nonvanishing of the Fermi golden rule*:
For a suitable choice of λ (which will be made precise later)

$$\Gamma(\lambda) \equiv \pi\,\epsilon^2 (W\psi_0, \delta(H_0-\lambda)(I-P_0)W\psi_0) \neq 0 \tag{7}$$

In most cases $\Gamma = \Gamma(\lambda_0)$. But in the case Γ is very small it turns out that the "correct" Γ will be

$$\Gamma(\lambda_0 + \delta)$$

with δ given in the proof of Proposition 12. See also Section 4.
Remark: Let $\mathcal{F}_c^{H_0}$ denote the (generalized) Fourier transform with respect to the continuous spectral part of H_0. The resonance condition (7) can be expressed as

$$\Gamma \equiv \pi\, \left|\mathcal{F}_c^{H_0}[W\psi_0](\lambda)\right|^2 > 0 \tag{8}$$

2. Main results

Theorem 1 *Let* H_0 *satisfy the conditions* **(H1)**...**(H5)** *and the perturbation satisfy the conditions* **(W1)...(W3)**. *Assume moreover that* ϵ *is sufficiently small and*
(i) we have good regularity, $\eta > 1$
or
(ii) we have lower regularity $0 < \eta < 1$ *supplemented by the conditions*

$$\Gamma > C\epsilon^n, \quad n \geq 2$$

and $\eta > \frac{n-2}{n}$.

Under (i) or (ii) we have
a) $H = H_0 + \epsilon W$ *has no eigenvalues in* Δ.
b) The spectrum of H *is purely absolutely continuous in* Δ, *and*

$$\|\langle x\rangle^{-\sigma}e^{-iHt}g_\Delta(H)\Phi_0\|_2 \le C_\epsilon\langle t\rangle^{-1-\eta}\|\langle x\rangle^\sigma\Phi_0\|_2 \tag{9}$$

c) For Φ_0 in the range of $g_\Delta(H)$ we have for $t\ge 0$,

$$e^{-iHt}\Phi_0 = (I+A_W)\Big(e^{-i\omega_* t}a(0)\psi_0 + e^{-iH_0t}\phi_d(0)\Big) + R(t) \tag{10}$$

where

$$\|A_W\| \le C\epsilon|||W|||,$$

$a(0)$ and $\phi_d(0)$ are determined by the initial data. The complex frequency ω_* is given by

$$-i\omega_* = -is_0 - \Gamma$$

where s_0 solves the equation

$$s_0 + \omega + \epsilon^2\Im\left\{F(\epsilon,-i\omega-is_0)\right\} = 0 \tag{11}$$

(see (45) and (46) below) and

$$\Gamma = \epsilon^2\Re\left\{F(\epsilon,-is_0)\right\} \tag{12}$$

Remark: ω_* can be found by solving the transcendental equation (11) by either expansion or iteration if sufficient regularity is present (see also Proposition 12 and note following it and Lemma 17).

In case we have enough regularity ($\eta > 1$) then

Theorem 2

$$\omega_* = \lambda_0 + \epsilon(\psi_0, W\psi_0) + (\Lambda + i\Gamma)(1+o(1)) \tag{13}$$

where

$$\Lambda = \epsilon^2(W\psi_0, P.V.(H_0-\lambda_0)^{-1}W\psi_0) \tag{14}$$

$$\Gamma = \pi\epsilon^2(W\psi_0, \delta(H_0-\lambda_0)(I-P_0)W\psi_0) \tag{15}$$

3. Decomposition and isolation of resonant terms

We begin with the following decomposition of the solution of (2):

$$e^{-iHt}\phi_0 = \phi(t) = a(t)\psi_0 + \tilde{\phi}(t) \tag{16}$$

$$\Big(\psi_0, \tilde{\phi}(t)\Big) = 0, \quad -\infty < t < \infty \tag{17}$$

Substitution into (2) yields

$$i\partial_t\tilde{\phi} = H_0\phi + \epsilon W\tilde{\phi} - (i\partial_t a - \lambda_0 a)\psi_0 + a\epsilon W\psi_0 \tag{18}$$

Recall now that $I = P_0 + P_1 + P_c^\sharp$. Taking the inner product of (18) with ψ_0 gives the amplitude equation:

$$i\partial_t a = (\lambda_0 + (\psi_0, \epsilon W\psi_0)\,)a + (\psi_0, \epsilon W P_1\tilde{\phi}) + (\psi_0, \epsilon W\phi_d), \tag{19}$$

where

$$\phi_d \equiv P_c^\sharp\tilde{\phi} \tag{20}$$

The following equation for ϕ_d is obtained by applying $P_c^\sharp$ to equation (18):

$$i\partial_t\phi_d = H_0\phi_d + P_c^\sharp\epsilon W(P_1\tilde{\phi} + \phi_d) + aP_c^\sharp\epsilon W\psi_0 \tag{21}$$

To derive a closed system for $\phi_d(t)$ and $a(t)$ we now propose to obtain an expression for $P_1\tilde{\phi}$, to be used in equations (19) and (21). Since $g_\Delta(H)\phi(\cdot,t) = \phi(\cdot,t)$ we find

$$(I - g_\Delta(H))\phi = (I - g_\Delta(H))\Big(a\psi_0 + P_1\tilde{\phi} + P_c^\sharp\tilde{\phi}\Big) = 0 \tag{22}$$

or

$$(I - g_\Delta(H)g_I(H_0))P_1\tilde{\phi} = -\overline{g}_\Delta(H)\Big(a\psi_0 + \phi_d\Big) \tag{23}$$

where $g_I(\lambda)$ is a smooth function which is identically equal to one on the support of $P_1(\lambda)$, and which has support disjoint from Δ. Therefore

$$P_1\tilde{\phi} = -B\overline{g}_\Delta(H)(a\psi_0 + \phi_d), \tag{24}$$

where

$$B = (I - g_\Delta(H)g_I(H_0))^{-1}. \tag{25}$$

This computation is justified in Appendix B of [11].

Proposition 3 ([11]) *The operator B in (25) is a bounded operator on $\mathcal{H}$.*

From (24) we get

$$\phi(t) = a(t)\psi_0 + \phi_d + P_1\tilde{\phi} = \tilde{g}_\Delta(H)(a(t)\psi_0 + \phi_d(t)), \tag{26}$$

with

$$\tilde{g}_\Delta(H) \equiv I - B\overline{g}_\Delta(H) = Bg_\Delta(H)(I - g_I(H_0)). \tag{27}$$

(see (3). Although $\tilde{g}_\Delta(H)$ is not really defined as a function of H, we indulge in this mild abuse of notation to emphasize its dependence on H. In fact, in some sense, $\tilde{g}_\Delta(H) \sim g_\Delta(H)$ to higher order in ϵ [11].

Substitution of (24) into (21) gives:

$$i\partial_t\phi_d = H_0\phi_d + aP_c^\sharp\epsilon W\tilde{g}_\Delta(H)\psi_0 + P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\phi_d \tag{28}$$

and

$$\begin{aligned} i\partial_t a = \Big(\lambda_0 + (\psi_0, \epsilon W\tilde{g}_\Delta(H)\psi_0)\Big)a + (\psi_0, \epsilon W\tilde{g}_\Delta(H)\phi_d) \\ = \omega a + (\omega_1 - \omega)a + (\psi_0, \epsilon W\tilde{g}_\Delta(H)\phi_d) \end{aligned} \tag{29}$$

where

$$\omega = \lambda_0 + (\psi_0, \epsilon W\psi_0) \tag{30}$$

$$\omega_1 = \lambda_0 + (\psi_0, \epsilon W\tilde{g}_\Delta(H)\psi_0) \tag{31}$$

We write (28) as an equivalent integral equation.

$$\begin{aligned} \phi_d(t) = e^{-iH_0t}\phi_d(0) - i\int_0^t e^{-iH_0(t-s)}a(s)P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\psi_0 ds \\ - i\int_0^t e^{-iH_0(t-s)}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\phi_d ds \end{aligned} \tag{32}$$

Proposition 4 ([11]) *Suppose $|a(t)| \le a_\infty\langle t\rangle^{-1-\alpha}$ and assume that $\eta > 0$ and $\alpha \ge \eta$. Then for some $C > 0$ we have*

$$\|\langle x\rangle^{-\sigma}\phi_d(t)\|_{L^2} \le C\langle t\rangle^{-1-\eta}\left(\|\langle x\rangle^{\sigma}\phi_d(0)\|_{L^2} + a_\infty|||W|||\right)$$

We define K as an operator acting on $C(\mathbb{R}^+, \mathcal{H})$, the space of continuous functions on $\mathbb{R}^+$ with values in $\mathcal{H}$ by

$$\Big(K\,f\Big)(t,x) = \int_0^t e^{-iH_0(t-s)}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)f(s,x)ds \tag{33}$$

We introduce on $C(\mathbb{R}^+, \mathcal{H})$ the norm

$$\|f\|_\beta = \sup_{t\geq 0} \langle t\rangle^\beta \|f(\cdot,t)\|_{\mathcal{H}} \tag{34}$$

and define the operator norm

$$\|A\|_{\beta;\sigma} = \|\langle x\rangle^{-\sigma} A \langle x\rangle^{\sigma}\|_\beta \tag{35}$$

Proposition 5 *If ϵ is small, $0 \leq \beta \leq r$, $r > 1$ and for some $\beta_1 > 0$ we have $\|\langle x\rangle^{-\sigma} e^{-iH_0 t} P_c^\sharp \langle x\rangle^{-\sigma}\| \leq C t^{-1-\beta_1}$, then*

$$\|Kf\|_{\beta;\sigma} \leq C_{\beta;\sigma;r}\epsilon \tag{36}$$

The proof is straightforward.

Using the definition of K given above we see that $K(1-K)^{-1} = \sum_{n=1}^\infty K^n$ is also bounded. We can now rewrite the equations for ϕ_d as

$$\phi_d(t) = K\big(a(t)\psi_0\big) + K\phi_d = (I-K)^{-1}\big(a(t)\psi_0\big) \tag{37}$$

and therefore

$$i\partial_t a = \omega a + \big(\psi_0, \epsilon W \tilde{g}_\Delta(H)(I-K)^{-1} K\big(a\psi_0\big)\big) \tag{38}$$

We now define two operators on L^∞ by

$$\tilde{j}(a) = \Big(v, \langle x\rangle^{-\sigma} K(a\psi_0)\Big); \quad \text{where } v = \langle x\rangle^\sigma \epsilon W \tilde{g}_\Delta(H)\psi_0 \tag{39}$$

and

$$j(a) = \Big(v, \langle x\rangle^{-\sigma}(I-K)^{-1} K(a\psi_0)\Big) \tag{40}$$

Proposition 6 *The operators $\tilde{j}$ and j are bounded from L^∞ into itself.*

The proof is immediate, using Proposition 5 for $\beta = 0$.

Remark. The equation for a can now be written in the equivalent integral form

$$a(t) = a(0)e^{-i\omega t} + e^{-i\omega t}\int_0^t e^{i\omega s} j(a)(s)ds \equiv a(0)e^{-i\omega t} + J(a) \tag{41}$$

Definition 1 *Consider the spaces $L^\infty_{T;\nu}$ and L^∞_ν to be the spaces of functions on $[0,T]$ and $\mathbb{R}^+$ respectively, in the norm*

$$\|a\|_\nu = \sup_s |e^{-\nu s} a(s)| \tag{42}$$

Remark 7 *We note that for $T \in \mathbb{R}^+$, the norm on $L^\infty_{T;\nu}$ is equivalent to the usual norm on $L^\infty[0,T]$.*

Proposition 8 *For some constants* c, C *and* $\tilde{c}$ independent of T *we have* $\|ja\|_\nu \le c\nu^{-1}\epsilon^2\|a\|_\nu$, $\|Ja\|_\nu \le C\nu^{-2}\epsilon^2\|a\|_\nu$ *and* $\|\tilde{j}a\|_\nu \le \tilde{c}\nu^{-1}\epsilon^2\|a\|_\nu$, *and thus* j, J, *and* $\tilde{j}$ *are defined on* $L^\infty_{T;\nu}$ *and* L^∞_ν *and their norms, in these spaces, are estimated by*

$$\|j\|_\nu \le c\nu^{-1}\epsilon^2; \quad \|\tilde{j}\|_\nu \le \tilde{c}\nu^{-1}\epsilon^2; \quad \|J\|_\nu \le C\nu^{-2}\epsilon^2 \tag{43}$$

This proof is also immediate.

Proposition 9 *The equation (38) has a unique solution in* $L^1_{loc}(\mathbb{R}^+)$, *and this solution belongs to* L^∞_ν *if* $\nu > \nu_0$ *with* ν_0 *sufficiently large. Thus, in the half-plane* $\Re(p) > \nu_0$ *the Laplace transform of* a

$$\hat{a} \equiv \int_0^\infty e^{-pt}a(t)dt \tag{44}$$

exists and is analytic in p.

Proof. By Proposition 8, and since $\|e^{-i\omega t}\|_\nu = 1$, for large ν the equation (41) is contractive in $L^\infty_{T;\nu}$ and has a unique solution there. It thus has a unique solution in L^1_{loc}, by Remark 7. Since by the same argument equation (41) is contractive in $L^\infty_{T;\nu}$ and since $L^\infty_\nu \subset L^1_{loc}$, the unique L^1_{loc} solution of (41) is in L^∞_ν as well. The rest is straightforward.

Proposition 10 *For* $\Re(p) > \nu_0$, *the Laplace transform* $\hat{a}$ *satisfies the equation*

$$\begin{aligned} &ip\hat{a} = \omega\hat{a} + ia(0) \\ &+\left(\psi_0, \epsilon W\tilde{g}_\Delta(H)\left[\left(I + \frac{iI}{p+iH_0}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\right)^{-1}\hat{a}(p)\frac{-iI}{p+iH_0}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\right]\psi_0\right) \end{aligned} \tag{45}$$

or

$$(ip - \omega + i\epsilon^2 F(p,\epsilon))\hat{a}(p) = ia(0) \tag{46}$$

Our assumptions easily imply that if ϵ is small enough, then:

(a) $F(p)$ is analytic except for a cut along $i\Delta$. $F(p)$ is Hölder continuous of order $\eta > 0$ (η depends on β) at the cut, i.e.

$$\lim_{\gamma\downarrow 0} F(i\tau \pm \gamma) \in H^\eta$$

the space of Hölder continuous functions of order η.

(b) $|F(p)| \le C|p|^{-1}$ for some $C > 0$ as $|p| \to \infty$.

To see it we write

$$B = B_1 B_2\langle x\rangle^{-\sigma}; \quad B_1 \equiv \frac{I}{p+iH_0}P_c^\sharp\langle x\rangle^{-\sigma}; \quad B_2 \equiv \epsilon\langle x\rangle^\sigma W\tilde{g}_\Delta(H)\langle x\rangle^\sigma \tag{47}$$

Noting that $P_c^\sharp$ projects on the interval Δ it is clear by the spectral theorem that $\langle x\rangle^{-\sigma}B$ is analytic in p on $\mathcal{D}\equiv\mathbb{C}\setminus(i\Delta)$. By the assumption on the decay rate and the Laplace transform of eq. (5) we have that

$$B_3(p)\equiv\langle x\rangle^{-\sigma}\frac{I}{p+iH_0}P_c^\sharp\langle x\rangle^{-\sigma} \tag{48}$$

is uniformly Hölder continuous, of order η, as $p\to i\Delta$. For $p_0\in i\Delta$, the two sided limits $\lim_{a\downarrow 0}B_3(p_0\pm a)=B_3^\pm$ will of course differ, in general. A natural closed domain of definition of B_3 is $\mathcal{D}$ together with the two sides of the cut, $\overline{\mathcal{D}}\equiv\mathcal{D}\cup\partial\mathcal{D}^+\cup\partial\mathcal{D}^-$. We then write

$$\|B_3\|\le C_1(p) \tag{49}$$

where we note that C_1 can be chosen so that:

Remark 11 $C_1(p)>0$ *is uniformly bounded for* $p\in\overline{\mathcal{D}}$ *and* $C_1(p)=O(p^{-1})$ *for large* p.

Hence for some C_2 we have uniformly in p (choosing ϵ small enough),

$$\|\langle x\rangle^{-\sigma}(B_1B_2)^n\|\le C_2^n\epsilon^n \tag{50}$$

and therefore the operator

$$\epsilon W\tilde{g}_\Delta(H)\left[\left(I-\frac{I}{p+iH_0}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\right)^{-1}\frac{I}{p+iH_0}P_c^\sharp\epsilon W\tilde{g}_\Delta(H)\right] \tag{51}$$

is analytic in $\mathcal{D}$ and in in $H^\eta(\overline{\mathcal{D}})$. Finally, part (c) follows from the above arguments and Remark 11.

4. General case

4.1. Definition of Γ. We have

$$\hat{a}(p)=\frac{ia(0)}{ip-\omega+i\epsilon^2F(\epsilon,p)} \tag{52}$$

We are most interested in the behavior of $\hat{a}$ for $p=is$, $s\in\mathbb{R}$. Γ will be defined in terms of the approximate zeros of the denominator in (52). Let $F=:F_1+iF_2$.

Proposition 12 *For ϵ small enough, the equation $s+\omega+\epsilon^2F_2(\epsilon,is)=0$ has at least one root s_0, and $s_0=-\omega+O(\epsilon^2)$. If $\eta\ge 1$, then for small enough ϵ the solution is unique. If $\eta<1$ then two solutions s_1 and s_2 differ by at most $O(\epsilon^{\frac{2}{1-\eta}})$.*

Proof. We write $s = -\omega + \delta$ and get for δ an equation of the form $\delta = \epsilon^2 G(\delta)$ where $G(x) = -F_2(\epsilon, ix - i\omega)$, and $G(x) \in H^\eta$. The existence of a solution for small ϵ is an immediate consequence of continuity and the fact that $\delta - \epsilon^2 G(\delta)$ changes sign in an interval of size $\epsilon^2 \|G\|_\infty$. If $\eta \geq 1$ we note that the equation $\delta = \epsilon^2 G(\delta)$ is contractive for small ϵ and thus has a unique root. If instead $0 < \eta < 1$ we have, if δ_1, δ_2 are two roots, then for some $K > 0$ independent of ϵ, $|\delta_1 - \delta_2| = \epsilon^2 |G(\delta_1) - G(\delta_2)| \leq \epsilon^2 K |\delta_1 - \delta_2|^\eta$ whence the conclusion.

Assumption If $\eta < 1$ then we assume that $\epsilon^2 F_1(\epsilon, -i\omega) \gg \epsilon^{\frac{2}{1-\eta}}$ for small ϵ. When $\eta > 1$ this restriction will not be needed, cf. § 4.3.

Definition We choose one solution $s_0 = -\omega + \delta$ and let Γ be defined by (12).
Note. In the case $\eta < 1$ the choice of s_0 yields, by the previous assumption a (possible) arbitrariness in the definition of Γ of order $O(\epsilon^{\frac{2}{1-\eta}}) = o(\Gamma)$.
Remarks on the verifiability of condition $\Gamma > 0$. We will look at various scenarios, which are motivated by concrete examples, in which the condition of positivity reduces to a condition on $F(\epsilon, -i\omega)$.

Let

$$\Gamma_0 = \epsilon^2 F_1(\epsilon, -i\omega); \quad \gamma_0 = \epsilon^2 F_2(\epsilon, -i\omega)$$

where we see that Γ_0 and γ_0 are $O(\epsilon^2)$. The equation for δ reads

$$\delta = -\epsilon^2 [F_2(\epsilon, -i\omega + i\delta) - F_2(\epsilon, -i\omega)] - \gamma_0 = \epsilon^2 H(\delta) - \gamma_0$$

where $H(0) = 0$. We write $\delta = -\gamma_0 + \zeta$ and get

$$\zeta = \epsilon^2 H(-\gamma_0 + \zeta)$$

and the definition of Γ becomes

$$\Gamma = \epsilon^2 F_1(\epsilon, -i\omega - i\gamma_0 + i\zeta)$$

Proposition 13 *(i) Assume that $\eta > 1$ and $\gamma_0 = o(\epsilon^{-2}\Gamma_0)$. Then as $\epsilon \to 0$ we have*

$$\Gamma = \Gamma_0 + o(\Gamma_0) \tag{53}$$

and thus the positivity of Γ follows from that of Γ_0.

(ii) Assume that $\eta < 1$, $\gamma_0 = o(\epsilon^{-2}\Gamma_0^{1/\eta})$ and $\Gamma_0 \gg \epsilon^{\frac{2}{1-\eta}}$ as $\epsilon \to 0$. Then again (53) holds.

Proof. (i) Since $\zeta = O(\epsilon^2 \gamma_0) + O(\epsilon^2 \zeta)$ we get $\zeta = O(\epsilon^2 \gamma_0)$, implying that

$$\Gamma = \epsilon^2 F_1 [\epsilon, -i\omega - i\gamma_0(1 + o(1))] = \Gamma_0 + O(\epsilon^2 \gamma_0) = \Gamma_0 + o(\Gamma_0)$$

(ii) We have

$$\zeta = O(\epsilon^2 \gamma_0^\eta) + O(\epsilon^2 \zeta^\eta) \tag{54}$$

If $\zeta \leq \text{const.}\gamma_0$ as $\epsilon \to 0$, then the proof is as in part (i). If on the contrary, for some large constant C we have $\zeta > C\gamma_0$ then by (54) we have $\zeta < \text{const.}\epsilon^2 \zeta^\eta$ so that $\zeta = O(\epsilon^{2/(1-\eta)})$ and $\epsilon^2 \zeta^\eta = O(\epsilon^{2/(1-\eta)}) = o(\Gamma_0)$. But then

$$\Gamma = \epsilon^2 F_1(\epsilon, -i\omega) + O(\epsilon^2 \gamma_0^\eta) + O(\epsilon^2 \zeta^\eta) = \Gamma_0 + o(\Gamma_0)$$

4.2. Exponential decay. We let $p = is_0 + v$ and get

Proposition 14 *Let $r \in \mathbb{R}^+$.*
(i) As $\epsilon \to 0$ and $t\Gamma = r$ we have

$$a(t) = e^{-is_0 t} e^{-\Gamma t} + O(\epsilon^2 \Gamma^{\eta-1})$$

(ii) (For much longer time) as $t \to \infty$ we have

$$a(t) = O(\Gamma^{-1} t^{-\eta-1})$$

Proof. (i) Note first that, taking $\Re(v) > 0$ we get

$$\begin{aligned} F(\epsilon, -is_0 + v) &= \int_0^\infty e^{-is_0 t - vt} f(t)dt = \int_0^\infty e^{-vt}\left(\int_0^t e^{-is_0 u} f(u)du\right)' \\ &= v\int_0^\infty e^{-vt}\int_0^t e^{-is_0 u} f(u)du = v\int_0^\infty e^{-vt}\left(\int_0^\infty - \int_t^\infty\right) e^{-is_0 u} f(u)du \\ &= \int_0^\infty e^{-is_0 u} f(u)du - v\int_0^\infty e^{-vt}\int_t^\infty e^{-is_0 u} f(u)du \\ &=: F(\epsilon, -is_0) - vL[g](v) \end{aligned} \quad (55)$$

so that with $h(v) = vL[g](v)$ we have

$$2\pi i a(t) = e^{-is_0 t}\int_{-i\infty}^{i\infty} \frac{e^{vt}}{v + \Gamma + \epsilon^2 h(v)} dv \quad (56)$$

where by construction we have $h \in H^\eta$, h is analytic in $\mathbb{C} \setminus i\Delta$ and $h(0) = 0$. We write

$$\begin{aligned} \int_{-i\infty}^{i\infty} \frac{e^{vt}}{v + \Gamma + \epsilon^2 h(v)} dv &= \int_{-i\infty}^{i\infty} \frac{e^{vt}}{(v+\Gamma)\left(1 + \epsilon^2 h(v+\Gamma)^{-1}\right)} dv \\ &= \int_{-i\infty}^{i\infty} \frac{e^{vt}dv}{v+\Gamma} - \epsilon^2 \int_{-i\infty}^{i\infty} \frac{1}{v+\Gamma}\frac{h(v+\Gamma)^{-1}}{1+\epsilon^2 h(v+\Gamma)^{-1}} e^{vt}dv \end{aligned} \quad (57)$$

We first need to estimate $L^{-1}\left[h(v+\Gamma)^{-1}\right]$ (the transformation is well defined, since the function is just $(v+\Gamma)^{-1}(F(\epsilon, -is_0 + v) - F(\epsilon, -is_0)))$. We need to write

$$vL[g](v) = (v+\Gamma)L[g_1](v) \quad \text{or} \quad L[g_1] = \left(1 - \frac{\Gamma}{v+\Gamma}\right) L[g] \quad (58)$$

and thus

$$g_1 = g - \Gamma e^{-\Gamma t}\int_0^t e^{\Gamma s} g(s)ds \quad (59)$$

Since $|g(t)| < \text{Const.} t^{-\eta}$ we have

$$|g_1(t)| \le \text{Const.} t^{-\eta} + e^{-\Gamma t} \int_0^{\Gamma t} e^u \left(\frac{u}{\Gamma}\right)^{-\eta} du \le \text{Const.} t^{-\eta} \tag{60}$$

A similar inequality holds for

$$Q = L^{-1} \left[\frac{\frac{h}{v+\Gamma}}{1 + \frac{\epsilon^2 h}{v+\Gamma}} \right] \tag{61}$$

Indeed, we have

$$Q = -L^{-1} \left[\frac{h}{v+\Gamma} \right] + \epsilon^2 L^{-1} \left[\frac{h}{v+\Gamma} \right] * Q \tag{62}$$

It is easy to check that for $t \le r\Gamma^{-1}$ and small enough ϵ this equation is contractive in the norm $\|Q\| = \sup_{s \le t} \langle s \rangle^{\eta} |Q(s)|$.

But now, for constants independent of ϵ,

$$\begin{aligned} \epsilon^2 L^{-1} \left[\frac{1}{v+\Gamma} \right] * Q &\le \text{Const.} e^{-\Gamma s} \int_0^t e^{\Gamma s} s^{-\eta} ds \\ &= \epsilon^2 \text{Const.} e^{-\Gamma s} \Gamma^{-1} \int_0^{\Gamma t} e^u \left(\frac{u}{\Gamma}\right)^{-\eta} du \le \text{Const.} \frac{\epsilon^2}{\Gamma^{1-\eta}} \end{aligned} \tag{63}$$

(ii) We now use the fact that

$$\frac{h}{v+\Gamma} = \frac{F}{v+\Gamma} - \frac{F_0}{v+\Gamma}$$

and get for some $C > 0$,

$$H_1 = L^{-1} \left[\frac{h}{v+\Gamma} \right] = e^{-\Gamma t} \int_0^t e^{\Gamma s} f(s) ds + C e^{-\Gamma t}$$

and thus, proceeding as in the proof of (i) we get for some $C > 0$ $H_1 \le C\Gamma^{-1} \langle t \rangle^{-\eta-1}$. To evaluate $a(t)$ for large t we resort again to Q as defined in (61) which satisfies (62). This time we note that the equation is contractive in the norm $\sup_{s \ge 0} |\langle s \rangle^{1+\eta} \cdot |$ when ϵ is small enough.

4.3. Regularity $\eta > 1$. In this case we obtain better estimates. We write $G(v) = L^{-1}[g](v)$ and (56) becomes

$$a(t) = e^{-is_0 t} \int_{-i\infty}^{i\infty} \frac{e^{vt}}{v + \Gamma + \epsilon^2 v G(v)} dv \tag{64}$$

Now

$$\begin{aligned} &L^{-1}\left[(v + \Gamma + \epsilon^2 v G(v))^{-1}\right] \\ &\qquad = L^{-1}\left[\frac{1}{v+\Gamma}\right] - \epsilon^2 L^{-1}\left[\frac{1}{v+\Gamma}\right] * L^{-1}\left[\frac{\frac{v}{v+\Gamma}G}{1+\epsilon^2 \frac{v}{v+\Gamma}G}\right] \end{aligned} \tag{65}$$

Proposition 15 *Let*

$$H_2(t) = L^{-1}\left[\frac{\frac{v}{v+\Gamma}G}{1+\epsilon^2 \frac{v}{v+\Gamma}G}\right]$$

We have

$$|H_2| \le Const. \langle t\rangle^{-\eta}; \quad \int_0^\infty H_2(t)dt = 0 \tag{66}$$

Proof. Consider first the function

$$h_1 = v(v+\Gamma)^{-1}G = G - \Gamma(v+\Gamma)^{-1}G$$

we see that (cf. (55))

$$H_1 = L^{-1}h_1 = \int_t^\infty e^{-is_0 u} f(u)du - \Gamma\ e^{-\Gamma t}\int_0^t e^{\Gamma s}\int_s^\infty e^{-is_0 u} f(u)duds \tag{67}$$

and thus, for some positive constants C_i,

$$|H_1| \le \mathrm{Const.}t^{-\eta} + \mathrm{Const.}e^{-\Gamma t}\int_0^{\Gamma t} e^v \Gamma^{-\eta}\langle v\rangle^{-\eta} dv \tag{68}$$

and thus, since $h_1(0) = 0$ we have

$$|H_1| \le \mathrm{Const.}\langle t\rangle^{-\eta}; \quad \int_0^\infty H_1(t)dt = 0$$

Note now that the function

$$\frac{v}{v+\Gamma}G\left(1+\epsilon^2\frac{v}{v+\Gamma}G\right)^{-1}$$

vanishes for $v = 0$. Note furthermore that

$$H_2 = H_1 - \epsilon^2 H_1 * H_2$$

It is easy to check that this integral equation is contractive in the norm $\|H\| = \sup_{s\le t}|\langle s\rangle^\eta H(s)|$ for small enough ϵ; the proof of the proposition is complete.

Proposition 16

$$L^{-1}\left[(v+\Gamma+\epsilon^2 G(v))^{-1}\right]=e^{-\Gamma t}+\Delta(t)$$

where for some constant C independent of ϵ, t, Γ we have

$$|\Delta|\le C\epsilon^2\langle t\rangle^{-\eta+1}$$

Proof. We have, by (65)

$$\Delta(t)=\epsilon^2 e^{-\Gamma t}\int_0^t e^{\Gamma s}\left(\int_s^\infty H_2(u)du\right)' ds$$
$$=\epsilon^2\int_t^\infty H_2(s)ds-\Gamma e^{-\Gamma t}\int_0^t e^{\Gamma s}\int_s^\infty H_2(u)du \quad (69)$$

The estimate of the last term is done as in (68).

5. Analytic case

Under the assumption

(**H6**) $\chi(H_0-z)^{-1}P_c^\sharp\chi$ has meromorphic continuation from $\Im(z)>0$ to a $\mathbb{C}\setminus\mathbb{R}$ neighborhood of Δ, and no poles accumulating to Δ, where χ is an appropriate weight.

we can prove stronger results on the existence of a unique resonance rate as well as give a more detailed description of the time behavior.

In this case, due to the spectral theorem, the function $F_\epsilon(p)$ will have meromorphic continuation in a neighborhood of Δ, with branch points at the end points of the interval $\overline{\Delta}$, corresponding to the thresholds of the operator $H_0 g_\Delta(H_0)$ (or $H_0 P_c^\sharp$). We use the same notation F_ϵ for this continuation.

5.1. Existence and uniqueness of a resonance. This follows from the following lemma, in which we impose conditions that were suggested by a number of examples.

Lemma 17 *Let $E(p,\epsilon)$ be a function with the following properties:*

(i) $E\in H^\eta(\overline{\mathcal{D}})$ and E is analytic in $\mathcal{D}$ (this allows for branch-points on the boundary of the domain, a more general setting that meromorphicity).

(ii) $|E(p,\epsilon)|\le C\epsilon^2$ for some C.

(iii) $\lim_{a\downarrow 0}\Re E(-i\omega-a,\epsilon)=-\Gamma_0<0$.

If (a) $\eta>1$ $E(-i\omega,\epsilon)=o(\Gamma_0/\epsilon^2)$ or (b) $\eta<1$ and $E(-i\omega,\epsilon)=O(\Gamma_0)$ and ϵ is small enough, then the function $G(p,\epsilon)=p+i\omega+E(p,\epsilon)$ has a unique zero $p=p_z$ in $\overline{\mathcal{D}}$ and furthermore $\Re(p_z)<0$. In fact, for some $C>0$

$$\Re(p_z)+\Gamma_0=o(\Gamma_0) \quad (70)$$

Remark If the condition For $\eta > 1$, $E(-i\omega, \epsilon) = o(\epsilon^{-2}\Gamma_0)$ is not satisfied, then we can replace $-i\omega$ by $-i\omega - is_0$ and the uniqueness of the complex zero will still be true.

Proof. The equation $G = 0$ can be written as

$$0 = p + i\omega + E(-i\omega) + [E(p) - E(-i\omega)]$$

or, taking $p = -i\omega + \zeta$, $\phi(\zeta, \epsilon) = E(p) - E(-i\omega)$,

$$\zeta = -E(-i\omega) + \epsilon^2\phi(\zeta, \epsilon)$$

Consider a square centered at $E(-i\omega)$ with side $2|\Re(E(-i\omega))| = 2\Gamma_0$. For both parts (a) and (b) of the lemma, note that in our assumptions and by the choice of the square we have

$$\left|\frac{\epsilon^2\phi(\zeta, \epsilon)}{\zeta + E(-i\omega)}\right| \to 0 \quad (\text{as } \epsilon \to 0) \tag{71}$$

(on all sides of the square). In case (a) on the boundary of the rectangle we have by construction of the rectangle, $|\zeta + E(-i\omega)| \geq \Gamma_0$. Also by construction, on the sides of the rectangle we have $|\zeta| \leq \Gamma_0$. Still by assumption, $\phi(\epsilon, \zeta) \leq C\zeta = o(\epsilon^{-2}\Gamma_0)$ and the ratio in (71) is $o(1)$. In case (b), we have

$$\epsilon^2\phi(\epsilon, \zeta) = O(\epsilon^2\zeta^\eta) = O(\epsilon^2\Gamma_0^\eta) = o(\Gamma_0)$$

Thus, on the boundary of the square, the variation of the argument of the functions $\zeta + E(-i\omega) - \epsilon^2\phi(\zeta)$ and that of $\zeta + E(-i\omega)$ differ by at most $o(1)$ and thus have to agree exactly (being integer multiples of $2\pi i$); thus $\zeta + E(-i\omega) - \epsilon^2\phi(\zeta)$ has exactly one root in the square. The same argument shows that $p + i\omega + E(p, \epsilon)$ has no root in any other region in its analyticity domain except in the square constructed in the beginning of the proof.

In the case there is meromorphic continuation of the resolvent and of the corresponding S–matrix in the interval Δ, for a Schrödinger operator of the type $-\Delta + V$, the localized Green's function is given by an expression of the form

$$\int_\Delta g_\Delta(k)\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2 - E}f(k)dkd\Omega$$

where f is analytic in a neighborhood of Δ. With the choice that g_Δ behaves like $e^{-1/(k-k_i)}$ near the endpoints k_i of the interval, direct calculation shows that the correction to the exponential behavior of type $e^{-i\omega_* t}$ of $a(t)$ is given by a remainder of the form

$$R_1(t) \sim e^{-2\sqrt{t}}\left(e^{i\theta_1 t}\sum_{k=0}^{\infty}\frac{c_{k;1}}{t^{k/4}} + e^{i\theta_2 t}\sum_{k=0}^{\infty}\frac{c_{k;2}}{t^{k/4}}\right)$$

where the power series are gotten by Watson's Lemma from Laplace integrals of the form $\int_0^a e^{-pt}F_{1,2}(p)dp$ (incomplete Borel summation). For newer results and references in Borel summability see e.g. [19], [20].

Theorem 18 *Under the additional assumptions of this section, the meromorphic continuation of*

$$\chi(H-z)^{-1}\chi,$$

exists for z near λ_0, and has a unique simple pole at ω_*.

The examples covered by the above approach include those discussed in [11] as well as the many cases where analytic continuation has been established, see e.g. [21]. Furthermore, following results of [21] it follows that under favorable assumptions on $V(x)$, $-\Delta + V(x)$ has no zero energy bound states in three or more dimensions extending the results of [11], where it was proved for 5 or more dimensions.

It is worth mentioning that the possible presence of thresholds inside Δ makes it necessary to allow for $\eta < \infty$, and that in the case where there are finitely many thresholds inside Δ of known structure, sharper results may be obtained.

Other applications of our methods involve numerical reconstruction of resonances from time dependent solutions data, in cases Borel summability is ensured. This and other implications will be discussed elsewhere.

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