

On the Dynamics of Crystal Electrons, high Momentum Regime

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Abstract

We study the quantum dynamics generated by $H^{SW} = -\frac{d^2}{dx^2} + V - x$ with V a real periodic function of weak regularity. We prove that the continuous spectrum of H^{SW} is never empty, and furthermore that for V small enough there are no bound states.

1 Introduction

Consider for $\alpha \geq 0$ potentials of the form

$$V : \mathbb{R} \rightarrow \mathbb{R}, V(x + \gamma) = V(x) \quad (x \in \mathbb{R}, \gamma \in 2\pi\mathbb{Z}) \text{ with}$$

$$\|V\|_\alpha^2 := \sum_{n \in \mathbb{Z}} |\widehat{V}(n)|^2 (1 + n^2)^\alpha < \infty$$

$$\text{where } \widehat{V}(n) := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} V(x) dx.$$

The Stark Wannier Hamiltonian is the selfadjoint operator

$$H^{SW} := -\frac{d^2}{dx^2} - x + V$$

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defined in $L^2(\mathbb{R})$ by extension from the core $C_0^\infty(\mathbb{R})$; this is a corollary of the Faris Lavine theorem, see [13].

We shall prove that there are always propagating states and that small potentials with some regularity cannot bind:

Theorem 1.1 (i) *Let $\alpha > 0$ then*

$$\sigma_{cont}(H^{SW}) \neq \emptyset;$$

(ii) *for $\alpha > 1/2$ there is a $c > 0$ such that for $\|V\|_\alpha < c$*

$$\sigma_{pp}(H^{SW}) = \emptyset.$$

This theorem is a consequence of our main result, Theorem 2.1, which asserts that the probability to be accelerated in the future grows with the momentum of the initial state.

Remark that our results are stated in terms of the Stark–Wannier problem but apply as well to the problem of driven quantum rings, see [4].

The dynamics of crystal electrons described by the present model have been studied since [15] both in mathematics and physics literature, see [11] for a review. The general problem is to understand how the reflections at the band edges accumulate to localize the electron or to create resonances; it is far from being settled. Our contribution is to the question: how do spectral properties change when α is diminishing? We refer to [1] for a physical discussion of this theme. Answers for two extreme cases are known: if $\alpha > 5/2$ then the spectrum of H^{SW} is absolutely continuous see [5] and [14, 7] for generalizations; on the other hand there are models (corresponding to $\alpha < 0$) for which the spectrum has no absolute continuous component, [12, 10] or is even pure point [3]. If V is analytic one has certain informations on existence and width of resonances, [9, 2].

The paper is organized as follows: in the next section we state our main result, Theorem 2.1, precisely and infer Theorem 1.1. The third section contains the dynamical information: the proof of Theorem 2.1 organized in several subsections.

2 Spectral properties

Denote the free Stark Hamiltonian by $H_0^{SW} = -\Delta - x$; by $D := -i\partial_x$ the momentum operator which is selfadjoint on $H^1(\mathbb{R})$; χ is the binary function with

values $\chi(True) = 1, \chi(False) = 0$; $\chi(D - t \in [a, b])$ is the cutoff function in Fourier space of the interval $[t + a, t + b]$; *cte.* a generic constant which may change from line to line;

$$\langle n \rangle := \begin{cases} 1 & n \leq 0 \\ n & n > 0 \end{cases} .$$

The main theorem is:

Theorem 2.1 *Let $\alpha > 0, M > 0$. There exists a $c = c_{\alpha, M} > 0$ such that for V with $\|V\|_{\alpha} \leq M$ and for all $t \in \mathbb{R}, n \in \mathbb{Z}$ it holds:*

(i)

$$\|\chi(D - t \in [n, n + 1))(e^{-iH^{SW}t} - e^{-iH_0^{SW}t})\| \leq c_{\alpha, M} \frac{\|V\|_{\alpha}}{\langle \text{sign}(t)n \rangle^{\min\{1, \alpha\}}};$$

(ii) for $\psi \in L^2(\mathbb{R})$ this implies:

$$\|\chi(D - t \in [n, n + 1))e^{-iH^{SW}t}\psi\| \geq \|\chi(D \in [n, n + 1))\psi\| - c_{\alpha, M} \frac{\|V\|_{\alpha}}{\langle \text{sign}(t)n \rangle^{\min\{1, \alpha\}}} \|\psi\|. \quad (1)$$

Remark 2.2 *The dynamical meaning of (ii) is that a large part of a state whose initial momentum is high enough is accelerated, i.e.: For any $\varepsilon > 0$ there is an n such that for $\psi = \chi(D \in [n, n + 1))\psi, \|\psi\| = 1$ it holds*

$$\|\chi(D - t \in [n, n + 1))e^{-iH^{SW}t}\psi\| \geq 1 - \varepsilon \quad (t > 0).$$

We show now that the result on the spectrum follows from this:

Proof of Theorem 1.1. For a state ψ in the pure point spectral subspace of H^{SW} it holds:

$$\lim_{R \rightarrow \infty} \sup_{\pm t \geq 0} \|\chi(D \geq R)e^{-iH^{SW}t}\psi\| = 0,$$

see, for exemple [8]. This implies for $n \in \mathbb{Z}$

$$\lim_{t \rightarrow \pm\infty} \|\chi(D - t \in [n, n + 1))e^{-iH^{SW}t}\psi\| = 0. \quad (2)$$

ad (i): Consider the initial state $\psi = \mathcal{F}^{-1}\chi(p \in [n, n+1))$ where \mathcal{F}^{-1} denotes the inverse unitary Fourier transform. By the inequality (1) it holds for $t > 0, n$ large enough

$$\|\chi(D - t \in [n, n+1))e^{-iH^{SW}t}\psi\| \geq 1 - c \frac{\|V\|_\alpha}{\langle \text{sign}(t)n \rangle^{\min\{1, \alpha\}}} \geq \frac{1}{2}$$

which contradicts (2). So ψ has a part in the continuous subspace of H^{SW} .

ad (ii): For ψ in the pure point subspace take the limits $t \rightarrow \text{sign}(n) \infty$ in (1); the equality (2) implies

$$\|\chi(D \in [n, n+1))\psi\| \leq c_{\alpha, M} \frac{\|V\|_\alpha}{\langle |n| \rangle^{\min\{1, \alpha\}}} \|\psi\| \quad (n \in \mathbb{Z})$$

which leads to

$$\|\psi\|^2 = \sum_{n \in \mathbb{Z}} \|\chi(D \in [n, n+1))\psi\|^2 \leq c_{\alpha, M}^2 \|V\|_\alpha^2 \sum_{n \in \mathbb{Z}} \frac{1}{\langle |n| \rangle^{\min\{2, 2\alpha\}}} \|\psi\|^2$$

which is a contradiction for $\|V\|_\alpha$ small enough and $\alpha > 1/2$; so there are no bound states. \square

3 Dynamics

To prove Theorem 2.1 we decompose the operator H^{SW} in the Bloch representation. Denote by \mathbb{V} the convolution operator $\mathbb{V}\psi(n) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} \widehat{V}(n-m)\psi(m)$ in $L^2(\mathbb{Z})$.

V is real, so it holds: $\overline{\widehat{V}(n)} = \widehat{V}(-n)$. We can always subtract a constant from H^{SW} so we suppose that $\widehat{V}(0) = 0$.

Consider in

$$L^2([0, 1), dk; L^2(\mathbb{Z})) \simeq \int_{[0, 1)}^\oplus L^2(\mathbb{Z}) dk$$

the time dependent operator

$$H(t)\psi(k, n) = (n + k + t)^2\psi(k, n) + \mathbb{V}\psi(k, n).$$

$H_0(t)$ denotes this operator for $V = 0$ which has constant domain $H^2(\mathbb{Z})$. \mathbb{V} is bounded from $H^2(\mathbb{Z})$ to $L^2(\mathbb{Z})$ for $\alpha \geq 0$ so the propagator \mathbb{U} generated by $H(t)$

is well defined in the strong sense. Its relation to the Wannier Stark propagator is:

$$\mathbb{U}(t) = \mathcal{B}e^{-itx}e^{-iH^{SW}t}\mathcal{B}^{-1} \quad (3)$$

where \mathcal{B} is the Fourier-Bloch transformation

$$\mathcal{B}\psi(k, n) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} \left\{ \sum_{\gamma \in 2\pi\mathbb{Z}} e^{-ik(x+\gamma)} \psi(x + \gamma) \right\} dx$$

which maps $L^2(\mathbb{R})$ unitarily onto $L^2([0, 1], dk; L^2(\mathbb{Z}))$.

Remark that $\mathcal{B}\psi(k, n) = \mathcal{F}\psi(k + n)$. We denote the quantities for the case $V = 0$ by a subscript 0.

The following statement is a reformulation of Theorem 2.1 in this representation.

Denote P_n the projection on the the site n in $L^2(\mathbb{Z})$ then:

Theorem 3.1 *For $\alpha > 0, M > 0$ there exists $c = c_{\alpha, M} > 0$ such that for V with $\|V\|_\alpha \leq M$ it holds in $\int_{[0,1]}^\oplus L^2(\mathbb{Z}) dk$ for $t \in \mathbb{R}, n \in \mathbb{Z}$:*

$$\begin{aligned} & \|P_n(\mathbb{U}(t) - \mathbb{U}_0(t))\|_\oplus = \\ & \left\| \int_0^t P_n \mathbb{U}_0^* V \mathbb{U} \right\|_\oplus \leq c_{\alpha, M} \frac{\|V\|_\alpha}{\langle \text{sign}(t)n \rangle^{\min\{1, \alpha\}}}. \end{aligned}$$

We first prove that this is indeed equivalent to Theorem 2.1:

Proof of Theorem 2.1. By the identity (3) it holds:

$$\begin{aligned} & \|\chi(D - t \in [n, n + 1])(e^{-iH^{SW}t} - e^{-iH_0^{SW}t})\|_{L^2(\mathbb{R})} = \\ & \|\mathcal{B}e^{-itx}\chi(D - t \in [n, n + 1])e^{itx}\mathcal{B}^{-1}\mathcal{B}e^{-itx}(e^{-iH^{SW}t} - e^{-iH_0^{SW}t})\mathcal{B}^{-1}\|_\oplus = \\ & \|\chi(\cdot + k \in [n, n + 1])(\mathbb{U}(t) - \mathbb{U}_0(t))\|_\oplus = \\ & \|P_n(\mathbb{U}(t) - \mathbb{U}_0(t))\|_\oplus. \end{aligned}$$

so part (i) is equivalent to Theorem 3.1. To see part (ii) observe that

$$e^{iH_0^{SW}t}\chi(D - t \in [n, n + 1])e^{-iH_0^{SW}t} = \chi(D \in [n, n + 1])$$

so by (i) and the unitarity of $e^{iH_0^{SW}t}$:

$$\begin{aligned} & \|\chi(D - t \in [n, n + 1])e^{-iH^{SW}t}\psi\| \geq \\ & \|\chi(D - t \in [n, n + 1])e^{-iH_0^{SW}t}\psi\| - c_{\alpha, M} \frac{\|V\|_\alpha}{\langle \text{sign}(t)n \rangle^{\min\{1, \alpha\}}} \|\psi\| = \\ & \|\chi(D \in [n, n + 1])\psi\| - c_{\alpha, M} \frac{\|V\|_\alpha}{\langle \text{sign}(t)n \rangle^{\min\{1, \alpha\}}} \|\psi\|. \end{aligned}$$

□

In the rest of the paper we prove Theorem 3.1.

3.1 Proof of Theorem 3.1

$\mathbb{U}(t) = \int^{\oplus} U^k(t) dk$. For $\psi \in H^2(\mathbb{Z})$, $\Omega^k := U_0^{k*} U^k$ it holds

$$\begin{aligned} \|P_n(U^k(t) - U_0^k(t))\psi\|_{L^2(\mathbb{Z})} &= \|P_n(\Omega^k(t) - \mathbb{I})\psi\| \leq \\ &\| \int_0^t P_n U_0^{k*} \nabla U_0^k \Omega^k \psi \| . \end{aligned}$$

Remark that

$$U^k(t, s) = U^0(t + k, s + k)$$

so it is sufficient to estimate

$$\| \int_0^t P_n U_0^{0*} \nabla U_0^0 \Omega^0 \psi \| . \quad (4)$$

We shall give the argument for nonnegative integers n and $t > 0$. For $n < 0, t < 0$ the result then follows from time reversal, i.e.: because for $\mathbb{U}(-t) = T\mathbb{U}(t)T^{-1}$ with $T\psi(n) := \overline{\psi}(-n)$ it holds: $\|P_{-n}(\mathbb{U}(-t) - \mathbb{U}_0(-t))\| = \|P_n(\mathbb{U}(t) - \mathbb{U}_0(t))\|$. For the remaining cases we shall give an argument later.

In the sequel we shall drop the superscript 0. We shall also suppress the states ψ ; the norm estimates below are to be understood as uniform estimates proven on the dense set $H^2(\mathbb{Z})$ and valid by extension in the operator norm.

Rapid oscillations are responsible for the smallness of (4). Remark firstly that with

$$E_n(t) := (n + t)^2 \text{ it holds}$$

$$U_0(t)P_n = e^{-i \int_0^t E_n} P_n,$$

secondly that in every time interval

$$I_l := \frac{l}{2} + [-\frac{1}{4}, \frac{1}{4}) \quad (l \in \mathbb{N})$$

there is exactly one degeneracy, namely

$$E_n\left(\frac{l}{2}\right) = E_{-n-l}\left(\frac{l}{2}\right)$$

corresponding to some point of stationary phase of $e^{-i \int_0^t (E_n - E_{-n-l})}$. Denote

$$\mathbb{P}_{n,l} := P_n + P_{-n-l}, \quad \mathbb{P}_{n,l}^\perp := \mathbb{I} - \mathbb{P}_{n,l}.$$

Heuristically, the contribution

$$\left\| \int_{I_l} P_n U_0^* \mathbb{V} U_0 \Omega \right\| \leq \left\| \int_{I_l} P_n U_0^* \mathbb{V} \mathbb{P}_{n,l} U_0 \Omega \right\| + \left\| \int_{I_l} P_n U_0^* \mathbb{V} \mathbb{P}_{n,l}^\perp U_0 \Omega \right\| \quad (5)$$

to (4) is small for two reasons; loosely speaking: the first term describes the probability that a reflection from momentum n to $-n-l$ takes place, i.e. that the electron behaves adiabatically. This are less likely for n large and \widehat{V} small. The second term describes transitions to the other states which is less probable for n large because the energetic distance to these states grows like $|2n+l|$.

We shall now proceed to the proof according to this intuition. We first treat the reflection to $-n-l$:

Lemma 3.2 *There is a numerical constant $c > 0$, independent of V , such that for all $\alpha < \beta \in I_l$*

$$\left\| \int_\alpha^\beta P_n U_0^* \mathbb{V} U_0 \mathbb{P}_{n,l} \Omega \right\| \leq c \sqrt{1 + \|V\|_0} \frac{|\widehat{V}(2n+l)|}{\sqrt{2n+l}}$$

Proof. We supposed that $\widehat{V}(0) = 0$ so $P_n \mathbb{V} P_n = 0$; with the notation

$$\varphi(t) := \int_0^t (E_n - E_{-n-l}) = \int_0^t (2n+l)(-l+2\tau) d\tau; \quad \mathbb{V}_{n,l} := P_n \mathbb{V} P_{-n-l};$$

we shall estimate

$$\left\| \int_{I_l} e^{i\varphi} \mathbb{V}_{n,l} \Omega \right\|.$$

It will be clear that the reasoning holds uniformly if we integrate only on $[\alpha, \beta] \subset I_l$. This estimate is done by a stationary phase calculation in the spirit of [6].

Decompose for an $a \in (0, 1/2)$

$$I_l = I_l \setminus \left\{ \frac{l}{2} + \left[-\frac{a}{2}, \frac{a}{2}\right] \right\} \cup \left\{ \frac{l}{2} + \left[-\frac{a}{2}, \frac{a}{2}\right] \right\} =: I_l^{NS} \cup I_l^S.$$

Then as $\|\Omega\| = 1$:

$$\left\| \int_{I_l^S} e^{i\varphi} \mathbb{V}_{n,l} \Omega \right\| \leq a \|\mathbb{V}_{n,l}\|.$$

On the other hand an integration by parts of $(\partial_t e^{i\varphi})\mathbb{V}_{n,l}\Omega/i\dot{\varphi}$ yields

$$\left\| \int_{I_i^{NS}} e^{i\varphi} \mathbb{V}_{n,l} \Omega \right\| \leq \left\| \frac{\mathbb{V}_{n,l}}{\dot{\varphi}} \right\|_{\partial I_i^{NS}} + \int_{I_i^{NS}} \left\| \frac{\mathbb{V}_{n,l} \dot{\Omega}}{\dot{\varphi}} \right\| + \left\| \frac{\varphi'' \mathbb{V}_{n,l} \Omega}{\dot{\varphi}^2} \right\|.$$

Now observe that $|\dot{\varphi}(t)| = |2n+l||-l+2t| \geq |2n+l|a$, and that $i\dot{\Omega} = U_0^* \mathbb{V} U_0 \Omega$, so the term on the last expression is smaller than

$$\begin{aligned} & 4 \frac{\|\mathbb{V}_{n,l}\|}{a|2n+l|} + \frac{\|\mathbb{V}_{n,l}\|}{|2n+l|} \int_{I_i^{NS}} \left(\frac{\|P_{-n-l}\mathbb{V}\|}{|l-2t|} + \frac{|2|}{|l-2t|^2} \right) dt \\ & \leq cte. \frac{\|\mathbb{V}_{n,l}\|}{a|2n+l|} (1 + \|V\|_0) \end{aligned}$$

where we have used that $\|P_{-n-l}\mathbb{V}\| \leq \|V\|_0$. Thus for $a \in (0, 1/2)$:

$$\left\| \int_{I_i} P_n U_0^* \mathbb{V} U_0 \mathbb{P}_{n,l} \Omega \right\| \leq a \|\mathbb{V}_{n,l}\| + \frac{1}{a} cte. \frac{\|\mathbb{V}_{n,l}\|}{|2n+l|} (1 + \|V\|_0)$$

The minimum of $a\alpha + \beta/a$ for positive a is $2\sqrt{\alpha\beta}$, thus

$$\left\| \int_{I_i} P_n U_0^* \mathbb{V} \mathbb{P}_{n,l} U_0 \Omega \right\| \leq c \frac{\|\mathbb{V}_{n,l}\|}{\sqrt{|2n+l|}} \sqrt{1 + \|V\|_0}$$

which implies our assertion as $\|\mathbb{V}_{n,l}\| \leq \frac{1}{\sqrt{2\pi}} |\widehat{V}(2n+l)|$. □

The other levels are separated by large gaps. We start the proof that the transition probability to them is small with a double integration by parts lemma.

We have $H_0(t)P_n = E_n(t)P_n$ Denote by

$$\widehat{R}_l(t) := (H_0(t) - E_n(t))^{-1} \mathbb{P}_{n,l}^\perp$$

the reduced resolvent. The Friedrichs twiddle operation is fundamental in adiabatic theories. The version needed here is defined for an operator A on $L^2(\mathbb{Z})$ by

$$\widetilde{A}_l := P_n A \widehat{R}_l.$$

Lemma 3.3 For $\beta > \alpha$ it holds on $H^2(\mathbb{Z})$

$$\begin{aligned} & \int_{\alpha}^{\beta} U_0^* P_n \mathbb{V} \mathbb{P}_{n,l}^{\perp} U_0 \Omega = \\ & \int_{\alpha}^{\beta} U_0^* \left(\widetilde{\mathbb{V}}_l \mathbb{V}_l \mathbb{V} + i \widetilde{\mathbb{V}}_l \dot{\mathbb{V}}_l - \widetilde{\mathbb{V}}_l \mathbb{V} \mathbb{P}_{n,l} - i \dot{\mathbb{V}}_l \right) U_0 \Omega \\ & + i U_0^* \left(\widetilde{\mathbb{V}}_l - \widetilde{\mathbb{V}}_l \mathbb{V}_l \right) U_0 \Omega \Big|_{\alpha}^{\beta} \end{aligned}$$

Proof. The twiddle operation is an inverse commutator. A dot $\dot{}$ or a prime \prime denotes differentiation. It holds:

$$i \partial_t U_0^* \widetilde{\mathbb{V}}_l U_0 = U_0^* ([\widetilde{\mathbb{V}}_l, H_0 - E_n] + i \dot{\widetilde{\mathbb{V}}}_l) U_0 = U_0^* (P_n \mathbb{V} \mathbb{P}_{n,l}^{\perp} + i \dot{\widetilde{\mathbb{V}}}_l) U_0$$

so an integration by parts and $i \partial_t \Omega = U_0^* \mathbb{V} U_0 \Omega$ yield

$$\begin{aligned} & \int_{\alpha}^{\beta} U_0^* P_n \mathbb{V} \mathbb{P}_{n,l}^{\perp} U_0 \Omega = \\ & - \int_{\alpha}^{\beta} U_0^* \left(\widetilde{\mathbb{V}}_l \mathbb{V} + i \dot{\widetilde{\mathbb{V}}}_l \right) U_0 \Omega + i U_0^* \widetilde{\mathbb{V}}_l U_0 \Omega \Big|_{\alpha}^{\beta}. \end{aligned}$$

The decomposition $\widetilde{\mathbb{V}}_l \mathbb{V} = \widetilde{\mathbb{V}}_l \mathbb{V} \mathbb{P}_{n,l} + \widetilde{\mathbb{V}}_l \mathbb{V} \mathbb{P}_{n,l}^{\perp}$, the identity

$$i \partial_t U_0^* \widetilde{\mathbb{V}}_l \mathbb{V}_l U_0 = U_0^* (\widetilde{\mathbb{V}}_l \mathbb{V} \mathbb{P}_{n,l}^{\perp} + i \dot{\widetilde{\mathbb{V}}}_l \mathbb{V}_l) U_0$$

and a second integration by parts imply

$$\begin{aligned} & - \int_{\alpha}^{\beta} U_0^* \widetilde{\mathbb{V}}_l \mathbb{V}_l \mathbb{P}_{n,l}^{\perp} U_0 \Omega = \\ & \int_{\alpha}^{\beta} U_0^* \left(\widetilde{\mathbb{V}}_l \mathbb{V}_l \mathbb{V} + i \dot{\widetilde{\mathbb{V}}}_l \mathbb{V}_l \right) U_0 \Omega - i U_0^* \widetilde{\mathbb{V}}_l \mathbb{V}_l U_0 \Omega \Big|_{\alpha}^{\beta}. \end{aligned}$$

Thus the assertion is proved. \square

Concerning the second contribution to (5) we shall now proceed to estimate the different terms of

$$\left\| \int_{I_i} P_n U_0^* \mathbb{V} \mathbb{P}_{n,l}^{\perp} U_0 \Omega \right\| \quad (6)$$

defined by Lemma 3.3 one after the other.

In the following lemma we collect facts that shall be used frequently and often without comment:

Lemma 3.4 Let $a \neq b \in \mathbb{Z}$. For $\alpha, \beta > 0$ there is a $cte. > 0$ such that it holds

$$\sup_{j \in \mathbb{Z} \setminus \{a, b\}} \frac{1}{|j-a|^\alpha |j-b|^\beta} = \sup_{j < \frac{a+b}{2}} \dots + \sup_{j \geq \frac{a+b}{2}} \dots \leq cte. \frac{1}{|a-b|^{\min\{\alpha, \beta\}}};$$

and for $\alpha, \beta > 1$:

$$\sum_{j \in \mathbb{Z} \setminus \{a, b\}} \frac{1}{|j-a|^\alpha} \frac{1}{|j-b|^\beta} \leq \sum_{j < \frac{a+b}{2}} \dots + \sum_{j \geq \frac{a+b}{2}} \dots \leq cte. \frac{1}{|a-b|^{\min\{\alpha, \beta\}}};$$

let A be the operator on $L^2(\mathbb{Z})$ whose kernel is $A(i, j) = f(i)g(i-j)$ for some $f, g \in L^2(\mathbb{Z})$, then

$$\|A\| = \sup_{\|\varphi\|=\|\psi\|=1} |\langle \varphi, A\psi \rangle| \leq \|f\|_{L^2(\mathbb{Z})} \|g\|_{L^2(\mathbb{Z})}.$$

The smallness of the terms in Lemma (3.3) results from the presence of the reduced resolvent; we shall use that for $m \neq n, m \neq -n-l, t \in I_l$ it holds as $|m+n+l| \geq 1$ and so $|m+n+l+\alpha| \geq \frac{1}{2}|m+n+l|$ for $\alpha < \frac{1}{2}$:

$$\inf_{t \in I_l} |E_m(t) - E_n(t)| = \inf_{t \in I_l} |(m-n)(m+n+2t)| \geq \frac{1}{2}|m-n||m+n+l|$$

and as in Lemma (3.4):

$$\inf_{m \in \mathbb{Z} \setminus \{n, -n-l\}} \inf_{t \in I_l} |E_m(t) - E_n(t)| \geq cte. |n - (-n-l)| = cte. |2n+l|.$$

The first relevant term in the integrand of Lemma (3.3) is $\widetilde{\mathbb{V}}_l \mathbb{V}_l \mathbb{V} = P_n \mathbb{V} \widehat{R}_l \mathbb{V} \widehat{R}_l \mathbb{V}$. Now

$$P_n \mathbb{V}(i, j) = \delta_{ni} \widehat{V}(n-j); \quad |\widehat{R}_l \mathbb{V}(i, j)| = \chi(i \neq n) \chi(i \neq -n-l) \frac{\widehat{V}(i-j)}{E_i - E_n}.$$

By the third point of Lemma (3.4) we get

$$\|\widehat{R}_l \mathbb{V}\| \leq \|V\|_0 \left(\sum_{\mathbb{Z} \setminus \{n, -n-l\}} \left| \frac{2}{(i-n)(i+n+l)} \right|^2 \right)^{1/2} \leq cte. \|V\|_0 \frac{1}{|2n+l|}$$

so

$$\sup_{t \in I_l} \|\widetilde{\mathbb{V}}_l \mathbb{V}_l \mathbb{V}\| \leq \sup_{I_l} \|P_n \mathbb{V}\| \|\widehat{R}_l \mathbb{V}\|^2 \leq cte. \|V\|_0^3 \frac{1}{|2n+l|^2}. \quad (7)$$

For the second term it holds

$$\|\widehat{\mathbb{V}}_l \dot{\mathbb{V}}_l\| = \|(P_n \mathbb{V} \widehat{R}_l \mathbb{V} \widehat{R}_l)'\| \leq 2\|\mathbb{V} \widehat{R}_l\| \|\mathbb{V} \widehat{R}_l\|$$

so for $j \in \{n, -n-l\}$:

$$\sqrt{2\pi} \mathbb{V} \widehat{R}_l(i, j) = \frac{\widehat{V}(i-j)(\dot{E}_j - \dot{E}_n)}{(E_j - E_n)^2} = \frac{2\widehat{V}(i-j)}{(j-n)(j+n+2t)^2}$$

so

$$\|\mathbb{V} \widehat{R}_l\| \leq cte. \|V\|_0 \frac{1}{|2n+l|}$$

and together with the estimate of $\mathbb{V} \widehat{R}_l$ above it results:

$$\sup_{t \in I_l} \|\widehat{\mathbb{V}}_l \dot{\mathbb{V}}_l\| \leq cte. \|V\|_0^2 \frac{1}{|2n+l|^2}. \quad (8)$$

Next we discuss $\dot{\mathbb{V}}_l = P_n \mathbb{V} \widehat{R}_l$.

$$\sqrt{2\pi} \dot{\mathbb{V}}_l(i, j) = 2\delta_{in} \frac{\widehat{V}(i-j)}{(j-n)(j+n+2t)^2} \frac{|i-j|^\alpha}{|i-j|^\alpha}$$

so:

$$\sup_I \|\dot{\mathbb{V}}_l\| \leq cte. \|V\|_\alpha \frac{1}{|2n+l|^{\min\{1+\alpha, 2\}}}. \quad (9)$$

The next contribution to (6) comes from

$$\widetilde{\mathbb{V}}_l \mathbb{V} P_{n,l} = P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_n + P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_{-n-l}.$$

The kernel of the second term is

$$2\pi P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_{-n-l}(i, j) = \delta_{in} \delta_{j(-n-l)} \sum_{\mathbb{Z} \setminus \{n, -n-l\}} \frac{\widehat{V}(n-m) \widehat{V}(m+n+l)}{(m-n)(m+n+2t)}$$

so

$$\begin{aligned} & \sup_{t \in I_l} \|P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_{-n-l}\| \\ & \leq cte. \sum_{\mathbb{Z} \setminus \{n, -n-l\}} \left| \frac{\widehat{V}(n-m) \widehat{V}(m+n+l)}{(m-n)(m+n+2t)} \right| \frac{|m-n|^\alpha |m+n+l|^\alpha}{|m-n|^\alpha |m+n+l|^\alpha} \\ & \leq cte. \|V\|_\alpha^2 \frac{1}{|2n+l|^{1+\alpha}} \end{aligned} \quad (10)$$

by the Cauchy-Schwartz inequality and Lemma (3.4).

The other term is more difficult, it is the part which is first scattered out of the state n , then back to it:

$$2\pi P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_n(i, j) = \delta_{in} \delta_{jn} \sum_{\mathbb{Z} \setminus \{n, -n-l\}} \frac{|\widehat{V}(n-m)|^2}{(m-n)(m+n+2t)}.$$

Changing coordinates we obtain

$$\begin{aligned} 2\pi P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_n(i, j) &= \delta_{in} \delta_{jn} \sum_{p \in \mathbb{Z} \setminus \{0, -(2n+l)\}} \frac{|\widehat{V}(p)|^2}{p(p+2n+2t)} = \\ &= \frac{|\widehat{V}(2n+l)|^2}{(2n+l)(4n+l+2t)} + \delta_{in} \delta_{jn} \sum_{p \in \mathbb{Z} \setminus \{0, \pm(2n+l)\}} \frac{|\widehat{V}(p)|^2}{p(p+2n+2t)} = \\ &= \frac{|\widehat{V}(2n+l)|^2}{(2n+l)(4n+l+2t)} + \delta_{in} \delta_{jn} \sum_{p \in \mathbb{Z} \setminus \{0, \pm(2n+l)\}} - \frac{|\widehat{V}(p)|^2}{p^2 - (2n+2t)^2} \end{aligned}$$

because only $-\frac{|\widehat{V}(p)|^2}{p^2 - (2n+2t)^2}$, the symmetric part of $\frac{|\widehat{V}(p)|^2}{p(p+2n+2t)}$, contributes to the symmetric sum. Physically speaking it is destructive interference of the contributions of the states $m = n \pm p$ which is at work here.

Now for $\alpha > 0$

$$\begin{aligned} \sup_{t \in I_t} \sum_{\mathbb{Z} \setminus \{0, \pm(2n+l)\}} \frac{|\widehat{V}(p)|^2}{|p^2 - (2n+2t)^2|} \frac{p^{2\alpha}}{p^{2\alpha}} &\leq \\ \|V\|_\alpha^2 \sup_{p, t} \frac{1}{|p^{2\alpha}| |p^2 - (2n+2t)^2|} &\leq \|V\|_\alpha^2 \frac{cte.}{|2n+l|^{\min\{1+2\alpha, 2\}}}; \end{aligned}$$

to see this recall that for $p \neq \pm(2n+l)$

$$\inf_{t \in I_t} |p^2 - (2n+t)^2| \geq \frac{1}{4} |p^2 - (2n+l)^2|,$$

and take the supremum over $|p| < \frac{2n+l}{2}$ and $\pm p \geq \frac{|2n+l|}{2}$ separately.

Furthermore

$$\frac{|\widehat{V}(2n+l)|^2}{|(2n+l)(2n+l+2n+2t)|} \leq \|V\|_0^2 \frac{cte.}{|2n+l|^2},$$

so the estimate for the backscattering term is

$$\begin{aligned} \sup_{t \in I_l} \|P_n \mathbb{V} \widehat{R}_l \mathbb{V} P_n\| &\leq cte. \sup_{t \in I_l} \left| \sum_{\mathbb{Z} \setminus \{n, -n-l\}} \frac{|\widehat{V}(n-m)|^2}{(m-n)(m+n+2t)} \right| \\ &\leq cte. \|V\|_\alpha^2 \frac{1}{|2n+l|^{\min\{1+2\alpha, 2\}}}. \end{aligned} \quad (11)$$

We are left with the boundary terms in (6). We first discuss

$$\begin{aligned} &\widetilde{\mathbb{V}}_l \mathbb{V}_l \left(t = \frac{l}{2} \pm \frac{1}{4} \right). \\ &\widetilde{\mathbb{V}}_l \mathbb{V}_l = P_n \mathbb{V}_l \widehat{R}_l \mathbb{V}_l \widehat{R}_l, \text{ so} \\ &\|\widetilde{\mathbb{V}}_l \mathbb{V}_l(\frac{l}{2} \pm \frac{1}{4})\| \leq \|\mathbb{V}_l \widehat{R}_l(\frac{l}{2} \pm \frac{1}{4})\|^2 \leq cte. \|V\|_0^2 \frac{1}{|2n+l|^2} \end{aligned} \quad (12)$$

where we used the estimate which led to (7).

For the other boundary term we have to be more careful: Consider for $t_{l+1} := \frac{l}{2} + \frac{1}{4}$

$$\begin{aligned} &\sqrt{2\pi} \left(\widetilde{\mathbb{V}}_l(t_{l+1}) - \widetilde{\mathbb{V}}_{l+1}(t_{l+1}) \right) (i, j) = \\ &P_n \mathbb{V} (H - E_n)^{-1}(t_{l+1}) (P_{-n-(l+1)} - P_{-n-l})(i, j) = \\ &\delta_{in} \delta_{j(-n-l)} \frac{\widehat{V}(2n+l)}{3/2(2n+l+1)} - \delta_{j(-n-(l+1))} \frac{\widehat{V}(2n+l+1)}{1/2(2n+l)}. \end{aligned}$$

So

$$\|\widetilde{\mathbb{V}}_l(t_{l+1}) - \widetilde{\mathbb{V}}_{l+1}(t_{l+1})\| \leq cte. \left(\frac{|\widehat{V}(2n+l)|}{|2n+l|} + \frac{|\widehat{V}(2n+l+1)|}{|2n+l+1|} \right). \quad (13)$$

Furthermore it holds for $x \in [-\frac{1}{4}, \frac{1}{4}]$:

$$\sqrt{2\pi} \widetilde{\mathbb{V}}_l \left(\frac{l}{2} + x \right) (i, j) = \delta_{in} \frac{\widehat{V}(n-j)}{(j-n)(j+n+l+2x)} \chi(j \neq n) \chi(j \neq -n-l),$$

so for $\alpha > 0$

$$\begin{aligned} \|\widetilde{\mathbb{V}}_l(\frac{l}{2} + x)\|^2 &\leq cte. \sum_{\mathbb{Z} \setminus \{n, -n-l\}} \frac{|\widehat{V}(n-j)|^2}{|(j-n)(j+n+l+2x)|^2} \\ &\leq \|V\|_0^2 \frac{cte.}{|2n+l|^2}. \end{aligned} \quad (14)$$

With these observations we have finished the proof of Theorem (2.1). We assemble the argument.

We have

$$\begin{aligned}
& \left\| \int_0^t P_n U_0^* \mathbb{V} U \right\|_{L^2(\mathbb{Z})} = \\
& \left\| \int_0^{\frac{1}{4}} \dots \right\| + \sum_{l=1}^{L-1} \left\| \int_{I_l} \dots \right\| + \left\| \int_{\frac{L}{2}-\frac{1}{4}}^t \dots \right\| \leq \\
& \sum_{l=0}^{\infty} \left\| \int_{I_l} \dots \right\| \leq \\
& \sum_{l=0}^{\infty} \left\| \int_{I_l} \dots \mathbb{P}_{n,l} \right\| + \sum_{l=0}^{\infty} \left\| \int_{I_l} \dots \mathbb{P}_{n,l}^\perp \right\|
\end{aligned}$$

where L is the integer such that $t \in [\frac{L}{2} - \frac{1}{4}, \frac{L}{2} + \frac{1}{4})$; $I_0 := [0, \frac{1}{4})$; for the index L we have redefined $I_L := [\frac{L}{2} - \frac{1}{4}, t)$.

Remark that for $n > 0, \beta > 1$

$$\sum_{l=0}^{\infty} \frac{1}{\langle 2n+l \rangle^\beta} \leq \frac{cte.}{\langle n \rangle^{\beta-1}}.$$

So by Lemma (3.2) it holds

$$\begin{aligned}
& \sum_{l=0}^{\infty} \left\| \int_{I_l} \dots \mathbb{P}_{n,l} \right\| \leq cte. \sqrt{1 + \|V\|_0} \sum \frac{|\widehat{V}(2n+l)|}{\sqrt{\langle 2n+l \rangle}} \\
& \leq cte. \sqrt{1 + \|V\|_0} \|V\|_\alpha \left(\sum \frac{1}{\langle 2n+l \rangle^{2\alpha+1}} \right)^{1/2} \\
& \leq cte. \|V\|_\alpha \frac{1}{\langle n \rangle^\alpha}.
\end{aligned}$$

By Lemma (3.3) and the estimates (7, 8, 9, 10, 11, 12, 13, 14) we have

$$\begin{aligned}
& \sum_{l=0}^{\infty} \left\| \int_{I_l} \dots \mathbb{P}_{n,l}^\perp \right\| \leq \\
& \sum \sup_{I_l} \left(\|\widetilde{\mathbb{V}}_l \mathbb{V}_l \mathbb{V}\| + \|\widetilde{\mathbb{V}}_l \dot{\mathbb{V}}_l\| + \|\widetilde{\mathbb{V}}_l \mathbb{V} \mathbb{P}_{n,l}\| + \|\dot{\mathbb{V}}_l\| \right) + \\
& \sum \left(\|\widetilde{\mathbb{V}}_l \mathbb{V}_l(t_{l+1})\| + \|\widetilde{\mathbb{V}}_l \mathbb{V}_l(t_l)\| \right) +
\end{aligned}$$

$$\sum \left(\|\tilde{V}_l(t_{l+1}) - \tilde{V}_{l+1}(t_{l+1})\| \right) + \|\tilde{V}_0(t_0)\| + \|\tilde{V}_L(t_L)\|$$

$$\leq c_{\alpha, M} \|V\|_{\alpha} \frac{1}{\langle n \rangle^{\min\{1, \alpha\}}}.$$

This proves the case $\text{sign}(nt) > 0$. From our calculations it is clear that for $\text{sign}(nt) < 0$ all the estimates give a bound proportional to $\|V_{\alpha}\|$ and no decay in n . Thus the proof of Theorem (2.1) is finished. \square

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